Calculations for the second-degree zonal harmonic

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2018-08-18

Notations:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, |\boldsymbol{z}| = 1, r = |\boldsymbol{r}|.$$

For oblateness along the z-axis, the potential is

$$\frac{J_2}{2r^5}(3z^2 - r^2).$$

For oblateness along $\boldsymbol{z}$,

$$U(\boldsymbol{r}) = \frac{J_2}{2r^5}(3(\boldsymbol{r}\cdot\boldsymbol{z})^2 - r^2).$$

Differentiating,

$$\frac{\mathrm{d}\,U}{\mathrm{d}\,\boldsymbol{r}} = \frac{J_2}{2}\left(-\frac{5}{r^6}\frac{\mathrm{d}\,r}{\mathrm{d}\,\boldsymbol{r}}(3(\boldsymbol{r}\cdot\boldsymbol{z})^2 - r^2) + \frac{1}{r^5}\left(3\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{z})^2 - 2r\frac{\mathrm{d}\,r}{\mathrm{d}\,\boldsymbol{r}}\right)\right).$$

Recall that $\frac{\mathrm{d}\,r}{\mathrm{d}\,\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$,

$$\begin{aligned}\frac{\mathrm{d}\,U}{\mathrm{d}\,\boldsymbol{r}} &= \frac{J_2}{2}\left(-\frac{5\boldsymbol{r}}{r^7}(3(\boldsymbol{r}\cdot\boldsymbol{z})^2 - r^2) + \frac{1}{r^5}\left(3\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{z})^2 - 2\boldsymbol{r}\right)\right) \\ &= \frac{J_2}{2}\left(-\frac{15\boldsymbol{r}}{r^7}(\boldsymbol{r}\cdot\boldsymbol{z})^2 + \frac{3\boldsymbol{r}}{r^5} + \frac{3}{r^5}3\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{z})^2\right).\end{aligned}$$

With $\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{z})^2 = 2(\boldsymbol{r}\cdot\boldsymbol{z})\frac{\mathrm{d}\,(\boldsymbol{r}\cdot\boldsymbol{z})}{\mathrm{d}\,\boldsymbol{r}} = 2(\boldsymbol{r}\cdot\boldsymbol{z})\boldsymbol{z}$,

$$\begin{aligned}\frac{\mathrm{d}\,U}{\mathrm{d}\,\boldsymbol{r}} &= \frac{J_2}{2}\left(-\frac{15\boldsymbol{r}}{r^7}(\boldsymbol{r}\cdot\boldsymbol{z})^2 + \frac{3\boldsymbol{r}}{r^5} + \frac{6\boldsymbol{z}}{r^5}(\boldsymbol{r}\cdot\boldsymbol{z})\right) \\ &= \frac{3J_2}{2r^5}\left(2\boldsymbol{z}(\boldsymbol{r}\cdot\boldsymbol{z}) + \boldsymbol{r}\left(1 - \frac{5(\boldsymbol{r}\cdot\boldsymbol{z})^2}{r^2}\right)\right).\end{aligned}$$

Note that this is invariant under $\boldsymbol{z} \mapsto -\boldsymbol{z}$ (but not under $\boldsymbol{r} \mapsto -\boldsymbol{r}$).