Celestial double ledger accounting

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1 Introduction

Consider a rocket with 9 000 m/s of vacuum Δv, *i.e.*, capable of reaching 9 000 m/s from a standstill in a vacuum. Launch it from Earth; it starts out going at 400 m/s eastward, yet it runs out of fuel in orbit going 7 800 m/s. What happened to 1 400 m/s? Consider another rocket with 12 000 m/s of vacuum Δv, launch it from Earth, it runs out of fuel reaching the same orbit as the other one. What happened to another 3 000 m/s?

The vacuum Δv of a rocket is

$$\Delta v_{\text{vac}} = \log \frac{m_{\text{final}}}{m_{\text{initial}}} I_{\text{sp vac}} = \int_{m_{\text{initial}}}^{m_{\text{final}}} \frac{I_{\text{sp vac}}}{m} \,\mathrm{d}\, m, \tag{1.1}$$

where we use specific impulse *stricto sensu*, impulse divided by mass[1], whose SI unit is the Newton-second per kilogram.

2 Losses

Consider a rocket of time-varying mass m with velocity $\boldsymbol{v}$, subject to an overall force $\boldsymbol{F}$. The rate of change of its speed is

$$\frac{\mathrm{d}\, v}{\mathrm{d}\, t} = \frac{\mathrm{d}}{\mathrm{d}\, t}\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \frac{\dot{\boldsymbol{v}}\cdot\boldsymbol{v}}{\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}} = \dot{\boldsymbol{v}}\cdot\hat{\boldsymbol{v}} = \frac{\boldsymbol{F}}{m}\cdot\hat{\boldsymbol{v}}. \tag{2.1}$$

Splitting up the force $\boldsymbol{F}$, working in an inertial reference frame so that we do not get fictitious forces, we get

$$\frac{\mathrm{d}\, v}{\mathrm{d}\, t} = \frac{\boldsymbol{F}_{\text{thrust}}}{m}\cdot\hat{\boldsymbol{v}} + \frac{\boldsymbol{F}_{\text{aerodynamic}}}{m}\cdot\hat{\boldsymbol{v}} + \frac{\boldsymbol{F}_{\text{gravitational}}}{m}\cdot\hat{\boldsymbol{v}}. \tag{2.2}$$

Aerodynamic forces along the direction of travel have a name, and they really go *against* the direction of travel:

$$\frac{\mathrm{d}\, v}{\mathrm{d}\, t} = \frac{\boldsymbol{F}_{\text{thrust}}}{m}\cdot\hat{\boldsymbol{v}} - \frac{F_{\text{drag}}}{m} + \frac{\boldsymbol{F}_{\text{gravitational}}}{m}\cdot\hat{\boldsymbol{v}}.$$

Only the thrust along the direction of travel contributes to the increase in velocity:

$$\frac{\mathrm{d}\, v}{\mathrm{d}\, t} = \frac{F_{\text{thrust}}}{m} - \frac{F_{\text{thrust}}}{m}\left(1 - \hat{\boldsymbol{F}}_{\text{thrust}}\cdot\hat{\boldsymbol{v}}\right) - \frac{F_{\text{drag}}}{m} + \frac{\boldsymbol{F}_{\text{gravitational}}}{m}\cdot\hat{\boldsymbol{v}}.$$

The thrust of a rocket depends on the current specific impulse and mass flow:

$$\frac{\mathrm{d}\, v}{\mathrm{d}\, t} = \frac{I_{\text{sp}}}{m}\dot{m} - \frac{F_{\text{thrust}}}{m}\left(1 - \hat{\boldsymbol{F}}_{\text{thrust}}\cdot\hat{\boldsymbol{v}}\right) - \frac{F_{\text{drag}}}{m} + \frac{\boldsymbol{F}_{\text{gravitational}}}{m}\cdot\hat{\boldsymbol{v}}.$$

[1] For specific impulse with a dimension of time, multiply it by standard gravity to give it the dimension of impulse divided by mass; we do not involve standard gravity in the equations here as it would only serve to obscure the physics.

In turn the specific impulse diminishes with atmospheric pressure:

$$\frac{\mathrm{d}\,v}{\mathrm{d}\,t} = \frac{I_{\text{sp vac}}}{m}\dot{m} - \frac{\Delta I_{\text{sp}}(P)}{m}\dot{m} - \frac{F_{\text{thrust}}}{m}\left(1 - \hat{\boldsymbol{F}}_{\text{thrust}} \cdot \hat{\boldsymbol{v}}\right) - \frac{F_{\text{drag}}}{m} + \frac{\boldsymbol{F}_{\text{gravitational}}}{m} \cdot \hat{\boldsymbol{v}}.$$

Compare with (1.1) and observe that the first term here is the rate of consumption of vacuum Δv. Integrating over the ascent, we get the relation between the actual change in speed and the expended vacuum Δv:

$$\begin{aligned}\Delta v_{\text{actual}} = \Delta v_{\text{vac}} &- \underbrace{\int \Delta I_{\text{sp}}(P)\frac{\mathrm{d}\,m}{m}}_{\text{backpressure losses}} - \underbrace{\int \frac{F_{\text{thrust}}}{m}\left(1 - \hat{\boldsymbol{F}}_{\text{thrust}} \cdot \hat{\boldsymbol{v}}\right)\mathrm{d}\,t}_{\text{steering losses}} \\ &- \underbrace{\int \frac{F_{\text{drag}}}{m}\,\mathrm{d}\,t}_{\text{drag losses}} + \underbrace{\int \frac{\boldsymbol{F}_{\text{gravitational}}}{m} \cdot \hat{\boldsymbol{v}}\,\mathrm{d}\,t}_{\text{gravity losses}}. \end{aligned} \tag{2.3}$$

The terms that account for the difference between the actual change in speed and the expended vacuum Δv are termed *losses*. The names *steering*, *drag*, and *gravity* losses are from [TODO CITE].

Backpressure and drag losses are reasonably self-explanatory; in order to better understand the other two terms, we will consider the case where they vanish, *i.e.*, a vacuum, in the remainder of this document:

$$\Delta v_{\text{actual}} = \Delta v_{\text{vac}} - \underbrace{\int \frac{F_{\text{thrust}}}{m}\left(1 - \hat{\boldsymbol{F}}_{\text{thrust}} \cdot \hat{\boldsymbol{v}}\right)\mathrm{d}\,t}_{\text{steering losses}} + \underbrace{\int \frac{\boldsymbol{F}_{\text{gravitational}}}{m} \cdot \hat{\boldsymbol{v}}\,\mathrm{d}\,t}_{\text{gravity losses}}. \tag{2.4}$$

3 Work and Oberth

While equation (2.3) accounts for the difference described in Section 1, not all terms represent actual losses. In particular, in free fall, gravity losses accrue when going up, but are recouped when going down; they cycle over the course of a revolution on an eccentric orbit.

Consider the specific kinetic energy $T = \frac{v^2}{2}$. Its rate of change is

$$\dot{T} = \frac{\mathrm{d}\,v}{\mathrm{d}\,t}v = \frac{\boldsymbol{F}}{m} \cdot \boldsymbol{v}, \tag{3.1}$$

the specific work done by $\boldsymbol{F}$, where we have used (2.1). Thus each term in (2.2) corresponds to the work done by one of the forces. In particular, steering losses correspond to thrust not doing work, *i.e.*, not contributing to the orbital energy.

The interpretation of gravity losses is more complex; gravity preserves the sum of the specific kinetic energy and the specific gravitational potential energy: work done by gravity is gained as potential energy. We could split and integrate (3.1) like (2.1) to get

$$\Delta T = \underbrace{\int \frac{F_{\text{thrust}}}{m}\,\mathrm{d}\,s}_{E_{\text{thrust}}} - \underbrace{\int \frac{F_{\text{thrust}}}{m}\left(1 - \hat{\boldsymbol{F}}_{\text{thrust}} \cdot \mathrm{d}\,\boldsymbol{s}\right)}_{E_{\text{thrust}} - W_{\text{thrust}}} + \Delta V_{\text{gravitational}}. \tag{3.2}$$

There is however an important difference between equations (3.2) and (2.4). The energy E_{thrust} is not a fixed property of the rocket like the Δv budget; it is path dependent, as the same Δv applied at greater speed results in a greater gain in kinetic energy; thus in order to maximize the gain in orbital energy, beyond thrusting along the direction of travel, one needs to maximize the speed at which thrust is applied: this is the Oberth effect [TODO CITE Oberth Ch. 12].

Conversely, whereas $\Delta V_{\text{gravitational}}$ is path-independent in (3.2), gravity losses are path-dependent, *i.e.*, they describe some property of the ascent profile. Specifically, gravity losses are lessened if the rocket has a greater speed (in any direction) when rising in the potential. In turn this means that this speed should have been gained lower in the potential, thus while going faster: this is a roundabout expression of the Oberth effect.

Summary:

1. steering losses are Δv expended not increasing the orbital energy;
2. gravity losses are a combination of:
   (a) Δv turned into gravitational potential energy (thus they can never be 0, even with a horizontal instant impulse, so long as the orbit is above the ground),
   (b) an expression of the Oberth effect (so they depend on the ascent profile).

[TODO(egg): Can we split gravity losses between these potential and Oberth parts?]

4 Thrust up bad, thrust sideways good

KSP players have a strong intuition that thrust expended upward is wasted.

There are cases where upward thrust is related to the losses described in preceding section; for instance, at launch on a rotating planet, thrust upward is pure steering losses, being orthogonal to the direction of motion. With low thrust, thrusting upward also causes the rocket to rise in the potential with a lower speed than thrusting sideways, so it causes gravity losses; but this is a fairly high order effect. Indeed, if one considers a launch with instant impulse from a non-rotating planet, there are no steering losses and the impulse takes full advantage of the Oberth effect; yet in a sense that impulse is still wasted, in that it will not help the vessel get into orbit.

The intuition that thrust should be sideways is instead related to the other invariant of the orbit, angular momentum: only horizontal thrust contributes to it. An accounting of horizontal speed similar to (2.3) may be used to express that.

Let $\hat{\boldsymbol{z}}$ be the unit zenithal vector. Vertical speed is $v_z \coloneqq \boldsymbol{v} \cdot \hat{\boldsymbol{z}}$. Horizontal velocity is $\boldsymbol{v}_{xy} \coloneqq \boldsymbol{v} - v_z \hat{\boldsymbol{z}}$, horizontal speed v_{xy} is its norm. The derivative of the unit zenithal vector is

$$\frac{\mathrm{d}\,\hat{\boldsymbol{z}}}{\mathrm{d}\,t} = \frac{\boldsymbol{v}_{xy}}{r},$$

where r is the distance from the centre of the planet. For the derivative of horizontal speed, we have

$$\begin{aligned}\frac{\mathrm{d}\,v_{xy}}{\mathrm{d}\,t} = \frac{\boldsymbol{v}_{xy} \cdot \frac{\mathrm{d}\,\boldsymbol{v}_{xy}}{\mathrm{d}\,t}}{v_{xy}} &= \hat{\boldsymbol{v}}_{xy} \cdot \left(\dot{\boldsymbol{v}} - \hat{\boldsymbol{z}} \frac{\mathrm{d}\,v_z}{\mathrm{d}\,t} - v_z \frac{\mathrm{d}\,\hat{\boldsymbol{z}}}{\mathrm{d}\,t}\right) \\ &= \hat{\boldsymbol{v}}_{xy} \cdot \left(\dot{\boldsymbol{v}} - v_z \frac{\mathrm{d}\,\hat{\boldsymbol{z}}}{\mathrm{d}\,t}\right)\end{aligned}$$

where we have used the orthogonality of $\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{v}}_{xy}$,

$$= \hat{\boldsymbol{v}}_{xy} \cdot \dot{\boldsymbol{v}} - \frac{v_z v_{xy}}{r}$$

[TODO(egg): Finish this.]