

# On an Algorithm by Fukushima

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This document describes an algorithm used by Fukushima in his implementation of the complete elliptic integrals of the second kind  $B$  and  $D$  ([Fuk18]). It follows the notation and conventions of [Fuk11a], but effectively replaces section 2.2.

## Definitions

Jacobi's nome  $q(m)$  is defined as a function of the elliptic integral  $K(m)$  as:

$$q(m) := \exp\left(\frac{-\pi K(1-m)}{K(m)}\right)$$

Changing the variable to be  $m_c = 1 - m$  and solving for  $K(m)$  yields:

$$K(m) = \left(\frac{K(m_c)}{\pi}\right)(-\log q(m_c))$$

Now let's split this expression in two terms to separate out the logarithmic part:

$$\begin{cases} X(m_c) & := -\log q(m_c) \\ K_X(m_c) & := \frac{K(m_c)}{\pi} \end{cases}$$

We have:  $K(m) = K_X(m_c)X(m_c)$ .

An expression for  $E(m)$  can be obtained from Legendre's relation ([Fuk11a] equation 2.11):

$$E(m) = \left(1 - \frac{E(m_c)}{K(m_c)}\right)K(m) + \frac{\pi}{2K(m_c)}$$

Similarly, let's split this expression using the terms:

$$\begin{cases} E_X(m_c) & := \left(1 - \frac{E(m_c)}{K(m_c)}\right)K_X(m_c) \\ E_0(m_c) & := \frac{1}{2K_X(m_c)} \end{cases}$$

We have:  $E(m) = E_X(m_c)X(m_c) + E_0(m_c)$ .

## Integrals of the second kind for $m$ close to 1

Fukushima defines  $B(m)$  as:

$$B(m) := \frac{E(m) - m_c K(m)}{m}$$

This expression can be rewritten using the definitions above:

$$\begin{aligned} B(m) &= \frac{1}{m}(E_X(m_c)X(m_c) + E_0(m_c) - m_c K_X(m_c)X(m_c)) \\ &= \frac{1}{m}\left(X(m_c)(E_X(m_c) - m_c K_X(m_c)) + \frac{1}{2K_X(m_c)}\right) \end{aligned}$$

Similarly the definition of  $D(m)$ :

$$D(m) := \frac{K(m) - E(m)}{m}$$

can be rewritten as:

$$\begin{aligned} D(m) &= \frac{1}{m} (K_X(m_c)X(m_c) - E_X(m_c)X(m_c) - E_0(m_c)) \\ &= \frac{1}{m} \left( X(m_c)(K_X(m_c) - E_X(m_c)) - \frac{1}{2K_X(m_c)} \right) \end{aligned}$$

These formulæ provide a means to compute  $B(m)$  and  $D(m)$  for  $m$  close to 1. First, a polynomial approximation of  $q(m_c)$  is computed, whose leading term is of order  $m_c/16$ . Then the log of that approximation is evaluated, yielding  $X(m_c)$  (this is the part that carries the logarithmic singularity). Finally,  $E_X(m_c)$  and  $K_X(m_c)$  are computed using Taylor or Maclaurin approximations.

It is easy to see that  $B(m) = X(m_c)K_X(m_c) - D(m)$ , which provides a simpler formula for computing  $B(m)$  once  $D(m)$  is known.

### Integrals of the second kind when $m$ tends towards 1

We are now interested in computing the leading term of  $B(m)$  and  $D(m)$  when  $m \rightarrow 1$ . First, we have  $B(1) = 1$  ([Fuk11a], equation 1.9). However,  $D(m) \rightarrow +\infty$  when  $m \rightarrow 1$ . To deal with this singularity we recall that:

$$B(m) = X(m_c)K_X(m_c) - D(m)$$

thus:

$$D(m) = X(m_c)K_X(m_c) - B(m)$$

Remember that  $X(m_c) = -\log q(m_c)$  and that  $q(m_c) = m_c/16 + \mathcal{O}(m_c^2)$ . Therefore:

$$X(m_c) = \log 16 - \log(m_c) + \mathcal{O}(m_c^2)$$

Furthermore  $K(0) = \pi/2$ , so  $K_X(0) = 1/2$ . Putting all these relations together we obtain the following equation:

$$D(m) = 2 \log 2 - 1 - \frac{\log m_c}{2} + \mathcal{O}(m_c^2)$$

### Integrals of the second kind for $m$ close to 0

The expressions defined in the first section can be rewritten by changing the variable to be  $m = 1 - m_c$ . In particular:

$$\begin{aligned} E_X(m) &= \left( 1 - \frac{E(m)}{K(m)} K_X(m) \right) \\ &= \frac{1}{\pi} (K(m) - E(m)) \end{aligned}$$

Thus:

$$D(m) = \frac{\pi}{m} E_X(m)$$

Similarly, define  $B_X^*(m)$  as follows:

$$B_X^*(m) := E_X(m) - mK_X(m)$$

We have:

$$\begin{aligned} B_X^*(m) &= \frac{1}{\pi} (K(m) - E(m) - mK(m)) \\ &= \frac{1}{\pi} (m_c K(m) - E(m)) \\ &= -\frac{mB(m)}{\pi} \end{aligned}$$

Therefore:

$$B(m) = -\frac{\pi}{m} B_X^*(m)$$

These formulæ make it possible, by computing a Maclaurin approximation of  $B_X^*(m)$  and  $E_X(m)$ , to evaluate  $B(m)$  and  $D(m)$  for  $m$  close to 0.

## Conclusion

We have demonstrated how [Fuk18] uses different techniques from the ones detailed in [Fuk11a] in order to handle the logarithmic singularities of the  $B$  and  $D$  complete integrals of the second kind: while [Fuk11a] divides the leading logarithmic term  $\log \frac{m_c}{16}$ , [Fuk18] divides the complete logarithmic term  $\log q(m_c)$ .

## References

- [Fuk11a] T. Fukushima. “Precise and fast computation of the general complete elliptic integral of the second kind”. In: *Mathematics of Computation* 80 (Feb. 2011), pp. 1725–1743.  
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- [Fuk18] T. Fukushima. *xelbdj.txt: Fortran test driver for “elbdj”/“relbdj”, subroutines to compute the double/single precision general incomplete elliptic integrals of all three kinds*. Software. Jan. 2018.  
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