Approximating roots and reciprocal roots of binary floating-point numbers

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In the following, $\mathbb{N} \coloneqq [0, \infty[\cap \mathbb{Z}$. For $x \in \mathbb{R}$, $\lfloor x \rfloor \coloneqq \max(]-\infty, x] \cap \mathbb{Z})$. We define $\rceil x \lceil \coloneqq x - \lfloor x \rfloor$, so that $\forall x \in \mathbb{R}, \rceil x \lceil \in [0, 1[$. $B \in \mathbb{N}$ is arbitrary.

Let $x > 0$. There are unique $F \in [0, 1[$, $K \in \mathbb{Z}$, such that $x = 2^K(1 + F)$; define

$$\text{定}x \coloneqq B + K + F.$$

Let $X \in \mathbb{R}$; define

$$\text{浮}X \coloneqq 2^{\lfloor X - B \rfloor}(1 + \rceil X \lceil).$$

Then $\text{定浮}X = X$, $\text{浮定}x = x$, $1 + \text{定}x = \text{定}(2x)$.

Let $n \in \mathbb{Z} \setminus \{0, 1\}$, $\gamma \in \mathbb{R}$. For $x > 0$, define

$$^n r(x) \coloneqq \text{浮}\left(C_{n,\gamma} + \frac{\text{定}x}{n}\right),$$

where

$$C_{n,\gamma} \coloneqq \frac{(n-1)B - \gamma}{n}.$$

For $n = -2$, which has received particular attention, γ here corresponds to $2t - 1$ in [cite Robertson here], $2r_0 - 1$ in [cite Lomont here], -3σ in [cite McEniry here]. For $n = 3$, $C_{n,\gamma}$ corresponds to Kahan's C [cite Kahan here].

Consider the signed relative error $\varepsilon(x)$ of $^n r(x)$ as an approximation of $\sqrt[n]{x}$. For $x = 2^K(1 + F)$, we have

$$\begin{aligned}
\varepsilon(x) &= \frac{^n r(x)}{\sqrt[n]{x}} - 1 \\
&= \frac{2^{\left\lfloor \frac{K+F-\gamma}{n} \right\rfloor}\left(1 + \left\rceil \frac{K+F-\gamma}{n} \right\lceil\right)}{2^{\frac{K}{n}} \sqrt[n]{1+F}} - 1 \\
&= 2^{\left\lfloor \frac{K+F-\gamma}{n} \right\rfloor - \frac{K}{n}} \frac{1 + \left\rceil \frac{K+F-\gamma}{n} \right\lceil}{\sqrt[n]{1+F}} - 1,
\end{aligned}$$

which is invariant under addition of n to K, so that

$$\varepsilon(x) = \varepsilon(2^n x).$$

in other words,

$$\varepsilon\text{浮} : X \mapsto \varepsilon(\text{浮}X)$$

is periodic with period n.

Consider the interval

$$I_{n,\gamma} \coloneqq \begin{cases} [2^{\lfloor \gamma \rfloor}(1 + \rceil \gamma \lceil), 2^{\lfloor \gamma \rfloor + n}(1 + \rceil \gamma \lceil)[& n > 0, \\ [2^{\lfloor \gamma \rfloor + n}(1 + \rceil \gamma \lceil), 2^{\lfloor \gamma \rfloor}(1 + \rceil \gamma \lceil)[& \text{otherwise.} \end{cases}$$

Note that

$$\text{定}I_{n,\gamma} = \begin{cases} [B + \gamma, B + n + \gamma[& n > 0, \\ [B + n + \gamma, B + \gamma[& \text{otherwise,} \end{cases}$$

so that it covers one period of the relative error.

Let $x \in I_{n,\gamma}$. Then, with $F \in [0,1[$, $K \in \mathbb{Z}$, such that $x = 2^K(1+F)$,

$$^n r(x) = 1 + \frac{K+F-\gamma}{n} = 1 + \frac{K + 2^{-K}x - 1 - \gamma}{n},$$

and $K \in [\lfloor\gamma\rfloor, \lfloor\gamma\rfloor + n - 1] \cap \mathbb{Z}$ if $n > 0$, $K \in [\lfloor\gamma\rfloor + n, \lfloor\gamma\rfloor] \cap \mathbb{Z}$ otherwise. For fixed K, *i.e.*, for $x \in [2^K, 2^{K+1}[$, $\varepsilon'(x) = 0$ at

$$x = 2^K\left(1 + \frac{K-\gamma}{n-1}\right),$$

which is in $[2^K, 2^{K+1}[$ unless $K = \lfloor\gamma\rfloor$ and $n > 0$, or $K = \lfloor\gamma\rfloor + n$ and $n > 0$.

It follows that the maximum for $x > 0$ of $|\varepsilon(x)|$ is the maximum of the absolute values of the following:

- the value $\varepsilon(2^{\lfloor\gamma\rfloor}(1 +]\gamma[)) = \frac{1}{\sqrt[n]{2^{\lfloor\gamma\rfloor}(1+]\gamma[)}} - 1$ at the endpoint of $I_{n,\gamma}$;
- the values at powers of two within $I_{n,\gamma}$, $\varepsilon(2^K) = 2^{-\frac{K}{n}}\left(1 + \frac{K-\gamma}{n}\right) - 1$ for $K \in [\lfloor\gamma\rfloor + 1, \lfloor\gamma\rfloor + n] \cap \mathbb{Z}$ if $n > 0$, $K \in [\lfloor\gamma\rfloor + n + 1, \lfloor\gamma\rfloor] \cap \mathbb{Z}$ otherwise;
- the smooth extrema, $\varepsilon\left(2^K\left(1 + \frac{K-\gamma}{n-1}\right)\right)$ where $K \in [\lfloor\gamma\rfloor + 1, \lfloor\gamma\rfloor + n - 2] \cap \mathbb{Z}$ if $n > 0$ and $K \in [\lfloor\gamma\rfloor + n, \lfloor\gamma\rfloor] \cap \mathbb{Z}$ otherwise.