On an Algorithm by Fukushima

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This document describes an algorithm used by Fukushima in his implementation of the complete elliptic integrals of the second kind B and D ([Fuk18]). It follows the notation and conventions of [Fuk11a], but effectively replaces section 2.2.

Definitions

Jacobi's nome $q(m)$ is defined as a function of the elliptic integral $K(m)$ as:

$$q(m) \coloneqq \exp\left(\frac{-\pi K(1-m)}{K(m)}\right)$$

Changing the variable to be $m_c = 1 - m$ and solving for $K(m)$ yields:

$$K(m) = \left(\frac{K(m_c)}{\pi}\right)(-\log q(m_c))$$

Now let's split this expression in two terms to separate out the logarithmic part:

$$\begin{cases} X(m_c) & \coloneqq -\log q(m_c) \\ K_X(m_c) & \coloneqq \dfrac{K(m_c)}{\pi} \end{cases}$$

We have: $K(m) = K_X(m_c)X(m_c)$.

An expression for $E(m)$ can be obtained from Legendre's relation ([Fuk11a] equation 2.11):

$$E(m) = \left(1 - \frac{E(m_c)}{K(m_c)}\right)K(m) + \frac{\pi}{2K(m_c)}$$

Similarly, let's split this expression using the terms:

$$\begin{cases} E_X(m_c) & \coloneqq \left(1 - \dfrac{E(m_c)}{K(m_c)}\right)K_X \\ E_0(m_c) & \coloneqq \dfrac{1}{2K_X(m_c)} \end{cases}$$

We have: $E(m) = E_X(m_c)X(m_c) + E_0(m_c)$.

Integrals of the second kind for m close to 1

Fukushima defines $B(m)$ as:

$$B(m) \coloneqq \frac{E(m) - m_c K(m)}{m}$$

This expression can be rewritten using the definitions above:

$$\begin{aligned} B(m) &= \frac{1}{m}(E_X(m_c)X(m_c) + E_0(m_c) - m_c K_X(m_c)X(m_c)) \\ &= \frac{1}{m}\left(X(m_c)(E_X(m_c) - m_c K_X(m_c)) + \frac{1}{2K_X(m_c)}\right) \end{aligned}$$

Similarly the definition of $D(m)$:

$$D(m) \coloneqq \frac{K(m) - E(m)}{m}$$

can be rewritten as:

$$\begin{aligned} D(m) &= \frac{1}{m}(K_X(m_c)X(m_c) - E_X(m_c)X(m_c) - E_0(m_c)) \\ &= \frac{1}{m}\left(X(m_c)(K_X(m_c) - E_X(m_c)) - \frac{1}{2K_X(m_c)}\right) \end{aligned}$$

These formulæ provide a means to compute $B(m)$ and $D(m)$ for m close to 1. First, a polynomial approximation of $q(m_c)$ is computed, whose leading term is of order $m_c/16$. Then the log of that approximation is evaluated, yielding $X(m_c)$ (this is the part that carries the logarithmic singularity). Finally, $E_X(m_c)$ and $K_X(m_c)$ are computed using Taylor or Maclaurin approximations.

It is easy to see that $B(m) = X(m_c)K_X(m_c) - D(m)$, which provides a simpler formula for computing $B(m)$ once $D(m)$ is known.

Integrals of the second kind when m tends towards 1

We are now interested in computing the leading term of $B(m)$ and $D(m)$ when $m \to 1$. First, we have $B(1) = 1$ ([Fuk11a], equation 1.9). However, $D(m) \to +\infty$ when $m \to 1$. To deal with this singularity we recall that:

$$B(m) = X(m_c)K_X(m_c) - D(m)$$

thus:

$$D(m) = X(m_c)K_X(m_c) - B(m)$$

Remember that $X(m_c) = -\log q(m_c)$ and that $q(m_c) = m_c/16 + \mathcal{O}(m_c^2)$. Therefore:

$$X(m_c) = \log 16 - \log(m_c) + \mathcal{O}(m_c^2)$$

Furthermore $K(0) = \pi/2$, so $K_X(0) = 1/2$. Putting all these relations together we obtain the following equation:

$$D(m) = 2\log 2 - 1 - \frac{\log m_c}{2} + \mathcal{O}(m_c^2)$$

Integrals of the second kind for m close to 0

The expressions defined in the first section can be rewritten by changing the variable to be $m = 1 - m_c$. In particular:

$$\begin{aligned} E_X(m) &= \left(1 - \frac{E(m)}{K(m)}K_X(m)\right) \\ &= \frac{1}{\pi}(K(m) - E(m)) \end{aligned}$$

Thus:

$$D(m) = \frac{\pi}{m}E_X(m)$$

Similarly, define $B_X^*(m)$ as follows:

$$B_X^*(m) \coloneqq E_X(m) - mK_X(m)$$

We have:

$$\begin{aligned} B_X^*(m) &= \frac{1}{\pi}(K(m) - E(m) - mK(m)) \\ &= \frac{1}{\pi}(m_cK(m) - E(m)) \\ &= -\frac{mB(m)}{\pi} \end{aligned}$$

Therefore:

$$B(m) = -\frac{\pi}{m}B_X^*(m)$$

These formulæ make it possible, by computing a Maclaurin approximation of $B_X^*(m)$ and $E_X(m)$, to evaluate $B(m)$ and $D(m)$ for m close to 0.

Conclusion

We have demonstrated how [Fuk18] uses different techniques from the ones detailed in [Fuk11a] in order to handle the logarithmic singularities of the B and D complete integrals of the second kind: while [Fuk11a] divides the leading logarithmic term $\log \frac{m_c}{16}$, [Fuk18] divides the complete logarithmic term $\log q(m_c)$.

References

[Fuk11a] T. Fukushima. "Precise and fast computation of the general complete elliptic integral of the second kind". In: *Mathematics of Computation* 80 (Feb. 2011), pp. 1725–1743. DOI: 10.1090/S0025-5718-2011-02455-5.

[Fuk18] T. Fukushima. *xelbdj.txt: Fortran test driver for "elbdj"/"relbdj", subroutines to compute the double/single precision general incomplete elliptic integrals of all three kinds*. Software. Jan. 2018. DOI: 10.13140/RG.2.2.11113.80489.