On a Formula by Cohen, Hubbard and Oesterwinter

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This document proves and generalizes a formula given in [CHO73] to compute the velocity of a body in the context of numerical integration of n-body systems.

Statement

[CHO73, p. 20] states that

the formula for a velocity component [is] of the form:

$$\dot{x}_{n+1} = \frac{1}{h}\left(x_n - x_{n-1} + h^2 \sum_{i=0}^{12} \beta_i \ddot{x}_{n-i}\right) \tag{1}$$

They then proceed to tabulate explicit values for the (rational) coefficients β_i without explaining how they are computed. This makes it impossible to use this formula for integrators of a different order or to construct similar formulæ for slightly different purposes.

Finite difference formulæ

In this section we prove a backward difference formula that makes it possible to compute an approximation of any derivative of a function on an equally-spaced grid at any desired order. We then derive a corollary based on the second derivative $f''(x)$ that is useful for the following sections.

Lemma. *Given two integers $m \leq n$, there exists a family of rational numbers $\lambda_{n,\nu}^m$, with $0 \leq \nu \leq n$, such that for any sufficiently regular function f:*

$$f^{(m)}(x_0) = \frac{(-1)^m}{h^m} \sum_{\nu=0}^{n} \lambda_{n,\nu}^m f(x_0 - \nu h) + \mathcal{O}(h^{n-m+1}) \tag{2}$$

Proof. [For88] gives finite difference formulæ for any order and for kernels of arbitrary size. Given a family $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_N)$ of points (which may not be equidistant), the m-th derivative at any point x_0 may be approximated to order $n - m + 1$ as:

$$f^{(m)}(x_0) \cong \sum_{\nu=0}^{n} \delta_{n,\nu}^m(\boldsymbol{\alpha}) f(\alpha_\nu)$$

where $m \leq n$ and the coefficients $\delta_{n,\nu}^m$ are dependent on $\boldsymbol{\alpha}$ but independent of f.

If the $\boldsymbol{\alpha}$ are equally spaced with step $-h$ (*i.e.*, $\alpha_\nu = x_0 - \nu h$) then we can write a backward formula as follows:

$$f^{(m)}(x_0) = \sum_{\nu=0}^{n} \delta_{n,\nu}^m(\boldsymbol{\alpha}) f(x_0 - \nu h) + \mathcal{O}(h^{n-m+1}) \tag{3}$$

In this case equation (3.8) from [For88] may be rewritten as follows, restoring x_0:

$$\delta_{n,\nu}^m = \frac{1}{\alpha_n - \alpha_\nu}((\alpha_n - x_0)\delta_{n-1,\nu}^m - m\delta_{n-1,\nu}^{m-1})$$

$$= \frac{1}{(\nu - n)h}(-nh\delta_{n-1,\nu}^m - m\delta_{n-1,\nu}^{m-1}) \qquad (4)$$

It is possible to extract the powers of h from the coefficients by defining, for $\nu < n$:

$$\lambda_{n,\nu}^m := \delta_{n,\nu}^m(\boldsymbol{\alpha})(-1)^m h^m$$

Substituting into equation (4) we obtain:

$$\lambda_{n,\nu}^m = \frac{1}{n - \nu}(n\lambda_{n-1,\nu}^m - m\lambda_{n-1,\nu}^{m-1})$$

and we see that $\lambda_{n,\nu}^m$ is independent from x_0 and h and therefore that $\delta_{n,\nu}^m$ is independent from x_0. Note that the definition of $\lambda_{n,\nu}^m$ extends immediately to $n = \nu$, that is to $\lambda_{n,n}^m$, using equation (3.10) of [For88].

Substituting:

$$\delta_{n,\nu}^m(\boldsymbol{\alpha}) = (-1)^m \frac{\lambda_{n,\nu}^m}{h^m}$$

into equation (3) yields equation (2), thus proving the lemma.

$\square$

Corollary. *Applying the lemma to the second derivative $f''(x)$ gives the following result when $m \geq 2$:*

$$f^{(m)}(x_0) = \frac{(-1)^m}{h^{m-2}} \sum_{\nu=0}^{n-2} \lambda_{n-2,\nu}^{m-2} f''(x_0 - \nu h) + \mathcal{O}(h^{n-m+1}) \qquad (5)$$

$\square$

A formula for symmetric linear multistep integrators

In this section we derive a formula suitable for computing the velocity of a body knowing its positions and accelerations at preceding times. This is the formula we use after integration using a symmetry linear multistep integrator.

Proposition. *There exists a family of rational numbers $\eta_{n,\nu}$, with $0 \leq \nu \leq n - 2$, such that for any sufficiently regular function f:*

$$f'(x_0) = \frac{1}{h}\left(f(x_0) - f(x_0 - h) + h^2 \sum_{\nu=0}^{n-2} \eta_{n,\nu} f''(x_0 - \nu h)\right) + \mathcal{O}(h^n) \qquad (6)$$

Proof. We start by writing the Taylor series of $f(x_0 - h)$ for h close to 0, extracting the leading terms:

$$f(x_0 - h) = \sum_{m=0}^{\infty} \frac{f^{(m)}(x_0)}{m!}(-1)^m h^m$$

$$= f(x_0) - hf'(x_0) + \sum_{m=2}^{\infty} \frac{f^{(m)}(x_0)}{m!}(-1)^m h^m$$

$$= f(x_0) - hf'(x_0) + \sum_{m=2}^{n} \frac{f^{(m)}(x_0)}{m!}(-1)^m h^m + \mathcal{O}(h^{n+1}) \qquad (7)$$

Injecting the expression from corollary (5) into this series we find:

$$f(x_0 - h) = f(x_0) - hf'(x_0) + h^2 \sum_{m=2}^{n} \sum_{\nu=0}^{n-2} \frac{\lambda_{n-2,\nu}^{m-2}}{m!} f''(x_0 - \nu h) + \mathcal{O}(h^{n+1})$$

The summations are independent and can be exchanged:

$$f(x_0 - h) = f(x_0) - hf'(x_0) + h^2 \sum_{\nu=0}^{n-2} \left(\sum_{m=2}^{n} \frac{\lambda_{n-2,\nu}^{m-2}}{m!} \right) f''(x_0 - \nu h) + \mathcal{O}(h^{n+1})$$

If we define:

$$\eta_{n,\nu} := \sum_{m=2}^{n} \frac{\lambda_{n-2,\nu}^{m-2}}{m!}$$

the equation (6) follows immediately.

The Cohen-Hubbard-Osterwinder formula

In this section we derive the Cohen-Hubbard-Osterwinder formula, equation (1).

Proposition. *There exists a family of rational numbers $\beta_{n,\nu}$, with $0 \leq \nu \leq n - 2$, such that for any sufficiently regular function f:*

$$f'(x_0 + h) = \frac{1}{h}(f(x_0) - f(x_0 - h)) + h \sum_{\nu=0}^{n-2} \beta_{n,\nu} f''(x_0 - \nu h) + \mathcal{O}(h^n)$$

Proof. We start by writing the Taylor series of $f'(x_0 + h)$ for h close to 0:

$$f'(x_0 + h) = f'(x_0) + \sum_{m=2}^{\infty} \frac{h^{m-1}}{(m-1)!} f^{(m)}(x_0)$$

$$= f'(x_0) + \sum_{m=2}^{n} \frac{h^{m-1}}{(m-1)!} f^{(m)}(x_0) + \mathcal{O}(h^n)$$

and we replace $f'(x_0)$ with its value from equation (??):

$$f'(x_0 + h) = \frac{1}{h}(f(x_0) - f(x_0 - h)) + \sum_{m=2}^{n} \frac{h^{m-1}}{(m-1)!} \left(1 + \frac{(-1)^m}{m} \right) f^{(m)}(x_0) + \mathcal{O}(h^n)$$

As we did above, we replace $f^{(m)}(x_0)$ using its expression from corollary (5) to obtain:

$$f'(x_0 + h) = \frac{1}{h}(f(x_0) - f(x_0 - h)) + h \sum_{m=2}^{n} \left(\frac{1}{(m-1)!} \left((-1)^m + \frac{1}{m} \right) \sum_{\nu=0}^{n-2} \lambda_{n-2,\nu}^{m-2} f''(x_0 - \nu h) \right) + \mathcal{O}(h^n)$$

If we exchange the summations and define:

$$\beta_{n,\nu} = \sum_{m=2}^{n} \frac{\lambda_{n-2,\nu}^{m-2}}{(m-1)!} \left((-1)^m + \frac{1}{m} \right)$$

we obtain the desired result, which is a reformulation of equation (1) where β_i in [CHO73] is our $\beta_{14,i}$. □

References

[CHO73]   C. J. Cohen, E. C. Hubbard and C. Oesterwinter. *Astronomical Papers Prepared for the Use of the American Ephemeris and Nautical Almanac – Elements of the Outer Planets for One Million Years*. Vol. XXII. I. United States Government Printing Office, 1973.

[For88]   B. Fornberg. "Generation of Finite Difference Formulas on Arbitrarily Spaced Grids". In: *Mathematics of Computation* 51.184 (Oct. 1988), pp. 699–706.
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