Homework 3 [25 pt]

Name:

Session (ROB 599 or ROB 498):

Reading Assignment: https://www.matthewpeterkelly.com/index.html

YouTube Playlist: https://www.youtube.com/playlist?list=PLEYKx4BGriISagN3Ihc-9L6Cw44vtzTD9

Submission: Export this file as a **PDF** and submit it with **two videos** to Canvas.

Please **export videos of quadrotor and pendulum motions by using the lower right button of animations to export.

*** Please submit files **separately** to Canvas instead of a zip file.

Deadline: 10/27/2025

- If you have any questions about the assignment or software, please use the Discussion function on Canvas to seek help!
- If your question was not addressed on Canvas, then reach out to GSI: Yulun Zhuang
- The live script for homework is still evolving, please check Canvas periodically to make sure you are working on the latest version

Written Assignment

Use LaTex for homework

Use LaTex syntax to anwer the written part of the assignment

Latex expressions can be brought up by pressing ctrl + shift + L in Windows or cmd + shift + L in Mac

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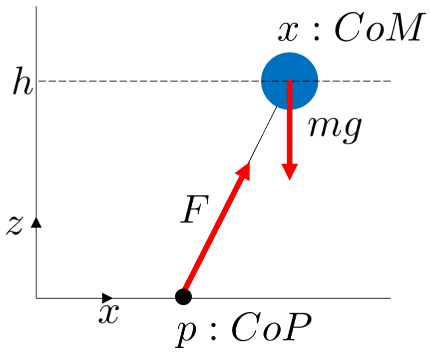
Problem 1: Template for Locomotion [3 pt]



a) [1 pt] What is the LIP dynamics expressed using the notation defined in the above figure?

$$\ddot{x} = \omega^2(x - p), \qquad \omega = \sqrt{\frac{g}{h}}.$$

b) [1 pt] What is the analytical solution to the differential equation a), given initial condition $x(0) = x_0, \dot{x}(0) = \dot{x}_0$?

Hint: the solution involves hyperbolic sine and cosine.

$\ddot{x} - \omega^2(x - p) = 0$

has the analytical solution:

$$x(t) = p + (x_0 - p)\cosh(\omega t) + \frac{\dot{x}_0}{\omega}\sinh(\omega t),$$

$$\dot{x}(t) = \omega(x_0 - p)\sinh(\omega t) + \dot{x}_0\cosh(\omega t).$$

c) [1 pt] The **capture point** ξ is the foot location such that, if you place the foot there *now* and keep it there, the robot will come to rest (i.e. $\dot{x} \to 0$). Can you derive the expression for capture point?

$$x(t) - \xi = (x(0) - \xi)e^{-\omega t}, \qquad \dot{x}(t) = \omega(\xi - x(t)) \to 0 \text{ as } t \to \infty.$$

Thus the system comes to rest, and the expression for the capture point is:

$$\xi = x + \frac{\dot{x}}{\omega}.$$

Problem 2: LQR [3 pt]

a) [1 pt] What is the HJB equation? Define each term in the recursive expression.

$$-\frac{\partial V}{\partial t}(x,t) = \min_u \left\{ l(x,u,t) + \nabla_x V(x,t)^\top f(x,u,t) \right\}, \quad V(x,T) = \phi(x).$$

b) [1 pt] What is the HJB equation for LQR problem?

$$0 = \min_u \left\{ x^\top Q x + u^\top R u + \nabla_x V(x)^\top (Ax + Bu) \right\}.$$

c) [1 pt] Derive the LQR linear feedback gain K_{lqr} from the HJB equation in b).

$$\frac{\partial}{\partial u}\{\cdots\} = 2Ru + 2B^\top Px = 0 \quad \Rightarrow \quad u^\star = -R^{-1}B^\top Px.$$

$$A^\top P + PA - PBR^{-1}B^\top P + Q = 0.$$

Thus, the LQR linear feedback gain is:

$$K_{\text{LQR}} = R^{-1}B^\top P, \qquad u^\star(t) = -K_{\text{LQR}}\, x(t).$$

Problem 3: Optimization [3 pt]

a) [1 pt] Newton's method: what is the Taylor expainsion that leads to the update rule? Can you write down the update rule?

$$\nabla^2 f(x_k)\,\Delta_k \;=\; -\nabla f(x_k), \qquad \Delta_k \;=\; -\nabla^2 f(x_k)^{-1}\nabla f(x_k),$$

and the update rule:

$$x_{k+1} = x_k - \alpha_k\,\nabla^2 f(x_k)^{-1}\nabla f(x_k)\,.$$

b) [1 pt] What are the KKT conditions for LP?

Primal feasibility Dual feasibility Stationarity Complementary slackness

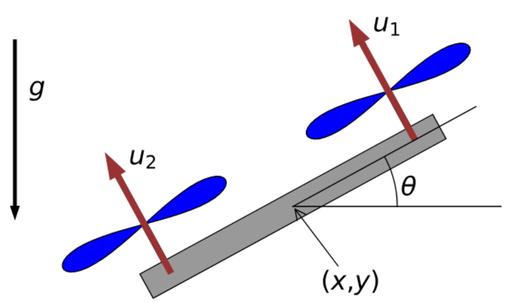
c) [1 pt] To encode inequality constraint $1 \le x^2 \le 2$ (donut) as part of the cost using log barrier function, can you write the penalty terms associated with the constraints?

$$g_1(x) = 1 - \|x\|^2 \le 0, \qquad g_2(x) = \|x\|^2 - 2 \le 0.$$

The logarithmic barrier term (penalty added to the objective) is

$$\{\ \Phi_\mu(x) \;=\; -\mu\Big[\log(\|x\|^2 - 1) \;+\; \log(2 - \|x\|^2)\Big]\,,$$

Problem 4: Quadrotor Dynamics [3 pt]



Reference: https://underactuated.csail.mit.edu/acrobot.html#section3

a) [1 pt] Read through the reference. Define the state space and input vector. Write down the quadrotor dynamics.

$$\dot{x} = \dot{x},\;\; \dot{y} = \dot{y}, \dot{\theta} = \omega, \ddot{x} = -\frac{1}{m}(u_1 + u_2)\sin\theta, \ddot{y} = \frac{1}{m}(u_1 + u_2)\cos\theta - g, \dot{\omega} = \frac{l}{I}(u_1 - u_2)$$

b) [1 pt] Derive the continuously time linear dynamics around the nominal pose where $\theta = 0$. Specifically, provide the expressions of the continuous time Jacobian matrices $A_c = \frac{\partial f}{\partial x}$ and $B_c = \frac{\partial f}{\partial u}$.

$$A_c = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/m & 1/m \\ l/I & -l/I \end{pmatrix}.$$

c) [1 pt] Based on the continuous time linear dynamics, what is the discrete-time linear dynamics. Specifically, what are the Jacobian matrices A_d, B_d, assuming forward Euler integration with time step dt?

$$A_d = \begin{pmatrix} 1 & 0 & 0 & dt & 0 & 0 \\ 0 & 1 & 0 & 0 & dt & 0 \\ 0 & 0 & 1 & 0 & 0 & dt \\ 0 & 0 & -gdt & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_d = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ dt/m & dt/m \\ dt\, l/I & -dt\, l/I \end{pmatrix}.$$

Coding Assignment

Codebase Structure & Setup

Execute the `setup.m` script **separately** to add paths of `spatical_v2` and check installation status of CasADi.

```
% clear; close all force;
% run ../../setup.m
```

Assignment 1: LQR Control [4 pts]

```
% parameters
p.g = 9.81;
p.mass = 1;
p.inertia = 1e-2;
p.halfWidth = 0.2;

tDuration = 1;
dt = 0.01;
tstart = 0;
tend = dt;
MAXITER = floor(tDuration/dt);

% state X = [x, z, th, dx, dz, dth]
% control U = [u1, u2]
X0 = [0.2 0.8 0.5 -2 0 0]';
```

```
XDes = [0 1 0 0 0 0]';
Uff = 0.5 * p.mass * p.g * [1; 1];      % feedforward propeller forces
U = Uff;

t_ = 0;
X_ = X0(:)';
U_ = U(:)';

% linearize dynamics around the desired state and control
[Ac, Bc, dc] = getJacobian(XDes, Uff, p);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% YOUR CODE STARTS
% - discretize dynamics using forward Euler integration
A = eye(6) + Ac * dt;
B = Bc * dt;
d = dc * dt;
% YOUR CODE ENDS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% weighting matrices for discrete time LQR controller
Q = diag([1, 1, 0.1, 0.01, 0.01, 0.01]);
R = 1e-3 * diag([1, 1]);
```

Function to be implemented

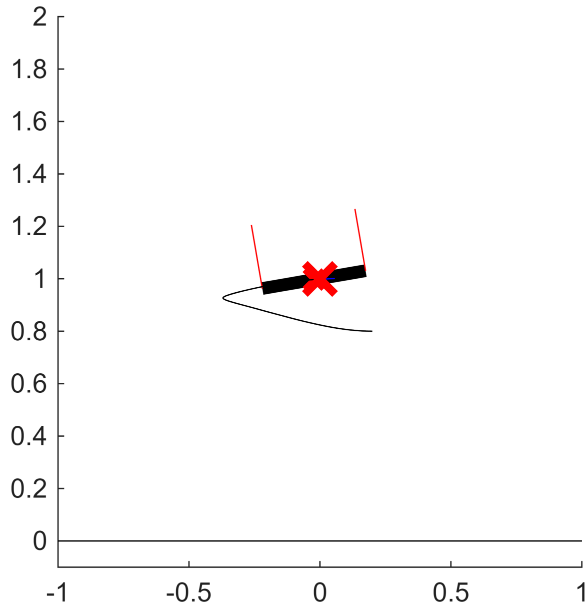
```
function [dXdt, U] = quad_dynamics(t,X,U,p)

[x, z, th] = deal(X(1), X(2), X(3));
[dx, dz, dth] = deal(X(4), X(5), X(6));
[u1, u2] = deal(U(1), U(2));

mass = p.mass;
halfWidth = p.halfWidth;
g = p.g;
inertia = p.inertia;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% YOUR CODE STARTS
ddx = -(u1 + u2)/mass * sin(th);
ddz = (u1 + u2)/mass * cos(th) - g;
ddth = halfWidth/inertia * (u1 - u2);
% YOUR CODE ENDS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

dXdt = [dx; dz; dth; ddx; ddz; ddth];

end

function [Ac, Bc, dc] = getJacobian(X, U, p)
[x, z, th] = deal(X(1), X(2), X(3));
```

```
[u1, u2] = deal(U(1), U(2));

mass = p.mass;
halfWidth = p.halfWidth;
g = p.g;
inertia = p.inertia;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% YOUR CODE STARTS
Ac = zeros(6);
Ac(1:3, 4:6) = eye(3);
Ac(4,3) = -(u1 + u2)/mass * cos(th);
Ac(5,3) = -(u1 + u2)/mass * sin(th);

Bc = zeros(6,2);
Bc(4,:) = -(1/mass) * sin(th);
Bc(5,:) = (1/mass) * cos(th);
Bc(6,:) = [halfWidth/inertia, -halfWidth/inertia];
% YOUR CODE ENDS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

dc = [0 0 0 0 -p.g 0]';

end
```

Main Control Loop

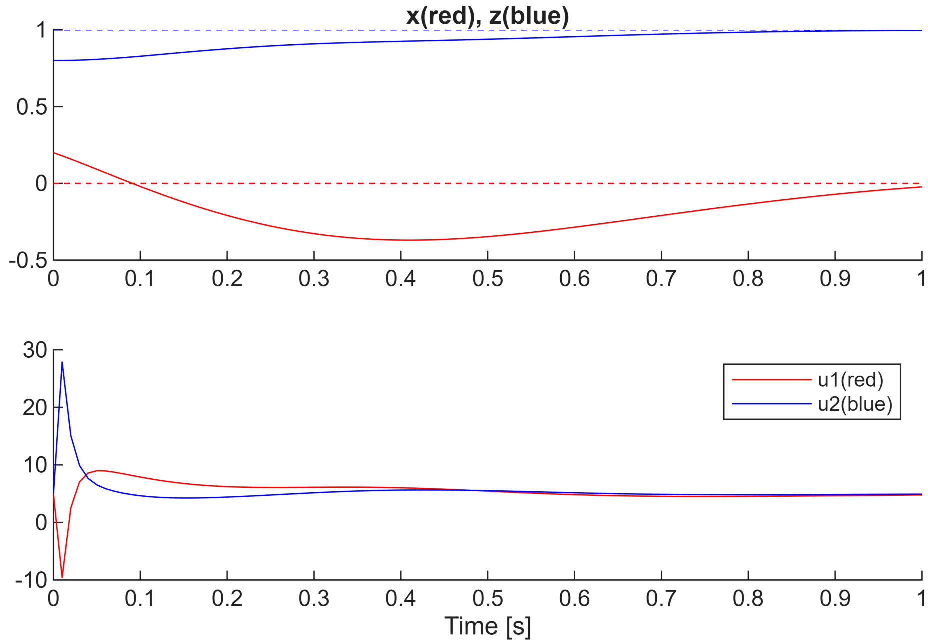
```
% Simulation Loop
for ii = 1:MAXITER
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % YOUR CODE STARTS
    % discrete-time LQR controller
    [K, ~, ~] = dlqr(A, B, Q, R);
    U = Uff - K * (X0 - XDes);
    % YOUR CODE ENDS
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    % integrator
    [t,X] = ode45(@(t,X)quad_dynamics(t, X, U, p), [tstart, tend], X0);

    % update
    tstart = tend;
    tend = tstart + dt;
    X0 = X(end,:)';

    % logging
    t_ = [t_; tstart];
    X_ = [X_; X0'];
    U_ = [U_; U'];
end
```

```
% animation
animate_quadrotor(t_, X_, U_, XDes, p)
```



I used my own implementation of `animate_quadrotor.m`

```
% Figures
clf
subplot(2,1,1)
hold on;
plot(t_, X_(:,1),'r')
plot(t_, X_(:,2),'b')
plot([t_(1),t_(end)], XDes(1)*[1 1],'r--')
plot([t_(1),t_(end)], XDes(2)*[1 1],'b--')
xlim([t_(1),t_(end)])
title('x(red), z(blue)')

subplot(2,1,2)
hold on
plot(t_, U_(:,1),'r')
plot(t_, U_(:,2),'b')
xlim([t_(1),t_(end)])
xlabel('Time [s]')
legend('u1(red)', 'u2(blue)')
```



Assignment 2: Constrained Optimization [4 pt]

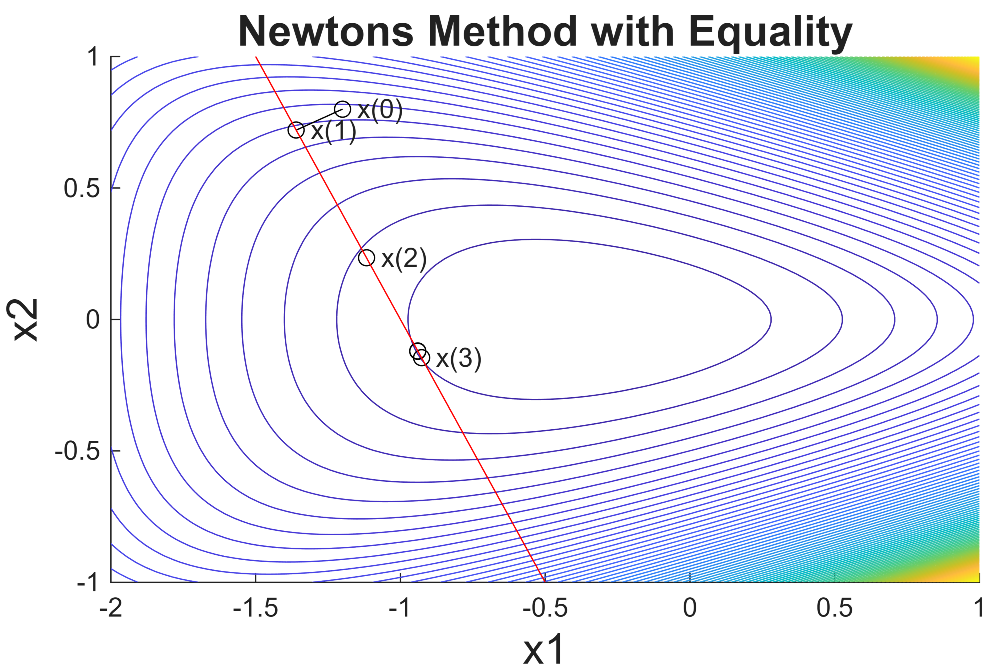
a) [2 pt] Newton's method with equality constraints

```
initialize_optimization()
```

```
ans = function_handle with value:
    @fcn_f_grad_Hess
```

```
maxRange = 1;
N = 100;
x = linspace(-2*maxRange,maxRange, N);
y = linspace(-maxRange,maxRange, N);
[X,Y] = meshgrid(x,y);
f = fcn_f_grad_Hess(X,Y);

x0 = [-1.2; 0.8];
lr = 0.17;
iter = 20;

figure; hold on
contour(X, Y, f, 90);
xlabel('x1','FontSize',16)
ylabel('x2','FontSize',16)
% Newton's method with equality constraints
A = [2,1];
b = -2;
path = newton_equality(x0, iter, A, b);

plot(path(1,:),path(2,:),'k-','Marker','o')
title('Newtons Method with Equality','FontSize',16)
```

```
for ii = 1:4
    xtext = ['x(',num2str(ii-1),')'];
    text(path(1,ii)+0.05,path(2,ii),xtext);
end

x1 = linspace(-2,1,51);
x2 = (b - A(1) * x1) / A(2);
plot(x1, x2, 'r-')
ylim([-1 1])
```



Function to be implemented

```
% Newton's method with equality constraints
function path = newton_equality(x0, iter, A, b)
path = x0;

% project x0 to the feasibility manifold
lam = -(A * x0 - b)/(A * A');
x0 = x0 + lam * A';
path = [path, x0];

for ii = 1:iter
    [~, grad, Hess] = fcn_f_grad_Hess(x0(1), x0(2));

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % YOUR CODE STARTS
    Amat = [Hess, A';
            A,    0];
    bvec = -[grad; A*x0 - b];

    dx_v = Amat \ bvec;
    dx = dx_v(1:2);
```

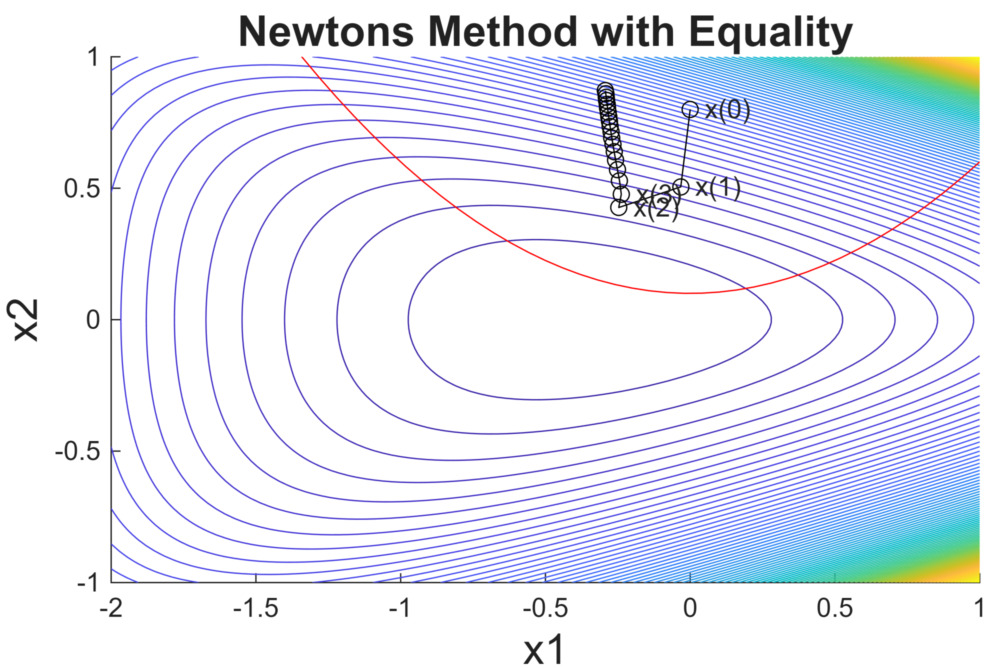
```
    x0 = x0 + dx;
    % YOUR CODE ENDS
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    path = [path, x0(:)];
end

end
```

b) [2 pt] Newton's method with inequality constraints as barrier penalty

```
init_inequality()
```

```
ans = function_handle with value:
    @fcn_ineq_grad_Hess
```

```
figure; hold on
contour(X, Y, f, 90);
xlabel('x1','FontSize',16)
ylabel('x2','FontSize',16)

x0 = [0; 0.8];
path = newton_inequality(x0, iter);

plot(path(1,:),path(2,:),'k-','Marker','o')
title('Newtons Method with Equality','FontSize',16)
for ii = 1:4
    xtext = ['x(',num2str(ii-1),')'];
    text(path(1,ii)+0.05,path(2,ii),xtext);
end

x1 = linspace(-2,1,51);
x2 = 0.5 * x1.^2 + 0.1;
plot(x1, x2, 'r-')
ylim([-1 1])
```



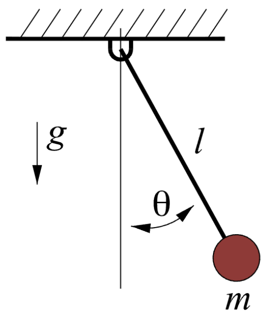
Function to be implemented

```
function init_inequality()
    syms x1 x2 t real
    x = [x1;x2];

    fi = - x2 + 0.5 * x1^2 + 0.1;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % YOUR CODE STARTS
    logBarrier = -t * log(-fi);
    grad_ineq = gradient(logBarrier, x);
    Hess_ineq = hessian(logBarrier, x);
    % YOUR CODE ENDS
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    input = {x, t};
    matlabFunction(fi,grad_ineq,Hess_ineq,"File","fcn_ineq_grad_Hess","Vars",input)
end

function path = newton_inequality(x0, iter)
    path = x0;
    for ii = 1:iter
        [~, grad, Hess] = fcn_f_grad_Hess(x0(1), x0(2));

        [~, grad_barrier, Hess_barrier] = fcn_ineq_grad_Hess(x0,ii);

        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        % YOUR CODE STARTS
        g = grad + grad_barrier;
        H = Hess + Hess_barrier;
```

```
        x0 = x0 + (-H \ g);
        % YOUR CODE ENDS
        %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
        path = [path, x0(:)];
    end
end
```

Assignment 3: Multiple Shooting [5 pt]

Multiple shooting for simple pendulum




```
% run ../../setup.m
% parameters
p.Nms = 20;
p.Nseg = 5;
p.L = 0.5;
p.mass = 1;
p.g = 9.81;
p.nco = 4;
tauMax = 3;

% boundary states
s0 = 0;                     % the integrator for cost
X0 = [0; 0; s0];
Xf = [pi; 0];

% setup optimization
opti = casadi.Opti();       % optimization problem

co_tau = opti.variable(1, p.nco);
tf = opti.variable(1, 1);
Xms = opti.variable(3, p.Nseg);
opt_var = [co_tau, tf, Xms(:)'];

% simulate forward
ds = 1/(p.Nseg*p.Nms);
for jj = 1:p.Nseg
    if jj == 1
        X = X0;
```

```
    else
        X = Xms(:,jj-1);
    end

    for kk = 1:p.Nms
        s = ds * (p.Nms * (jj-1) + (kk-1));
        X = sim_forward(X, s, ds, opt_var, p);
        tau = polyval(opt_var(1:p.nco)', s);
        opti.subject_to(-tauMax <= tau <= tauMax);
    end
    opti.subject_to(X == Xms(:,jj));
end

opti.subject_to(Xms(1:2,end) == Xf(1:2));     % terminal state
opti.subject_to(tf >= 0.5);

obj = Xms(3,end) + 1 * tf^2;
opti.minimize(obj);

opti.set_initial(tf, 1);

opti.solver('ipopt');
sol = opti.solve();
```

```
This is Ipopt version 3.14.11, running with linear solver MUMPS 5.4.1.

Number of nonzeros in equality constraint Jacobian...:      112
Number of nonzeros in inequality constraint Jacobian.:      398
Number of nonzeros in Lagrangian Hessian.............:       67

Total number of variables............................:       20
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:       17
Total number of inequality constraints...............:      101
        inequality constraints with only lower bounds:        1
   inequality constraints with lower and upper bounds:      100
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  1.0000000e+00 3.14e+00 5.00e-01  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  9.1601019e-01 2.84e+00 2.39e+01  -1.0 4.59e+01    -  2.78e-01 9.54e-02h  3
   2  1.2364657e+00 2.83e+00 2.07e+01  -1.0 5.99e+02  -2.0 1.06e-02 3.88e-03h  4
   3  1.6139704e+00 2.74e+00 1.76e+01  -1.0 2.51e+02    -  2.41e-01 3.36e-02h  4
   4  1.9549537e+00 2.61e+00 1.56e+01  -1.0 2.14e+02    -  2.97e-01 4.70e-02h  4
   5  2.6367924e+00 2.35e+00 1.44e+01  -1.0 2.38e+02    -  2.32e-01 9.80e-02h  3
   6  2.7263019e+00 2.13e+00 1.48e+01  -1.0 2.53e+01  -1.6 3.08e-01 9.55e-02h  3
   7  3.1716295e+00 1.87e+00 1.36e+01  -1.0 1.40e+01  -2.1 7.08e-01 1.21e-01h  3
   8  3.2038000e+00 1.87e+00 1.36e+01  -1.0 1.28e+03    -  2.50e-02 1.55e-03h  6
   9  3.8507507e+00 1.51e+00 1.78e+01  -1.0 2.04e+01  -1.6 6.94e-01 1.94e-01h  2
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  10  4.5634909e+00 1.17e+00 2.39e+01  -1.0 5.10e+01    -  5.69e-01 2.24e-01H  1
  11  5.7103044e+00 5.36e-01 1.54e+02  -1.0 3.08e+01    -  3.83e-01 6.68e-01H  1
  12  5.7664562e+00 5.19e-01 1.49e+02  -1.0 7.31e+00    -  7.80e-01 3.13e-02h  1
  13  7.2456963e+00 3.96e-01 8.32e+01  -1.0 3.61e+00    -  3.32e-01 7.19e-01H  1
  14  8.6728448e+00 1.95e-01 1.59e+01  -1.0 1.47e+01    -  6.29e-02 1.00e+00f  1
  15  9.1132449e+00 6.07e-03 1.60e+00  -1.0 6.41e-01    -  1.00e+00 1.00e+00h  1
```

```
  16  9.2048660e+00 1.46e-03 1.44e-02  -1.0 1.98e+00    -  1.00e+00 1.00e+00h  1
  17  8.9898846e+00 7.42e-03 4.09e-01  -2.5 5.73e+00    -  8.11e-01 1.00e+00f  1
  18  8.7721968e+00 1.34e-02 3.89e-02  -2.5 6.98e+00    -  8.65e-01 1.00e+00f  1
  19  8.7396866e+00 2.73e-03 4.03e-02  -2.5 2.31e+00    -  8.41e-01 9.71e-01h  1
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
  20  8.7500406e+00 5.70e-06 4.82e-01  -2.5 1.75e-01    -  4.82e-01 1.00e+00f  1
  21  8.7428466e+00 8.83e-04 2.82e-03  -2.5 1.16e+00    -  1.00e+00 1.00e+00h  1
  22  8.7314259e+00 6.57e-04 4.77e-02  -3.8 1.26e+00    -  9.11e-01 6.80e-01h  1
  23  8.7309561e+00 1.23e-05 8.02e-04  -3.8 1.24e-01    -  1.00e+00 1.00e+00h  1
  24  8.7302657e+00 1.18e-05 7.05e-04  -5.7 1.10e-01    -  9.77e-01 9.60e-01h  1
  25  8.7302320e+00 1.57e-06 4.04e-06  -5.7 4.34e-02    -  1.00e+00 1.00e+00h  1
  26  8.7302278e+00 6.08e-10 9.88e-09  -8.6 7.83e-04    -  1.00e+00 1.00e+00h  1

Number of Iterations....: 26

                                   (scaled)                 (unscaled)
Objective...............:   8.7302278167879699e+00    8.7302278167879699e+00
Dual infeasibility......:   9.8755483790569087e-09    9.8755483790569087e-09
Constraint violation....:   6.0755311892535246e-10    6.0755311892535246e-10
Variable bound violation:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity.........:   5.7618783684177403e-09    5.7618783684177403e-09
Overall NLP error.......:   9.8755483790569087e-09    9.8755483790569087e-09


Number of objective function evaluations             = 71
Number of objective gradient evaluations             = 27
Number of equality constraint evaluations            = 71
Number of inequality constraint evaluations          = 71
Number of equality constraint Jacobian evaluations   = 27
Number of inequality constraint Jacobian evaluations = 27
Number of Lagrangian Hessian evaluations             = 26
Total seconds in IPOPT                               = 0.840

EXIT: Optimal Solution Found.
      solver  :   t_proc      (avg)   t_wall      (avg)    n_eval
       nlp_f  |        0 (        0)  61.00us (859.15ns)        71
       nlp_g  |  22.00ms (309.86us)  20.80ms (292.96us)        71
  nlp_grad_f  |        0 (        0)  83.00us (  2.96us)        28
  nlp_hess_l  | 218.00ms (  8.38ms) 213.84ms (  8.22ms)        26
   nlp_jac_g  |  75.00ms (  2.68ms)  81.07ms (  2.90ms)        28
       total  | 871.00ms (871.00ms) 869.89ms (869.89ms)         1
```

```
% solution
co_tau_val = sol.value(co_tau)
```

```
co_tau_val = 1×4
  -43.0558   43.9169   -1.2874   -2.9915
```

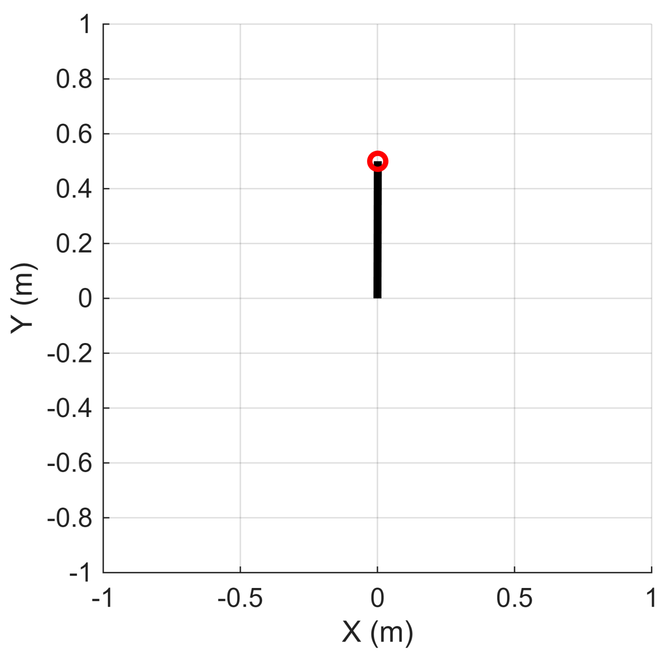
```
tf_val = sol.value(tf)
```

```
tf_val =
2.0615
```

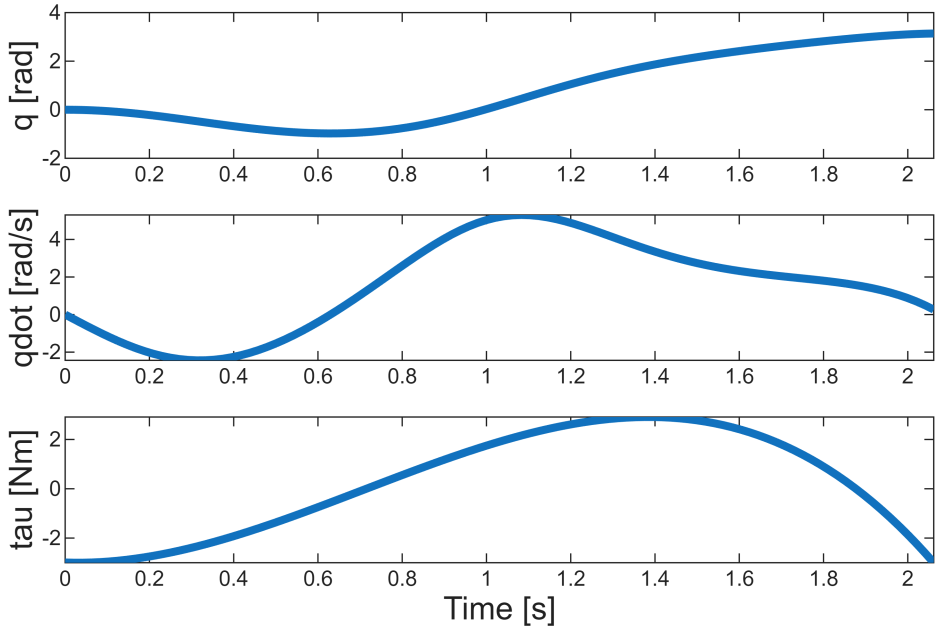
Function to be implemented

```
function [dXdt, tau] = dynamics(s, X, opt_var, p)
    L = p.L;
    mass = p.mass;
    g = p.g;
    nco = p.nco;
```

```
    q = X(1);
    tf = opt_var(nco+1);
    tau = polyval(opt_var(1:nco)', s);

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % YOUR CODE STARTS
    ddq = (tau - mass * g * L * sin(q)) / (mass * L^2);
    ds = 0.5 * tau^2;  % integrator for the cost
    dq = X(2);
    dXdt = tf * [dq;
                 ddq;
                 ds];
    % YOUR CODE ENDS
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

end

% Animation
opt_sol = [co_tau_val, tf_val];
[Xout, tauout] = animate_pendulum(X0, opt_sol, p.Nms*p.Nseg, p);
```




```
tout = tf_val * linspace(0,1,p.Nms*p.Nseg);
make_figures(tout, Xout, tauout)
```



I used my own implementation of `animate_pendulum.m`

Helper Functions

```
% LQR control for quadrotor
function animate_quadrotor(t_, X_, U_, XDes, p)
nt = length(t_);
for ii = 1:nt
    clf
    hold on;
    axis equal;
    xlim([-1, 1])
    ylim([-0.1, 2]);

    plotQuadrotor(X_(ii,:), U_(ii,:), p)
    % CoM trajectory
    plot(X_(1:ii,1), X_(1:ii,2), 'k')
    % Desired CoM
    plot(XDes(1),XDes(2),'rx','MarkerSize',16, 'LineWidth',3)

    pause(0.01);
end
end

function plotQuadrotor(X, U, p)
r = p.halfWidth;
[x, z, th] = deal(X(1), X(2), X(3));
[u1, u2] = deal(U(1), U(2));

% rotor body
```

```
p1 = [x;z] + rot(th) * [r; 0];
p2 = [x;z] + rot(th) * [-r; 0];
chain_rotor = [p1, p2];

% propelling force
scale = 0.05;
p1u = p1 + rot(th) * [0; u1] * scale;
p2u = p2 + rot(th) * [0; u2] * scale;
chain_u1 = [p1, p1u];
chain_u2 = [p2, p2u];

% CoM velocity
pcom  = X(1:2)';
vel = X(4:5)';
chain_vel = [pcom, pcom+vel*0.2];

hold on;
plot(chain_rotor(1,:), chain_rotor(2,:),'k','linewidth',5)
plot(chain_u1(1,:), chain_u1(2,:), 'r');
plot(chain_u2(1,:), chain_u2(2,:), 'r');
plot(chain_vel(1,:), chain_vel(2,:), 'b');
plot(x, z, 'rx','linewidth',2,'markersize',10)
plot([-10, 10],[0 0],'k-')

end

% optimization
function initialize_optimization()
syms x1 x2 t real
x = [x1;x2];

f = exp(x1+3*x2-0.1) + exp(x1-3*x2-0.1) + exp(-x1-0.1);
grad = jacobian(f,x)';
Hess = hessian(f,x);

matlabFunction(f,grad,Hess,"File","fcn_f_grad_Hess","Vars",x)

end

% single and multiple shooting
function [Xout, tauout] = animate_pendulum(X0, opt_sol, Num, p)
ds = 1 / Num;
co_tau_val = opt_sol(1:p.nco);
Xout = [];
tauout = [];
X = X0;
for kk = 1:Num
    s = ds * (kk-1);
    tau = polyval(co_tau_val, s);
    Xout = [Xout; X'];
```

```
    tauout = [tauout; tau];

    X = sim_forward(X, s, ds, opt_sol, p);
end

for ii = 1:Num
    clf;
    hold on; grid on; axis equal;
    plotFrame(Xout(ii,:), tauout(ii,:), p);
    pause(0.01)
end
end

function plotFrame(X, tau, p)
L = p.L;
q = X(1);

pMass = rot(q)*[0;-L];
chain_r = [[0;0], pMass];
plot(chain_r(1,:), chain_r(2,:),'k-','LineWidth',3)
plot(pMass(1),pMass(2),'ro','LineWidth',2)

xlim([-1, 1]);
ylim([-1, 1]);
xlabel("X (m)");
ylabel("Y (m)");

end

function make_figures(tout, Xout, tauout)
figure
subplot(3,1,1)
plot(tout, Xout(:,1),'LineWidth',3)
ylabel('q [rad]','FontSize',12)
xlim([tout(1), tout(end)])

subplot(3,1,2)
plot(tout, Xout(:,2),'LineWidth',3)
ylabel('qdot [rad/s]','FontSize',12)
xlim([tout(1), tout(end)])

subplot(3,1,3)
plot(tout, tauout,'LineWidth',3)
ylabel('tau [Nm]','FontSize',12)
xlabel('Time [s]','FontSize',12)
xlim([tout(1), tout(end)])

end

function [Xf, tau] = sim_forward(X, s, ds, opt_var, p)
```

```
% Runge-Kutta 4 integration
k1 = dynamics(s, X, opt_var, p);
k2 = dynamics(s+ds/2, X+ds/2*k1, opt_var, p);
k3 = dynamics(s+ds/2, X+ds/2*k2, opt_var, p);
[k4, tau] = dynamics(s+ds, X+ds*k3, opt_var, p);
Xf = X + ds/6 * (k1 + 2*k2 + 2*k3 + k4);

end

function R = rot(a)
R = [cos(a), -sin(a);
     sin(a), cos(a)];
end
```