Acknowledgements

Test

Contents

Chapter 1

Introduction

Chapter 2

Bornholdt model

We now present the model proposed by Bornholdt in [1]. The idea is to formulate a model with maximum simplicity, which includes the possibility of strategic interaction in the market. The model is based on the Ising model, which is a model of ferromagnetism in statistical mechanics.

2.1 Theoretical background: the Ising model

The Ising model is a simple mathematical model of ferromagnetic materials. In its description, we will mostly follow the notation and tools presented in [2]. It consists of Ising spins (that is, spins which can take binary values) on a d-dimensional cubic lattice. Mathematically, given a cubic lattice $\mathbb{L} = \{1, \ldots, L\}^n$, we define an an Ising spin $\sigma_i \in \{-1, 1\}$ for each site $i \in \mathbb{L}$. Then, we can have any configuration $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathcal{X}_N = \{+1, -1\}^{\mathbb{L}}$. The energy of a configuration $\underline{\sigma}$ is given by:

$$H(\underline{\sigma}) = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i \tag{2.1}$$

Where the sum over $\langle i, j \rangle$ is a sum over all nearest neighbors, and B is an external magnetic field. At equilibrium, the probability of a configuration $\underline{\sigma}$ is given by the Boltzmann distribution:

$$P(\underline{\sigma}) = \frac{e^{-\beta H(\underline{\sigma})}}{Z} \tag{2.2}$$

Where β is the inverse temperature, and Z is the partition function:

$$Z = \sum_{\underline{\sigma} \in \mathcal{X}_N} e^{-\beta H(\underline{\sigma})} \tag{2.3}$$

Interestingly, despite its simplicity, an analytical solution has been found only in the $d = 1$ and $d = 2$ cases. Higher dimensions remain unsolved, but numerical methods can be used to study the model in these cases.

One important quantity in the Ising model is the magnetization, which is defined as:

$$m = \frac{1}{N} \sum_i \langle \sigma_i \rangle \tag{2.4}$$

where $\langle \cdot \rangle$ denotes the average.

2.1.1 Solution of the Ising model in the one-dimensional case

For simplicity, assume $B = 0$. In the one-dimensional case, the Ising model can be solved exactly. Recall that:

$$H(\underline{\sigma}) = -\sum_{\langle i,j\rangle} \sigma_i \sigma_j \tag{2.5}$$

Then, the partition function is given by:

$$Z = \sum_{\underline{\sigma}\in\mathcal{X}_N} e^{-\beta H(\underline{\sigma})} = \sum_{\underline{\sigma}\in\mathcal{X}_N} e^{\beta \sum_{\langle i,j\rangle} \sigma_i\sigma_j} \tag{2.6}$$

Since each spin is connected to its nearest neighbors, we can write:

$$Z = \sum_{\underline{\sigma}\in\mathcal{X}_N} e^{\beta \sum_n \sigma_n\sigma_{n+1}} \tag{2.7}$$

Let us define $\tau_n = \sigma_{n-1}\sigma_n \implies \sigma_n = \tau_n\tau_{n-1}\dots\tau_2\sigma_1$. Then, we can write:

$$\begin{aligned} Z &= \sum_{\sigma_1\in\{-1,1\}} \sum_{\tau_2,\dots\tau_n} e^{\beta\sum_n \tau_n} = 2\sum_{\tau_2,\dots,\tau_N} e^{\beta\sum_n \tau_n} \\ &= 2\sum_{\tau_2,\dots,\tau_N} \prod_n e^{\beta\tau_n} = 2(\sum_{\tau_2} e^{\beta\tau_2})\dots(\sum_{\tau_N} e^{\beta\tau_N}) \\ &= 2(2\cosh(\beta))^N \end{aligned} \tag{2.8}$$

Thus, we have found an analytical expression for the partition function in the one-dimensional case. The magnetization can be computed as:

$$m = \frac{1}{N}\sum_i \langle\sigma_i\rangle = \frac{1}{N}\frac{\partial}{\partial\beta}\log Z = \tanh(\beta) \tag{2.9}$$

2.1.2 The Curie-Weiss model

The Curie-Weiss model is remarkably similar to the Ising model, but with all the spins interacting with each other, then, the model represents a fully connected graph of spins rather than a lattice. The Hamiltonian is given by:

$$H(\underline{\sigma}) = -\frac{1}{N}\sum_{i,j}\sigma_i\sigma_j - B\sum_i \sigma_i \tag{2.10}$$

The scaling factor $1/N$ is introduced to have a non-trivial free energy. This model is interesting because it introduced the concept of mean-field approximations. To compute the partition function, we first notice that the empirical magnetization is:

$$m(\underline{\sigma}) = \frac{1}{N}\sum_i \sigma_i \tag{2.11}$$

Then, we can write:

$$H(\underline{\sigma}) = \frac{1}{2}N - \frac{1}{2}Nm(\underline{\sigma})^2 - NBm(\underline{\sigma}) \tag{2.12}$$

2.1.3 Mean-field approximation of the Ising model in the two-dimensional case

Chapter 3

Simulations of the Bornholdt model

Chapter 4

Conclusion

In this thesis, we have explored [briefly summarize the main topics or findings]. The results indicate that [summarize key findings].

The contributions of this work include [list contributions]. Future work could focus on [suggest future research directions].

Bibliography

[1] Stefan Bornholdt. Expectation bubbles in a spin model of markets: Intermittency from frustration across scales. *International Journal of Modern Physics C*, 12(05):667–674, 2001.

[2] Marc Mézard and Andrea Montanari. *Information, Physics, and Computation*. Oxford University Press, 01 2009.