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Chapter 1

Introduction

The relationship between physics and finance is a long-standing one. [Bachelier, 1900] is perhaps the first example of a model developed by physicists, in this case Brownian motion, being applied to financial markets, specifically to the pricing of derivative products. This work would remain largely unnoticed until [Black and Scholes, 1973] was published, which is now considered the foundation of modern quantitative finance.

Beyond derivative pricing, econophysics as a field has been active since the 1990s, with the aim of applying methods from statistical physics to a wide range of problems in economics and finance. [Bouchaud and Mézard, 2000], for instance, examines the distribution of wealth in a simplified model of an economy, mapping this problem to the random 'directed polymer' problem.

For an introduction to the field of econophysics, refer to [Mantegna and Stanley, 1999] and [Sharma et al., 2011].

1.1 Motivation

The scope of this work is to review the spin model of financial price introduced in [Bornholdt, 2001], after going through the relevant core background in statistical physics. From simulations (and analytical results?), we will see that the price time-series generated by this model exhibits properties similar to those observed in real financial data, challenging the assumptions of commonly used financial models.

Chapter 2

Bornholdt model

We now present the model proposed in [Bornholdt, 2001]. The idea is to formulate a model with maximum simplicity, which includes the possibility of strategic interaction in the market. The model is based on the Ising model, which is a model of ferromagnetism in statistical mechanics.

2.1 Theoretical background: the Ising model

The Ising model is a simple mathematical model of ferromagnetic materials. In its description, we will mostly follow the notation and tools presented in [Mézard and Montanari, 2009] and from professor Mézard's lecture notes. The model consists of Ising spins (that is, spins which can take binary values) on a d-dimensional cubic lattice (see figure 2.1). Mathematically, given a cubic lattice $\mathbb{L} = \{1, \ldots, L\}^n$, we define an an Ising spin $\sigma_i \in \{-1, 1\}$ for each site $i \in \mathbb{L}$. Then, we can have any configuration $\underline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \mathcal{X}_N = \{+1, -1\}^{\mathbb{L}}$.

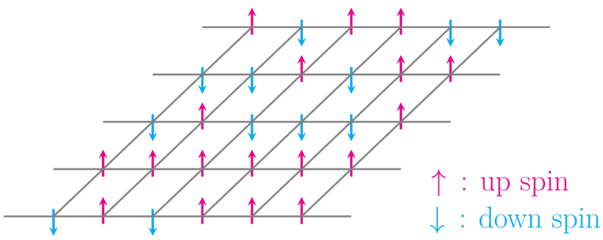

Figure 2.1: The Ising model on a 2D lattice.

The energy of a configuration $\underline{\sigma}$ is given by:

$$H(\underline{\sigma}) = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i \tag{2.1}$$

Where the sum over $\langle i, j \rangle$ is a sum over all nearest neighbors, and B is an external magnetic field. At equilibrium, the probability of a configuration $\underline{\sigma}$ is given by the Boltzmann distribution:

$$P(\underline{\sigma}) = \frac{e^{-\beta H(\underline{\sigma})}}{Z} \tag{2.2}$$

Where β is the inverse temperature, and Z is the partition function:

$$Z = \sum_{\underline{\sigma} \in \mathcal{X}_N} e^{-\beta H(\underline{\sigma})} \tag{2.3}$$

Interestingly, despite its simplicity, an analytical solution has been found only in the $d = 1$ and $d = 2$ cases. Higher dimensions remain unsolved, but numerical methods and mean-field approximations can be used to study the model in these cases.

One important quantity in the Ising model is the magnetization, which is defined as:

$$m = \frac{1}{N} \sum_i \langle \sigma_i \rangle \tag{2.4}$$

where $\langle \cdot \rangle$ denotes the average.

2.1.1 Solution of the Ising model in the one-dimensional case

For simplicity, assume $B = 0$. In the one-dimensional case, the Ising model can be solved exactly. Recall that:

$$H(\underline{\sigma}) = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{2.5}$$

Then, the partition function is given by:

$$Z = \sum_{\underline{\sigma} \in \mathcal{X}_N} e^{-\beta H(\underline{\sigma})} = \sum_{\underline{\sigma} \in \mathcal{X}_N} e^{\beta \sum_{\langle i,j \rangle} \sigma_i \sigma_j} \tag{2.6}$$

Since each spin is connected to its nearest neighbors, we can write:

$$Z = \sum_{\underline{\sigma} \in \mathcal{X}_N} e^{\beta \sum_n \sigma_n \sigma_{n+1}} \tag{2.7}$$

Let us define $\tau_n = \sigma_{n-1}\sigma_n \implies \sigma_n = \tau_n \tau_{n-1} \dots \tau_2 \sigma_1$. Then, we can write:

$$\begin{aligned} Z &= \sum_{\sigma_1 \in \{-1,1\}} \sum_{\tau_2, \dots \tau_n} e^{\beta \sum_n \tau_n} = 2 \sum_{\tau_2, \dots, \tau_N} e^{\beta \sum_n \tau_n} \\ &= 2 \sum_{\tau_2, \dots, \tau_N} \prod_n e^{\beta \tau_n} = 2 (\sum_{\tau_2} e^{\beta \tau_2}) \dots (\sum_{\tau_N} e^{\beta \tau_N}) \\ &= 2 (2 \cosh(\beta))^N \end{aligned} \tag{2.8}$$

Thus, we have found an analytical expression for the partition function in the one-dimensional case. The magnetization can be computed as:

$$m = \frac{1}{N}\sum_i \langle \sigma_i \rangle = \frac{1}{N}\frac{\partial}{\partial \beta}\log Z = \tanh(\beta) \tag{2.9}$$

2.1.2 Mean-field approximation of the Ising model in higher dimensions

While a closed-form solution for the Ising model in two dimensions exists, it does not for $d \geq 3$ so we will study its mean-field approximation. The method we will see can be applied to a more general Ising model, and then be reconduced to the original one. The hamiltonian we focus on is:

$$H(\underline{\sigma}) = -\sum_{\langle i,j \rangle} J_{i,j}\sigma_i\sigma_j - \sum_i B_i\sigma_i \tag{2.10}$$

Which differs from the standard Ising model by having arbitrary $J_{i,j}$ and B_i for every i, j. The idea is to approzimate the Boltzmann distribution $P(\underline{\sigma}) = (1/Z)e^{-\beta H(\underline{\sigma})}$ with a probability with independent variables $Q(\underline{\sigma}) = \prod_{i=1}^N q_i(\sigma_i)$. The idea is to find the q_i such that the "distance" between P and Q is minimized. We will use the Kullback-Leibler divergence as notion of distance.

Definition 1 *Given $p(x)$ and $q(x)$ probability distributions over the same finite space $\mathcal{X}$, the Kullback–Leibler (KL) divergence between them is:*

$$D(q||p) = \sum_{x\in\mathcal{X}} q(x)\log\frac{q(x)}{p(x)}$$

Remark 1

1. *$D(q||p)$ is convex in $q(x)$.*
2. *$D(q||p) \geq 0$ with equality $\iff$ $p(x) = q(x)\ \forall x \in \mathcal{X}$.*
3. *In general, the KL divergence is not symmetric.*

Then, the KL divergence lacks the symmetry property to be properly defined as a distance between probability distributions.

We will define Q as the most general joint binary probability distribution:

$$Q(\underline{\sigma}) = \prod_{i=1}^N q_i(\sigma_i); \quad q_i(\sigma_i) = \frac{1 + m_i\sigma_i}{2} \tag{2.11}$$

Where m_i is the mean of each q_i, and it is the parameter which we want to find. Then,

$$\begin{aligned} D(Q||P) &= \sum_{\underline{\sigma}\in\mathcal{X}} Q(\underline{\sigma}) \log \frac{Q(\underline{\sigma})}{P(\underline{\sigma})} \\ &= \sum_{\underline{\sigma}\in\mathcal{X}} Q(\underline{\sigma}) \log Q(\underline{\sigma}) - \sum_{\underline{\sigma}\in\mathcal{X}} Q(\underline{\sigma}) \log P(\underline{\sigma}) = (A) + (B) \end{aligned} \tag{2.12}$$

We can split this in the first term, depending only on Q, and the second term, depending on P as well. Then:

$$\begin{aligned} (A) &= \sum_{\underline{\sigma}\in\mathcal{X}} Q(\underline{\sigma}) \log Q(\underline{\sigma}) = \sum_{i=1}^{N} \left(\frac{1+m_i}{2} \log \frac{1+m_i}{2} + \frac{1-m_i}{2} \log \frac{1-m_i}{2} \right) \\ (B) &= \beta \sum_{i<j} J_{ij} m_i m_j + \beta \sum_i B_i m_i - \log Z \end{aligned} \tag{2.13}$$

In the second term, we see that the $\log Z$ term is independent of m_i, so we can ignore it. Then, we are interested in finding the values of m_i that solve:

$$\frac{\partial D(Q||P)}{\partial m_i} = 0 \iff \frac{1}{2} \log \frac{1+m_i}{1-m_i} - \beta \sum_{j\in\partial_i} J_{ij} m_j - \beta B_i = 0 \tag{2.14}$$

Then, we find the mean field equation:

$$m_i = \tanh\left(\beta \sum_{j\in\partial_i} J_{ij} m_j + \beta B_i \right) \tag{2.15}$$

Now, going back to the original Ising model, we can set $J_{i,j} = J$ and $B_i = B \implies m_i = m$. Then, we have the mean field equation for the Ising model in d dimensions:

$$m = \tanh(\beta(B + 2dJm)) \tag{2.16}$$

Let us consider the case $B = 0$. Then, depending on the value of β, we can have one or three solutions to the mean-field equation, depending on the slope of the function $f(m) = \tanh(\beta 2dJm)$. The critical value of β is then:

$$\beta_c = \frac{1}{2dJ} \tag{2.17}$$

2.1.3 The Curie-Weiss model

TBA: definition and mean-field approximation of the curie weiss model

2.2 Financial background: option pricing

TBA: introduction

2.2.1 Setting and options

Let $\mathcal{T}$ be the set of trading times, and let Ω be the state space of the price of a stock. Let $\mathcal{P} = \{\mathcal{P}_t\}_{t\in\mathcal{T}}$ be a filtration on Ω. On this setting, we can define a locally riskless asset $\{B(t)\}_{t\in\mathcal{T}}$, described by a $\mathcal{P}_t$-measurable stochastic process $\{r(t)\}_{t\in\mathcal{T}}$, which is the return of $B(t)$ at time t. The return is defined as:

$$r(t) = \frac{B(t+1) - B(t)}{B(t)} \quad \forall t \in \mathcal{T} \tag{2.18}$$

We can also define N risky asset as $\mathcal{P}_{t+1}$-measurable stochastic processes $\{S_i(t)\}_{t\in\mathcal{T}}$ $i \in \{1, \ldots, N\}$, which is the price of each asset at time t. Intuitively, the locally riskless asset is a bond, for which we know the return at time $t + 1$ given the return at time t. The risky assets are stocks, for which we do not know the return at time $t+1$ given the return at time t.

Let $\theta_j = \{\theta_j(t)\}_{t\in\mathcal{T}}$ be stochastic process representing the value of the position held in the j-th asset, which might be positive or negative (in the case of short selling).

We can now go ahead and define what a european call option is.

Definition 2 *A european call option is a contract that gives the holder the right, but not the obligation, to buy an asset at a specified price (the strike price) at a specified time (the expiration date). The payoff of a european call option at time t is given by:*

$$X(t) = \begin{cases} max(S(t) - K, 0) & if\, t = T \\ 0 & if\, t < T \end{cases} \tag{2.19}$$

Where $S(t)$ is the price of the underlying asset at time t, K is the strike price, and T is the expiration date. The payoff is zero if the option is not exercised, and it is equal to the difference between the price of the underlying asset and the strike price if the option is exercised.

A european put option is defined similarly, but it is defined as the right to sell an asset at a specified price at a specified time.

2.2.2 Assumptions of the Black-Scholes model

The Black-Scholes model is a mathematical model for the pricing of european options, indtroduced in [Black and Scholes, 1973]. The assmuptions for deriving the price of the option are:

1. The locally riskless asset is known and constant over time: $r(t) = r \; \forall t \in \mathcal{T}$.

2. The risky asset follows a geometric Brownian motion, meaning that the price of the asset at time t is given by:

$$S(t) = S(0)e^{(r-\frac{\sigma^2}{2})t+\sigma W(t)} \tag{2.20}$$

TBA: check this equation Where $W(t)$ is a standard Brownian motion, and σ is the volatility of the asset.

3. The market is frictionless, meaning that there are no transaction costs, and the assets can be traded continuously.

4. The asset pays no dividends.

5. It is possible to short-sell the asset without any restrictions or penalties.

6. It is possible to borrow a fraction of the security at the risk-free rate.

TBA: Black-Scholes model (if deemed useful)
This model is widely used in the financial industry, and it is the basis for many other models. part about how useful the model is and how important derivatives are The key assumption of this model is 2, which requires that the price of the underlying asset follows a geometric Brownian motion. This would imply that the distribution of prices in any finite period is log-normal. This assumption has more to do with the ease of computation that such a distribution allows rather than the real-world features of financial data. As a matter of fact, it has been shown that the distribution of returns is fat-tailed, meaning that there are more extreme events than would be predicted by a normal distribution, and the returns show autocorrelation, which means that returns are not independent over time. This is a key observation, and it is one of the main motivations for studying models such as the Bornholdt model.

2.3 The Bornholdt model

2.3.1 Financial motivation

In [Bornholdt, 2001], a simple spin model, which we will refer to as the Bornholdt model, is proposed to model trading in financial markets. In the model, agents are seen as interacting spins, which have two possible actions: buy or sell an asset. The choice of each agent is influenced by two contrasting forces:

- "Do what your neighbors do": this is the strategy that momentum traders follow. They try to follow the trend in the market, buying when the price is rising and selling when it is falling.

- "Do what the minority does": this is the strategy that mean-reversion traders follow. They try to buy when the price is falling and sell when it is rising, betting on a reversal of the trend.

We will see how these two strategies are implemented in the model.

2.3.2 Model definition

Consider a model with N spins with orientations $\sigma_i \in \{-1, +1\}$, representing the decision of agent i to buy or sell a stock. We will consider updates following a heat-bath dynamics:

$$\begin{aligned} \sigma_i(t+1) &= +1 \text{ with } \; p = \frac{1}{1+e^{-2\beta h_i(t)}} \\ \sigma_i(t+1) &= -1 \text{ with } \; 1-p \end{aligned} \tag{2.21}$$

Where $h_i(t)$ is the local field of agent i at time t:

$$h_i(t) = \sum_{j=1}^{N} J_{ij}\sigma_j - \alpha C_i(t) \frac{1}{N} \sum_{j=1}^{N} \sigma_j(t) \tag{2.22}$$

Where J_{ij} is the coupling between agents i and j, σ_j is the agent's action at t, $C_i(t)$ is the strategy of i at time t, and α is a parameter. The first term in the local field pushes the agent to follow the trend (do what your neighbors do), while the second term pushes the agent to follow the minority (do what the minority does), assuming the strategy $C_i(t)$ is positive. If we consider the case in which $C_i(t) = 1 \; \forall i, t$, we have that each trader follows both a momentum and a mean-reversion strategy simultaneously. This leads to near-vanishing magnetization even for temperatures below the critical temperature. We will focus on the more interesting case in which the strategy is updated according to:

$$C_i(t+1) = -C_i(t) \quad \text{if } \alpha\sigma_i(t)C_i(t) \sum_{j=1}^{N} \sigma_j(t) < 0 \tag{2.23}$$

When we assume the strategy adjustment to be done instantaneously, we can write the local field as:

$$h_i = \sum_{j=1}^{N} J_{ij}\sigma_j - \alpha\sigma_i \left| \frac{1}{N} \sum_{j=1}^{N} \sigma_j \right| \tag{2.24}$$

Then, the hamiltonian of the model is:

$$H(\underline{\sigma}) = \sum_{i,j} J_{i,j}\sigma_i\sigma_j - \frac{\alpha}{N} \sum_{i=1} \sigma_i \left| \sum_{j=1}^{N} \sigma_j \right| \tag{2.25}$$

Then, the magnetization $M = \frac{1}{N}\sum_{j=1}^{N}\sigma_j$ can be interpreted as the price of the security that is being traded. We seek to study the dynamics of the model.

2.3.3 Analysis of the model

We will focus on the fully connected case in which $J_{i,j} = \frac{J}{N}\ \forall i,j$. The hamiltonian of the model is then:

$$H(\underline{\sigma}) = \frac{J}{N}\sum_{i,j}\sigma_i\sigma_j - \frac{\alpha}{N}\sum_{i=1}\sigma_i\left|\sum_{j=1}^{N}\sigma_j\right| \tag{2.26}$$

TBA: Analysis of the model

Chapter 3

Simulations of the Bornholdt model

3.1 Results

In this study, we conducted an extensive numerical simulation of the Bornholdt model, employing a system consisting of 1024 spins arranged in a two-dimensional square lattice. The coupling constant was set to $J = 1.0$, the external field parameter was $\alpha = 4.0$, and the temperature of the system was defined as $T = 1/\beta = 1.5$. The simulation was carried out over a sufficiently large number of time steps to allow the results to be representative of the model's long-term dynamics.

3.1.1 Dynamics of the strategies

The simulations indicate that the initial distribution of agents' strategies has a negligible long-term impact on the system's behavior. Regardless of the starting configuration, the system quickly converges to a stable state where approximately 50% to 70% of agents adopt the strategy $C_i(t) = 1$. This convergence is evident from the dynamics shown in figure 3.1. Consequently, for the purposes of simulating the model, it is not necessary to distinguish between different initial distributions of strategies, as the system's evolution is largely independent of these initial conditions.

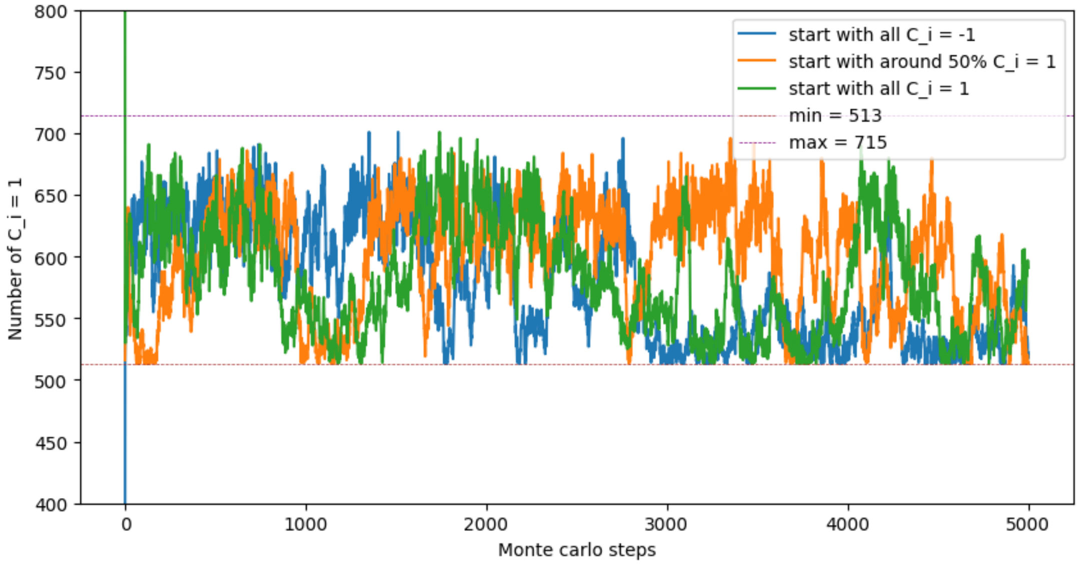

Figure 3.1: Dynamics of agent's strategy based on different starting distributions.

3.1.2 Dynamics of the price

Following the interpretation proposed in [Bornholdt, 2001], the magnetization of the system is treated as the price of a financial asset being traded by the agents. By plotting the magnetization as a function of time, we obtain a time-series that mimics the price dynamics of a financial asset. This is illustrated in figure 3.2, where the temporal evolution of the magnetization is shown.

In addition to examining the magnetization (i.e., the price), it is also insightful to analyze the underlying configuration of spins that produces this magnetization. The spin configuration provides a microscopic view of the system's state, which complements the macroscopic perspective offered by the magnetization. As shown in figure 3.3, the system alternates between two distinct regimes: metastable states, where the spin configuration remains relatively stable over time, and turbulent states, characterized by rapid and chaotic changes in the spin configuration. These alternating regimes highlight the complex dynamics of the system and their influence on the observed price behavior.

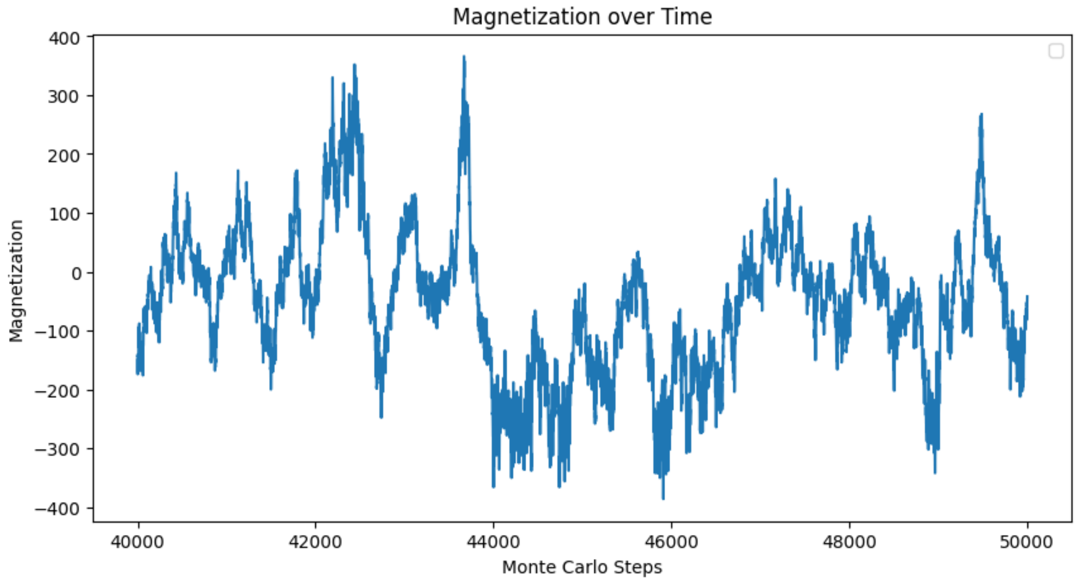

Figure 3.2: Dynamics of the magnetization of the system.

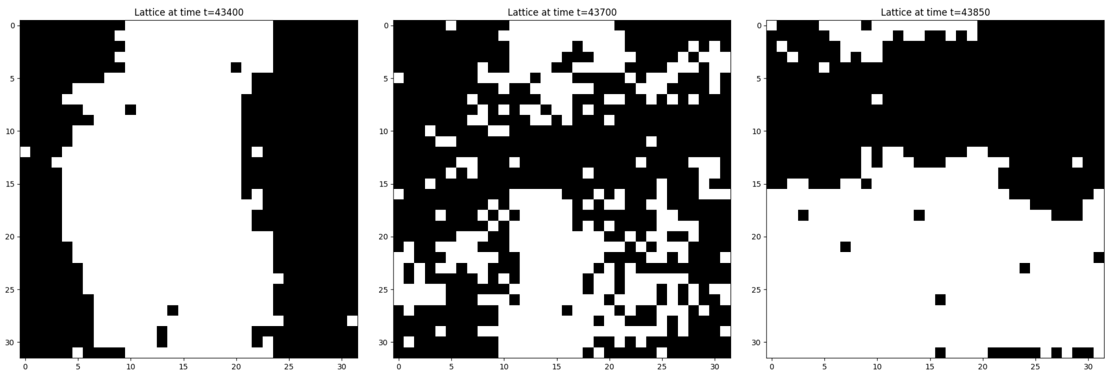

Figure 3.3: Snapshots of the lattices at times $t = 43400$, $t = 43700$ and $t = 43850$.

3.1.3 Dynamics and statistical properties of the returns

To analyze the dynamics of the returns of the asset, we define the log-returns as the logarithm of the ratio of the magnetization at consecutive time steps. Since the magnetization $M(t)$ can take on both positive and negative values, we use its absolute value to ensure that the log-returns are well-defined. Additionally, to avoid issues with division by zero or taking the logarithm of zero, we introduce a small positive constant ϵ, where $\epsilon \ll 1$. The log-returns are then computed as:

$$R(t) = \log\left(\frac{|M(t)| + \epsilon}{|M(t-1)| + \epsilon}\right) \tag{3.1}$$

Here, $R(t)$ represents the return at time t, $M(t)$ is the magnetization at time t, and ϵ is a small constant added for numerical stability.

The resulting time-series of the returns is shown in figure 3.4.

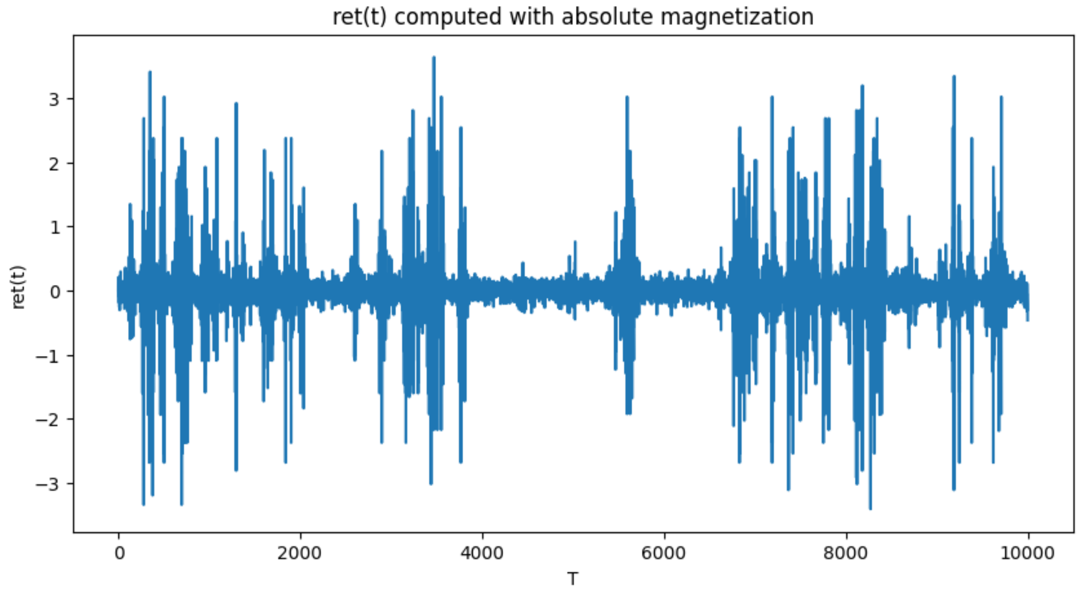

Figure 3.4: Time-series of the returns of the asset.

From a financial perspective, we are interested in checking wether the returns exhibit the statistical properties of real financial data, namely fat tails and autocorrelation, as noted in [Bouchaud and Potters, 2000].

Fat tails

The analysis of the distribution of the returns, as illustrated in figures 3.5 and 3.6, provides strong evidence that the returns exhibit fat tails. This characteristic is a hallmark of financial time-series and indicates that extreme events (large positive or negative returns) occur more frequently than would be expected under a normal distribution.

To further investigate this property, we compare the returns to a normal distribution using a QQ-plot, shown in figure 3.7. The QQ-plot clearly demonstrates significant deviations from the straight line that would be expected if the returns were normally distributed. These deviations, particularly in the tails, confirm the presence of fat tails in the distribution.

Additionally, we perform a statistical test to formally assess whether the returns follow a normal distribution. Specifically, we use the Shapiro-Wilk test, which tests the null hypothesis that the data is drawn from a normal distribution. The results of the test yield a p-value on the order of 10^{-192}. Such an extremely small p-value indicates that we can reject the null hypothesis with extremely high confidence, confirming that the returns are not normally distributed.

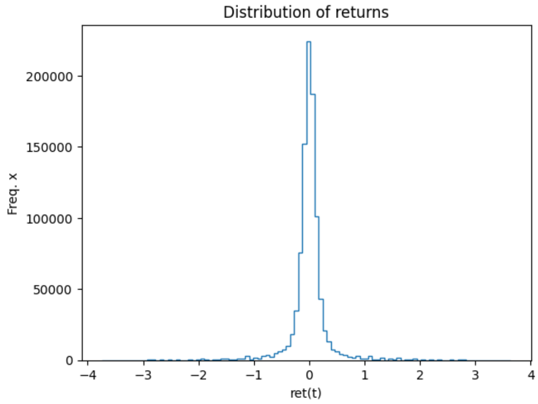

Figure 3.5: Distribution of the returns of the asset.

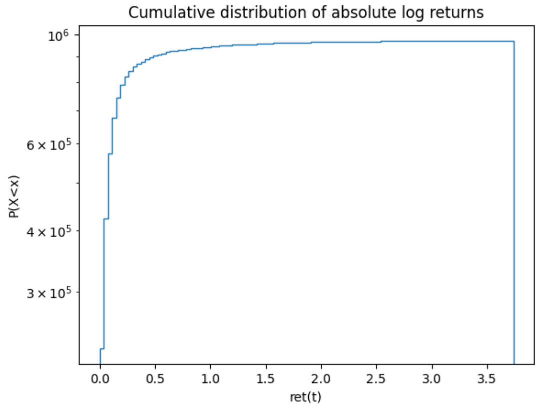

Figure 3.6: Cumulative distribution of the returns of the asset.

Autocorrelation

From figure 3.4, we can already seen that the periods of high volatility (i.e. large positive or negative returns) tend to be clustered together. This is a sign of autocorrelation in the returns. To quantify this, we can compute the autocorrelation function of the returns, defined as

$$\rho(k) = \frac{\sum_{t=1}^{N-k}(R(t) - \bar{R})(R(t+k) - \bar{R})}{\sum_{t=1}^{N}(R(t) - \bar{R})^2} \tag{3.2}$$

where N is the number of returns, $R(t)$ is the return at time t, and $\bar{R}$ is the mean of the returns. The autocorrelation function measures the correlation between the returns at time t and time $t+k$. A positive value of $\rho(k)$ indicates that the returns at time t and time $t+k$ are positively correlated, while a negative value indicates that they are negatively correlated. A value close to zero indicates that there is no correlation between the returns at time t and time $t+k$. The autocorrelation function is computed for lags $k = 1, 2, \ldots, 100$ and is shown in figure 3.8. The results show that the autocorrelation is significant for a large number of lags, which is consistent with the empirical observation that financial returns are autocorrelated.

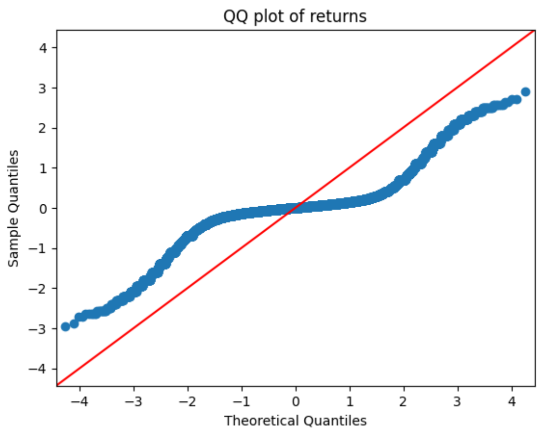

Figure 3.7: QQ-plot of the returns against a normal distribution.

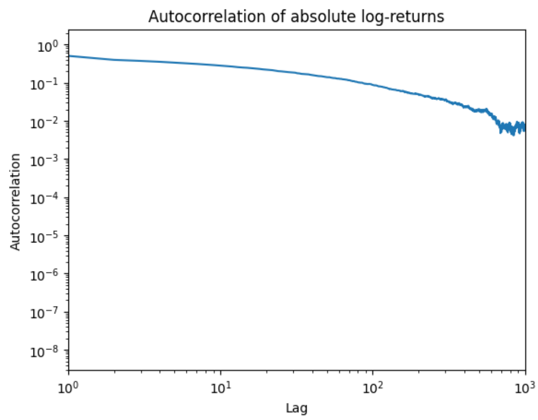

Figure 3.8: Autocorrelation function of the returns.

3.2 Discussion

The results of the simulation suggest that the Bornholdt model is able to reproduce some of the statistical properties of real financial data, such as fat tails and autocorrelation of returns. This is interesting because the model is based on a simple spin model, and does not rely on any assumptions about the rationality of agents or the efficiency of markets.

As seen in 2.2, one of the main assumptions of the model introduced in [Black and Scholes, 1973] is that the price of the underlying asset follows a geometric Brownian motion. The Bornholdt model, however, results in price dynamics which exhibit fat tails, and do not have independent increments, as shown by the autocorrelation of the returns, which is the case also in real financial data.

The key observation here is that, unlike in a geometric Brownian motion, the price (i.e. magnetization) at time t is not enough to describe the distribution of prices at times $t + s, \ \ s \geq 0$. This might be counterintuitive, as by equation 2.21, the future configuration of spins is determined by the current configuration of spins. However, it is important to note that the current magnetization is not enough to determine the current configuration of spins. In fact, the magnetization is a global property of the system, while the configuration can vary locally. In figures 3.9 and 3.10, we can see that the magnetization is the same at times $t = 129722$ and $t = 130293$, but the configuration of spins is completely different.

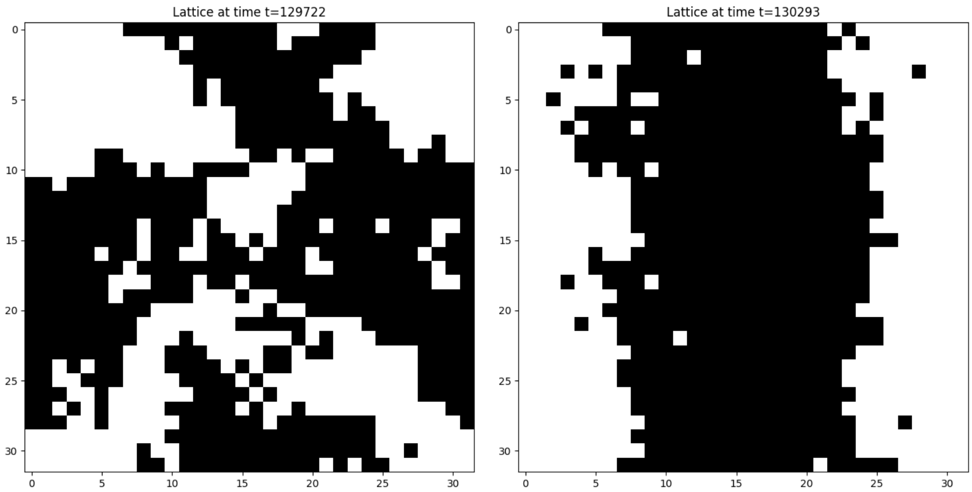

Figure 3.9: Snapshots of the lattices at times $t = 129722$, and $t = 130293$.

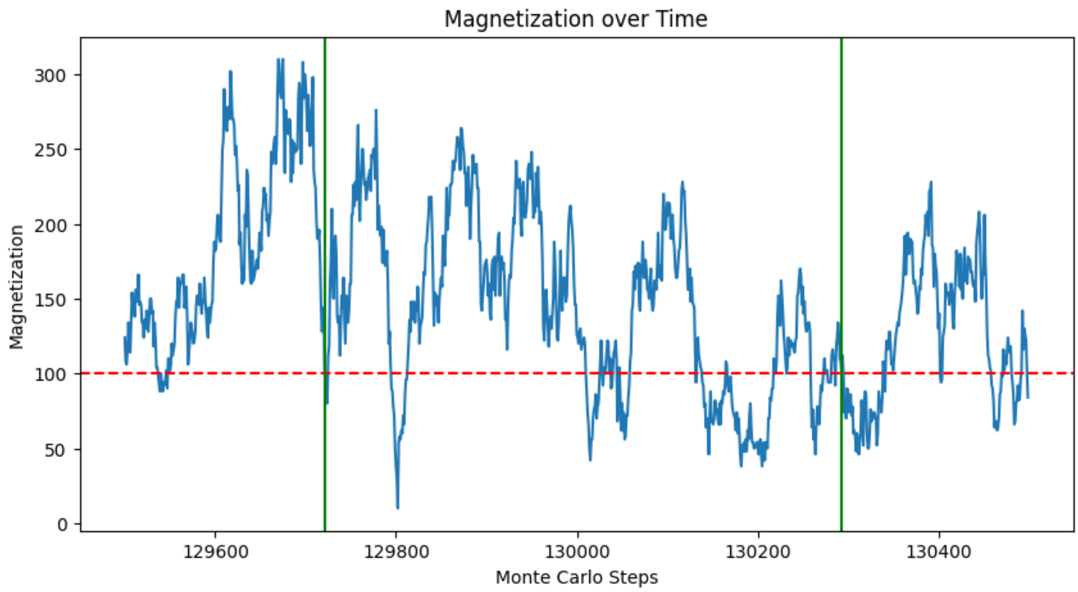

Figure 3.10: Dynamics of the magnetization of the system. The green lines represent the times at which the snapshots in figure 3.9 were taken.

Chapter 4

Conclusion

In this thesis, we have explored [briefly summarize the main topics or findings]. The results indicate that [summarize key findings].

The contributions of this work include [list contributions]. Future work could focus on [suggest future research directions].

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