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Contents

Chapter 1

Introduction

The relationship between physics and finance is a long-standing one. [Bachelier, 1900] is perhaps the first example of a model developed by physicists, in this case Brownian motion, being applied to financial markets, specifically to the pricing of derivative products. This work would remain largely unnoticed until [Black and Scholes, 1973] was published, which is now considered the foundation of modern quantitative finance.

Beyond derivative pricing, econophysics as a field has been active since the 1990s, with the aim of applying methods from statistical physics to a wide range of problems in economics and finance. [Bouchaud and Mézard, 2000], for instance, examines the distribution of wealth in a simplified model of an economy, mapping this problem to the random 'directed polymer' problem.

For an introduction to the field of econophysics, refer to [Mantegna and Stanley, 1999] and [Sharma et al., 2011].

1.1 Motivation

The scope of this work is to review the spin model of financial price introduced in [Bornholdt, 2001], after going through the relevant core background in statistical physics. From simulations (and analytical results?), we will see that the price time-series generated by this model exhibits properties similar to those observed in real financial data, challenging the assumptions of commonly used financial models.

Chapter 2

Bornholdt model

We now present the model proposed in [Bornholdt, 2001]. The idea is to formulate a model with maximum simplicity, which includes the possibility of strategic interaction in the market. The model is based on the Ising model, which is a model of ferromagnetism in statistical mechanics.

2.1 Theoretical background: the Ising model

The Ising model is a simple mathematical model of ferromagnetic materials. In its description, we will mostly follow the notation and tools presented in [Mézard and Montanari, 2009] and from professor Mézard's lecture notes. The model consists of Ising spins (that is, spins which can take binary values) on a d-dimensional cubic lattice (see figure 2.1). Mathematically, given a cubic lattice $\mathbb{L} = \{1, \dots, L\}^n$, we define an an Ising spin $\sigma_i \in \{-1, 1\}$ for each site $i \in \mathbb{L}$. Then, we can have any configuration $\underline{\sigma} = (\sigma_1, \dots, \sigma_n) \in \mathcal{X}_N = \{+1, -1\}^{\mathbb{L}}$.

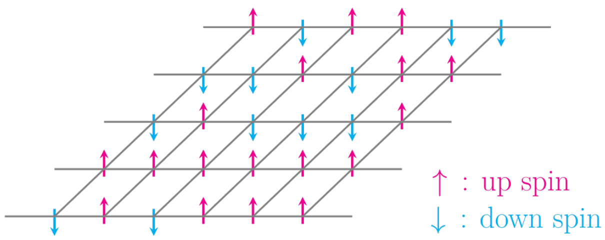

Figure 2.1: The Ising model on a 2D lattice.

The energy of a configuration $\underline{\sigma}$ is given by:

$$H(\underline{\sigma}) = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j - B \sum_i \sigma_i \tag{2.1}$$

Where the sum over $\langle i, j \rangle$ is a sum over all nearest neighbors, and B is an external magnetic field. At equilibrium, the probability of a configuration $\underline{\sigma}$ is given by the Boltzmann distribution:

$$P(\underline{\sigma}) = \frac{e^{-\beta H(\underline{\sigma})}}{Z} \tag{2.2}$$

Where β is the inverse temperature, and Z is the partition function:

$$Z = \sum_{\underline{\sigma} \in \mathcal{X}_N} e^{-\beta H(\underline{\sigma})} \tag{2.3}$$

Interestingly, despite its simplicity, an analytical solution has been found only in the $d = 1$ and $d = 2$ cases. Higher dimensions remain unsolved, but numerical methods and mean-field approximations can be used to study the model in these cases.

One important quantity in the Ising model is the magnetization, which is defined as:

$$m = \frac{1}{N} \sum_i \langle \sigma_i \rangle \tag{2.4}$$

where $\langle \cdot \rangle$ denotes the average.

2.1.1 Solution of the Ising model in the one-dimensional case

For simplicity, assume $B = 0$. In the one-dimensional case, the Ising model can be solved exactly. Recall that:

$$H(\underline{\sigma}) = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{2.5}$$

Then, the partition function is given by:

$$Z = \sum_{\underline{\sigma} \in \mathcal{X}_N} e^{-\beta H(\underline{\sigma})} = \sum_{\underline{\sigma} \in \mathcal{X}_N} e^{\beta \sum_{\langle i,j \rangle} \sigma_i \sigma_j} \tag{2.6}$$

Since each spin is connected to its nearest neighbors, we can write:

$$Z = \sum_{\underline{\sigma} \in \mathcal{X}_N} e^{\beta \sum_n \sigma_n \sigma_{n+1}} \tag{2.7}$$

Let us define $\tau_n = \sigma_{n-1}\sigma_n \implies \sigma_n = \tau_n \tau_{n-1} \ldots \tau_2 \sigma_1$. Then, we can write:

$$\begin{aligned} Z &= \sum_{\sigma_1 \in \{-1,1\}} \sum_{\tau_2, \ldots \tau_n} e^{\beta \sum_n \tau_n} = 2 \sum_{\tau_2, \ldots, \tau_N} e^{\beta \sum_n \tau_n} \\ &= 2 \sum_{\tau_2, \ldots, \tau_N} \prod_n e^{\beta \tau_n} = 2 (\sum_{\tau_2} e^{\beta \tau_2}) \ldots (\sum_{\tau_N} e^{\beta \tau_N}) \\ &= 2 (2 \cosh(\beta))^N \end{aligned} \tag{2.8}$$

Thus, we have found an analytical expression for the partition function in the one-dimensional case. The magnetization can be computed as:

$$m = \frac{1}{N}\sum_i \langle\sigma_i\rangle = \frac{1}{N}\frac{\partial}{\partial\beta}\log Z = \tanh(\beta) \tag{2.9}$$

2.1.2 Mean-field approximation of the Ising model in higher dimensions

While a closed-form solution for the Ising model in two dimensions exists, it does not for $d \geq 3$ so we will study its mean-field approximation. The method we will see can be applied to a more general Ising model, and then be reconduced to the original one. The hamiltonian we focus on is:

$$H(\underline{\sigma}) = -\sum_{\langle i,j\rangle} J_{i,j}\sigma_i\sigma_j - \sum_i B_i\sigma_i \tag{2.10}$$

Which differs from the standard Ising model by having arbitrary $J_{i,j}$ and B_i for every i, j. The idea is to approzimate the Boltzmann distribution $P(\underline{\sigma}) = (1/Z)e^{-\beta H(\underline{\sigma})}$ with a probability with independent variables $Q(\underline{\sigma}) = \prod_{i=1}^N q_i(\sigma_i)$. The idea is to find the q_i such that the "distance" between P and Q is minimized. We will use the Kullback-Leibler divergence as notion of distance.

Definition 1 *Given $p(x)$ and $q(x)$ probability distributions over the same finite space $\mathcal{X}$, the Kullback–Leibler (KL) divergence between them is:*

$$D(q||p) = \sum_{x\in\mathcal{X}} q(x)\log\frac{q(x)}{p(x)}$$

Remark 1

1. *$D(q||p)$ is convex in $q(x)$.*
2. *$D(q||p) \geq 0$ with equality $\iff p(x) = q(x)\ \forall x \in \mathcal{X}$.*
3. *In general, the KL divergence is not symmetric.*

Then, the KL divergence lacks the symmetry property to be properly defined as a distance between probability distributions.

We will define Q as the most general joint binary probability distribution:

$$Q(\underline{\sigma}) = \prod_{i=1}^N q_i(\sigma_i); \quad q_i(\sigma_i) = \frac{1+m_i\sigma_i}{2} \tag{2.11}$$

Where m_i is the mean of each q_i, and it is the parameter which we want to find. Then,

$$\begin{aligned} D(Q||P) &= \sum_{\underline{\sigma}\in\mathcal{X}} Q(\underline{\sigma}) \log \frac{Q(\underline{\sigma})}{P(\underline{\sigma})} \\ &= \sum_{\underline{\sigma}\in\mathcal{X}} Q(\underline{\sigma}) \log Q(\underline{\sigma}) - \sum_{\underline{\sigma}\in\mathcal{X}} Q(\underline{\sigma}) \log P(\underline{\sigma}) = (A) + (B) \end{aligned} \tag{2.12}$$

We can split this in the first term, depending only on Q, and the second term, depending on P as well. Then:

$$\begin{aligned} (A) &= \sum_{\underline{\sigma}\in\mathcal{X}} Q(\underline{\sigma}) \log Q(\underline{\sigma}) = \sum_{i=1}^{N} \left(\frac{1+m_i}{2} \log \frac{1+m_i}{2} + \frac{1-m_i}{2} \log \frac{1-m_i}{2} \right) \\ (B) &= \beta \sum_{i<j} J_{ij} m_i m_j + \beta \sum_i B_i m_i - \log Z \end{aligned} \tag{2.13}$$

In the second term, we see that the $\log Z$ term is independent of m_i, so we can ignore it. Then, we are interested in finding the values of m_i that solve:

$$\frac{\partial D(Q||P)}{\partial m_i} = 0 \iff \frac{1}{2} \log \frac{1+m_i}{1-m_i} - \beta \sum_{j\in\partial_i} J_{ij} m_j - \beta B_i = 0 \tag{2.14}$$

Then, we find the mean field equation:

$$m_i = \tanh\left(\beta \sum_{j\in\partial_i} J_{ij} m_j + \beta B_i \right) \tag{2.15}$$

Now, going back to the original Ising model, we can set $J_{i,j} = J$ and $B_i = B \implies m_i = m$. Then, we have the mean field equation for the Ising model in d dimensions:

$$m = \tanh(\beta(B + 2dJm)) \tag{2.16}$$

Let us consider the case $B = 0$. Then, depending on the value of β, we can have one or three solutions to the mean-field equation, depending on the slope of the function $f(m) = \tanh(\beta 2dJm)$. The critical value of β is then:

$$\beta_c = \frac{1}{2dJ} \tag{2.17}$$

2.2 The Bornholdt model

2.2.1 Financial motivation

In [Bornholdt, 2001], a simple spin model, which we will refer to as the Bornholdt model, is proposed to model trading in financial markets. In the model, agents are seen as interacting spins, which have two possible actions: buy or sell an asset. The choice of each agent is influenced by two contrasting forces:

- "Do what your neighbors do": this is the strategy that momentum traders follow. They try to follow the trend in the market, buying when the price is rising and selling when it is falling.
- "Do what the minority does": this is the strategy that mean-reversion traders follow. They try to buy when the price is falling and sell when it is rising, betting on a reversal of the trend.

We will see how these two strategies are implemented in the model.

2.2.2 Model definition

Consider a model with N spins with orientations $\sigma_i \in \{-1, +1\}$, representing the decision of agent i to buy or sell a stock. We will consider updates following a heat-bath dynamics:

$$\begin{aligned} \sigma_i(t+1) &= +1 \text{ with } \; p = \frac{1}{1+e^{-2\beta h_i(t)}} \\ \sigma_i(t+1) &= -1 \text{ with } \; 1-p \end{aligned} \tag{2.18}$$

Where $h_i(t)$ is the local field of agent i at time t:

$$h_i(t) = \sum_{j=1}^{N} J_{ij}\sigma_j - \alpha C_i(t)\frac{1}{N}\sum_{j=1}^{N}\sigma_j(t) \tag{2.19}$$

Where J_{ij} is the coupling between agents i and j, σ_j is the agent's action at t, $C_i(t)$ is the strategy of i at time t, and α is a parameter. The first term in the local field pushes the agent to follow the trend (do what your neighbors do), while the second term pushes the agent to follow the minority (do what the minority does), assuming the strategy $C_i(t)$ is positive. If we consider the case in which $C_i(t) = 1 \; \forall i, t$, we have that each trader follows both a momentum and a mean-reversion strategy simultaneously. This leads to near-vanishing magnetization even for temperatures below the critical temperature. We will focus on the more interesting case in which the strategy is updated according to:

$$C_i(t+1) = -C_i(t) \quad \text{if } \alpha\sigma_i(t)C_i(t)\sum_{j=1}^{N}\sigma_j(t) < 0 \tag{2.20}$$

When we assume the strategy adjustment to be done instantaneously, we can write the local field as:

$$h_i = \sum_{j=1}^{N} J_{ij}\sigma_j - \alpha\sigma_i \left| \frac{1}{N} \sum_{j=1}^{N} \sigma_j \right| \tag{2.21}$$

Then, the hamiltonian of the model is:

$$H(\underline{\sigma}) = \sum_{i,j} J_{i,j}\sigma_i\sigma_j - \frac{\alpha}{N} \sum_{i=1} \sigma_i \left| \sum_{j=1}^{N} \sigma_j \right| \tag{2.22}$$

Then, the magnetization $M = \frac{1}{N}\sum_{j=1}^{N} \sigma_j$ can be interpreted as the price of the security that is being traded. We seek to study the dynamics of the model.

2.2.3 Analysis of the model

We will focus on the case in which $J_{i,j} = 1\ \forall i, j$. The hamiltonian of the model is then:

$$H(\underline{\sigma}) = \sum_{i,j} \sigma_i\sigma_j - \frac{\alpha}{N} \sum_{i=1} \sigma_i \left| \sum_{j=1} \sigma_j \right| \tag{2.23}$$

2.3 Comparison with other financial models

Chapter 3

Simulations of the Bornholdt model

3.1 Results

We performed a numerical simulation of the Bornholdt model, with 1024 spins arranged on a square lattice. The values for the parameters are: $J = 1.0$, $\alpha = 4.0$, $T = 1/\beta = 1.5$.

3.1.1 Dynamics of the strategies

The simulations have shown that the initial distribution of the agent's strategy is irrelevant, as the system quickly converges to a state where between 50% and 70% of the agents follow the strategy $C_i(t) = 1$, as can be seen from figure 3.1.

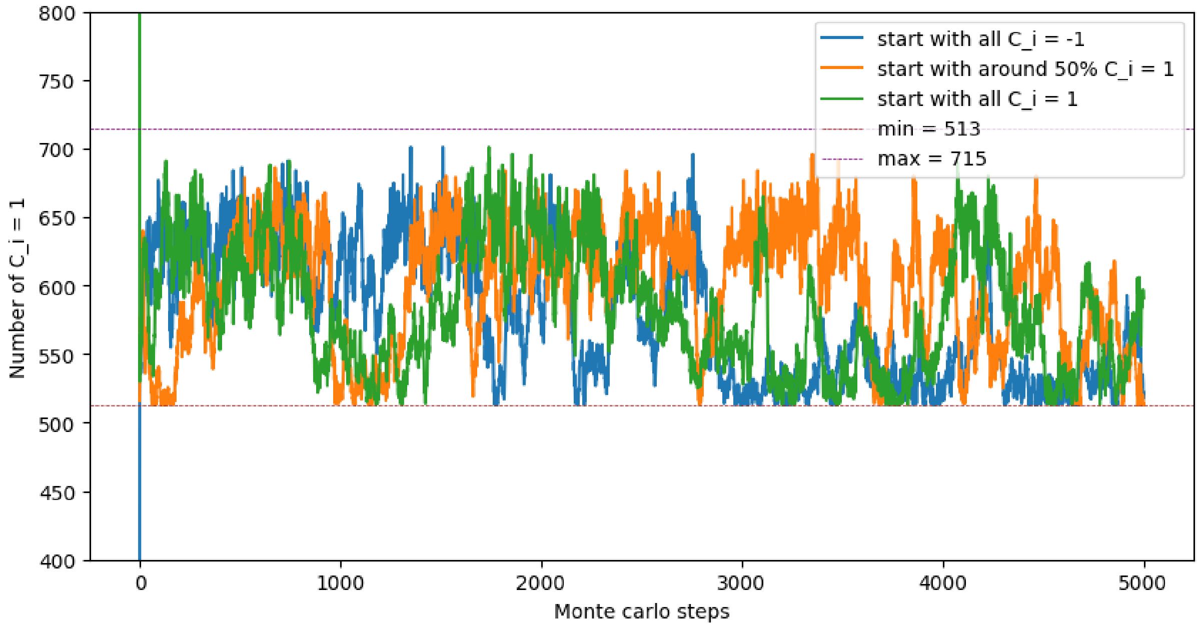

Figure 3.1: Dynamics of agent's strategy based on different starting distributions.

3.1.2 Dynamics of the price

We can see the dynamics of the price (i.e. the magnetization) in figure 3.2.

By looking at the spin configurations at different times (figure 3.3), we can see that the system alternates between metastable and turbulent states.

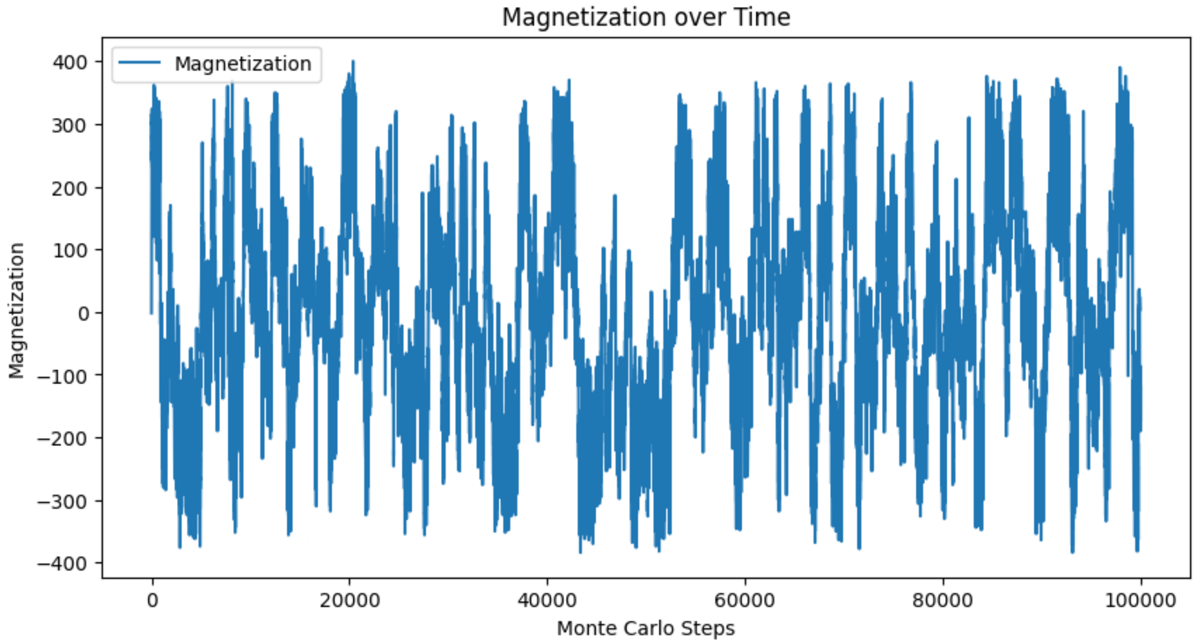

Figure 3.2: Dynamics of the magnetization of the system.

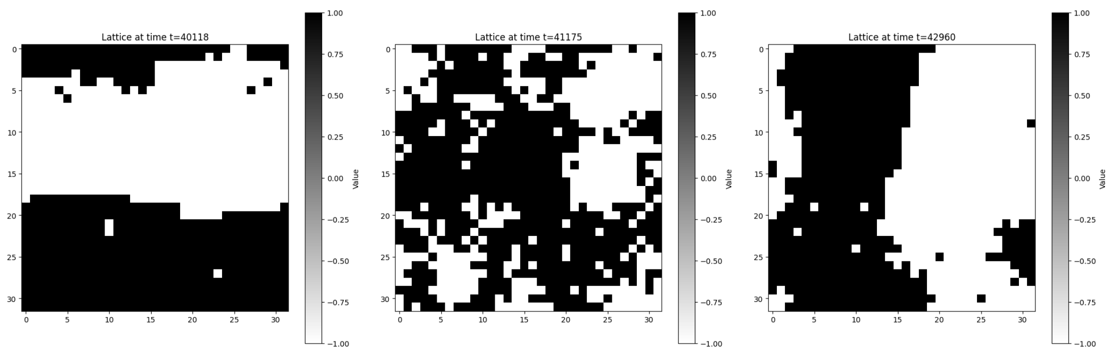

Figure 3.3: Snapshots of the lattices at times $t = 40118$, $t = 41175$ and $t = 42960$.

3.1.3 Dynamics and statistical properties of the returns

By interpreting the magnetization as the price of the financial asset, we can calculate the log-returns of the asset as $\log(M(t)/M(t-1))$. Actually, to make sure the returns are well-defined, we use the absolute value of the magnetization, $|M(t)|$. The time-series of the returns is shown in figure 3.4.

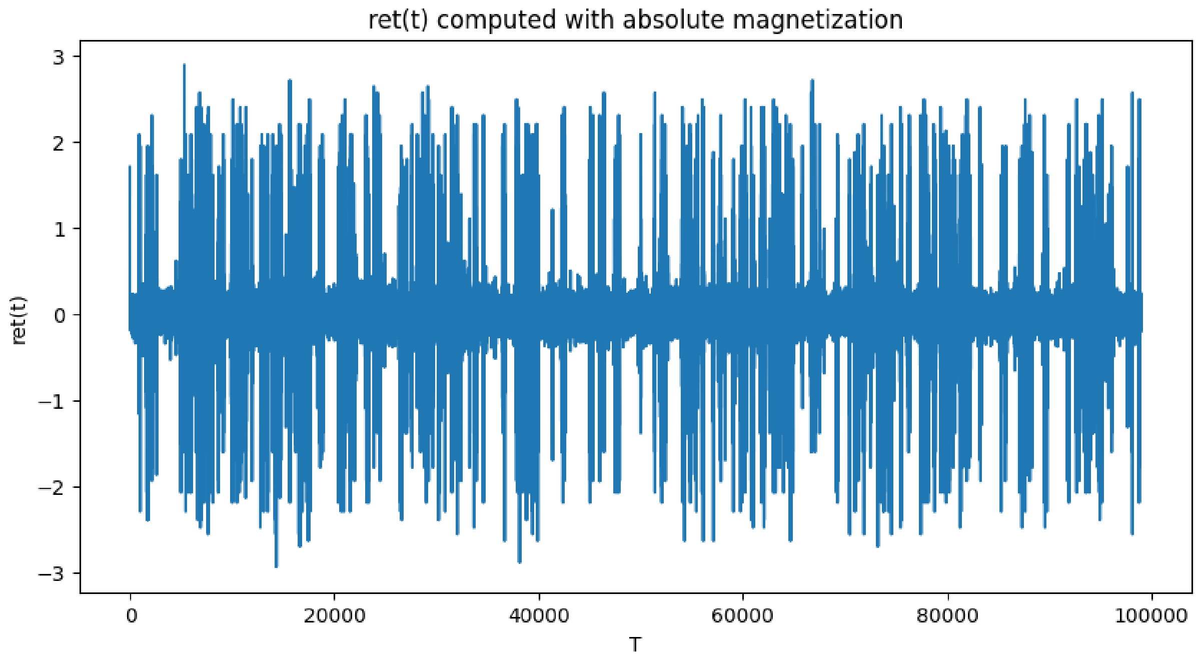

Figure 3.4: Time-series of the returns of the asset.

From a financial perspective, we are interested in checking wether the returns exhibit the statistical properties of real financial data, namely fat tails and autocorrelation, as noted in [Bouchaud and Potters, 2000].

Fat tails

A plot of the distribution of the returns, as shown in figures 3.5 and 3.6, indicates that the returns indeed seem fat-tailed. A QQ-plot of the returns against a normal distribution (figure 3.7) confirms this.

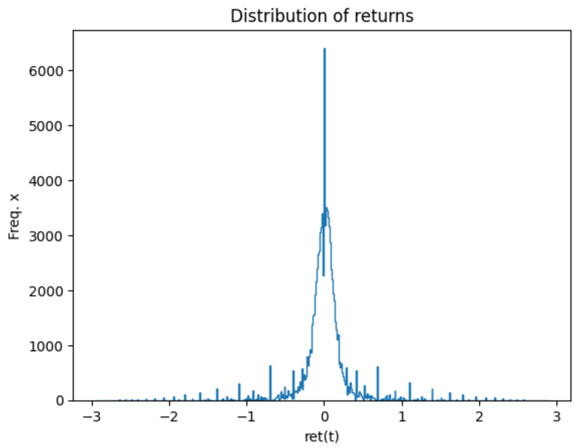

Figure 3.5: Distribution of the returns of the asset.

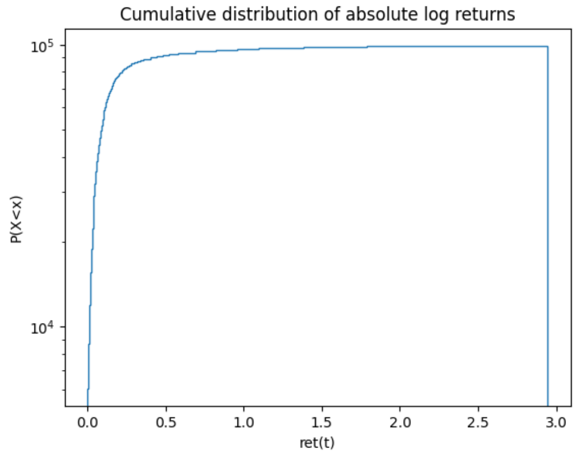

Figure 3.6: Cumulative distribution of the returns of the asset.

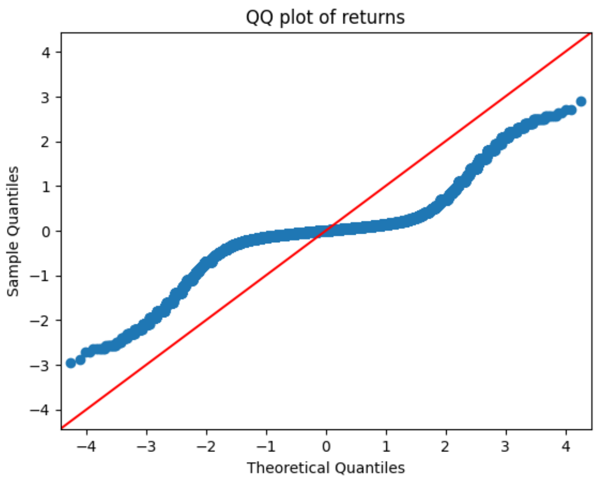

Figure 3.7: QQ-plot of the returns against a normal distribution.

Autocorrelation

The autocorrelation function of the returns is shown in figure 3.8. We can see that the autocorrelation is significant for a large number of lags, which is consistent with the empirical observation that financial returns are autocorrelated.

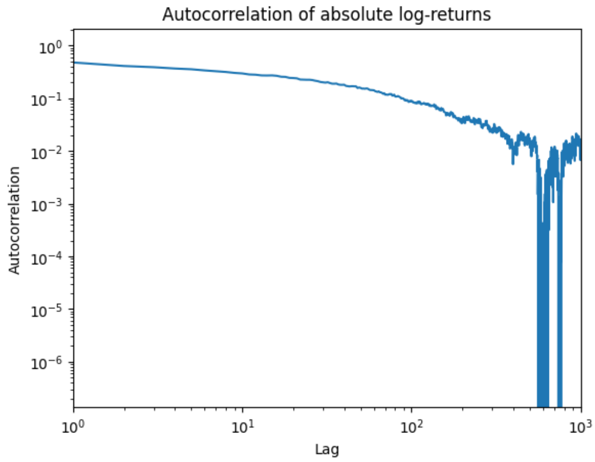

Figure 3.8: Autocorrelation function of the returns.

3.2 Financial implications

The results of the simulation suggest that the Bornholdt model is able to reproduce some of the statistical properties of real financial data, such as fat tails and autocorrelation of returns. This is interesting because the model is based on a simple spin model, and does not rely on any assumptions about the rationality of agents or the efficiency of markets.

The key assumptions of models such as [Black and Scholes, 1973] are that the price of the underlying asset follows a geometric Brownian motion, meaning that the distribution of prices in any finite period is log-normal. The Bornholdt model, however, results in price dynamics which exhibit fat tails, and do not have independent increments, as shown by the autocorrelation of the returns. This is the case also in real financial data.

Chapter 4

Simple option pricing with the Bornholdt model

Chapter 5

Conclusion

In this thesis, we have explored [briefly summarize the main topics or findings]. The results indicate that [summarize key findings].

The contributions of this work include [list contributions]. Future work could focus on [suggest future research directions].

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