Notes on Inference Devices

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Notation and Definitions

U Set of possible histories of the universe.

u A history of the universe in U.

X Setup function of an ID that maps $U \to X(U)$. A binary question concerning $\Gamma(u)$.

x A binary question and a member of image $X(U)$.

Y Conclusion function of an ID that maps $U \to \{-1, 1\}$. A binary answer of an ID for $X(u) = x$.

y A single-valued answer, and member of image $Y(U) = \{0, 1\}$.

Γ A function of the actual values of a physical variable over U, equivalent to $\Gamma(u) = S(t_i)(u)$.

γ Possible value of a physical variable, a member of the image $\Gamma(U)$.

δ Probe of any variable V parameterized by $v \in V$ such that :

$$\delta_v(v') = \begin{cases} 1 & \text{if } v = v' \\ -1 & \text{otherwise} \end{cases}$$

$\wp$ Set of probes over $\Gamma(U)$.

$\mathcal{D} = (X, Y)$ An inference device, consisting of functions X and Y.

$\bar{F}$ Inverse. Given a function F over U, $F^{-1} = \bar{F} \equiv \{\{u : F(u) = f\} : f \in F(U)\}$.

$>$ Weak inference: a device $\mathcal{D}$ weakly infers Γ *iff* $\forall \gamma \in \Gamma(U), \exists x \in X(U)$ s.t. $\forall u \in U$, $X(u) = x \implies Y(u) = \delta_\gamma(\Gamma(u))$.

$>>$ Strong inference: a device (X_1, Y_1) strongly infers a device (X_2, Y_2) *iff* $\forall \delta \in \wp(Y_2)$ and all x_2, $\exists x_1$ *such that* $X_1 = x_1 \implies X_2 = x_2, Y_1 = \delta(Y_2)$.

Turing Machines

Arora and Barak denote a Turing Machine T as $T = (\Gamma, Q, \delta)$ containing:

1. An *alphabet* Γ of a finite set of symbols that T's tapes can contain. We assume that Γ contains a special blank symbol B, start symbol S, and the numbers 0 and 1.

2. A finite set Q of possible states that T's register can be in. We assume that Q contains a special start state q_s and a special halt state q_h.

3. A transition function function $\delta : Q \times \Gamma^k \to Q \times \Gamma^{k-1} \times \{L, S, R\}^k$, where $k \geq 2$, describing the rules T use in performing each step. The set $\{L, S, R\}$ denote the actions *Left, Stay,* and *Right*, respectively.

Suppose T is in state $q \in Q$ and $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ are the symbols on the k tapes. Then $\delta(q, (\sigma_1, \ldots, \sigma_k)) = (q', (\sigma_2^{'}, \ldots, \sigma_k^{'}), z)$ where $z \in \{L, S, R\}^k$ and at the next step the σ symbols in the last $k-1$ tapes will be replaced by the σ' symbols, the machine will be in state q, and the k heads will move *Left, Right* or *Stay.*

Remark: Γ can be reduced to $\mathbb{B} = \{0, 1\}$ and k can be reduced to 1 without loss of computational power.

Inference of Turing Machines

Theorem *Every Turing Machine can be weakly inferred by an inference device.*
Proof Recall the definition of weak inference:

$$\mathcal{D} > \Gamma \textit{ iff } \forall \gamma \in \Gamma(U), \exists x \in X(U) \text{ such that } \forall u \in U, X(u) = x \implies Y(u) = \delta_\gamma(\Gamma(u))$$

Theorem *Every Turing Machine can be strongly inferred by an inference device.*
Proof Recall the definition of strong inference:

$$(X_1, Y_1) >> (X_2, Y_2) \textit{ iff } \forall \delta \in \wp(Y_2) \text{ and all } x_2, \exists x_1 \textit{ such that } X_1 = x_1 \implies X_2 = x_2, Y_1 = \delta(Y_2)$$