exercises35

August 29, 2025

0.1 Exercise 1

a)

Consider the expression

$\frac{\delta(\mathbf{a}^T\mathbf{x})}{\delta\mathbf{x}}$

Both $\mathbf{a}$ and $\mathbf{x}$ are column vectors with length n, so $a, x \in \mathbb{R}^n$.

The dot product of $\mathbf{a}^T \cdot \mathbf{x}$ is a scalar and the shape is (1×1).

We are taking the derivative with respect to $\mathbf{x}$ as in $\delta\mathbf{x}$, which has the shape $(n \times 1)$.

The result of the expression is $\mathbf{a}$ with shape $(n \times 1)$.

b)

Show that

$\frac{\delta(\mathbf{a}^T\mathbf{x})}{\delta\mathbf{x}} = \mathbf{a}^T$

Writing the scalar as a sum and differentiating by $\mathbf{x}$'s component j.

$a^T x = \sum_{i=1}^n a_i x_i \; \frac{\delta}{\delta x_j}(a^T x) = \sum_{i=1}^n a_i \frac{\delta x_i}{\delta x_j} = \sum_{i=1}^n a_i \delta_i = a_j$

c)

Show that

$\frac{\delta(\mathbf{a}^T\mathbf{A}\mathbf{a})}{\delta\mathbf{a}} = \mathbf{a}^T \left(\mathbf{A}+\mathbf{A}^T\right)$

0.2 Exercise 2

a)

We minimize the squared error between the model $\mathbf{X}\theta$ and the true values y. This is a quadratic and convex function. Therefore, taking the derivate and setting it equal to 0, we can therefore find the global minima for each parameter θ.

b)

If $\mathbf{X}$ is invertible, we can solve the model $y = X\theta$ by multiplying by X^{-1} on both sides yielding $\hat{\theta} = X^{-1}y$.

c)

Show that

$\frac{\delta(x-As)^T(x-As)}{\delta s} = -2(x-As)^T A$

Expand first

$(x-As)^T(x-As) = x^T x - 2x^T As + s^T A^T As$

Differensiate

$\frac{\delta}{\delta s}((x-As)^T(x-As)) = -2x^T a + 2s^T A^T A = -2(x-As)^T A$

d)

Using the equation from c)

$-2(x-As)^T A$

Substituting $\theta = s$, $y = x$, $X = A$, gives the gradient.

$-2(y-X\theta)^T X$

Setting equal to 0.

$-2(y-X\theta)^T X = 0$

This gives

$X^T y = X^T X\theta$

Which in turn give

$\hat{\theta}_{ols} = (X^T X)^{-1} X^T y$

0.3 Exercise 3

[247]:
```
import numpy as np

n = 20
income = np.array([116., 161., 167., 118., 172., 163., 179., 173., 162., 116.,␣
 ↪101., 176., 178., 172., 143., 135., 160., 101., 149., 125.])
children = np.array([5, 3, 0, 4, 5, 3, 0, 4, 4, 3, 3, 5, 1, 0, 2, 3, 2, 1, 5,␣
 ↪4])
spending = np.array([152., 141., 102., 136., 161., 129.,  99., 159., 160., 107.
 ↪,  98., 164., 121.,  93., 112., 127., 117.,  69., 156., 131.])
```

[248]:
```
# a)

X = np.zeros((n, 3))
# first try: X[:, 0] = np.array([1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,␣
 ↪1, 1, 1, 1, 1])
# second try:
X[:, 0] = np.ones(n)
X[:, 1] = income
X[:, 2] = children

print(X)
```

```
[[  1. 116.   5.]
 [  1. 161.   3.]
 [  1. 167.   0.]
 [  1. 118.   4.]
 [  1. 172.   5.]
 [  1. 163.   3.]
 [  1. 179.   0.]
 [  1. 173.   4.]
 [  1. 162.   4.]
 [  1. 116.   3.]
 [  1. 101.   3.]
 [  1. 176.   5.]
 [  1. 178.   1.]
 [  1. 172.   0.]
 [  1. 143.   2.]
 [  1. 135.   3.]
 [  1. 160.   2.]
 [  1. 101.   1.]
 [  1. 149.   5.]
 [  1. 125.   4.]]
```

[249]:
```
# b)

def OLS_parameters(X, y):
    return np.linalg.pinv(X) @ y

beta = OLS_parameters(X, spending)
print(beta)
```

```
[ 9.12808583  0.5119025  14.60743095]
```

0.4 Exercise 4

[250]:
```
import numpy as np
from sklearn.model_selection import train_test_split
import matplotlib.pyplot as plt
```

[251]:
```
n = 100
x = np.linspace(-3, 3, n)
y = np.exp(-x**2) + 1.5 * np.exp(-(x-2)**2) + np.random.normal(0, 0.1)
```

[252]:
```
# a)

def polynomial_features(x, p):
    n = len(x)
    X = np.zeros((n, p + 1))
    for j in range(p + 1):
        X[:, j] = x**j
```

```python
    return X

X = polynomial_features(x, 5)

print(X)
```

```
[[ 1.00000000e+00 -3.00000000e+00  9.00000000e+00 -2.70000000e+01
   8.10000000e+01 -2.43000000e+02]
 [ 1.00000000e+00 -2.93939394e+00  8.64003673e+00 -2.53964716e+01
   7.46502347e+01 -2.19426447e+02]
 [ 1.00000000e+00 -2.87878788e+00  8.28741965e+00 -2.38577232e+01
   6.86813245e+01 -1.97718964e+02]
 [ 1.00000000e+00 -2.81818182e+00  7.94214876e+00 -2.23824192e+01
   6.30777269e+01 -1.77764503e+02]
 [ 1.00000000e+00 -2.75757576e+00  7.60422406e+00 -2.09692239e+01
   5.78242235e+01 -1.59454677e+02]
 [ 1.00000000e+00 -2.69696970e+00  7.27364555e+00 -1.96168016e+01
   5.29059195e+01 -1.42685662e+02]
 [ 1.00000000e+00 -2.63636364e+00  6.95041322e+00 -1.83238167e+01
   4.83082440e+01 -1.27358098e+02]
 [ 1.00000000e+00 -2.57575758e+00  6.63452709e+00 -1.70889334e+01
   4.40169497e+01 -1.13376992e+02]
 [ 1.00000000e+00 -2.51515152e+00  6.32598714e+00 -1.59108162e+01
   4.00181133e+01 -1.00651618e+02]
 [ 1.00000000e+00 -2.45454545e+00  6.02479339e+00 -1.47881292e+01
   3.62981354e+01 -8.90954232e+01]
 [ 1.00000000e+00 -2.39393939e+00  5.73094582e+00 -1.37195370e+01
   3.28437400e+01 -7.86259231e+01]
 [ 1.00000000e+00 -2.33333333e+00  5.44444444e+00 -1.27037037e+01
   2.96419753e+01 -6.91646091e+01]
 [ 1.00000000e+00 -2.27272727e+00  5.16528926e+00 -1.17392938e+01
   2.66802131e+01 -6.06368480e+01]
 [ 1.00000000e+00 -2.21212121e+00  4.89348026e+00 -1.08249715e+01
   2.39461490e+01 -5.29717842e+01]
 [ 1.00000000e+00 -2.15151515e+00  4.62901745e+00 -9.95940117e+00
   2.14278025e+01 -4.61022418e+01]
 [ 1.00000000e+00 -2.09090909e+00  4.37190083e+00 -9.14124718e+00
   1.91135168e+01 -3.99646261e+01]
 [ 1.00000000e+00 -2.03030303e+00  4.12213039e+00 -8.36917383e+00
   1.69919590e+01 -3.44988258e+01]
 [ 1.00000000e+00 -1.96969697e+00  3.87970615e+00 -7.64184545e+00
   1.50521198e+01 -2.96481148e+01]
 [ 1.00000000e+00 -1.90909091e+00  3.64462810e+00 -6.95792637e+00
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 [ 1.00000000e+00 -1.84848485e+00  3.41689624e+00 -6.31608092e+00
   1.16751799e+01 -2.15813931e+01]
 [ 1.00000000e+00 -1.78787879e+00  3.19651056e+00 -5.71497343e+00
   1.02176798e+01 -1.82679729e+01]
```

```
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[ 1.00000000e+00 -1.42424242e+00  2.02846648e+00 -2.88902802e+00
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[ 1.00000000e+00 -4.54545455e-01  2.06611570e-01 -9.39143501e-02
  4.26883410e-02 -1.94037913e-02]
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  5.53240899e-03  1.50883882e-03]
[ 1.00000000e+00  3.33333333e-01  1.11111111e-01  3.70370370e-02
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  1.63991531e-01  1.04358247e-01]
[ 1.00000000e+00  6.96969697e-01  4.85766758e-01  3.38564710e-01
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[ 1.00000000e+00  7.57575758e-01  5.73921028e-01  4.34788658e-01
  3.29385347e-01  2.49534354e-01]
[ 1.00000000e+00  8.18181818e-01  6.69421488e-01  5.47708490e-01
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[ 1.00000000e+00  8.78787879e-01  7.72268136e-01  6.78659877e-01
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[ 1.00000000e+00  1.06060606e+00  1.12488522e+00  1.19306008e+00
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[ 1.00000000e+00  1.12121212e+00  1.25711662e+00  1.40949439e+00
  1.58034220e+00  1.77189883e+00]
```

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[ 1.00000000e+00  1.30303030e+00  1.69788797e+00  2.21239948e+00
  2.88282356e+00  3.75640646e+00]
[ 1.00000000e+00  1.36363636e+00  1.85950413e+00  2.53568745e+00
  3.45775562e+00  4.71512130e+00]
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[ 1.00000000e+00  1.48484848e+00  2.20477502e+00  3.27375685e+00
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[ 1.00000000e+00  1.66666667e+00  2.77777778e+00  4.62962963e+00
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  8.90109965e+00  1.53746267e+01]
[ 1.00000000e+00  1.78787879e+00  3.19651056e+00  5.71497343e+00
  1.02176798e+01  1.82679729e+01]
[ 1.00000000e+00  1.84848485e+00  3.41689624e+00  6.31608092e+00
  1.16751799e+01  2.15813931e+01]
[ 1.00000000e+00  1.90909091e+00  3.64462810e+00  6.95792637e+00
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[ 1.00000000e+00  1.96969697e+00  3.87970615e+00  7.64184545e+00
  1.50521198e+01  2.96481148e+01]
[ 1.00000000e+00  2.03030303e+00  4.12213039e+00  8.36917383e+00
  1.69919590e+01  3.44988258e+01]
[ 1.00000000e+00  2.09090909e+00  4.37190083e+00  9.14124718e+00
  1.91135168e+01  3.99646261e+01]
[ 1.00000000e+00  2.15151515e+00  4.62901745e+00  9.95940117e+00
  2.14278025e+01  4.61022418e+01]
[ 1.00000000e+00  2.21212121e+00  4.89348026e+00  1.08249715e+01
  2.39461490e+01  5.29717842e+01]
[ 1.00000000e+00  2.27272727e+00  5.16528926e+00  1.17392938e+01
  2.66802131e+01  6.06368480e+01]
[ 1.00000000e+00  2.33333333e+00  5.44444444e+00  1.27037037e+01
  2.96419753e+01  6.91646091e+01]
[ 1.00000000e+00  2.39393939e+00  5.73094582e+00  1.37195370e+01
  3.28437400e+01  7.86259231e+01]
[ 1.00000000e+00  2.45454545e+00  6.02479339e+00  1.47881292e+01
  3.62981354e+01  8.90954232e+01]
[ 1.00000000e+00  2.51515152e+00  6.32598714e+00  1.59108162e+01
  4.00181133e+01  1.00651618e+02]
[ 1.00000000e+00  2.57575758e+00  6.63452709e+00  1.70889334e+01
  4.40169497e+01  1.13376992e+02]
```

```
 [ 1.00000000e+00  2.63636364e+00  6.95041322e+00  1.83238167e+01
   4.83082440e+01  1.27358098e+02]
 [ 1.00000000e+00  2.69696970e+00  7.27364555e+00  1.96168016e+01
   5.29059195e+01  1.42685662e+02]
 [ 1.00000000e+00  2.75757576e+00  7.60422406e+00  2.09692239e+01
   5.78242235e+01  1.59454677e+02]
 [ 1.00000000e+00  2.81818182e+00  7.94214876e+00  2.23824192e+01
   6.30777269e+01  1.77764503e+02]
 [ 1.00000000e+00  2.87878788e+00  8.28741965e+00  2.38577232e+01
   6.86813245e+01  1.97718964e+02]
 [ 1.00000000e+00  2.93939394e+00  8.64003673e+00  2.53964716e+01
   7.46502347e+01  2.19426447e+02]
 [ 1.00000000e+00  3.00000000e+00  9.00000000e+00  2.70000000e+01
   8.10000000e+01  2.43000000e+02]]
```

[253]:
```
# b)

beta = OLS_parameters(X, y)
print(beta)
```

```
[ 0.93971968  0.27464654 -0.02326439  0.05342623 -0.0034652  -0.0087781 ]
```

[254]:
```
# c)

from sklearn.model_selection import train_test_split

X_train, X_test, y_train, y_test = train_test_split(X, y, test_size = 0.3,
 ↪random_state = 1705)

print(X_train, X_test, y_train, y_test)
```

```
[[ 1.00000000e+00 -9.09090909e-02  8.26446281e-03 -7.51314801e-04
   6.83013455e-05 -6.20921323e-06]
 [ 1.00000000e+00 -2.09090909e+00  4.37190083e+00 -9.14124718e+00
   1.91135168e+01 -3.99646261e+01]
 [ 1.00000000e+00  1.12121212e+00  1.25711662e+00  1.40949439e+00
   1.58034220e+00  1.77189883e+00]
 [ 1.00000000e+00 -9.39393939e-01  8.82460973e-01 -8.28978490e-01
   7.78737370e-01 -7.31541165e-01]
 [ 1.00000000e+00  2.87878788e+00  8.28741965e+00  2.38577232e+01
   6.86813245e+01  1.97718964e+02]
 [ 1.00000000e+00 -2.93939394e+00  8.64003673e+00 -2.53964716e+01
   7.46502347e+01 -2.19426447e+02]
 [ 1.00000000e+00 -3.93939394e-01  1.55188246e-01 -6.11347636e-02
   2.40833917e-02 -9.48739674e-03]
 [ 1.00000000e+00  1.84848485e+00  3.41689624e+00  6.31608092e+00
   1.16751799e+01  2.15813931e+01]
 [ 1.00000000e+00  2.75757576e+00  7.60422406e+00  2.09692239e+01
   5.78242235e+01  1.59454677e+02]
```

```
[ 1.00000000e+00 -2.57575758e+00  6.63452709e+00 -1.70889334e+01
  4.40169497e+01 -1.13376992e+02]
[ 1.00000000e+00 -2.45454545e+00  6.02479339e+00 -1.47881292e+01
  3.62981354e+01 -8.90954232e+01]
[ 1.00000000e+00 -2.33333333e+00  5.44444444e+00 -1.27037037e+01
  2.96419753e+01 -6.91646091e+01]
[ 1.00000000e+00 -2.69696970e+00  7.27364555e+00 -1.96168016e+01
  5.29059195e+01 -1.42685662e+02]
[ 1.00000000e+00 -5.75757576e-01  3.31496786e-01 -1.90861786e-01
  1.09890119e-01 -6.32700686e-02]
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  5.29059195e+01  1.42685662e+02]
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[ 1.00000000e+00  1.78787879e+00  3.19651056e+00  5.71497343e+00
  1.02176798e+01  1.82679729e+01]
[ 1.00000000e+00  3.03030303e-02  9.18273646e-04  2.78264741e-05
  8.43226488e-07  2.55523178e-08]
[ 1.00000000e+00 -1.42424242e+00  2.02846648e+00 -2.88902802e+00
  4.11467627e+00 -5.86029651e+00]
[ 1.00000000e+00 -1.60606061e+00  2.57943067e+00 -4.14272199e+00
  6.65346258e+00 -1.06858641e+01]
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[ 1.00000000e+00 -2.12121212e-01  4.49954086e-02 -9.54448062e-03
  2.02458680e-03 -4.29457806e-04]
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  7.78737370e-01  7.31541165e-01]
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  1.58034220e+00 -1.77189883e+00]
[ 1.00000000e+00  6.96969697e-01  4.85766758e-01  3.38564710e-01
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  1.16751799e+01 -2.15813931e+01]
[ 1.00000000e+00  4.54545455e-01  2.06611570e-01  9.39143501e-02
  4.26883410e-02  1.94037913e-02]
[ 1.00000000e+00  8.78787879e-01  7.72268136e-01  6.78659877e-01
  5.96398074e-01  5.24107398e-01]
[ 1.00000000e+00  1.72727273e+00  2.98347107e+00  5.15326822e+00
  8.90109965e+00  1.53746267e+01]
[ 1.00000000e+00 -4.54545455e-01  2.06611570e-01 -9.39143501e-02
  4.26883410e-02 -1.94037913e-02]
[ 1.00000000e+00 -7.57575758e-01  5.73921028e-01 -4.34788658e-01
  3.29385347e-01 -2.49534354e-01]
[ 1.00000000e+00 -1.00000000e+00  1.00000000e+00 -1.00000000e+00
  1.00000000e+00 -1.00000000e+00]
```

```
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  1.23456790e-02 -4.11522634e-03]
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[ 1.00000000e+00  2.45454545e+00  6.02479339e+00  1.47881292e+01
  3.62981354e+01  8.90954232e+01]
[ 1.00000000e+00 -1.18181818e+00  1.39669421e+00 -1.65063862e+00
  1.95075473e+00 -2.30543741e+00]
[ 1.00000000e+00  3.33333333e-01  1.11111111e-01  3.70370370e-02
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[ 1.00000000e+00  1.42424242e+00  2.02846648e+00  2.88902802e+00
  4.11467627e+00  5.86029651e+00]
[ 1.00000000e+00  1.66666667e+00  2.77777778e+00  4.62962963e+00
  7.71604938e+00  1.28600823e+01]
[ 1.00000000e+00 -2.21212121e+00  4.89348026e+00 -1.08249715e+01
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[ 1.00000000e+00 -1.30303030e+00  1.69788797e+00 -2.21239948e+00
  2.88282356e+00 -3.75640646e+00]
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[ 1.00000000e+00 -1.51515152e-01  2.29568411e-02 -3.47830926e-03
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  1.02176798e+01 -1.82679729e+01]
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  1.50521198e+01 -2.96481148e+01]
[ 1.00000000e+00  2.51515152e+00  6.32598714e+00  1.59108162e+01
  4.00181133e+01  1.00651618e+02]
[ 1.00000000e+00  2.27272727e+00  5.16528926e+00  1.17392938e+01
  2.66802131e+01  6.06368480e+01]
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[ 1.00000000e+00 -2.39393939e+00  5.73094582e+00 -1.37195370e+01
  3.28437400e+01 -7.86259231e+01]
```

```
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   1.09890119e-01  6.32700686e-02]
 [ 1.00000000e+00  1.36363636e+00  1.85950413e+00  2.53568745e+00
   3.45775562e+00  4.71512130e+00]
 [ 1.00000000e+00  2.03030303e+00  4.12213039e+00  8.36917383e+00
   1.69919590e+01  3.44988258e+01]
 [ 1.00000000e+00  2.21212121e+00  4.89348026e+00  1.08249715e+01
   2.39461490e+01  5.29717842e+01]
 [ 1.00000000e+00  1.48484848e+00  2.20477502e+00  3.27375685e+00
   4.86103290e+00  7.21789734e+00]
 [ 1.00000000e+00 -2.51515152e+00  6.32598714e+00 -1.59108162e+01
   4.00181133e+01 -1.00651618e+02]
 [ 1.00000000e+00  2.12121212e-01  4.49954086e-02  9.54448062e-03
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 [ 1.00000000e+00  9.09090909e-02  8.26446281e-03  7.51314801e-04
   6.83013455e-05  6.20921323e-06]
 [ 1.00000000e+00 -6.96969697e-01  4.85766758e-01 -3.38564710e-01
   2.35969344e-01 -1.64463482e-01]
 [ 1.00000000e+00 -2.81818182e+00  7.94214876e+00 -2.23824192e+01
   6.30777269e+01 -1.77764503e+02]
 [ 1.00000000e+00  2.15151515e+00  4.62901745e+00  9.95940117e+00
   2.14278025e+01  4.61022418e+01]
 [ 1.00000000e+00  7.57575758e-01  5.73921028e-01  4.34788658e-01
   3.29385347e-01  2.49534354e-01]] [[ 1.00000000e+00  2.33333333e+00
5.44444444e+00  1.27037037e+01
   2.96419753e+01  6.91646091e+01]
 [ 1.00000000e+00 -1.90909091e+00  3.64462810e+00 -6.95792637e+00
   1.32833140e+01 -2.53590540e+01]
 [ 1.00000000e+00 -2.15151515e+00  4.62901745e+00 -9.95940117e+00
   2.14278025e+01 -4.61022418e+01]
 [ 1.00000000e+00 -1.24242424e+00  1.54361800e+00 -1.91782842e+00
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 [ 1.00000000e+00 -1.06060606e+00  1.12488522e+00 -1.19306008e+00
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 [ 1.00000000e+00 -2.75757576e+00  7.60422406e+00 -2.09692239e+01
   5.78242235e+01 -1.59454677e+02]
 [ 1.00000000e+00 -8.78787879e-01  7.72268136e-01 -6.78659877e-01
   5.96398074e-01 -5.24107398e-01]
```

```
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 [ 1.00000000e+00 -2.87878788e+00  8.28741965e+00 -2.38577232e+01
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 [ 1.00000000e+00  2.39393939e+00  5.73094582e+00  1.37195370e+01
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 [ 1.00000000e+00  1.24242424e+00  1.54361800e+00  1.91782842e+00
   2.38275652e+00  2.96039447e+00]
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 [ 1.00000000e+00  8.18181818e-01  6.69421488e-01  5.47708490e-01
   4.48125128e-01  3.66647832e-01]
 [ 1.00000000e+00 -1.36363636e+00  1.85950413e+00 -2.53568745e+00
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   7.04271195e-02 -3.62806373e-02]
 [ 1.00000000e+00  1.30303030e+00  1.69788797e+00  2.21239948e+00
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 [ 1.00000000e+00 -2.72727273e-01  7.43801653e-02 -2.02854996e-02
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 [ 1.00000000e+00 -3.03030303e-02  9.18273646e-04 -2.78264741e-05
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 [ 1.00000000e+00  1.90909091e+00  3.64462810e+00  6.95792637e+00
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0.80396184 0.11094093
 0.97188404 1.60953517 0.9562309  0.11207825 0.1131821  0.1150843
 0.11145763 0.83058381 1.03429556 0.62330191 1.1261846  1.58567155
 1.14083154 0.24231326 0.18658458 0.73158603 1.07800886 1.01156205
 0.39532529 1.00059398 0.14357873 1.06176008 0.99943776 1.55385895
 0.92772664 0.6748239  0.47882861 1.01208373 1.422541   1.33318537
 0.35823772 1.09886815 1.31907383 1.51519955 0.11825937 1.11500801
 1.63004417 0.29386123 0.87912653 1.10271523 1.18885094 1.08074376
 0.22104736 0.15166962 0.77920578 0.11172229 0.13142116 1.2629237
 1.50895353 0.16138227 0.11400806 1.05609888 1.02591813 1.26702045
```

```
 1.62559722 1.55226215 1.37140984 0.11255325 1.12812296 1.14173033
 0.7270297  0.11111949 1.58648539 0.9944867 ] [1.45734326 0.13689552 0.12052845
0.32441168 1.61104566 0.66270662
 1.03046265 0.43558199 0.12697413 0.11126239 0.57310563 1.01139319
 1.11222885 0.11101571 1.39838902 1.16933943 0.17294275 1.04309068
 1.47096217 0.99388901 0.26653219 0.88036148 1.21667174 1.04764988
 1.13416111 1.1372899  0.11088746 0.20254294 0.11647548 1.62454957]
```

[255]:
```
# d)

bh = OLS_parameters(X_train, y_train)

yh_train = X_train @ bh
yh_test = X_test @ bh

MSE_train = np.mean((y_train - yh_train)**2)
MSE_test = np.mean((y_test - yh_test)**2)

print(MSE_train)
print(MSE_test)
```

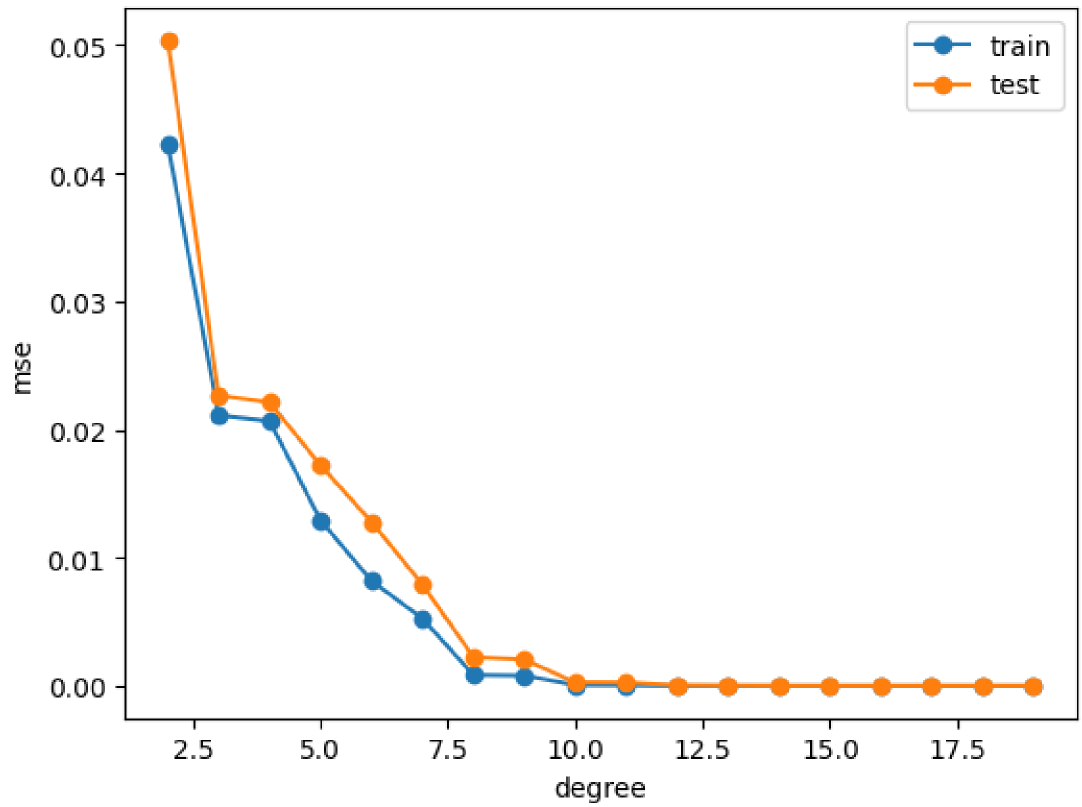
```
0.012923446821553504
0.017212234759576596
```

[256]:
```
# e)

degrees = range(2, 20)
train_mse, test_mse = [], []

for p in degrees:
    X = polynomial_features(x, p)
    X_train, X_test, y_train, y_test = train_test_split(X, y, test_size=0.3,␣
 ↪random_state=1705)
    beta = OLS_parameters(X_train, y_train)
    yhat_tr = X_train @ beta
    yhat_te = X_test  @ beta
    train_mse.append(np.mean((y_train - yhat_tr)**2))
    test_mse.append(np.mean((y_test  - yhat_te)**2))

plt.plot(degrees, train_mse, marker='o', label='train')
plt.plot(degrees, test_mse, marker='o', label='test')
plt.xlabel('degree')
plt.ylabel('mse')
plt.legend()
plt.show()
```



f)

Well, it does'nt seem as though I was able to reproduce the test decreasing steadily before increasing for some time.

Although in general that shape of train mse always decreasing and test mse at some point increasing again speaks to the fact that the model at some point starts capturing patterns in the noise that aren't really part of the true estimated value.

0.5 Exercise 5

[257]:
```
import numpy as np
from sklearn.preprocessing import PolynomialFeatures
from sklearn.linear_model import LinearRegression
```

[258]:
```
# a)

p = 10

x1 = polynomial_features(x,p)
x2 = PolynomialFeatures(degree=p, include_bias=True).fit_transform(x.
 ↪reshape(-1,1))
```

```python
print(x1.shape, x2.shape)
print(np.allclose(x1, x2))
print(np.max(np.abs(x1-x2)))
```

```
(100, 11) (100, 11)
True
1.4551915228366852e-11
```

[259]:

```python
# b)

p = 15
X = polynomial_features(x, p)

b1 = OLS_parameters(X, y)
lr = LinearRegression(fit_intercept=False)
lr.fit(X, y)

b2 = lr.coef_

print(b1.shape, b2.shape)
print(np.allclose(b1, b2))
print(np.max(np.abs(b1-b2)))
```

```
(16,) (16,)
True
3.1086244689504383e-15
```