STK2130
Oblig 1

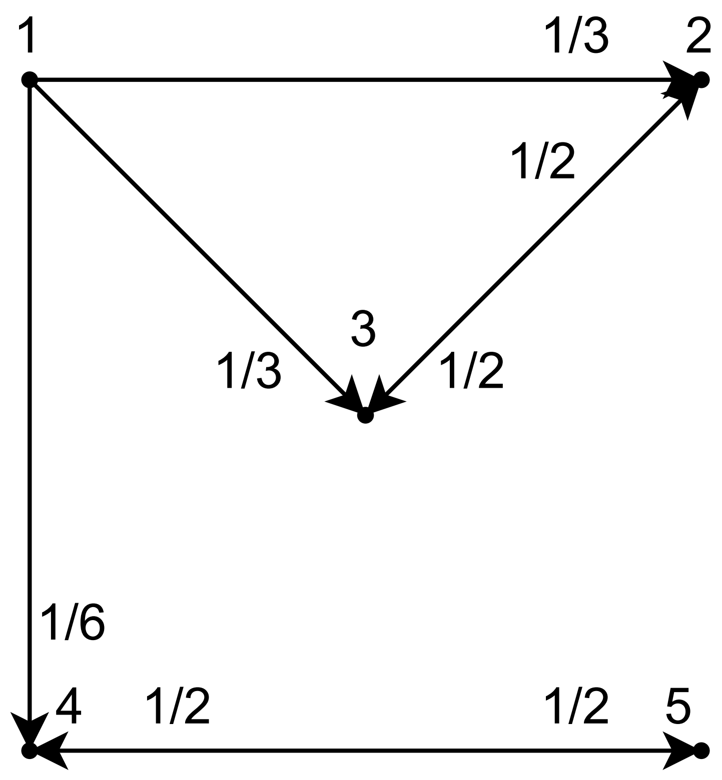
Egil Furnes
Studentnummer: 693784

Task 1

(i)

Below is a figure representation of the transition matrix P, where I have not drawn the probability of $i \to i$.

$$P = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



(ii)

Two states i and j communicates if it is possible to go from i to j and j to i with positive probability in some number of steps. Therefore, the communicating classes are $\{1\}$, $\{2,3\}$, and $\{4,5\}$, and both $\{2,3\}$ and $\{4,5\}$ are closed classes, if one enters them, one will not escape them.

(iii)

Determine for all $i, j = 1, \dots, 5$ the probability that $X_3 = j$ given that $X_0 = i$.

$$P^3(i,j) = \sum_{k=1}^{5}\sum_{l=1}^{5} P(i,k)\cdot P(k,l)\cdot P(l,j)$$

$P^3(1,1)$:

$$\left(\frac{1}{6}\right)^3 = \frac{1}{216}$$

$P^3(1,2) = P^3(1,3)$:

$$\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{4}{12} = \frac{1}{3}$$

$P^3(1,4) = P^3(1,5)$:

$$\frac{1}{6}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{6}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{2}{24} = \frac{1}{12}$$

$P^3(2,1) = P^3(2,4) = P^3(2,5) = P^3(3,1) = P^3(3,4) = P^3(3,5)$:

$$0$$

$P^3(2,2) = P^3(2,3) = P^3(3,2) = P^3(3,3) = P^3(4,5) = P^3(4,4) = P^3(5,4) = P^3(5,5)$:

$$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{4}{8} = \frac{1}{2}$$

The matrix of probabilities becomes:

$$P^3 = \begin{pmatrix} \frac{1}{216} & \frac{1}{3} & \frac{1}{3} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Task 2

(i)

Let $f_i = \mathbb{P}_i(S_n \text{ hits 0 for some } n \geq 0)$ for $i = 0, 1, 2, 3$.

Because of 0 being absorbing, we can let $f_0 = 1$. I write up the hitting probabilities for $i = 1, 2, 3$.

$$\begin{aligned}
f_1 &= 0.05 \cdot f_0 + 0.4 \cdot f_1 + 0.35 \cdot f_2 + 0.2 \cdot f_3 \\
f_2 &= 0.05 \cdot f_0 + 0.35 \cdot f_1 + 0.35 \cdot f_2 + 0.25 \cdot f_3 \\
f_3 &= 0.02 \cdot f_0 + 0.4 \cdot f_1 + 0.5 \cdot f_2 + 0.08 \cdot f_3
\end{aligned}$$

Substituting $f_0 = 1$ and rearranging:

$$\begin{aligned}
0.6f_1 - 0.35f_2 - 0.2f_3 &= 0.05 \\
-0.35f_1 + 0.65f_2 - 0.25f_3 &= 0.05 \\
-0.4f_1 - 0.5f_2 + 0.92f_3 &= 0.02
\end{aligned}$$

Solving this system yields:

$$f_1 = f_2 = f_3 = 1$$

Hence, starting from any nonzero state, the stock will eventually hit \$0 with probability 1.

(ii)

Let $h_i = \mathbb{E}_i[H_0]$ be the expected number of steps until the process reaches state 0, starting from state i. Clearly, $h_0 = 0$.

From the transition matrix, we obtain the following system of equations for h_1, h_2, and h_3:

$$\begin{aligned}
h_1 &= 1 + 0.4h_1 + 0.35h_2 + 0.2h_3 \\
h_2 &= 1 + 0.35h_1 + 0.35h_2 + 0.25h_3 \\
h_3 &= 1 + 0.4h_1 + 0.5h_2 + 0.08h_3
\end{aligned}$$

Rewriting:

$$
\begin{aligned}
0.6h_1 - 0.35h_2 - 0.2h_3 &= 1 \\
-0.35h_1 + 0.65h_2 - 0.25h_3 &= 1 \\
-0.4h_1 - 0.5h_2 + 0.92h_3 &= 1
\end{aligned}
$$

Solving this system yields:

$$
\begin{aligned}
h_1 &= 22.66 \\
h_2 &= 22.69 \\
h_3 &= 23.27
\end{aligned}
$$

(iii)

Define $H_{\{3\}} = \inf\{n \geq 0 : S_n = 3\}$ as the hitting time of state 3, and let $k_i = \mathbb{E}_i[H_{\{3\}}]$ and $k_3 = 0$.

For $i = 1, 2$ I use the hitting time recurrence:

$$
\begin{aligned}
k_1 &= 1 + 0.05 \cdot k_0 + 0.4 \cdot k_1 + 0.35 \cdot k_2 \\
k_2 &= 1 + 0.05 \cdot k_0 + 0.35 \cdot k_1 + 0.35 \cdot k_2
\end{aligned}
$$

Assuming $k_0 = 0$ (i.e., stopping once we hit 0 or 3), the system becomes:

$$
\begin{aligned}
0.6k_1 - 0.35k_2 &= 1 \\
-0.35k_1 + 0.65k_2 &= 1
\end{aligned}
$$

Solving this system yields:

$$
\begin{aligned}
k_1 &= 3.74 \\
k_2 &= 3.55
\end{aligned}
$$

Thus, the expected time to reach the maximal stock price of \$3 starting from \$1 is approximately 3.74 weeks.

(iv)

A state is *recurrent* if the process returns to it with probability 1, and *transient* if there is a positive probability that the process never returns. State 0 is recurrent and absorbing, since $P(0,0) = 1$ and once entered, the process never leaves. States 1, 2, and 3 are transient. Each of them has a non-zero probability of transitioning to state 0, which is absorbing. Once the process reaches state 0, it cannot return to any other state. The only recurrent state is 0. All other states are transient.