Modeling by Stochastic Processes (STK 2130)
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2 Problems out of 5 are supposed to be solved.
Deadline: Thursday, 3. April, 14:30, electronic submission via Canvas

Problem 1 Consider a Markov chain $X_n, n \geq 0$ with state space $I = \{1, 2, 3, 4, 5\}$ and transition matrix P given by

$$P = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(i) Describe the Markov chain by a diagram.
(ii) Indentify the communicating classes of the Markov chain. Which classes are closed?

(iii) Determine for all $i, j = 1, ..., 5$ the probability that $X_3 = j$ given that $X_0 = i$.

Problem 2 Let us consider a stock price process S_n with values

$$S_n = 0\$ \text{ or } 1\$ \text{ or } 2\$ \text{ or } 3\$$$

for all $n = 0, 1, 2, ...$ (in weeks).

Assume that $(S_n)_{n \geq 0}$ is a Markov chain with transition matrix

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.05 & 0.4 & 0.35 & 0.2 \\ 0.05 & 0.35 & 0.35 & 0.25 \\ 0.02 & 0.4 & 0.5 & 0.08 \end{pmatrix}.$$

(i) Determine for all $i = 1, 2, 3$ the probability that the stock with initial price i\$ ever reaches 0\$.
(ii) Compute for all $i = 1, 2, 3$ the expected time that the stock price starting in i\$ ever hits 0\$.
(iii) What is the mean time that the stock with initial price 1\$ attains its maximal price (i.e. 3\$)? Or more precisely: Find

$$k_1^A \overset{def}{=} E\left[H^A \middle| S_0 = 1\right],$$

where H^A is the hitting time given by

$$H^A = \inf\{n \geq 0 : S_n \in A\}$$

for $A = \{3\}$ (see e.g. Section 3.2 in the manuscript (on the course webpage)).

(iv) Which states of $(S_n)_{n\geq 0}$ are recurrent and which are transient?

Problem 3 Suppose that a population consists of a fixed number, say m, of genes in any generation. Each gene is one of two possible genetic types. If exactly i (of the m) genes of any generation are of type 1, then the next generation will have j type 1 (and $m-j$ type 2) genes with probability

$$\binom{m}{j}(\frac{i}{m})^j(\frac{m-i}{m})^{m-j}, j = 0, ..., m.$$

Denote by X_n the number of type 1 genes in the nth generation.

(i) Compute $E[X_3\,|X_0 = i]$, $i = 0, ..., m$ for $m = 4$.

(ii) Calculate the probabilities that eventually all the genes will be of type 1 provided that $X_0 = i$ ($i = 0, ..., m$ for $m = 4$).

Problem 4 A gambler has 3\$ and needs to increase it to 15\$ in a hurry. He can play a game with the following rules: A fair coin is tossed. If a player bets on the right side, he wins a sum equal to his stake, and his stake is returned. Otherwise he loses his stake. The gambler decides to use a bold strategy in which he stakes all his money if he has 7\$ or less, and otherwise stakes just enough to increase his capital, if he wins, to 15\$.

(i) Find the probability that the gambler with initial capital 3\$ will achieve his aim.

(ii) What is the expected number of tosses until the gambler either achieves his aim or loses his capital?

Problem 5 Consider a Markov chain $X_n, n \geq 0$ which is only observed at certain times. In the first instance suppose that J is some subset of the state space I and that we observe the chain only when it takes values in J. For example, in Problem 2 we may think of a trader who only wants to deal with stock price observations in $J = \{1\$, 2\$, 3\$\}$. The resulting chain $Y_m, m \geq 0$ may be obtained formally by setting

$$Y_m = X_{T_m},$$

where

$$T_0 \overset{def}{=} \inf\{n \geq 0 : X_n \in J\}$$

and, for $m = 0, 1, 2, ...$

$$T_{m+1} \overset{def}{=} \inf\{n > T_m : X_n \in J\}.$$

Assume that

$$P(T_m < \infty) = 1$$

for all m.

(i) Show that $Y_m, m \geq 0$ is a Markov chain with state space J, that is

$$\begin{aligned} & P\left(Y_{m+1} = i_{m+1} \left| Y_0 = i_0, ..., Y_m = i_m\right.\right) \\ = \; & P\left(Y_{m+1} = i_{m+1} \left| Y_m = i_m\right.\right) \end{aligned}$$

for all $i_0, ..., i_{m+1} \in J$.

(ii) Prove that the transition probabilities of $Y_m, m \geq 0$, denoted by $\overline{p}_{i,j}$, are given by

$$\overline{p}_{i,j} = P\left(X_{T_1} = j \left| X_0 = i\right.\right), i, j \in J.$$

(iii) Show that

$$\overline{p}_{i,j} = h_i^j,$$

where for $j \in J$ the vector $\left(h_i^j\right)_{i \in J}$ is the minimal non-negative solution to the linear equations

$$h_i^j = p_{ij} + \sum_{k \notin J} p_{ik} h_k^j.$$