UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: 17 Nov 2025

Examination hours: 12.00 – 14.00

This problem set consists of 3subsection.4.3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Consider a portfolio consisting of three stock positions. Their market values are

$$V_1 = 1000 \text{ NOK}, \qquad V_2 = 3500 \text{ NOK}, \qquad V_3 = 5500 \text{ NOK}.$$

The expected returns of the three assets are

$$\mathbb{E}[R_1] = 10\%, \qquad \mathbb{E}[R_2] = 15\%, \qquad \mathbb{E}[R_3] = 5\%.$$

Compute the expected return of the whole portfolio.

b (weight 10p)

A portfolio consists of two risky assets with current market values $V_1 = 400$ NOK, $V_2 = 600$ NOK. The variances of the asset returns are

$$\sigma_1^2 = 0.0004, \qquad \sigma_2^2 = 0.0009,$$

and the correlation coefficient between the two asset returns is $\rho_{12} = 0.5$. Compute the variance of the portfolio return.

(Continued on page 2.)

c (weight 10p)

Write the payoff functions (as functions of the stock price S) for the following securities:

1. A *strip*: long 1 call option and long 2 put options with the same strike K.
2. A *bull spread*: long 1 call option with strike K_1 and short 1 call option with strike K_2, where $K_1 < K_2$.

For each case, draw the payoff diagram.

Problem 2

Suppose a stock price dynamics is given by a two-period binomial model. Today's stock price is $S(0) = 1$, and in each period it goes up by a factor u and down by d, with $0 < d < 1 < u$. The interest rate is $r = 0$. Assume there is a European call option written on the stock with strike K, paying $(S_2 - K)^+$ in period two. Suppose that the strike satisfies

$$d^2 < K < ud.$$

a (weight 15p)

Derive the replicating portfolio of the call option. Argue by arbitrage that the price of the option must be equal to the cost of the replicating portfolio.

b (weight 15p)

What is the risk-neutral probability q for a stock price increase in each period? Show that the present expected value of the option payoff is equal to the option price, where the expectation is taken with respect to the risk-neutral probability.

Problem 3

Consider a two-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, probability measure $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$, interest rate $r = 0$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1,2}$ with prices given by

$$S_1(0) = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 4 \\ 2 \\ 4 \\ 2 \end{pmatrix}, \quad S_1(2) = \begin{pmatrix} 7 \\ 3 \\ 3 \\ 1 \end{pmatrix}.$$

a (weight 10p)

Find $\{\mathcal{F}_t\}_{t=0,1,2}$, the filtration generated by the price process $S_1 = \{S_1(t)\}_{t=0,1,2}$. Calculate $\mathbb{E}\left[S_1(2,\omega) \middle| \mathcal{F}_1\right]$.

(Continued on page 3.)

b (weight 20p)

Let $Q = \left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}\right)^T$ be the unique martingale measure on this market (you do not have to prove this). Consider the following optimal portfolio problem

$$\max_{H\in\mathbb{H}} \mathbb{E}\left[U\left(V\left(2\right)\right)\right]$$
$$\text{subject to } V\left(0\right) = v,$$

where v is a given strictly positive real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U\left(u\right) = 2u^{1/2}$. Compute the optimal attainable wealth, the optimal objective value and the optimal trading strategy.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion and let $a > 0$ be a constant. Define the process

$$X(t) := \frac{1}{a}\, B(a^2 t), \qquad t \geq 0.$$

The time scale of the Brownian motion is compressed by a factor a^2, while its magnitude is scaled by the factor $1/a$. Find the expected value and the variance of $X(t)$. Determine the probability distribution and the probability density function of $X(t)$. Finally, prove that the process $\{X(t) : t \geq 0\}$ is itself a standard Brownian motion.

b (weight 10p)

Let $\{X_n,\ n \geq 1\}$ be independent identically distributed random variables such that $\mathbb{E}[X_i] = a,\ i \geq 1$. Set $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1$. Find the conditional expectations:

$$\mathbb{E}[X_1 \cdot X_2 \cdot ... \cdot X_n | \mathcal{F}_k],\ n > k.$$

c (weight 10p)

Define what is a martingale with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,...,T}$ under the probability measure P. Let $Z = \{Z(t)\}_{t=0,...,T}$ be a martingale and $H = \{H(t)\}_{t=1,...,T}$ be a predictable process. Prove that the process $G = \{G(t)\}_{t=0,...,T}$ defined by

$$G(0) = 0,$$

$$G(t) = \sum_{u=1}^{t} H(u)(Z(u) - Z(u-1)),\ t = 1, ..., T.$$

is also a martingale.

(Continued on page 4.)

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: Thursday January 18, 2024

Examination hours: 15.00 – 19.00

This problem set consists of 0 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All exercises count equally.

Problem 1

a

Let Ω be a finite probability space (sample space). Define what is an algebra on Ω and what is a filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T\}$.

b

Let Ω be a finite probability space (sample space) with a filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T\}$. Furthermore, let the asset price be given by an $\mathbb{F}$-adapted stochastic process $(S(t))_{t=0}^T$ on Ω. with a filtration $\mathbb{F}$. What does it mean that $S(t)$ is $\mathcal{F}_t$-measurable and that $(S(t))_{t=0}^T$ is $\mathbb{F}$-adapted?

c

Consider a forward contract delivering the asset at time T. Argue that the forward price $F_t(T)$ of the contract at time $t \leq T$ must be

$$F_t(T) = S(t)(1+r)^{T-t}$$

where $r > 0$ is the interest rate.

d

Define what is a risk-neutral probability Q. If we have such Q, show that

$$F_t(T) = \mathbb{E}_Q[S(T) \mid \mathcal{F}_t].$$

(Continued on page 2.)

Problem 2

Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ with a probability $P(\{\omega_i\}) = 1/6$ for $i = 1, ..., 6$. Moreover, assume that $\mathcal{F} = \{\emptyset, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \Omega\}$.

a

Show that $\mathcal{F}$ is an algebra.

b

Let Z be a random variable such that $Z(\omega_i) = 1$ for $i = 1, 2, 6$ and zero otherwise. Show that Z is not $\mathcal{F}$-measurable.

c

Compute $\mathbb{E}[Z|\mathcal{F}]$, and show that $\mathbb{E}[Z|\mathcal{F}]$ is $\mathcal{F}$-measurable.

Problem 3

In this exercise you can argue by words and graphics.

a

Suppose you manage a portfolio of certain classes of stocks traded at an exchange. Explain, using the Markowitz theory, what happens with your portfolio's risk and expected return if you add another class of stocks traded at the exchange to your portfolio.

b

Suppose that you have access to bonds that give a return $r > 0$. Explain, using the Markowitz theory, how you can optimally construct a portfolio of stocks and bonds which achieves an expected return higher than r, but not more than r_T with $r_T > r$. Here r_T is the expected return of the tangent portfolio of the efficient frontier and σ_T is its risk.

Problem 4

In this exercise you are going to price options based on the Black & Scholes formula for at-the-money call options. We suppose that the continuously compounding interest rate is zero, $r = 0$ and we denote by S today's price of the underlying asset.

a

State the Black & Scholes formula for the price of a call option with exercise time T and strike $K = S$ (at-the-money call option). Denote this price by C.

(Continued on page 3.)

b

Denote the price of a put option with the same exercise time T and strike $K = S$ by P. Argue that buying the call and selling the put yields the same payoff as if you buy the stock and and finance this by borrowing S in the bank. What is then the price of the put?

Problem 5

In a one-period market, suppose $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and that you have a price of an asset being $S(0) = 1$, $S(1)(\omega_1) = u, S(1)(\omega_2) = m$ and $S(1)(\omega_3) = d$. Here, $0 < d < 1 + r \leq m < u$, and the interest rate is given by $r > 0$.

a

Show that a claim paying X at time 1 is in general not possible to replicate.

b

Characterise all the risk-neutral probabilities Q on Ω.

Problem 6

Suppose the price of an asset $(S(t))_{t=0,1,2}$ is defined by a two-period binomial tree model, where the initial price $S(0) = 1$, and in each step the price goes up by a factor u or it remains the same. The interest rate is $r > 0$, and $u > 1 + r$. The probability space (sample space) is $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ in this model, where ω_1 corresponds to the asset price going up twice, ω_2 up and then down, ω_3 down and then up and finally ω_4 down twice.

a

Find $\mathcal{F}_1$, the algebra generated by $S(0)$ and $S(1)$. The risk-neutral probability Q is given by $Q(\omega_1) = q^2, Q(\omega_2) = q(1-q) = Q(\omega_3)$, $Q(\omega_4) = (1-q)^2$ and $q = \frac{r}{u-1}$. Suppose that you buy an option paying you 1 if the asset has gone down twice at time 2, and zero otherwise. Assume that the option is attainable. Under this assumption, find the value of the option at time 1 (without resorting to the replicating portfolio).

b

Find the replicating strategy for the option, and show that the value of this strategy at time 1 is the same as the one you derived in **b** above.

END

(Continued on page 4.)

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Financ and Investment Theory

Day of examination: Monday December 4, 2023

Examination hours: 15.00 – 19.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All exercises count equally.

Problem 1

Assume that a stock price today is $s > 0$ and at time 1 it is equal to su or sd for positive constants u and d, $0 < d < u$. The interest rate is $r > 0$ in the bank. Suppose you have a claim paying X, being x if the stock is su and y if the stock is sd, where $x \neq y$. In the exercises below, you may need to make additional assumptions.

a

Find a portfolio investing in the stock and the bank that replicates the claim. What is the value of this portfolio today?

b

If the price of the claim is bigger than the value of the replicating portfolio today, show how you can create an arbitrage opportunity.

c

Find a probability q for stock price going to su which is such that the value of the replicating portfolio is equal to $\mathbb{E}_q[X]/(1+r)$. Here, $\mathbb{E}_q[\cdot]$ means expectation with respect to this probability q.

Problem 2

Consider a forward contract delivering an asset at time T.

(Continued on page 2.)

a

Argue that the forward price at time zero is

$$F_0(T) = S(0)(1 + r)^T$$

where $S(0)$ is the asset price at time zero and r is the interest rate.

b

Let Ω be a finite probability space (sample space) with a filtration $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_T\}$. Furthermore, let the asset price be given by an $\mathbb{F}$-adapted stochastic process $(S(t))_{t=0}^{T}$ on Ω. Define what is a risk-neutral probability Q, and show that if we have such Q then

$$F_0(T) = \mathbb{E}_Q[S(T)]$$

Problem 3

Suppose that $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ with a probability $P(\{\omega_i\}) = 1/5$ for $i = 1, ..., 5$. Moreover, assume that $\mathcal{F} = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \Omega\}$.

a

Show that $\mathcal{F}$ is an algebra.

b

Compute $\mathbb{E}[Y|\mathcal{F}]$, when $Y(\omega_i) = 0$ for $i = 1, 2, 3$ and $Y(\omega_i) = 1$ for $i = 4, 5$.

c

Show that $\mathbb{E}[Y|\mathcal{F}]$ is $\mathcal{F}$-measurable.

Problem 4

In this exercise you are going to price options based on the Black & Scholes formula for call options.

a

State the Black & Scholes formula for the price of a call option with exercise time T and strike K. Denote this price by $C(0)$.

b

Denote the price of a put option with the same exercise time T and strike K by $P(0)$. Using the put-call parity $C(0) - P(0) = S(0) - Ke^{-rT}$, derive a Black & Scholes formula for $P(0)$. Here, $r > 0$ is the continuously compounding interest rate and $S(0)$ today's price of the underlying asset.

(Continued on page 3.)

Problem 5

Consider two investors in the stock market, investor A and B. Investor A wants to find an optimal mix of return and risk taking *all* stocks in the market into account, while investor B focuses on a selection of only a few of the available stocks. Sketch the efficient portfolio frontier for the two investors. For a fixed risk, which one of the two will get the highest expected return on their investment according to the optimal portfolio suggested by Markowitz' theory?

Problem 6

Suppose the price of an asset $(S(t))_{t=0,1,2}$ is defined by a two-period binomial tree model, where the initial price $S(0) = 100$, and in each step the price goes up by a factor u or it remains the same. The interest rate is $r > 0$, and $u > 1 + r$. The probability space (sample space) is $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ in this model, where ω_1 corresponds to the asset price going up twice, ω_2 up and then down, ω_3 down and then up and finally ω_4 down twice.

a

Find $\mathcal{F}_1$, the algebra generated by $S(0)$ and $S(1)$.

b

The risk-neutral probability Q is given by $Q(\omega_1) = q^2, Q(\omega_2) = q(1-q) = Q(\omega_3)$, $Q(\omega_4) = (1-q)^2$ and $q = \frac{r}{u-1}$. Suppose that you buy a digital option paying you 100 if the asset has gone up twice at time 2, and zero otherwise. Assume that you can replicate the option (i.e., that the digital option is attainable). Find the value of the option at time 1 without resorting to the replicating portfolio.

c

Find the replicating strategy for the digital option, and show that the value of this strategy at time 1 is the same as the one you derived in **b** above.

END

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Financ

Day of examination: Trial exam, Autumn 2023

Examination hours: XX – YY

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All exercises count equally.

Problem 1

Consider a stock price dynamics described by a one-period binomial model. The price goes either up from 1 to $1+\xi$, or it remains the same. The interest rate is $r = \xi/2$.

a

Let X be the payoff from a claim paying 1 a time 1 if the stock goes up, zero otherwise. Find a replicating portfolio for this claim.

b

Find the arbitrage opportunities if the price of the claim at time zero is not equal to the value of the replicating portfolio at time zero.

c

Find the risk-neutral probability Q in this market, and show that the price of the claim at time zero is equal to $\mathbb{E}_Q[X]/(1 + (\xi/2))$.

Problem 2

a

In a finite probability space (sample space) Ω, define what an algebra and a filtration is.

(Continued on page 2.)

b

Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \{\emptyset, \{\omega_1, \omega_2, \omega_5\}, \{\omega_3, \omega_4\}, \Omega\}$, and $\mathcal{F}_2$ being the set of all subsets of Ω. Are $\mathcal{F}_i, i = 0, 1, 2$ algebras, and is $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ a filtration?

c

A random variable Y takes the values $Y(\omega_i) = 2$ for $i = 1, 2$ and $Y(\omega_i) = 1$ for $i = 3, 4, 5$. Argue that Y is not $\mathcal{F}_1$-measurable? What is a partition of $\mathcal{F}_1$? Calculate $\mathbb{E}[Y|\mathcal{F}_1]$ when $P(\{\omega_i\}) = 1/5$, $i = 1, ..., 5$.

Problem 3

Assume that the stock price dynamics $(S(t))_{t=0,1,2}$ follows a 2-period binomial tree model. The initial stock price is 1, and it goes up by a factor u and down by a factor d in each period. The interest rate is $r > 0$ and we suppose that $0 < d < 1 + r < u$.

a

Describe the probability space (path space) Ω for this model. Find $\mathcal{F}_1$, the algebra generated by $S(0)$ and $S(1)$. What is $\mathcal{F}_0$ and $\mathcal{F}_2$?

b

Let X be an option paying 1 if the stock price has moved downwards twice, and zero otherwise. Find the price dynamics $(C(t))_{t=0,1,2}$ for this option assuming that it is attainable. You can assume that the risk-neutral probability Q is defined by $Q(\omega_1) = q^2$, $Q(\omega_2) = Q(\omega_3) = q(1-q)$ and $Q(\omega_4) = (1-q)^2$, where $q = (1 + r - d)/(u - d)$.

c

Find the replicating strategy of the claim in **b** above. Is the value of the replicating portfolio at times 0 and 1 the same as $C(0)$ and $C(1)$?

Problem 4

In this exercise you are going to price options based on the Black & Scholes formula for call options.

a

State the Black & Scholes formula for the price of a call option with exercise time T and strike K.

(Continued on page 3.)

b

A 'straddle' is a long position in a call and a put option, were both contracts have the same exercise time T and strike K. Recall the put-call parity $C(0) - P(0) = S(0) - Ke^{-rT}$, where $r > 0$ is the continuously compounding interest rate and $S(0)$ today's price of the underlying asset. Find a formula for the price of this position.

Problem 5

Let $(S(t))_{t=0}^{T}$ be the price process of an asset, defined on a finite probability space (sample space) Ω. On the asset, there is a forward contract traded with delivery of the asset at time T.

a

Argue by arbitrage that the forward price at time $t = 0, 1, ..., T$ is

$$F_t(T) = S(t)(1+r)^{T-t}$$

where $r > 0$ is the interest rate.

b

Equip Ω with a filtration $\mathbb{F} = \{\mathcal{F}_0, ..., \mathcal{F}_T\}$. Assume that Q is a risk-neutral probability. What does this mean for $(S(t))_{t=0}^{T}$? Show that

$$F_t(T) = \mathbb{E}_Q[S(T)|\mathcal{F}_t]$$

Problem 6

Make a sketch of the efficient portfolio frontier in an (σ, r)-coordinate system and the tangent portfolio. Explain (in words and graphics) what the tangent portfolio is used for.

END

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: november 2022

Examination hours: 0.00 − 00.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Consider a loan of **1000** to be paid back in 5 equal instalments due at yearly intervals. The instalments include both the interest payable each year calculated at 10% of the current outstanding balance and the repayment of a fraction of the loan. What is the amount of interest included in each instalment? How much of the loan is repaid as part of each instalment?

b (weight 10p)

Suppose that at time $t = 0$, the market price of the underlying asset will be 1000 NOK, the price of a forward contract with a delivery time of one year will be 1080 NOK, under periodic compounding with r = 8%, and short-selling requires a 30% security deposit attracting interest at d = 4%. Is there an arbitrage opportunity? Find the highest rate d for which there is no arbitrage opportunity.

c (weight 10p)

Find the total payout function that depends on the share price of the following securities: 1 call option with a strike price K and 1 put options with a strike price $3K$ are purchased. Construct a graph of the payout function.

(Continued on page 2.)

Problem 2

Consider a one-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3\}$, interest rate $r = \frac{1}{10}$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1}$ with prices given by

$$S_1(0) = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} \frac{33}{5} \\ \frac{22}{5} \\ \frac{33}{10} \end{pmatrix}.$$

a (weight 10p)

Find all risk neutral measures in this market. Is this market arbitrage-free? Justify your answer.

b (weight 10p)

Find all contingent claims $X = (X_1, X_2, X_3)^T$ that are attainable in this market. Is this market complete? Justify your answer.

c (weight 10p)

Compute the arbritrage-free price (or prices) for the contingent claims $X = (1, 5, 2)^T$ and $Y = (6, 2, 0)^T$.

Problem 3

Consider a two-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, probability measure $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$, interest rate $r = 0$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1,2}$ with prices given by

$$S_1(0) = \begin{pmatrix} 4 \\ 4 \\ 4 \\ 4 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 6 \\ 3 \\ 3 \\ 6 \end{pmatrix}, \quad S_1(2) = \begin{pmatrix} 8 \\ 2 \\ 5 \\ 5 \end{pmatrix}.$$

a (weight 10p)

Find $\{\mathcal{F}_t\}_{t=0,1,2}$, the filtration generated by the price process $S_1 = \{S_1(t)\}_{t=0,1,2}$. Discuss carefully the partitions associated to the price process and how they generate the algebras in the filtration. Calculate $\mathbb{E}\left[S_1(2,\omega) | \mathcal{F}_1\right]$

b (weight 10p)

Find risk neutral probability measure $Q = (Q(\omega_1), Q(\omega_2), Q(\omega_3), Q(\omega_4))$ for given market.

(Continued on page 3.)

c (weight 10p)

Consider the following optimal portfolio problem

$$\max_{H \in \mathbb{H}} \mathbb{E}\left[U\left(V\left(2\right)\right)\right]$$
$$\text{subject to } V\left(0\right) = v,$$

where v is a given strictly positive real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U\left(u\right) = 2u^{1/2}$. Compute the optimal attainable wealth, the optimal objective value and the optimal trading strategy.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Suppouse that X, Y, Z are random variables with $X, Y \in \mathcal{F}$. Prove that

$$\mathbb{E}\left[X + YZ \middle| \mathcal{F}\right] = X + Y\mathbb{E}\left[Z \middle| \mathcal{F}\right].$$

You may use, without having to prove it, that the conditional expectation is a linear operator.

b (weight 10p)

Let $\{X_n,\ n \geq 1\}$ be independent identically distributed random variables such that $\mathbb{E}[X_i] = a, var[X_i] = \sigma^2$, $i \geq 1$. Set $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1$. Find the conditional expectations:

$$\mathbb{E}(X_1 \cdot X_2 \cdot ... \cdot X_n | \mathcal{F}_k).$$

c (weight 10p)

Define what is a martingale with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,....,T}$ under the probability measure P. Let $Z = \{Z(t)\}_{t=0,...,T}$ be a martingale and $H = \{H(t)\}_{t=1,...,T}$ be a predictable process. Prove that the process $G = \{G(t)\}_{t=0,...,T}$ defined by

$$G(0) = 0,$$

$$G(t) = \sum_{u=1}^{t} H(u)(Z(u) - Z(u-1)),\ t = 1, ..., T.$$

is also a martingale.

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: Tuesday 30. november 2021

Examination hours: 9.00 – 13.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

The return of a zero coupon bond over 9 months is 2%. Find the implied continuous compounding annual rate.

b (weight 10p)

Let $S(0)$ be the price at time zero of a stock paying no dividends. Let $C^E(0)$ and $P^E(0)$ denote the prices of a European call option and a European put option at time zero, respectively. Both options with the same expiry time T and the same strike price K. Let $r \geq 0$ be the continuously compounded interest rate. Show that, if

$$C^E(0) - P^E(0) - S(0) + Ke^{-rT} > 0,$$

then you can make a sure risk-less profit.

c (weight 10p)

Explain how to construct a straddle. Write a table showing the profits given by this strategy in terms of the price of the stock at expiry time. You may assume that the interest rate is zero.

(Continued on page 2.)

Problem 2

Consider a one-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3\}$, interest rate $r = \frac{1}{10}$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1}$ with prices given by

$$S_1(0) = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} \frac{33}{5} \\ \frac{22}{5} \\ \frac{33}{10} \end{pmatrix}.$$

a (weight 10p)

Find all risk neutral measures in this market. Is this market arbitrage-free? Justify your answer.

b (weight 10p)

Find all contingent claims $X = (X_1, X_2, X_3)^T$ that are attainable in this market. Is this market complete? Justify your answer.

c (weight 10p)

Compute the arbritrage-free price (or prices) for the contingent claims $X = (1, 5, 2)^T$ and $Y = (6, 2, 0)^T$.

Problem 3

Consider a two-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, probability measure $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$, interest rate $r = 0$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1,2}$ with prices given by

$$S_1(0) = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 4 \\ 2 \\ 4 \\ 2 \end{pmatrix}, \quad S_1(2) = \begin{pmatrix} 7 \\ 3 \\ 3 \\ 1 \end{pmatrix}.$$

a (weight 10p)

Find $\{\mathcal{F}_t\}_{t=0,1,2}$, the filtration generated by the price process $S_1 = \{S_1(t)\}_{t=0,1,2}$. Is the the random variable $Y = (3, 1, 3, 2)^T$ measurable with respect to $\mathcal{F}_1$?

(Continued on page 3.)

b (weight 20p)

Let $Q = \left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}\right)^T$ be the unique martingale measure on this market (you do not have to prove this). Consider the following optimal portfolio problem

$$\max_{H \in \mathbb{H}} \mathbb{E}\left[U\left(V\left(2\right)\right)\right]$$
$$\text{subject to } V\left(0\right) = v,$$

where v is a given strictly positive real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U\left(u\right) = 2u^{1/2}$. Compute the optimal attainable wealth, the optimal objective value and the optimal trading strategy.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Given $\mathcal{G}$ an algebra and X a random variable on Ω, give the abstract definition of conditional expectation of X given $\mathcal{G}$.

Define what is a martingale with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\ldots,T}$ under the probability measure P.

b (weight 10p)

Let Y be a $\mathcal{G}$-measurable random variable. Prove that

$$\mathbb{E}\left[XY\right|\mathcal{G}] = Y\mathbb{E}\left[X\right|\mathcal{G}].$$

You may use, without having to prove it, that the conditional expectation is a linear operator.

c (weight 10p)

Assume that $\mathcal{F} = \mathcal{F}_T$, let Q be another probability measure on Ω equivalent to P, i.e., $Q > 0$, and define the process $L = \left\{L\left(t\right) = \mathbb{E}\left[\frac{Q}{P}\middle|\mathcal{F}_t\right]\right\}_{t=0,\ldots,T}$. Let $X = \{X\left(t\right)\}_{t=0,\ldots,T}$ be an $\mathbb{F}$-adapted process.

Show that X is a F-martingale under Q if and only if $Z = \{Z\left(t\right) = L\left(t\right)X\left(t\right)\}_{t=0,\ldots,T}$ is an $\mathbb{F}$-martingale under P.

Hint: Recall that, given a random variable Y, $\mathbb{E}_Q\left[Y\right] = \mathbb{E}\left[Y\frac{Q}{P}\right]$. Moreover, you can use, without having to prove it, that:

1. The process L is strictly positive with $L\left(0\right) = 1$.
2. If W is a random variable, then $\mathbb{E}_Q\left[W\right|\mathcal{F}_t] = \frac{\mathbb{E}[WL(T)|\mathcal{F}_t]}{L(t)}$.

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — An Introduction to Mathematical Finance

Day of examination: Thursday 26. November 2020

Examination hours: 15.00 – 19.00

This problem set consists of 4 pages.

Appendices: All

Permitted aids: All

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

After how many days will a zero-coupon bond purchased at time zero for $B(0, 1) = 0.93$ produce a 4% return? Assume there are 365 days in a year.

b (weight 10p)

Let $F(t, T), t \in [0, T]$ denote the forward price in a forward contract (on a non-paying dividends stock) starting at time t and with delivery date T. Let $V(t)$ be the time t value of a forward contract initiated at time zero. Show that if

$$V(t) < (F(t, T) - F(0, T))\, e^{-r(T-t)},$$

then you can make a risk free profit.

c (weight 10p)

Explain how to construct a butterfly spread. Write a table showing the profits given by this strategy in terms of the price of the stock at expiry time. You may assume that the interest rate is zero.

(Continued on page 2.)

Problem 2

Consider a one-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and prices given by $B(0) = 1, S_1(0) = 7,\ S_2(0) = 8$ and

$$B(1) = \begin{pmatrix} \frac{9}{8} \\ \frac{9}{8} \\ \frac{9}{8} \\ \frac{9}{8} \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 9 \\ \frac{45}{4} \\ \frac{27}{4} \\ \frac{27}{4} \end{pmatrix} \quad S_2(1) = \begin{pmatrix} \frac{27}{2} \\ \frac{27}{4} \\ \frac{27}{4} \\ \frac{45}{4} \end{pmatrix}.$$

a (weight 10p)

Find all risk neutral measures in this market. Is this market arbitrage-free? Justify your answer.

b (weight 10p)

Find all contingent claims $X = (X_1, X_2, X_3, X_4)^T$ that are attainable in this market. Is this market complete? Justify your answer.

c (weight 10p)

Compute the arbritrage-free price (or prices) for $X = \max(0, S_2(1) - S_1(1) - 9/4)$.

d (weight 5p)

Are the contingent claims of the form $Y = (Y_1, Y_2, Y_3, Y_4)^T$, for non-negative numbers $Y_1 = Y_2 = Y_3 \neq Y_4$, measurable with respecto to $\mathfrak{a}(S_1(1))$? And with respect to $\mathfrak{a}(S_2(1))$?

Problem 3

Consider a two-period market, with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, probability measure $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$, interest rate $r = 0$, and one risky asset $S_1 = \{S_1(t)\}_{t=0,1,2}$ with prices given by

$$S_1(0) = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}, \quad S_1(1) = \begin{pmatrix} 4 \\ 4 \\ 2 \\ 2 \end{pmatrix}, \quad S_1(2) = \begin{pmatrix} 6 \\ 1 \\ 4 \\ 1 \end{pmatrix}.$$

(Continued on page 3.)

a (weight 5p)

Find the filtration generated by the price process $S_1 = \{S_1(t)\}_{t=0,1,2}$. Discuss carefully the partitions associated to the price process and how they generate the algebras in the filtration.

b (weight 20p)

Let $Q = \left(\frac{3}{10}, \frac{1}{5}, \frac{1}{6}, \frac{1}{3}\right)^T$ be the unique martingale measure on this market (you do not have to prove this). Consider the following optimal portfolio problem

$$\max_{H \in \mathbb{H}} \mathbb{E}\left[U\left(V\left(2\right)\right)\right]$$
$$\text{subject to } V(0) = v,$$

where v is a given non-negative real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U(u) = 2u^{1/2}$. Compute the optimal attainable wealth, the optimal expected utility and the optimal trading strategy.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Let $\mathcal{G}$ be an algebra and X be a random variable on Ω.

Give the abstract definition of conditional expectation of X given $\mathcal{G}$.

Prove that the following two statements are equivalent:

1. The random variable Z is the conditional expectation of X given $\mathcal{G}$.
2. The random variable Z is $\mathcal{G}$-measurable and satisfies $\mathbb{E}\left[(X - Z)Y\right] = 0$, for all random variables Y that are $\mathcal{G}$-measurable.

b (weight 10p)

Define what is a martingale with respect to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\ldots,T}$ under the probability measure P.

Let $\mathbb{G} = \{\mathcal{G}_t\}_{t=0,\ldots,T}$ be another filtration such that $\mathcal{G}_t \subseteq \mathcal{F}_t$. Prove that if M is an $\mathbb{F}$-martingale and M is $\mathbb{G}$-adapted, then M is also a $\mathbb{G}$-martingale.

c (weight 10p)

Consider the market model of Problem 3. Compute the variables of the stochastic process $A = \{A(t)\}_{t=0,1,2}$, such that

- $A(0) = 0$;

(Continued on page 4.)

- A is predictable with respect to $\mathbb{F}$, the filtration generated by the price process;
- $M = \left\{ M(t) := S_1^2(t) - A(t) \right\}_{t=0,1,2}$ is a martingale with respect to $\mathbb{F}$.

1

a)

$$S_0 \begin{array}{l} \nearrow S_0 u \\ \searrow S_0 d \end{array}$$

Interest rate $r > 0$.
$0 < d < 1+r < u$.

Call option with strike $K = S_0(1+r)$. By assumed inequality, $S_0 d < K < S_0 u$, so option pays when stock price is $S(1) = S_0 u$, paying $S_0 u - S_0(1+r)$. Otherwise it pays zero.

a = # stocks, b position of money in bank.

Replicating portfolio

$$a S_0 u + b(1+r) = S_0 u - S_0(1+r)$$
$$a S_0 d + b(1+r) = 0$$

$$\Rightarrow b(1+r) = -a S_0 d \Rightarrow a S_0 u - a S_0 d = S_0 u - S_0(1+r)$$

$$\Rightarrow a = \frac{u-(1+r)}{u-d}, \quad b = -S_0 \frac{d(u-(1+r))}{(1+r)(u-d)}$$

b) If price of option C_0 is not V_0, then arbitrage.

$C_0 > V_0$: Sell option at time 0, and buy replicating strategy. You have available $C_0 - V_0$, that you put in bank. At time 1, you settle option, covered exactly by selling replicating portfolio. You are left with $(C_0-V_0)(1+r)>0$, an arbitrage.

$\underline{C_0<V_0}$. Buy option and sell (go short) the replicating portfolio. Place V_0-C_0 in bank. At time 1, settle short position in replicating strategy by exercising the option. You are left with $(V_0-C_0)(1+r)>0$, an arbitrage.

To argue above, students do not need to know V_0.

$$V_0 = aS_0+b = \frac{u-(1+r)}{u-d}S_0 - S_0\frac{d(u-(1+r))}{(1+r)(u-d)}$$

$$= S_0\frac{(1+r)(u-(1+r))-d(u-(1+r))}{(1+r)(u-d)}$$

$$= S_0\frac{(1+r-d)(u-(1+r))}{(1+r)(u-d)}$$

c) By definition, a risk neutral probability for price going up is $\frac{1}{1+r}E_q[S(1)] = S(0)$.

$$E_q[S(1)] = S_0uq + S_0d(1-q) = S_0(u-d)q + S_0d$$

$$= (1+r)S_0 \Rightarrow q = \frac{1+r-d}{u-d}$$

$u > d$ and $d < 1+r \Rightarrow \underline{q > 0}$.

$u > 1+r \Rightarrow u-d > 1+r-d > 0 \Rightarrow \underline{q < 1}$.

Thus $q \in (0,1)$ is a probability!
If conditions are not satisfied, we will not have that q is a probability, and a risk neutral probability will not exist.

2

a) $\phi^c = \Omega \in \mathcal{F}$. $\Omega^c = \phi \in \mathcal{F}$
$\{\omega_1, \omega_3\}^c = \{\omega_2, \omega_4\} \in \mathcal{F}$
$\{\omega_2, \omega_4\}^c = \{\omega_1, \omega_3\} \in \mathcal{F}$
$\mathcal{F}$ is closed under complements

$\phi \cup A = A \in \mathcal{F}$ for any $A \in \mathcal{F}$
$\Omega \cup A = \Omega \in \mathcal{F}$ for any $A \in \mathcal{F}$
$\{\omega_1, \omega_3\} \cup \{\omega_2, \omega_4\} = \Omega \in \mathcal{F}$
$\mathcal{F}$ is closed under unions.

Since also $\phi \in \mathcal{F}$, $\mathcal{F}$ is therefore an algebra.

The partition is $\mathcal{A} = \{ \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\} \}$, since

- $\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\} \in \mathcal{F}$ (all sets in $\mathcal{F}$)
- $\{\omega_1, \omega_3\} \cap \{\omega_2, \omega_4\} = \phi$ (sets are disjoint)
- $\{\omega_1, \omega_3\} \cup \{\omega_2, \omega_4\} = \Omega$ (union of all sets is Ω)

b) $$\mathcal{X}(\omega) = \begin{cases} a, & \omega = \omega_1, \omega_2 \\ b, & \omega = \omega_3 \\ 0, & \omega = \omega_4 \end{cases}$$, $a \neq b$, a and b are both different than zero.

$\{\omega \in \Omega \mid \mathcal{X}(\omega) = a\} = \{\omega_1, \omega_2\} \notin \mathcal{F}$. Hence, not $\mathcal{F}$-measurable. One may also check
$\{\omega \in \Omega \mid \mathcal{X}(\omega) = b\} = \{\omega_3\} \notin \mathcal{F}$, or
$\{\omega \in \Omega \mid \mathcal{X}(\omega) = 0\} = \{\omega_4\} \notin \mathcal{F}$.

c) We have given $P(\omega_1) = \frac{1}{3}$, $P(\omega_2) = \frac{1}{6}$, $P(\omega_3) = \frac{1}{3}$, $P(\omega_4) = \frac{1}{6}$.

We have that $E[\mathcal{X} \mid \mathcal{F}]$ is constant on the partition $\mathcal{A}$.

Let $A = \{\omega_1, \omega_3\}$. Then $P(A) = P(\omega_1) + P(\omega_3) = \underline{\frac{2}{3}}$

$$E[\mathcal{X} \mid A] = \sum_{i=1}^{4} \mathcal{X}(\omega_i) P(\omega_i \mid A), \quad \omega_i \cap A = \begin{cases} \phi, & i = 2,4 \\ \omega_i, & i = 1,3 \end{cases}$$

$$= \mathcal{X}(\omega_1) \frac{P(\omega_1)}{P(A)} + \mathcal{X}(\omega_3) \frac{P(\omega_3)}{P(A)}$$

$$= a \frac{1/3}{2/3} + b \frac{1/3}{2/3} = \underline{\frac{1}{2}(a+b)}$$

Let next $A = \{\omega_2, \omega_4\}$. $P(A) = P(\omega_2) + P(\omega_4) = \frac{1}{3}$

Also, $\omega_i \cap A = \begin{cases} \phi, & i = 1,3 \\ \omega_i, & i = 2,4 \end{cases}$

$$E[X|A] = X(\omega_2)\frac{P(\omega_2)}{P(A)} + X(\omega_4)\frac{P(\omega_4)}{P(A)}$$
$$= a\,\frac{1/6}{2/6} + b\,\frac{1/6}{2/6} = \underline{\frac{1}{2}a}\,.$$

$$\underline{\underline{E[X|\mathcal{F}](\omega)}} = \underline{\underline{\begin{cases} \frac{1}{2}(a+b), & \omega = \omega_1, \omega_3 \\ \frac{1}{2}a, & \omega = \omega_2, \omega_4 \end{cases}}}$$

Since
$$\{\omega \in \Omega \mid E[X|\mathcal{F}](\omega) = \tfrac{1}{2}(a+b)\} = \{\omega_1, \omega_3\} \in \mathcal{F}$$
and
$$\{\omega \in \Omega \mid E[X|\mathcal{F}](\omega) = \tfrac{1}{2}a\} = \{\omega_2, \omega_4\} \in \mathcal{F}$$
the conditional expectation is $\mathcal{F}$-measurable.
One can also argue that $E[X|\mathcal{F}]$ is constant on A.

3

Black & Scholes formula says
$$C_0 = S_0\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$$
where S_0 is the current stock price, K is strike, r is risk-free interest rate and T is exercise time of option. Moreover, Φ is the cumulative standard normal probability distribution function, and

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

with σ being the volatility of the stock.

$r = 0$, $K = S_0$ yields $\underline{d_1} = \dfrac{\ln(1) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}} = \underline{\frac{1}{2}\sigma\sqrt{T}}$

$\Rightarrow \underline{d_2 = -\frac{1}{2}\sigma\sqrt{T}}$

But; $\underline{\Phi}\left(-\frac{1}{2}\sigma\sqrt{T}\right) = P\left(X \leq -\frac{1}{2}\sigma\sqrt{T}\right)$, $X \sim N(0,1)$

$= P\left(X \geq \frac{1}{2}\sigma\sqrt{T}\right) = 1 - P\left(X \leq \frac{1}{2}\sigma\sqrt{T}\right)$

$$\underline{\underline{C_0}} = S_0\left(\underline{\Phi}\left(\tfrac{1}{2}\sigma\sqrt{T}\right) - \Phi\left(-\tfrac{1}{2}\sigma\sqrt{T}\right)\right)$$

$$\underline{\underline{= S_0\left(2\underline{\Phi}\left(\tfrac{1}{2}\sigma\sqrt{T}\right) - 1\right)}}$$

4

$$\begin{tikzcd}
 & \frac{5}{4} \arrow[r] \arrow[dr] & \left(\frac{5}{4}\right)^2 \\
1 \arrow[ur] \arrow[dr] & & \frac{5}{4}\cdot\frac{3}{4} \\
 & \frac{3}{4} \arrow[ur] \arrow[r] & \left(\frac{3}{4}\right)^2
\end{tikzcd}$$

$, \; r = 0.$

Digital option paying 1 if $S(2) = \left(\frac{5}{4}\right)^2$, zero otherwise.

a) We go backwards in time:

<u>At time 1:</u> If $\underline{S(1) = \frac{5}{4}}$, choose stocks a and bank

b that replicates at time 2:

$$a\left(\frac{5}{4}\right)^2 + b = 1$$

$$a\left(\frac{5}{4}\right)\left(\frac{3}{4}\right) + b = 0$$

Solving; $b = -a\left(\frac{5}{4}\right)\left(\frac{3}{4}\right)$ gives $a\left(\frac{5}{4}\right)^2 - a\left(\frac{5}{4}\right)\left(\frac{3}{4}\right)$

$= a\left(\frac{5}{4}\right)\frac{1}{2} = 1 \Rightarrow \underline{a = \frac{8}{5}}$. Thus, $\underline{b} = -\frac{8}{5}\cdot\frac{5}{4}\cdot\frac{3}{4} = \underline{-\frac{3}{2}}$

If $\underline{S(1) = \frac{3}{4}}$, then option pays zero, so

$$\left.\begin{aligned} a\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right) + b &= 0 \\ a\left(\tfrac{3}{4}\right)^2 + b &= 0 \end{aligned}\right\} \Rightarrow \underline{a = b = 0}.$$

Hence,

$$\underline{a = \begin{cases} \frac{8}{5}, & S(1) = \frac{5}{4} \\ 0, & S(1) = \frac{3}{4} \end{cases}}, \quad \underline{b = \begin{cases} -\frac{3}{2}, & S(1) = \frac{5}{4} \\ 0, & S(1) = \frac{3}{4} \end{cases}}$$

<u>At time 0</u>, we replicate V_1.

$$V_1 = aS(1) + b = \begin{cases} \frac{8}{5}\cdot\frac{5}{4} - \frac{3}{2} = \frac{1}{2}, & S(1) = \frac{5}{4} \\ 0, & S(1) = \frac{3}{4} \end{cases}$$

$$\left.\begin{aligned} a\left(\tfrac{5}{4}\right) + b &= \tfrac{1}{2} \\ a\cdot\tfrac{3}{4} + b &= 0 \end{aligned}\right\} \Rightarrow b = -\frac{3}{4}a, \text{ yielding}$$

$a\left(\frac{5}{4}\right) - \frac{3}{4}a = \frac{1}{2}a = \frac{1}{2} \Rightarrow \underline{a=1}$, and $\underline{b = -\frac{3}{4}}$

Value of portfolio is;

$$\underline{V_0} = a\,S_0 + b = 1 \cdot 1 - \frac{3}{4} = \underline{\frac{1}{4}}$$

b) A risk neutral prob. Q is such that $Q(\omega) > 0 \ \forall \omega \in \Omega$ and $E_Q[S^*(2) | \mathcal{F}_1] = S^*(1)$ and $E_Q[S^*(1)] = S^*(0)$.

$S^*(t) = \frac{S(t)}{(1+r)^t}$, discounted value, but $r=0$ so

$S^*(t) = S(t)$.

To calculate conditional expectation, we need $\mathcal{F}_1$.

$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, ω_1: up-up, ω_2: up-down, ω_3: down-up, ω_4: down-down.

$\mathcal{F}_1$ is generated by $S(1)$:

$\{\omega \in \Omega \mid S(1)(\omega) = \frac{5}{4}\} = \{\omega_1, \omega_2\}$

$\{\omega \in \Omega \mid S(1)(\omega) = \frac{3}{4}\} = \{\omega_3, \omega_4\}$

These two sets is a partition $\mathcal{A} = \{\{\omega_1,\omega_2\}, \{\omega_3,\omega_4\}\}$ a partition of $\mathcal{F}_1$. And since $\{\omega_1,\omega_2\} \cap \{\omega_3,\omega_4\} = \phi$ and $\{\omega_1,\omega_2\} \cup \{\omega_3,\omega_4\} = \Omega$, $\mathcal{F}_1 = \{\phi, \{\omega_1,\omega_2\}, \{\omega_3,\omega_4\}, \Omega\}$

is the algebra:

(i) $\phi \in \mathcal{F}_1$

(ii) $\phi^c = \Omega$, $\{\omega_1, \omega_2\}^c = \{\omega_3, \omega_4\} \in \mathcal{F}_1$. closed under complements

(iii) $\{\omega_1, \omega_2\} \cup \{\omega_3, \omega_4\} = \Omega \in \mathcal{F}_1$

$\phi \cup A = A \in \mathcal{F}_1$ for all $A \in \mathcal{F}_1$

$\Omega \cup A = \Omega \in \mathcal{F}_1$ for all $A \in \mathcal{F}_1$.

We find a q for upward movement such that

$Q(\omega_1) = q^2$, $Q(\omega_2) = Q(\omega_3) = q(1-q)$, $Q(\omega_4) = (1-q)^2$.

Calculating $E[S(2) | \mathcal{F}_1]$:

$A = \{\omega_1, \omega_2\}$. $\quad \omega_i \cap A = \begin{cases} \phi, & i = 3,4 \\ \omega_i, & i = 1,2 \end{cases}$

$Q(A) = Q(\omega_1) + Q(\omega_2) = q^2 + q(1-q) = q$.

$$E_Q[S(2) | A] = S(2)(\omega_1) \frac{Q(\omega_1)}{Q(A)} + S(2)(\omega_2) \frac{Q(\omega_2)}{Q(A)}$$

$$= \left(\frac{5}{4}\right)^2 \frac{q^2}{q} + \left(\frac{5}{4}\right)\left(\frac{3}{4}\right) \frac{q(1-q)}{q}$$

$$= \frac{1}{2} \cdot \frac{5}{4} q + \frac{5}{4} \cdot \frac{3}{4}.$$

$A = \{\omega_3, \omega_4\}$ $\quad Q(A) = q(1-q) + (1-q)^2 = 1 - q$

$$E_Q[S(2) | A] = S(2)(\omega_3) \frac{Q(\omega_3)}{Q(A)} + S(2)(\omega_4) \frac{Q(\omega_4)}{Q(A)}$$

$$= \frac{5}{4}\cdot\frac{3}{4}\,\frac{q(1-q)}{1-q} + \left(\frac{3}{4}\right)^2 \frac{(1-q)^2}{1-q}$$

$$= \frac{1}{2}\,\frac{3}{4}\,q + \left(\frac{3}{4}\right)^2$$

$$E_Q[S(2)|\mathcal{F}_1](\omega) = \begin{cases} \frac{1}{2}\cdot\frac{5}{4}q + \frac{5}{4}\cdot\frac{3}{4}, & \omega = \omega_1,\omega_2 \\ \frac{1}{2}\cdot\frac{3}{4}q + \left(\frac{3}{4}\right)^2, & \omega = \omega_3,\omega_4 \end{cases}$$

$$\equiv S(1)(\omega) = \begin{cases} \frac{5}{4}, & \omega = \omega_1,\omega_2 \\ \frac{3}{4}, & \omega = \omega_3,\omega_4 \end{cases}$$

Thus, $\frac{1}{2}\cdot\frac{\cancel{5}}{\cancel{4}}q + \frac{\cancel{5}}{\cancel{4}}\frac{3}{4} = \frac{\cancel{5}}{\cancel{4}} \Rightarrow \frac{1}{2}q = 1 - \frac{3}{4} \Rightarrow \underline{q = \frac{1}{2}}$.

Second eqn is ok for this q, so is $E_Q[S(1)] = S(0) = 1$.

$\Rightarrow$ $\underline{\underline{q = \frac{1}{2}}}$ defines a risk-neutral probability.

We know from theory that if X is payoff of claim

$C_0 = E_Q[X^*]$. But $r = 0 \Rightarrow X^* = X = \begin{cases} 1, & \omega_1 \\ 0 & \text{otherwise} \end{cases}$

$\underline{E_Q[X]} = 1 \cdot Q(\omega_1) = q^2 = \left(\frac{1}{2}\right)^2 = \underline{\underline{\frac{1}{4}}}$.

Corresponds to V_0 computed above!

c) We need only to check what happens at time $t=1$ with the replicating strategy, since this is where we re-allocate the portfolio.

At time 0 we buy 1 stock, and at time 1 we <u>either</u> reallocate to $\frac{8}{5}$ stocks, or to 0 stocks.

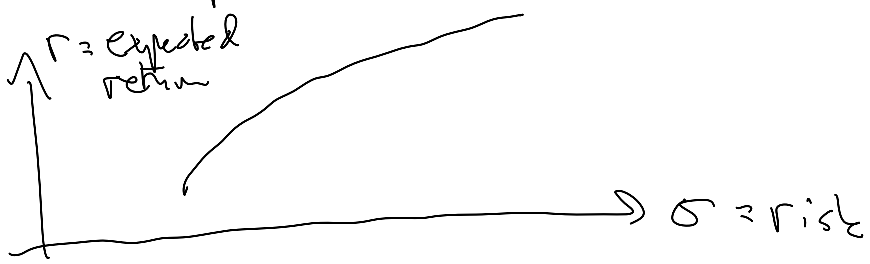
I.e, if $\underline{S(1) = \frac{5}{4}}$, we buy $\frac{3}{5} = \frac{8}{5} - 1$ more stocks. This costs $S(1) \cdot \frac{3}{5} = \frac{3}{5} \cdot \frac{5}{4} = \frac{3}{4}$. But at time 0 we have a bank loan of $\frac{3}{4}$ (i.e, position is $-\frac{3}{4}$). So we borrow additionally $\frac{3}{4}$ to have altogether a bank position $-\frac{6}{4} = -\frac{3}{2}$. But this is exactly what the new position in the hedge is, so we have not introduced any money, nor withdrawn, from the hedge.

If $\underline{S(1) = \frac{3}{4}}$, our stock is sold to have a new position $a = 0$. But selling 1 stock gives us $1 \cdot S(1) = \frac{3}{4}$. We can use this to settle the bank loan, hence yielding a position zero in the bank. But this is consistent with the replicating portfolio.

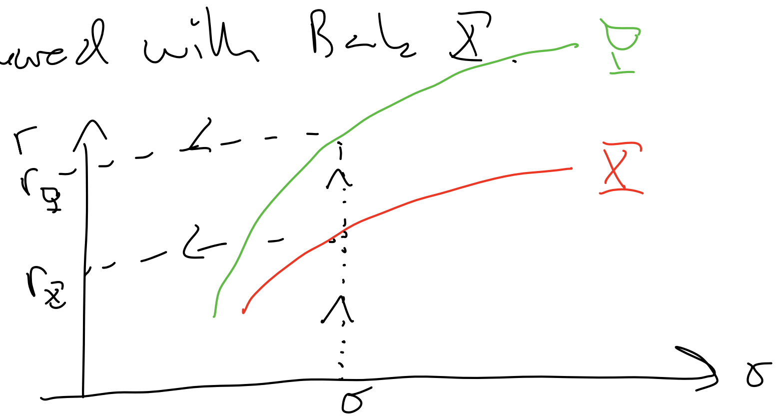
$\Rightarrow$ Strategy is self-financing.

(5)

Efficient frontier describes the best risk-return combination possible to achieve in the market.



Since Norway ⊆ World, then Bank II has more assets to invest in. All possible portfolios of Bank X is also allowed for Bank II. Therefore, Bank II can obtain lower risk σ for given expected return r, compared with Bank X.



$r_{II} > r_X$!

The fund of Bank II is preferable, you gain on average more return that Bank X, for the same risk.

SUGGESTED SOLUTIONS EXAM AUTUMN 2023

STK-MAT 3700/4700

1

$$
\begin{tikzcd}[row sep=small]
 & su \\
S \arrow[ur] \arrow[dr] & \\
 & sd \\
S(0) & S(1)
\end{tikzcd}
$$

Interest rate is r.

$$\mathbb{X} = \begin{cases} x, & \text{if } S(1) = su \\ y, & \text{if } S(1) = sd \end{cases}$$

a) Let a be the number of assets we buy or sell initially, and b how much money we borrow or deposit in the bank. At time 1 we require that

$$b(1+r) + a\,S(1) = \mathbb{X}$$

or, as equations

$$b(1+r) + a\,su = x$$
$$b(1+r) + a\,sd = y$$

A solution of this 2x2 system of equations, is

$$\underline{\underline{a = \frac{x-y}{s(u-d)}}}, \quad \underline{\underline{b = \frac{yu - xd}{(1+r)(u-d)}}}$$

The value of this portfolio today is

$$b + aS(0) = \frac{yu - xd}{(1+r)(u-d)} + \frac{x-y}{S(u-d)} \cdot S$$

$$= \frac{yu - xd + x(1+r) - y(1+r)}{(1+r)(u-d)}$$

$$= \underline{\underline{\frac{x(1+r-d) + y(u-(1+r))}{(1+r)(u-d)}}} \qquad (= V_0)$$

Some may prefer to use the notation $H_0(1)$ for b and $H_1(1)$ for a.

b) We denote the price of the claim c, and $c > V_0$, where V_0 is today's value of the replicating portfolio.

Today:

Sell claim X,	$+\ c$
Buy replicating portfolio	$-\ V_0$
Place rest in bank	$-\ (c - V_0)$

Net position $\qquad 0$

Time 1

Cover claim $\begin{cases} -x, & S(1)=su \\ -y, & S(1)=sd \end{cases}$

Sell replicating portfolio $\begin{cases} +x, & S(1)=su \\ +y, & S(1)=sd \end{cases}$

Withdraw money in bank $\qquad +(c-V_0)(1+r)$

Net position $\qquad +(c-V_0)(1+r) > 0$

This is a sure gain, hence an arbitrage opportunity.

c) We have, by how q is defined,

$$\frac{1}{1+r} E_q[X] = \frac{1}{1+r}\left(x \cdot q + y(1-q)\right)$$

$$\stackrel{!}{=} \frac{x(1+r-d) + y(u-(1+r))}{(1+r)(u-d)}$$

or

$$(x-y)\,q = \frac{x(1+r-d) + y(u-(1+r))}{(u-d)} - y$$

or
$$(x-y)q = \frac{(x-y)(1+r-d)}{(u-d)}$$

or
$$q = \frac{1+r-d}{u-d}$$
(This is known from the course!)

In the exercise, we have assumed $u>d$.

We check if q is a probability.

$q>0$ iff $1+r-d>0 \Leftrightarrow d<1+r$

$q<1$ iff $1+r-d<u-d \Leftrightarrow u>1+r$.

We need to ADD the assumption that

$$\boxed{d<1+r<u}$$

2

a) We do the buy & hold strategy to argue for the relationship:

$F_0(T) > S(0)(1+r)^T$

Time 0:

Sell forward $\qquad 0$

Buy asset $\qquad -S(0)$

Borrow in bank $\qquad +S(0)$

$\qquad 0$

Time T :

Deliver asset, receive forward price $\qquad +F_0(T)$

Pay loan in bank $\qquad -S(0)(1+r)^T$

$$F_0(T)-S(0)(1+r)^T>0$$

$$F_0(T)<S(0)(1+r)^T$$

Time 0

Buy forward $\qquad 0$

Short sell asset $\qquad +S(0)$

Deposit money in bank $\qquad -S(0)$

$\qquad 0$

Time T :

Receive asset, which

close the short position. Pay forward price $\qquad - F_0(T)$

Withdraw money in bank $\qquad + S(0)(1+r)^T$

$$S(0)(1+r)^T - F_0(T) > 0$$

We have arbitrage opportunities whenever $F_0(T) \neq S(0)(1+r)^T$, but no arbitrage when $F_0(T) = S(0)(1+r)^T$

b)

Q is a risk-neutral probability if

(i) $Q(\omega) > 0$ for all $\omega \in \Omega$

(ii) $(S^*(t))_{t=0}^{T}$ is a Q-martingale, i.e

$(S^*(t))_{t=0}^{T}$ is $\mathbb{F}$-adapted and

$E_Q[S^*(u) | \mathcal{F}_t] = S^*(t)$, $u \geq t$, $u,t = 0,\dots,T$.

where $S^*(t) = S(t)/(1+r)^t$.

If we have such a Q ;

$$E_Q[S^*(T)] = E_Q[S^*(T) \mid \mathcal{F}_0] = S^*(0) = S(0)$$

or, $S(0) = \frac{1}{(1+r)^T} E_Q[S(T)]$

$$\Rightarrow \underline{\underline{F_0(T) = E_Q[S(T)]}}.$$

3

a) $\{\omega_1, \omega_2\}^c = \{\omega_3, \omega_4, \omega_5\} \in \mathcal{F}$

$\{\omega_3, \omega_4, \omega_5\}^c = \{\omega_1, \omega_2\} \in \mathcal{F}$

$\Rightarrow$ $\mathcal{F}$ is closed under complement

$\{\omega_1, \omega_2\} \cup \{\omega_3, \omega_4, \omega_5\} = \Omega \in \mathcal{F}$

$\Rightarrow$ $\mathcal{F}$ is closed under union.

$\phi \in \mathcal{F}$. Hence, $\mathcal{F}$ is an algebra.

b) The partition of $\mathcal{F}$ is $A_1 = \{\omega_1, \omega_2\}$ and $A_2 = \{\omega_3, \omega_4, \omega_5\}$. This is a partition because $A_1, A_2 \in \mathcal{F}$, $A_1 \cap A_2 = \phi$, $A_1 \cup A_2 = \Omega$.

$$E[\Xi \mid \mathcal{F}](\omega) = \begin{cases} E[\Xi \mid A_1], & \omega \in A_1 \\ E[\Xi \mid A_2], & \omega \in A_2 \end{cases}$$

$$E[\Xi \mid A_1] = \sum_{i=1}^{5} \Xi(\omega_i) P(\omega_i \mid A_1)$$

$$= \sum_{i=1}^{5} \underbrace{\Xi(\omega_i)}_{=0,\ i=1,2,3} \frac{P(\{\omega_i\} \cap A_1)}{P(A_1)} \qquad \text{Bayes' Formula}$$

$$= \underline{\underline{0}}, \quad \text{since } \{\omega_i\} \cap A_1 = \begin{cases} \{\omega_i\}, & i = 1,2 \\ \phi, & i = 3,4,5 \end{cases}$$

Likewise

$$E[\Xi \mid A_2] = \sum_{i=1}^{5} \underbrace{\Xi(\omega_i)}_{=0,\ i=1,2,3} \frac{P(\{\omega_i\} \cap A_2)}{P(A_2)}$$

$$= 1 \cdot \frac{P(\{\omega_4\} \cap A_2)}{P(A_2)} + 1 \cdot \frac{P(\{\omega_5\} \cap A_2)}{P(A_2)}$$

$$= \frac{P(\omega_4)}{P(A_2)} + \frac{P(\omega_5)}{P(A_2)}$$

$$P(A_2) = P(\{\omega_3, \omega_4, \omega_5\}) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}.$$

$$= \frac{1/5}{3/5} + \frac{1/5}{3/5} = \underline{\underline{2/3}}$$

Hence

$$E[\Psi|\mathcal{F}](\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ 2/3, & \omega = \omega_3, \omega_4, \omega_5 \end{cases}$$

c)

$$\{\omega \in \Omega \mid E[\Psi|\mathcal{F}](\omega) = 0\} = \{\omega_1, \omega_2\} \in \mathcal{F}$$

$$\{\omega \in \Omega \mid E[\Psi|\mathcal{F}](\omega) = 2/3\} = \{\omega_3, \omega_4, \omega_5\} \in \mathcal{F}$$

Thus, $E[\Psi|\mathcal{F}]$ is constant on sets in $\mathcal{F}$, and therefore $\mathcal{F}$-measurable.

4

a) $$C(0) = S(0)\,\Phi(d_1) - K e^{-rT}\,\Phi(d_2)$$

where $\Phi(d) = P(X \leq d)$, $X \sim N(0,1)$, the cumulative standard normal distribution function,

and
$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T} = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

b) From $C(t_0) - P(t_0) = S(t_0) - Ke^{-rT}$ we get

$$\underline{\underline{P(t_0)}} = C(t_0) - S(t_0) + Ke^{-rT}$$

$$= S(t_0)\left(\Phi(d_1) - 1\right) - Ke^{-rT}\left(\Phi(d_2) - 1\right)$$

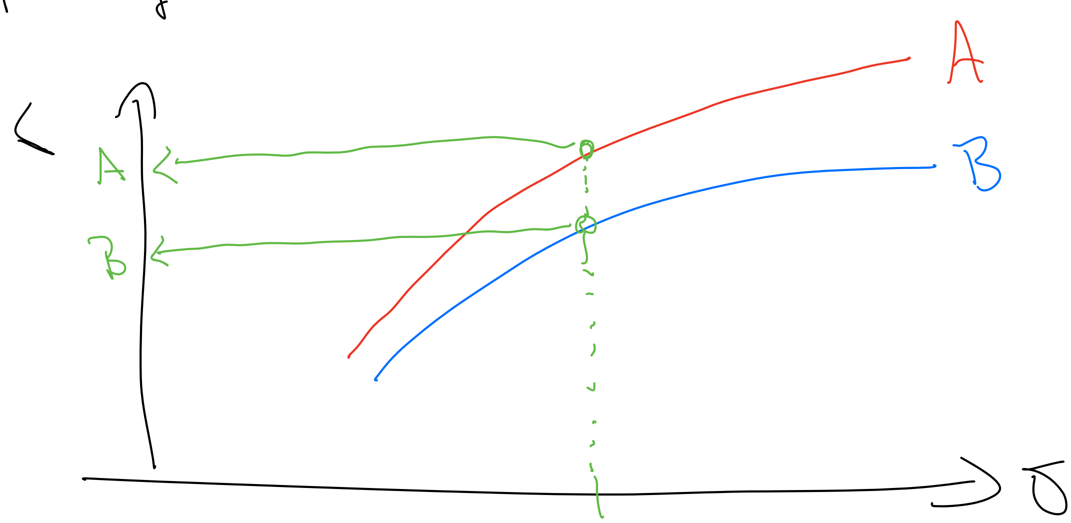
$$1 - \Phi(d) = 1 - P(X \leq d) = P(X \geq d)$$

$$= P(X \leq -d) = \Phi(-d)$$

$$\underline{\underline{= Ke^{-rT}\Phi(-d_2) - S(t_0)\Phi(-d_1)}}$$

5

The more assets, the lower risk for a given expected return according to the Markowitz theory.



Investor A will get the highest returns.

It is expected that students elaborate and discuss more!

(6)

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & 100u^2 \\
 & 100u \arrow[ur] \arrow[r] & 100u \\
100 \arrow[ur] \arrow[r] & 100 \arrow[ur] \arrow[r] & 100
\end{tikzcd}$$

ω_1: up - up

ω_2: up - down

ω_3: down - up

ω_4: down - down.

a) $S(0)$ generates only Ω, since

$$\{\omega \in \Omega \mid S(0) = 100\} = \Omega.$$

$$\Rightarrow \quad \mathcal{F}_0 = \{\emptyset, \Omega\}.$$

$$\{\omega \in \Omega \mid S(1)(\omega) = 100u\} = \{\omega_1, \omega_2\}$$

$$\{\omega \in \Omega \mid S(1)(\omega) = 100\} = \{\omega_3, \omega_4\}$$

The set $\mathcal{F}_1 = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$ is an algebra:

(a) $\emptyset \in \mathcal{F}_1$

(b) $\{\omega_1, \omega_2\}^c = \{\omega_3, \omega_4\} \in \mathcal{F}_1$

$\{\omega_3, \omega_4\}^c = \{\omega_1, \omega_2\} \in \mathcal{F}_1$

(c) $\{\omega_1, \omega_2\} \cup \{\omega_3, \omega_4\} = \Omega \in \mathcal{F}_1$.

$\Rightarrow \quad \mathcal{F}_1 = \{\phi, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$

b) X is the payoff of the digital:

$$X(\omega) = \begin{cases} 100, & \omega = \omega_1 \\ 0, & \omega = \omega_2, \omega_3, \omega_4 \end{cases}$$

According to the pricing theorem for attainable claims,

$$C^*(1) = E_Q[X^* \mid \mathcal{F}_1]$$

where $C(1)$ is the price of the digital at time 1, and $C^*(1)$, X^* are discounted values.

I.e,

$$\frac{C(1)}{(1+r)} = E_Q\left[\frac{X}{(1+r)^2} \mid \mathcal{F}_1\right]$$

$$\Leftrightarrow \quad C(1) = \frac{1}{1+r} E_Q[X \mid \mathcal{F}_1].$$

We must calculate $E_Q[X \mid \mathcal{F}_1]$: Partition of

$\mathcal{F}_1$ is $A_1 = \{\omega_1, \omega_2\}$, $A_2 = \{\omega_3, \omega_4\}$, since $A_1, A_2 \in \mathcal{F}_1$, $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 = \Omega$.

$$E_Q[\bar{X} | \mathcal{F}_1](\omega) = \begin{cases} E_Q[\bar{X} | A_1], & \omega \in A_1 \\ E_Q[\bar{X} | A_2], & \omega \in A_2 \end{cases}$$

$\bar{X}(\omega_1) = 100$, while $\bar{X}(\omega_i) = 0$, $i = 2, 3, 4$.

Hence;

$$E_Q[\bar{X} | A_k] = 100 \frac{Q(\{\omega_1\} \cap A_k)}{Q(A_k)}, \quad k = 1, 2.$$

But $\{\omega_1\} \cap A_2 = \phi$, so

$$E_Q[\bar{X} | A_2] = 0 \quad \text{and}$$

$$E_Q[\bar{X} | A_1] = 100 \frac{Q(\{\omega_1\})}{Q(\omega_1) + Q(\omega_2)} = 100 \frac{q^2}{q^2 + q(1-q)}$$

$$= 100\, q = \frac{100 r}{u - 1}.$$

$$\Rightarrow C(1) = \begin{cases} 100 \dfrac{r}{(1+r)(u-1)}, & \omega = \omega_1, \omega_2 \\ 0, & \omega = \omega_3, \omega_4. \end{cases}$$

c)

We start with $t=1$, and finding an investment in stock (a) and bank (b) replicating the digital at time $t=2$.

$\underline{S(1) = 100u}$ $(\omega = \omega_1, \omega_2)$

$$a \cdot 100u^2 + b(1+r) = 100$$
$$a \cdot 100u + b(1+r) = 0$$

$$\Rightarrow b = -\frac{a \cdot 100u}{1+r}$$

$$\Rightarrow a100u^2 - a \cdot 100u = \cancel{100}u \cdot a(u-1) = \cancel{100}$$

$$\underline{a = \frac{1}{u(u-1)}}, \quad \underline{b = -\frac{100}{(1+r)(u-1)}}$$

$\underline{S(1) = 100}$ $(\omega = \omega_3, \omega_4)$

$$\left.\begin{array}{l} a100u + b(1+r) = 0 \\ a100 + b(1+r) = 0 \end{array}\right. \Rightarrow \underline{a = b = 0}.$$

At time $t=1$, select

$$a = \begin{cases} \frac{1}{u(u-1)}, & \omega = \omega_1, \omega_2 \\ 0, & \omega = \omega_3, \omega_4 \end{cases} \qquad b = \begin{cases} -\frac{100}{(1+r)(u-1)}, & \omega = \omega_1, \omega_2 \\ 0, & \omega = \omega_3, \omega_4 \end{cases}$$

We find $V(1)$:

$$V(1) = a \cdot S(1) + b$$

Inserting, we find

$$V_1(\omega) = \begin{cases} -\frac{100}{(1+r)(u-1)} + \frac{1}{u(u-1)} \cdot 100u & , \omega = \omega_1, \omega_2 \\ 0 & , \omega = \omega_3, \omega_4 \end{cases}$$

$$= \begin{cases} 100 \frac{r}{(1+r)(u-1)} & , \omega = \omega_1, \omega_2 \\ 0 & , \omega = \omega_3, \omega_4 \end{cases}$$

Note: $\boxed{C(1) = V_1}$!

We next go to time $t=0$, and find a (stocks) and b (bank) that replicates $V(1)$!

i.e.,

$$a\, 100u + b(1+r) = \frac{100r}{(1+r)(u-1)}$$

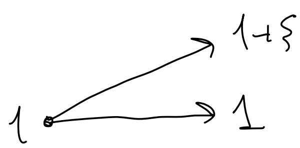
$$a\, 100 + b(1+r) = 0$$

$$\Rightarrow \quad a = \frac{r}{(1+r)(u-1)^2}, \quad b = -\frac{100r}{(1+r)^2(u-1)^2}$$

We have found the replicating strategy.

SOLUTIONS (SUGGESTED) TO THE TRIAL EXAM 2023

1



, $r = \xi/2$.

a) X pays 1 if stock goes up, zero otherwise. Let a be # of stocks and b be the position in bank at time 0. To replicate X, we must have that

$$b\left(1+\tfrac{\xi}{2}\right) + a\cdot(1+\xi) = 1$$
$$b\left(1+\tfrac{\xi}{2}\right) + a \qquad = 0$$

$$\Rightarrow \quad a = -b\left(1+\tfrac{\xi}{2}\right) \quad \Rightarrow \quad -a + a + a\xi = 1$$

$$\Rightarrow \quad \underline{\underline{a = 1/\xi}}, \quad \underline{\underline{b}} = -\frac{1/\xi}{1+\xi/2} = -\underline{\underline{\frac{2}{2\xi+\xi^2}}}$$

We assume that $\xi > 0$!

b) The replicating portfolio has the initial cost

$$\underline{V_0} = b + a \cdot 1 = -\frac{2}{(2+\xi)\xi} + \frac{1}{\xi} = \underline{\frac{1}{2+\xi}}$$

If price C_0 of claim X is not V_0, we find an arbitrage possibility;

$\underline{V_0 > C_0}$ ie, claim is cheap, so buy it!

<u>At time 0</u>

Buy claim	$-\ C_0$
Sell hedge (go short!)	$+\ V_0$
Deposit in bank	$-\ (V_0 - C_0)$
	0

<u>At time 1</u>

Receive money from X	$\begin{cases} +1 \\ 0 \end{cases}$
Settle hedge	$\begin{cases} -1 \\ 0 \end{cases}$
Withdraw deposit	$+ (V_0 - C_0)(1 + \frac{\xi}{2})$

$$+(V_0 - C_0)(1+\tfrac{\xi}{2}) > 0$$

$\Rightarrow$ Arbitrage!

$V_0 < C_0$, i.e., claim is expensive, so sell it!

At time 0

Sell claim	$+C_0$
Buy hedge	$-V_0$
Deposit money in bank	$-(C_0 - V_0)$
	0

At time 1

Pay money to claim	$\begin{cases} -1 \\ 0 \end{cases}$
Sell hedge	$\begin{cases} +1 \\ 0 \end{cases}$
withdraw money in bank	$+(C_0 - V_0)(1+\tfrac{\xi}{2})$
	$+(C_0 - V_0)(1+\tfrac{\xi}{2}) > 0$

$\Rightarrow$ Arbitrage!

c) Risk-neutral probability Q is such that discounted stock price is martingale: In one-period market, this means

$$1 = S(0) = E_Q[S^*(1)] = E_Q[S(1)] / (1+\tfrac{\xi}{2})$$

$$= \frac{1}{1+\frac{\xi}{2}}\left(q\cdot(1+\xi) + (1-q)\cdot 1\right)$$

$$\Rightarrow \quad 1+\frac{\xi}{2} = q\xi + 1 \quad \Rightarrow \quad \underline{\underline{q = \frac{1}{2}}}.$$

q is the risk-neutral probability for moving up.

$$E_Q[X] = q\cdot 1 + (1-q)\cdot 0 = \frac{1}{2}.$$

$$\frac{1}{1+\frac{\xi}{2}} E_Q[X] = \underline{\underline{\frac{1}{2+\xi}}} = V_0 \, .$$

2

a) An algebra $\mathcal{F}$ is a collection of subsets of Ω which is closed under complement and union:

(i) $\phi \in \mathcal{F}$

(ii) If $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$

(iii) If $A_1, A_2 \in \mathcal{F} \Rightarrow A_1 \cup A_2 \in \mathcal{F}$

A filtration $\mathbb{F}$ is a family of algebras $\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T\}$, such that

- $\mathcal{F}_0 = \{\phi, \Omega\}$
- $\mathcal{F}_s \subseteq \mathcal{F}_t$ when $s \leq t$
- $\mathcal{F}_T$ = all subsets of Ω.

b) $\mathcal{F}_0$ and $\mathcal{F}_2$ are trivially algebras, and we also have $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$. Show that $\mathcal{F}_1$ is an algebra:

(i) $\phi \in \mathcal{F}_1$ ✓

(ii) $\{\omega_3, \omega_4\}^c = \{\omega_1, \omega_2, \omega_5\} \in \mathcal{F}_1$

$\{\omega_1, \omega_2, \omega_5\}^c = \{\omega_3, \omega_4\} \in \mathcal{F}_1$ ✓

(iii) $\{\omega_3, \omega_4\} \cup \{\omega_1, \omega_2, \omega_5\} = \Omega \in \mathcal{F}_1$ ✓

$\mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\}$ is a filtration.

c) We check what sets Ψ is constant on:

$\{\omega \in \Omega \mid \Psi(\omega) = 2\} = \{\omega_1, \omega_2\} \notin \mathcal{F}_1$.

$\Rightarrow \Psi$ is not $\mathcal{F}_1$-measurable.

It is strictly speaking not necessary, but we could also check that

$\{\omega \in \Omega \mid \Psi(\omega) = 1\} = \{\omega_3, \omega_4, \omega_5\} \notin \mathcal{F}_1$

Partition of $\mathcal{F}_1$: Let $A_1 = \{\omega_1, \omega_2, \omega_5\}$, $A_2 = \{\omega_3, \omega_4\}$. We note that $A_1, A_2 \in \mathcal{F}_1$, $A_1 \cup A_2 = \Omega$ and $A_1 \cap A_2 = \emptyset$. Hence, A_1 and A_2 is a partition of $\mathcal{F}_1$.

Let us compute the conditional expectation $E[\Psi \mid \mathcal{F}_1]$. By definition

$$E[\Psi \mid \mathcal{F}_1](\omega) = \begin{cases} E[\Psi \mid A_1], & \omega \in A_1 \\ E[\Psi \mid A_2], & \omega \in A_2. \end{cases}$$

$$E[\Psi \mid A_1] = \sum_{i=1}^{5} \Psi(\omega_i) \cdot P(\{\omega_i\} \mid A_1)$$

Bayes' Formula

$$= \sum_{i=1}^{5} \underline{Q}(\omega_i) \frac{P(\{\omega_i\} \cap A_1)}{P(A_1)}$$

$$\{\omega_i\} \cap A_1 = \begin{cases} \{\omega_i\}, & i = 1, 2, 5 \\ \phi, & i = 3, 4 \end{cases}$$

$$= \underline{Q}(\omega_1) \frac{P(\{\omega_1\})}{P(A_1)} + \underline{Q}(\omega_2) \frac{P(\{\omega_2\})}{P(A_1)} + \underline{Q}(\omega_5) \frac{P(\{\omega_5\})}{P(A_1)}$$

$$P(A_1) = P(\{\omega_1\} \cup \{\omega_2\} \cup \{\omega_5\}) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$$

$$= 2 \cdot \frac{1/5}{3/5} + 2 \cdot \frac{1/5}{3/5} + 1 \cdot \frac{1/5}{3/5}$$

$$= \underline{\underline{\frac{5}{3}}}$$

$$E[\underline{Q} | A_2] = \sum_{i=1}^{5} \underline{Q}(\omega_i) \cdot \frac{P(\{\omega_i\} \cap A_2)}{P(A_2)}$$

$$\{\omega_i\} \cap A_2 = \begin{cases} \{\omega_i\}, & i = 3, 4 \\ \phi, & i = 1, 2, 5 \end{cases}$$

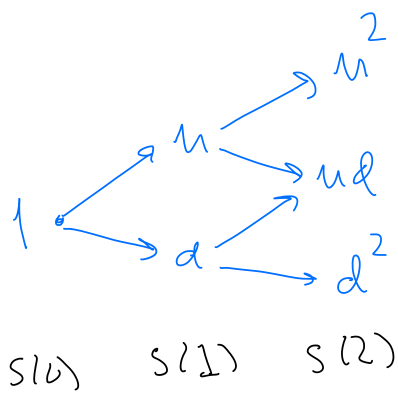
$$P(A_2) = P(\{\omega_3\} \cup \{\omega_4\}) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$$

$$= \underline{Q}(\omega_3) \frac{P(\{\omega_3\})}{P(A_2)} + \underline{Q}(\omega_4) \frac{P(\{\omega_4\})}{P(A_2)}$$

$$= 1 \cdot \frac{1/5}{2/5} + 1 \cdot \frac{1/5}{2/5} = \underline{\underline{1}}$$

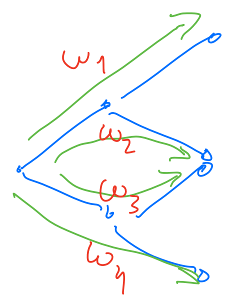
$$E[Y \mid \mathcal{F}_1](\omega) = \begin{cases} 5/3, & \omega = \omega_1, \omega_2, \omega_5 \\ 1, & \omega = \omega_3, \omega_4 \end{cases}$$

3



$r > 0, \quad 0 < d < 1+r < u.$

a)



ω_1 : stock goes up and up

ω_2 : —"— up —"— down

ω_3 : —"— down —"— up

ω_4 : —"— down —"— down.

$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}.$

$\mathcal{F}_1$ is generated by $S(0)$ and $S(1)$. Since $S(0) = 1$, we find only the sets Ω and $\emptyset$ from this, which

We knew in any case are sets in $\mathcal{F}_1$.

$$\{\omega \in \Omega \mid S(\omega) = u\} = \{\omega_1, \omega_2\}$$
$$\{\omega \in \Omega \mid S(\omega) = d\} = \{\omega_3, \omega_4\}.$$

$\mathcal{F}_1$ must consist of the sets $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$. We see that since

- $\{\omega_1, \omega_2\}^c = \{\omega_3, \omega_4\} \in \mathcal{F}_1$
  $\{\omega_3, \omega_4\}^c = \{\omega_1, \omega_2\} \in \mathcal{F}_1$
- $\{\omega_1, \omega_2\} \cup \{\omega_3, \omega_4\} = \Omega \in \mathcal{F}_1$

$\mathcal{F}_1 = \{\phi, \{\omega_1,\omega_2\}, \{\omega_3, \omega_4\}, \Omega\}$ is an algebra.

$\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_2 = \{\phi, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\},$
$\{\omega_1,\omega_2\}, \{\omega_1,\omega_3\}, \{\omega_1,\omega_4\}, \{\omega_2,\omega_3\}, \{\omega_2,\omega_4\},$
$\{\omega_3,\omega_4\}, \{\omega_1,\omega_2,\omega_3\}, \{\omega_1,\omega_2,\omega_4\}, \{\omega_1,\omega_3,\omega_4\},$
$\{\omega_2,\omega_3,\omega_4\}, \Omega\}$. ie, all subsets of Ω!

b) By the general pricing formula;

$$C^*(t) = E_Q[X^* \mid \mathcal{F}_t] \quad , \quad t = 0, 1, 2.$$

$t=2$:

this is $\mathcal{F}_2$-measurable!

$$C^*(2) = E_Q[X^* \mid \mathcal{F}_2] = X^* \quad \text{(by properties of cond. expectation!)}$$

$$\Leftrightarrow \quad \frac{C(2)}{(1+r)^2} = \frac{X}{(1+r)^2} \quad \Leftrightarrow \quad \underline{\underline{C(2) = X = \begin{cases} 0, & \omega = \omega_1, \omega_2, \omega_3 \\ 1, & \omega = \omega_4 \end{cases}}}$$

(this we could also show by no-arbitrage arguments!)

$t=1$

$$C^*(1) = E_Q[X^* \mid \mathcal{F}_1]$$

$$\Leftrightarrow \quad \frac{C(1)}{1+r} = E_Q\left[\frac{X}{(1+r)^2} \mid \mathcal{F}_1\right]$$

$$\Leftrightarrow \quad C(1) = \frac{1}{1+r} E_Q[X \mid \mathcal{F}_1].$$

Let us calculate the conditional expectation:

Notice that $A_1 = \{\omega_1, \omega_2\}$ and $A_2 = \{\omega_3, \omega_4\}$ is a partition of $\mathcal{F}_1$ since $A_1 \cup A_2 = \Omega$ and $A_1 \cap A_2 = \phi$, and obviously $A_1, A_2 \in \mathcal{F}_1$.

We have $Q(A_1) = Q(\{\omega_1\} \cup \{\omega_2\}) = q^2 + q(1-q) = q$

and $Q(A_2) = Q(\{\omega_3\} \cup \{\omega_4\}) = (1-q)q + (1-q)^2 = 1-q$

$$\{\omega_i\} \cap A_1 = \begin{cases} \{\omega_i\}, & i=1,2 \\ \phi, & i=3,4 \end{cases} \qquad \{\omega_i\} \cap A_2 = \begin{cases} \{\omega_i\}, & i=3,4 \\ \phi, & i=1,2 \end{cases}$$

We find

$$E_Q[X \mid A_1] = \sum_{i=1}^{4} X(\omega_i) \frac{Q(\{\omega_i\} \cap A_1)}{Q(A_1)}$$

$$= \underbrace{X(\omega_1)}_{=0} \frac{Q(\{\omega_1\})}{Q(A_1)} + \underbrace{X(\omega_2)}_{=0} \frac{Q(\{\omega_2\})}{Q(A_1)}$$

$$= 0.$$

$$E_Q[X \mid A_2] = \underbrace{X(\omega_3)}_{=0} \frac{Q(\{\omega_3\})}{Q(A_2)} + \underbrace{X(\omega_4)}_{=1} \frac{Q(\{\omega_4\})}{Q(A_2)}$$

$$= 1 \cdot \frac{Q(\{\omega_4\})}{Q(A_2)} = \frac{(1-q)^2}{1-q} = 1-q.$$

$$E_Q[X \mid F_1](\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ 1-q, & \omega = \omega_3, \omega_4. \end{cases}$$

$$C(1)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ \frac{1-q}{1+r}, & \omega = \omega_3, \omega_4 \end{cases}$$

$\underline{t=0}$ $\quad C^*(0) = E_q[X^* \mid \mathcal{F}_0] = E_q[X^*]$

$$\Rightarrow \quad C(0) = E_q\left[\frac{X}{(1+r)^2}\right] = \frac{1}{(1+r)^2} \cdot E_q[X]$$

$$= \frac{1}{(1+r)^2}\left(0 \cdot q^2 + 0 \cdot q(1-q) + 0 \cdot (1-q)q + 1 \cdot (1-q)^2\right)$$

$$= \frac{(1-q)^2}{(1+r)^2}$$

We can now spell out what q is, to find that $1-q = \frac{u-(1+r)}{u-d}$.

Thus

$$C(0) = \frac{(u-(1+r))^2}{(u-d)^2(1+r)^2}$$

$$C(1)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ \frac{u-(1+r)}{(u-d)(1+r)}, & \omega = \omega_3, \omega_4 \end{cases}$$

$$C(2)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2, \omega_3 \\ 1, & \omega = \omega_4 \end{cases}$$

c) We start with creating a portfolio at time 1 which is so that we replicate the claim at time 2.

$H_1(2)$ is the investment in the asset done at time 1, while $H_0(2)$ is the bank position. Both $H_1(2)$ and $H_0(2)$ are $\mathcal{F}_1$-measurable random variables. This means that $H_1(2)$ and $H_0(2)$ are constants on $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$.

$$\Rightarrow \quad H_1(2)(\omega_1) = H_1(2)(\omega_2)$$
$$H_0(2)(\omega_1) = H_0(2)(\omega_2)$$
$$H_1(2)(\omega_3) = H_1(2)(\omega_4)$$
$$H_0(2)(\omega_3) = H_0(2)(\omega_4)$$

To be replicating, we must have

$$H_0(2)(\omega)(1+r) + H_1(2)(\omega)S(2)(\omega) = X(\omega)$$

for $\omega = \omega_1, \omega_2, \omega_3, \omega_4$.

Or,

$$\begin{cases} H_0(2)(\omega_1)(1+r) + H_1(2)(\omega_1)u^2 = 0 \end{cases}$$

$$\mathrm{I}\quad \left\{ H_0(2)(\omega_1)(1+r) + H_1(2)(\omega_1)\,ud = 0 \right.$$

$H_0(2)(\omega_1) = H_0(2)(\omega_2)$, $H_1(2)(\omega_1) = H_1(2)(\omega_2)$

$$\mathrm{II}\quad \begin{cases} H_0(2)(\omega_3)(1+r) + H_1(2)(\omega_3)\,ud = 0 \\ H_0(2)(\omega_3)(1+r) + H_1(2)(\omega_3)\,d^2 = 1 \end{cases}$$

I is a system of 2 equations with 2 unknowns, $H_0(2)(\omega_1)$ and $H_1(2)(\omega_1)$. The solution is obviously

$$\underline{H_0(2)(\omega_1) = H_1(2)(\omega_1) = 0.}$$

II is also a 2x2 system of equations, and we solve to find

$$H_1(2)(\omega_3)(ud - d^2) = -1 \Rightarrow \underline{H_1(2)(\omega_3) = -\frac{1}{d(u-d)}}$$

$$\Rightarrow \underline{H_0(2)(\omega_3) = \frac{u}{(1+r)(u-d)}}$$

This portfolio has the value at time 1 given by

$$V_1(\omega) = H_0(2)(\omega) + H_1(2)(\omega)\,S(1)(\omega)$$

or

$$V_1(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ \dfrac{u-(1+r)}{(1+r)(u-d)}, & \omega = \omega_3, \omega_4. \end{cases}$$

Next, we find a position in the bank $H_0(1)$ and in the asset $H_1(1)$, taken at time 0, such that we replicate V_1 (this ensures self-financing!) i.e.,

$$H_0(1)(1+r) + H_1(1) \cdot S(1)(\omega) = V_1(\omega)$$

or

$$\begin{cases} H_0(1)(1+r) + H_1(1)\, u = 0 \\ H_0(1)(1+r) + H_1(1)\, d = \dfrac{u-(1+r)}{(1+r)(u-d)} \end{cases}$$

Again we have a 2×2 system, where we find

$$H_1(1)(u-d) = -\frac{u-(1+r)}{(1+r)(u-d)} \Rightarrow \underline{H_1(1)} = \underline{-\frac{u-(1+r)}{(1+r)(u-d)^2}}$$

$$\Rightarrow H_0(1) = \frac{u(u-(1+r))}{(1+r)^2(u-d)^2}$$

The replicating strategy is therefore

$$\underline{\underline{H_0(1) = \frac{u(u-(1+r))}{(1+r)^2(u-d)^2},}} \quad H_0(2)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ \dfrac{u}{(1+r)(u-d)}, & \omega = \omega_3, \omega_4 \end{cases}$$

$$H_1(1) = -\frac{u-(1+r)}{(1+r)(u-d)^2}, \quad H_1(2)(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2 \\ -\frac{1}{d(u-d)}, & \omega = \omega_3, \omega_4 \end{cases}$$

By no-arbitrage, we must have that $V_0 = C(0)$, $V_1 = C(1)$, and $V_2 = C(2) = X$.

Indeed, $V_2 = X$ is by construction of the replicating portfolio. We also see above that $V_1 = C(1)$, so we have a sanity check of our calculations.

We can also check our calculations of the replicating portfolio by noting that

$$\begin{aligned} \underline{V_0} &= H_0(1) + H_1(1) S(0) \\ &= \frac{u(u-(1+r))}{(1+r)^2(u-d)^2} - \frac{u(1+r)}{(1+r)(u-d)^2} \\ &= \frac{(u-(1+r))^2}{(1+r)^2(u-d)^2} = \underline{C(0)} \; ! \end{aligned}$$

4

a) Recall from lectures, learn by heart for the exam!!

b) The price, call it $S(t)$ for "straddle", of a long call and put position, is

$$\underline{\underline{S(t)}} = \underset{\text{price of call}}{C(t)} + \underset{\text{price of put}}{P(t)}$$

put-call parity

$$= C(t) + \left(C(t) - S(t) + Ke^{-rT}\right)$$

$$\underline{\underline{= 2C(t) - S(t) + Ke^{-rT}}}$$

5

a) Suppose $\underline{F_t(T) > S(t)(1+r)^{T-t}}$.

Then we sell forward, and buy the asset on spot at time t:

Time t:

Sell forward	0
Buy spot	$-S(t)$
Finance this by borrowing money	$+S(t)$
	0

Time T

Deliver asset, which you have sold forwards, receive forward price	$+F_t(T)$
Settle the loan	$-S(t)(1+r)^{T-t}$
	$F_t(T)-S(t)(1+r)^{T-t}>0.$

Sure profit from zero investment $\Rightarrow$ Arbitrage opportunity.

Suppose $F_t(T) < S(t)(1+r)^{T-t}$

Time t

Buy forward	0
Short asset	$+S(t)$

Deposit money in bank	$-S(t)$
	0

Time T

Receive the asset from buying forward, pay forward price	$-F_t(T)$
Settle short position with asset from forward contract	0
Withdraw money from bank	$+S(t)(1+r)^{T-t}$
	$S(t)(1+r)^{T-t} - F_t(T) > 0$

Again, sure profit from zero investment $\Rightarrow$ Arbitrage opportunity.

Unless $F_t(T) = S(t)(1+r)^{T-t}$ we have an arbitrage opportunity.

b) By definition of a risk-neutral probability Q,

(i) $Q(\omega) > 0 \quad \forall\, \omega \in \Omega$

(ii) $S^*(t) := S(t)/(1+r)^t$ is a Q-martingale.

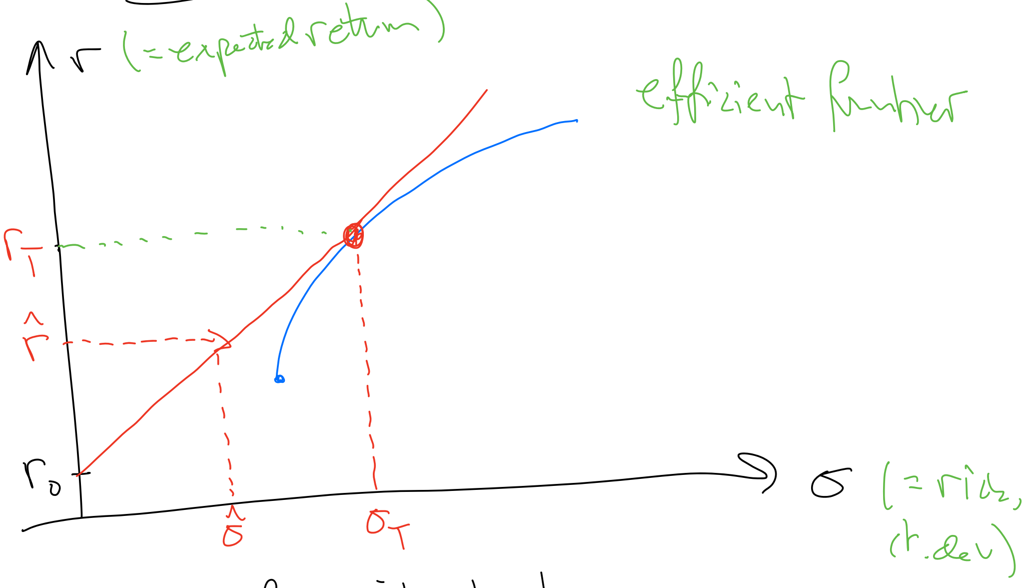
The last point means that $S(t)$ is $\mathcal{F}_t$-measurable for all $t = 0, .., T$ (or, $(S(t))_{t=0}^T$ is $\mathbb{F}$-adapted)

and
$$E_Q[S^*(s) \mid \mathcal{F}_t] = S^*(t), \quad s \geq t.$$

By this last property, we see that
$$E_Q\left[\frac{S(T)}{(1+r)^T} \,\Big|\, \mathcal{F}_t\right] = \frac{S(t)}{(1+r)^t} \quad \text{when}$$
letting $s = T$.

or $\quad E_Q[S(T) \mid \mathcal{F}_t] = S(t)(1+r)^{T-t} = F_t(T)$

6



r_0 is the risk-free interest rate

The tangent portfolio is the portfolio on the efficient frontier which is tangential to the frontier and the tangent passes through r_0 on the r-axis. We can have a portfolio with any risk $\hat{\sigma}$, with $0 \le \hat{\sigma} \le \sigma_T$, by mixing a bank deposit with the tangent portfolio. The bank deposit has risk-return $(0, r_0)$, while tangent portfolio has (σ_T, r_T).

Choose x so that $x r_0 + (1-x) r_T = \hat{r}$, where $\hat{r}$ is the desired return. This means

$$x = \frac{r_T - \hat{r}}{r_T - r_0}$$, invested in the bank.

Risk is then $\hat{\sigma}^2 = (1-x)^2 \sigma_T^2$.

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: Tuesday 29. november 2022

Examination hours: 15.00 – 19.00

This problem set consists of 13 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

In general, let assume N years, interest rate r and amount borrowed P. Then

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + ... + \frac{C}{(1+r)^N} = C \cdot \sum_{i=1}^{N} \frac{1}{(1+r)^i}$$

and using the formula $S = \frac{b_1(1-q^n)}{1-q}$ with $b_1 = \frac{1}{1+r}$, q=$\frac{1}{1+r}$ we get

$$P = C \cdot \frac{\frac{1}{1+r}\left(1 - \frac{1}{(1+r)^N}\right)}{1 - \frac{1}{1+r}}$$

Then the formula for each instalment is

$$C = \frac{rP}{1-(1+r)^{-N}}.$$

(Continued on page 2.)

The outstanding balance remaining after $n-1$ instalments is

$$P - \left(\frac{C}{1+r} + \frac{C}{(1+r)^2} + ... + \frac{C}{(1+r)^{n-1}}\right) =$$

$$= P - \frac{rP}{1-(1+r)^{-N}} \cdot \frac{1-(1+r)^{1-n}}{r} =$$

$$= P \cdot \frac{(1+r)^N - (1+r)^{n-1}}{(1+r)^N - 1}.$$

The interest included in the nth instalment is

$$P\frac{(1+r)^N - (1+r)^{n-1}}{(1+r)^N - 1}r.$$

The capital repaid as part of the nth instalment is

$$C - Pr \cdot \frac{(1+r)^N - (1+r)^{n-1}}{(1+r)^N - 1} =$$

$$= Pr \cdot \frac{(1+r)^N}{(1+r)^N - 1} - Pr \cdot \frac{(1+r)^N - (1+r)^{n-1}}{(1+r)^N - 1} = Pr \cdot \frac{(1+r)^{n-1}}{(1+r)^N - 1}.$$

b (weight 10p)

$S(0) = 1000$ NOK, $F(0,1) = 1070$ NOK.

There will be an arbitrage opportunity.

At time $t = 0$***:***

- Sell short one share for the price $S(0)$, investing 60 % of the proceeds at 8% and the remaining 30% as a security deposit to attract interest at 4%
- Enter a long forward contract with forward price $F(0,1)$

At time $t = 1$***:***

- Collect cash from investments and deposit.
- Buy a share for the price $F(0,1)$ and close the short position in stock.

This leaves an arbitrage profit of 10 NOK:

$$0.6 \cdot S(0) \cdot (1+r) + 0.4 \cdot S(0) \cdot (1+d) > F(0,1);$$

$$0.6 \cdot 1000 \cdot 1.1 + 0.4 \cdot 1000 \cdot 1.05 > 1070.$$

The rate d for the security deposit such that there is no arbitrage opportunity should satisfy

$$0.6 \cdot S(0) \cdot (1+r) + 0.4 \cdot S(0) \cdot (1+d) \leq F(0,1),$$

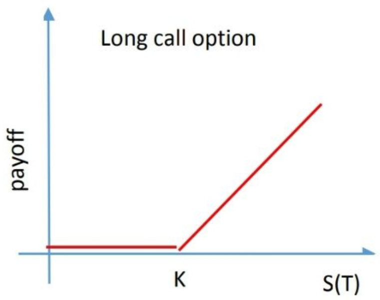
$$d \leq 0.025.$$

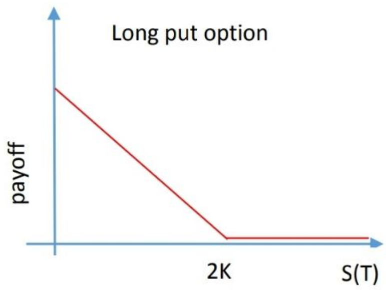
Hence, the highest rate is $d = 2.5\%$.

(Continued on page 3.)

c (weight 10p)

In this strategy you buy a call option with strike k and two put option with the strike $2K$ and the same expiry time T.



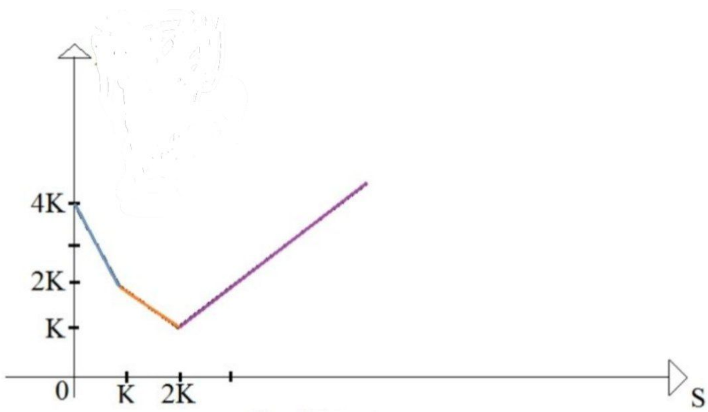


The full payoff function:

$$P\left(S(T)\right) = \left(S(T) - K\right)^{+} + 2\left(2K - S(T)\right)^{+}.$$

In this case, the table of profits is given by

$S(T)$	Profit
$S(T) < K$	$4K - 2S(T)$
$K \leq S(T) \leq 2K$	$3K - S(T)$
$S(T) > 2K$	$S(T) - K$



Problem 2

a (weight 10p)

Let B denote the price process for the bank account. We have that $B\left(0\right) = 1$ and $\mathrm{B}(1) = 1 + r = \frac{10}{9}$. The discounted price process for the risky asset is given by

(Continued on page 4.)

$S_1^*(0) = S_1(0)/B(0) = 7$ and $S_1^*(1) = S_1(1)/B(1) = (8,2,5)^T$. A risk neutral probability measure Q=$(Q_1,Q_2,Q_3)^T$ must satisfy the following system of equations

$$\begin{aligned} 7 &= S_1^*(0) = \mathbb{E}_Q[S_1^*(1)] = 8Q_1 + 2Q_2 + 5Q_3, \\ 1 &= Q_1 + Q_2 + Q_3. \end{aligned} \tag{1}$$

We have that $Q_1 = 1 - Q_2 - Q_3$ and substituting this value in equation (1) we obtain

$$1 = 6Q_2 + 3Q_3 \Longleftrightarrow Q_2 = \frac{1-3Q_3}{6}.$$

Moreover,

$$Q_1 = 1 - \frac{1-3Q_3}{6} - Q_3 = \frac{6-1+3Q_3-6Q_3}{6} = \frac{5-3Q_3}{6}.$$

Hence, setting $Q_3 = \lambda$, we get $Q_\lambda = \left(\frac{5-3\lambda}{6}, \frac{1-3\lambda}{6}, \lambda\right)^T$. Finally, as $Q_1 > 0, Q_2 > 0$, and $Q_3 > 0$ we have the following conditions on the parameter λ

$$\begin{aligned} Q_1 &= \frac{5-3\lambda}{6} > 0 \Longleftrightarrow \lambda < \frac{5}{3}, \\ Q_2 &= \frac{1-3\lambda}{2} > 0 \Longleftrightarrow \lambda < \frac{1}{3}, \\ Q_3 &= \lambda > 0, \end{aligned}$$

which yield that $\lambda \in \left(0, \frac{1}{3}\right)$. Therefore, the set of risk neutral measures $\mathbb{M}$ is given by

$$\mathbb{M} = \left\{ Q_\lambda = \left(\frac{5-3\lambda}{6}, \frac{1-3\lambda}{6}, \lambda\right)^T : 0 < \lambda < \frac{1}{3} \right\}$$

By the first fundamental theorem of asset pricing we know that the market is arbitrage free because the set of risk neutral probability measures is non empty. Alternative parametrizations of $\mathbb{M}$ are

$$\begin{aligned} \mathbb{M} &= \left\{ Q_\lambda = \left(\lambda, \frac{3\lambda-2}{3}, \frac{5-6\lambda}{3}\right)^T, \frac{2}{3} < \lambda < \frac{5}{6} \right\} \\ &= \left\{ Q_\lambda = (2+3\lambda, \lambda, -1-4\lambda)^T, 0 < \lambda < \frac{1}{4} \right\}. \end{aligned}$$

b (weight 10p)

A contingent claim $X = (X_1, X_2, X_3)^T$ is attainable if there exists a portfolio $H = (H_0, H_1)^T$such that $X = H_0 B(1) + H_1 S_1(1)$. This translates to the following system of equations

$$\begin{aligned} X_1 &= \frac{10}{9} H_0 + \frac{80}{9} H_1, \\ X_2 &= \frac{10}{9} H_0 + \frac{20}{9} H_1, \\ X_3 &= \frac{10}{9} H_0 + \frac{50}{9} H_1. \end{aligned}$$

(Continued on page 5.)

From the first equation we get that $\frac{10}{9}H_0 = X_1 - \frac{80}{9}H_1$. Substituting in the second and third equations we obtain

$$\begin{aligned} X_3 - X_2 &= \frac{30}{9}H_1, \\ X_1 - X_3 &= \frac{30}{9}H_1. \end{aligned}$$

Hence

$$X_1 - X_3 = X_3 - X_2 \iff X_1 + X_2 - 2X_3 = 0.$$

An alternative way of characterizing the attainable claims, when $\mathbb{M} \neq \emptyset$, is to find $X = (X_1, X_2, X_3)^T$ such that $\mathbb{E}_Q\left[\frac{X}{B(1)}\right]$ does not depend on λ. Hence, since $\lambda \in (0, 1/3)$, we have that

$$\begin{aligned} \mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] &= \frac{9}{10}\left\{X_1\lambda + X_2\frac{3\lambda - 2}{3} + X_3\frac{5 - 6\lambda}{3}\right\} \\ &= \frac{9}{10}\left\{(X_1 + X_2 - 2X_3)\lambda - \frac{2}{3}X_2 + \frac{5}{3}X_3\right\}, \end{aligned}$$

does not depend on λ (that is, on Q_λ) if and only if $X_1 + X_2 - 2X_3 = 0$.

In addition, by the second fundamental theorem of asset pricing we can conclude that the market is not complete because there are infinitely many risk neutral measures in this market.

c (weight 10p)

The contingent claim $Y = (2, 4, 3)^T$ is attainable because

$$X_1 + X_2 - 2X_3 = 2 + 4 - 6 = 0.$$

Therefore, for this claim the upper and lower hedging price coincide and are equal to

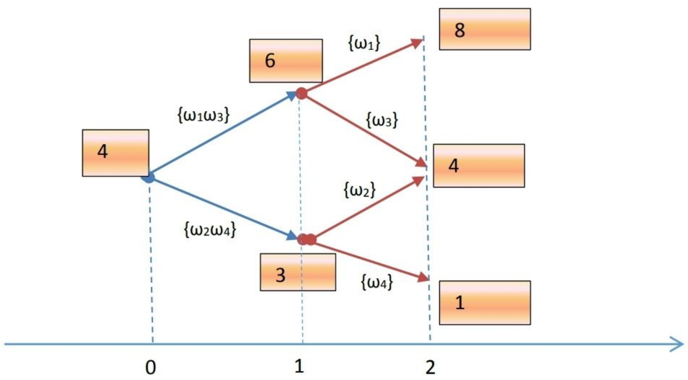
$$\begin{aligned} \mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] &= \frac{9}{10}\left\{(X_1 + X_2 - 2X_3)\lambda - \frac{2}{3}X_2 + \frac{5}{3}X_3\right\} = \\ &= \frac{9}{10}\left\{-\frac{2}{3}X_2 + \frac{5}{3}X_3\right\} = 2.1. \end{aligned}$$

Problem 3

a (weight 10p)

We have two-period market with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $P = \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right)^T$, $r = 0$, $B(t) = (1 + r)^t = 1$ and one risky asset $S_1 = \{S_(t)\}_{t=0,1,2}$

(Continued on page 6.)



The filtrations are given by

$$\mathcal{F}_0 = \sigma\left(S_1\left(0\right)\right) = \sigma\left(\left\{\Omega\right\}\right) = \left\{\emptyset, \Omega\right\},$$
$$\mathcal{F}_1 = \sigma\left(S_1\left(0\right), S_1\left(1\right)\right) = \sigma\left(\left\{\{\omega_1,\omega_3\},\{\omega_2,\omega_4\}\right\}\right) = \left\{\emptyset, \Omega, \{\omega_1,\omega_3\},\{\omega_2,\omega_4\}\right\},$$
$$\mathcal{F}_2 = \sigma\left(S_1\left(0\right), S_1\left(1\right), S_1\left(2\right)\right) = \sigma\left(\left\{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\right\}\right) = \mathcal{P}\left(\Omega\right),$$

where $\mathcal{P}\left(\Omega\right)$ is the set of all subsets of Ω.

$$\mathbb{E}\left[S_1(2,\omega)\middle|\,\mathcal{F}_1\right] = \mathbb{E}\left[S_1(2,\omega)\middle|\,\{\omega_1,\omega_2\}\right]\mathbf{1}_{\{\omega_1,\omega_2\}} + \mathbb{E}\left[S_1(2,\omega)\middle|\,\{\omega_2,\omega_4\}\right]\mathbf{1}_{\{\omega_2,\omega_4\}} =$$
$$= \left(S_1(2,\omega_1)\frac{p(\omega_1)}{p(\omega_1)+p(\omega_3)} + S_1(2,\omega_3)\frac{p(\omega_3)}{p(\omega_1)+p(\omega_3)}\right)\mathbf{1}_{\{\omega_1,\omega_3\}}+$$
$$+\left(S_1(2,\omega_2)\frac{p(\omega_2)}{p(\omega_2)+p(\omega_4)} + S_1(2,\omega_4)\frac{p(\omega_4)}{p(\omega_2)+p(\omega_4)}\right)\mathbf{1}_{\{\omega_2,\omega_4\}} =$$

$$= \left(8\cdot\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}} + 4\cdot\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}}\right)\mathbf{1}_{\{\omega_1,\omega_3\}} + + \left(4\cdot\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}} + 1\cdot\frac{\frac{1}{6}}{\frac{1}{3}+\frac{1}{6}}\right)\mathbf{1}_{\{\omega_2,\omega_4\}=}$$
$$= 6\cdot\mathbf{1}_{\{\omega_1,\omega_3\}} + 3\cdot\mathbf{1}_{\{\omega_2,\omega_4\}}$$

b (weight 10p)

By definition a risk neutral probability measure (martingale measure) is a probability measure Q such that

1. $Q\left(\omega\right) > 0, \omega \in \Omega$.

(Continued on page 7.)

2. S_n^*, $n = 1, ..., N$ are martingales under Q, that is,

$$\mathbb{E}_Q \left[S_n^* (t+s) | \mathcal{F}_t \right] = S_n^* (t), \qquad t, s \geq 0, n = 1, ..., N. \tag{2}$$

It suffices to check (2) for $s = 1$ and $t = 0, ..., T-1$, that is,

$$\mathbb{E}_Q \left[S_n^* (t+1) | \mathcal{F}_t \right] = S_n^* (t).$$

If $B(t) = (1+r)^t$, then (2) is equivalent to

$$\mathbb{E}_Q \left[S_n (t+1) | \mathcal{F}_t \right] = (1+r) S_n (t). \tag{3}$$

We will find $Q = (Q_1, Q_2, Q_3, Q_4)$ satisfying (3) for $t = 0, 1$.
At time $t = 0$:

$$S(0) = \mathbb{E}_Q \left[\frac{S(1)}{1+r} | \mathfrak{F}_0 \right] = \mathbb{E}_Q \left[\frac{S(1)}{1+r} \right]$$

For given market we will have

$$6(Q_1 + Q_3) + 3(Q_2 + Q_4) = 4$$

At time $t = 1$:

$$\begin{aligned}
(1+r)S(1) = \mathbb{E}_Q[S(2)|\mathfrak{F}_1] &= E_Q[S(2)|\omega_1\omega_2]\mathbf{1}_{\{\omega_1,\omega_2\}} + \mathbb{E}_Q[S(2)|\omega_3\omega_4]\mathbf{1}_{\{\omega_3,\omega_4\}} = \\
&= \left(S(2,\omega_1)\frac{Q(\omega_1)}{Q(\omega_1,\omega_3)} + S(2,\omega_3)\frac{Q(\omega_3)}{Q(\omega_1,\omega_3)} \right) \mathbf{1}_{\{\omega_1,\omega_3\}} + \\
&+ \left(S(2,\omega_3)\frac{Q(\omega_3)}{Q(\omega_3,\omega_4)} + S(2,\omega_4)\frac{Q(\omega_4)}{Q(\omega_3,\omega_4)} \right) \mathbf{1}_{\{\omega_3,\omega_4\}} = \\
&= \left(8\frac{Q(\omega_1)}{Q(\omega_1,\omega_3)} + 4\frac{Q(\omega_3)}{Q(\omega_1,\omega_3)} \right) \mathbf{1}_{\{\omega_1,\omega_3\}} + \\
&+ \left(4\frac{Q(\omega_2)}{Q(\omega_2,\omega 4)} + 1\frac{Q(\omega_4)}{Q(\omega_2,\omega_4)} \right) \mathbf{1}_{\{\omega_2,\omega_4\}}.
\end{aligned}$$

Note that

$$S(1) = S(1,\omega)\mathbf{1}_{\{\omega_1,\omega_3\}} + S(1,\omega)\mathbf{1}_{\{\omega_2,\omega_4\}},$$
$$S(1) = 6 \cdot \mathbf{1}_{\{\omega_1,\omega_3\}} + 3 \cdot \mathbf{1}_{\{\omega_2,\omega_4\}}.$$

Hence, we get

$$8Q_1 + 4Q_3 = 6(Q_1 + Q_3),$$
$$4Q_2 + Q_4 = 3(Q_2 + Q_4).$$

By definition RNPM we also have

$$Q_1 + Q_2 + Q_3 + Q_4 = 1.$$

(Continued on page 8.)

As a result, we get the system with four equations

$$Q_1 + Q_2 + Q_3 + Q_4 = 1,$$
$$8Q_1 + 4Q_3 = 6(Q_1 + Q_3),$$
$$4Q_2 + Q_4 = 3(Q_2 + Q_4),$$
$$6(Q_1 + Q_3) + 3(Q_2 + Q_4) = 4$$

We have four variable, four equations and we obtain

$$Q_1 = \frac{1}{6};\ \ Q_3 = \frac{1}{6};$$
$$Q_2 = \frac{4}{9};\ \ Q_4 = \frac{2}{9}.$$

An alternative way. Consider three single period market:

- From $t = 0$ to $t = 1$.
$$6Q(\omega_1\omega_3|\Omega) + 3Q(\omega_2\omega_4|\Omega) = 4;$$
$$Q(\omega_1\omega_3|\Omega) + Q(\omega_2\omega_4|\Omega) = 1.$$
Then
$$Q(\omega_1\omega_3|\Omega) = \frac{1}{3};\ Q(\omega_2\omega_4|\Omega) = \frac{2}{3}.$$
- From t=1 (S(1)=6) to $t = 2$.
$$8Q(\omega_1|\omega_1\omega_3) + 4Q(\omega_3|\omega_1\omega_3) = 6;$$
$$Q(\omega_1|\omega_1\omega_3) + Q(\omega_3|\omega_1\omega_3) = 1.$$
Then
$$Q(\omega_1|\omega_1\omega_3) = \frac{1}{2};\ Q(\omega_3|\omega_1\omega_3) = \frac{1}{2}.$$
- From t=1 (S(1)=3) to $t = 2$
$$4Q(\omega_2|\omega_2\omega_4) + Q(\omega_4|\omega_2\omega_4) = 3;$$
$$Q(\omega_2|\omega_2\omega_4) + Q(\omega_4|\omega_2\omega_4) = 1.$$
Then
$$Q(\omega_2|\omega_2\omega_4) = \frac{2}{3};\ Q(\omega_4|\omega_2\omega_4) = \frac{1}{3}.$$
Hence,
$$Q_1 = Q(\omega_1|\omega_1\omega_3)\cdot Q(\omega_1\omega_3|\Omega) = \frac{1}{6};\ \ Q_3 = Q(\omega_3|\omega_1\omega_3)\cdot Q(\omega_1\omega_3|\Omega) = \frac{1}{6};$$
$$Q_2 = Q(\omega_2|\omega_2\omega_4)\cdot Q(\omega_2\omega_4|\Omega) = \frac{4}{9};\ \ Q_4 = Q(\omega_4|\omega_2\omega_4)\cdot Q(\omega_2\omega_4|\Omega) = \frac{2}{9}.$$

(Continued on page 9.)

c (weight 10p)

Since the market is arbitrage free and complete, due to the first and second fundamental theorem of asset pricing. Then, we can use the martingale method to solve the optimal portfolio problem. In this setup, $M = \{Q\}$, the martingale method consists in the following two steps:

1. We first solve the constrained optimization problem

$$\max_{W} \mathbb{E}\left[U\left(W\right)\right]$$
$$\text{subject to } \mathbb{E}_Q\left[\frac{W}{B\left(2\right)}\right] = v,$$

   and obtain the optimal attainable wealth $\widehat{W}$.
2. Given $\widehat{W}$, we find the optimal trading strategy $\widehat{H}$ such that its associated value process $\widehat{V}$ replicates $\widehat{W}$, that is, $\widehat{V}\left(2\right) = \widehat{W}$.

The previous constrained problem can be solved using the Lagrange multipliers method. The optimal attainable wealth $\widehat{W}$ is given by

$$\widehat{W} = I\left(\frac{\widehat{\lambda}L}{B\left(2\right)}\right),$$

where I is the inverse of $U'\left(u\right)$, L is the state-price density vector $L = \frac{Q}{P}$, $B\left(2\right)$ is the price of the risk-less asset at time 2 and $\widehat{\lambda}$ is the optimal Lagrange multiplier associated to the constraint $\mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v$. Taking into account that $r = 0$, $U\left(u\right) = log(u)$, $P = \left(\frac{1}{4}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}\right)^T$ and $Q = \left(\frac{1}{6}, \frac{4}{9}, \frac{1}{6}, \frac{2}{9}\right)^T$,we have that

$$i = U'\left(u\right) = \frac{1}{u} \Longleftrightarrow I\left(i\right) = \frac{1}{i},$$
$$L = \left(\frac{\frac{1}{6}}{\frac{1}{4}}, \frac{\frac{4}{9}}{\frac{1}{3}}, \frac{\frac{1}{6}}{\frac{1}{4}}, \frac{\frac{2}{9}}{\frac{1}{6}}\right)^T = \left(\frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)^T,$$
$$B\left(2\right) = 1,$$

We get that $\widehat{W} = \left(\widehat{\lambda}L\right)^{-1}$. The optimal Lagrange multiplier $\widehat{\lambda}$ satisfies the equation

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B\left(2\right)}\right];$$
$$v = \frac{1}{\widehat{\lambda}}\mathbb{E}_Q\left[L^{-1}\right];$$

Therefore, we get

$$\widehat{\lambda} = v^{-1}\mathbb{E}_Q\left[L^{-1}\right].$$

(Continued on page 10.)

Note, that

$$\mathbb{E}_Q\left[L^{-1}\right] = \sum_{i=1}^{4} Q_i \cdot \frac{1}{L(\omega_i)} = \sum_{i=1}^{4} P(\omega_i)L(\omega_i) \cdot \frac{1}{L(\omega_i)} = 1.$$

Hence,

$$\widehat{W} = \frac{v}{L} = \left(\frac{3}{2}v; \frac{3}{4}v; \frac{3}{2}v; \frac{3}{4}v\right)^T.$$

and the optimal objective value is given by

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right] = \mathbb{E}\left[log\left(\widehat{W}\right)\right] =$$

$$= \mathbb{E}\left[log\left(\frac{v}{L}\right)\right] = \frac{1}{4}log\left(\frac{3}{2}v\right) + \frac{1}{3}log\left(\frac{3}{4}v\right) + \frac{1}{4}log\left(\frac{3}{2}v\right) + \frac{1}{6}log\left(\frac{3}{4}v\right).$$

Finally, we have to compute the optimal trading strategy $\widehat{H} = \left\{(H_0(t), H_1(t))^T\right\}_{t=1,2}$, that is, a self-financing and predictable process such that its asociated value process V satisfies $V(2) = \widehat{W}$. We first compute the discounted increments of the risky asset

$$\Delta S_1^*(2) = \Delta S_1(2) = (2, 1, -2, -2)^T,$$
$$\Delta S_1^*(1) = \Delta S_1(1) = (2, -1, 2, -1)^T.$$

- For $t = 2$, using that $\widehat{H}$ must be self-financing we have that

$$\frac{\widehat{W}}{B(2)} = \widehat{W} = \widehat{V}^*(1) + \widehat{H}_1(2)\,\Delta S_1^*(2).$$

  * Assuming that $\omega \in \{\omega_1, \omega_3\}$ and the predictability of $\widehat{H}$ we get the equations

$$\frac{3}{2}v = \widehat{W}_1 = \widehat{V}^*(1,\omega_1) + \widehat{H}_1(2,\omega_1) \times 2,$$
$$\frac{3}{2}v = \widehat{W}_3 = \widehat{V}^*(1,\omega_3) + \widehat{H}_1(2,\omega_3) \times (-2),$$
$$\widehat{V}^*(1,\omega_1) = \widehat{V}^*(1,\omega_3),$$
$$\widehat{H}_1(2,\omega_1) = \widehat{H}_1(2,\omega_3),$$

    which, using that $r = 0$, yield

$$\widehat{V}^*(1,\omega_1) = \widehat{V}^*(1,\omega_3) = V(1,\omega_1) = V(1,\omega_3) = \frac{3}{2}v,$$
$$\widehat{H}_1(2,\omega_1) = \widehat{H}_1(2,\omega_3) = 0.$$

(Continued on page 11.)

- $*$ Assuming that $\omega \in A_{1,2} = \{\omega_1, \omega_3\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned}
\frac{3}{4}v = \widehat{W_2} &= \widehat{V}^* (1, \omega_2) + \widehat{H}_1 (2, \omega_2) \times 1, \\
\frac{3}{4}v = \widehat{W_4} &= \widehat{V}^* (1, \omega_4) + \widehat{H}_1 (2, \omega_4) \times (-2), \\
\widehat{V}^* (1, \omega_1) &= \widehat{V}^* (1, \omega_3), \\
\widehat{H}_1 (2, \omega_1) &= \widehat{H}_1 (2, \omega_3),
\end{aligned}$$

which, using that $r = 0$, yield

$$\begin{aligned}
\widehat{V}^* (1, \omega_2) &= \widehat{V}^* (1, \omega_4) = V (1, \omega_2) = V (1, \omega_4) = \frac{3}{4}v, \\
\widehat{H}_1 (2, \omega_2) &= \widehat{H}_1 (2, \omega_4) = 0.
\end{aligned}$$

- For $t = 1$, the predictability assumption yields that $\widehat{H}_1 (1)$ is constant. Moreover, using that $\widehat{H}$ must be self-financing we have that $\widehat{V}^* (1) = \widehat{V}^* (0) + \widehat{H}_1 (1) \Delta S_1^* (1)$ and we get the following two equations

$$\begin{aligned}
\frac{3}{2}v &= \widehat{V}^* (1, \omega) = \widehat{V}^* (0) + \widehat{H}_1 (1) \times 2, \qquad (\text{for } \omega \in \{\omega_1, \omega_3\}) \\
\frac{3}{4}v &= \widehat{V}^* (1, \omega) = \widehat{V}^* (0) + \widehat{H}_1 (1) \times (-1), \qquad (\text{for } \omega \in \{\omega_2, \omega_4\})
\end{aligned}$$

which, using that $r = 0$, yield

$$\widehat{V}^* (0) = V (0) = v, \qquad \widehat{H}_1 (1) = \frac{1}{4}v.$$

- Finally we compute $\widehat{H}_0 (1)$ and $\widehat{H}_0 (2)$ from the definition of value process. We have

$$\widehat{H}_0 (1) = \widehat{V}^* (0) - \widehat{H}_1 (1) S_1^* (0) = v - \frac{1}{4}v \times 4 = 0,$$

and

$$\begin{aligned}
\widehat{H}_0 (2, \omega) &= \widehat{V}^* (1, \omega) - \widehat{H}_1 (2, \omega) S_1^* (1, \omega) \\
&= \begin{cases} \frac{3}{2}v - 0 \times 6 = \frac{3}{2}v & \text{if} \quad \omega \in \{\omega_1, \omega_3\} \\ \frac{3}{4}v - 0 \times 3 = \frac{3}{4}v & \text{if} \quad \omega \in \{\omega_2, \omega 4\} \end{cases}
\end{aligned}$$

Problem 4

a (weight 10p)

If X_1 and X_2 are random variable that are $\mathcal{F}$-measurable, in order to prove that, if Y is $\mathcal{F}$-measurable then

$$\mathbb{E} [(X_1 Y_1 + X_2 Y_2) | \mathcal{F}] = X_1 \mathbb{E} [Y_1 | \mathcal{F}] + X_2 \mathbb{E} [Y_2 | \mathcal{F}].$$

(Continued on page 12.)

we have to prove first that $X_i \mathbb{E}\left[Y_i|\,\mathcal{F}\right], i = 1,2$ is $\mathcal{F}$-measurable and secondly that

$$\mathbb{E}\left[X_i Y_i \mathbf{1}_B\right] = \mathbb{E}\left[X_i \mathbb{E}\left[Y_i|\,\mathcal{F}\right]\mathbf{1}_B\right], \qquad B \in \mathcal{F}. \tag{4}$$

Let $\{A_1, A_2, ..., A_m\}$be the partition that generates $\mathcal{F}$. That $X_i \mathbb{E}\left[Y_i|\,\mathcal{F}\right]$ is a $\mathcal{F}$-measurable random variable follows from the fact that the product of a $\mathcal{F}$-measurable r.v is a $\mathcal{F}$-measurable r.v., because it is constant over the subsets of the partition generating $\mathcal{F}$.

To prove (4), first note that by the linearity of the conditional expectation we can assume that $X_i = \mathbf{1}_{A_i}$ for some $i \in \{1, ..., m\}$ (Recall that an arbitrary $\mathcal{F}$-measurable r.v. is of the form $\sum_{i=1}^m a_i \mathbf{1}_{A_i}$ with $a_i \in \mathbb{R}$). Moreover, for all $B \in \mathcal{F}$ and $i = 1,2$

$$\begin{aligned}\mathbb{E}\left[X_i Y_i \mathbf{1}_B\right] &= \mathbb{E}\left[Y_i \mathbf{1}_{A_i}\mathbf{1}_B\right] = \mathbb{E}\left[Y_i \mathbf{1}_{A_i \cap B}\right] = \mathbb{E}\left[\mathbb{E}\left[Y_i|\,\mathcal{F}\right]\mathbf{1}_{A_i \cap B}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[Y_i|\,\mathcal{F}\right]\mathbf{1}_{A_i}\mathbf{1}_B\right] = \mathbb{E}\left[\mathbb{E}\left[Y_i|\,\mathcal{F}\right]X_i \mathbf{1}_B\right],\end{aligned}$$

which proves (4). In the third equality we have used the definition of conditional expectation and the fact that $A_i \cap B \in \mathcal{F}$.

b (weight 10p)

We want to find $\mathbb{E}\left[X_1^2 \cdot X_2^2 \cdot ... \cdot X_n^2 | \mathcal{F}_k\right]$. As $n > k$, then $X_1, X_2, ..., X_k$ are $\mathcal{F}_k$-measurable, $X_{k+1}, X_{k+2}, ..., X_n$ are independent. So,

$$\begin{aligned}\mathbb{E}\left[X_1^2 \cdot X_2^2 \cdot ... \cdot X_n^2 | \mathcal{F}_k\right] &= \mathbb{E}\left[X_1^2 \cdot X_2^2 \cdot ... \cdot X_k^2 | \mathcal{F}_k\right] \cdot \mathbb{E}\left[X_{k+1}^2 \cdot X_{k+2}^2 \cdot ... \cdot X_n^2 | \mathcal{F}_k\right] = \\ &= X_1^2 \cdot X_2^2 \cdot ... \cdot X_k^2 \cdot (\mathbb{E}[\left[X_{k+1}^2\right] \cdot ... \cdot \mathbb{E}[\left[X_n^2\right]) = \\ &\quad X_1^2 \cdot X_2^2 \cdot ... \cdot X_k^2 \cdot (\sigma^2 + a^2)^{n-k}.\end{aligned}$$

c (weight 10p)

We say that a process $X = \{X(t)\}_{t=0,...,T}$ is a martingale with respect to the filtration $\mathcal{F}$ under the probability measure P if X is $\mathcal{F}$-adapted, that is, $X(t)$ is $\mathcal{F}_t$-measurable for all $t = 0, ..., T$, and

$$\mathbb{E}\left[X(t+s)|\mathcal{F}_t\right] = X(t), \; t, s \geq 0.$$

Or, equivalently,

$$\mathbb{E}[X(t+1)|\mathcal{F}_t] = X(t), t \geq 0.$$

To prove that the process $G = \{G(t)\}_{t=0,...,T}$ defined by

$$G(0) = 0,$$

(Continued on page 13.)

$$G(t) = \sum_{u=1}^{t} H(u)(Z(u) - Z(u-1)),$$

is a martingale first we have to prove that $G(t)$ is $\mathcal{F}$-adapted. First note that if X and Y are $\mathcal{G}$-measurable with respect to an algebra $\mathcal{G}$ on , then XY and $X+Y$ are $\mathcal{G}$-measurable. The process H is predictable and, in particular, adapted to $\mathcal{F}$. The process Z is adapted to $\mathcal{F}$ because it is an $\mathcal{F}$-martingale, moreover $Z(u-1)$ is also $\mathcal{F}_u$-measurable because $\mathcal{F}_{u-1} \subseteq \mathcal{F}_u$. Therefore, $H(u)(Z(u) - Z(u-1))$ is $\mathcal{F}_u$-measurable for $u \leq t$. As $\mathcal{F}$ is a filtration, $\mathcal{F}_u \subseteq \mathcal{F}_t$, and we can conclude that $G(t)$ is $\mathcal{F}_t$-measurable and, hence, G is $\mathcal{F}$-adapted. The result To prove the martingale property, first note that

$$G(t+1) = G(t) + H(t+1)(Z(t+1) - Z(t)).$$

Then,

$$\begin{aligned} \mathbb{E}[G(t+1)|\mathcal{F}_t] &= \mathbb{E}[G(t) + H(t+1)(Z(t+1) - Z(t))|\mathcal{F}_t] = \\ &= \mathbb{E}[G(t)|\mathcal{F}_t] + \mathbb{E}[H(t+1)(Z(t+1) - Z(t))|\mathcal{F}_t] = \\ &= [G(t) + H(t+1)\mathbb{E}[(Z(t+1) - Z(t))|\mathcal{F}_t] = G(t), \end{aligned}$$

where in the second equality we have used the linearity of conditional expectation, in the third equality we have used that $H(t+1)$ is $\mathcal{F}_t$-measurable and the property proved in the previous section, and in the fourth equality we have used that Z is a martingale and, therefore,

$$\mathbb{E}[Z(t+1)|\mathcal{F}_t] = Z(t) \Leftrightarrow \mathbb{E}[(Z(t+1) - Z(t))|\mathcal{F}_t] = 0.$$

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: Tuesday 30. november 2021

Examination hours: 9.00 – 13.00

This problem set consists of 10 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Let $B(t,T)$ denote the price at time t of a a zero coupon bond with maturity time T. The return of the bond over a period $[s,t]$ with $0 \leq s < t \leq T$ is given by the formula

$$R(s,t) = \frac{B(t,T) - B(s,T)}{B(s,T)}.$$

Moreover the formula for the price of the bond is given by $B(t,T) = e^{-r(T-t)}$. Hence, we get

$$R(s,t) = \frac{e^{-r(T-t)} - e^{-r(T-s)}}{e^{-r(T-s)}} = e^{r(t-s)} - 1.$$

and

$$r = \frac{\log(1 + R(s,t))}{t-s}.$$

In this problem we have $t - s = 9/12 = 3/4$ and $R(s,t) = 0.02$, which yields

$$r = \frac{\log(1+0.02)}{3/4} = \frac{4}{3}\log(1.02) \approx 0.0264 = 2.64\%.$$

b (weight 10p)

Suppose that

$$C^E(0) - P^E(0) > S(0) - Ke^{-rT}.$$

At time 0

(Continued on page 2.)

- Buy one share for $S(0)$.
- Buy one put option for $P^E(0)$.
- Write and sell one call option for $C^E(0)$.
- Invest/borrow the amount $C^E(0) - P^E(0) - S(0)$, depending on the sign, risk free at rate r.

The value of this portfolio is zero.

At time T:

- Close the money market position, collecting (or paying) the amount

$$\left(C^E(0) - P^E(0) - S(0)\right) e^{rT}.$$

- Sell the share for K, either by:
  - exercising the put option if $S(T) \leq K$
  - settling the short position in the call option if $S(T) > K$.

This will give a total profit of

$$\left(C^E(0) - P^E(0) - S(0)\right) e^{rT} + K,$$

which is positive by assumption. Hence, we have a sure risk-less profit.

c (weight 10p)

In this strategy you buy a call option and a put option with the same strike K and the same expiry time T. The profit of the straddle as a function of the final price of the stock S_T is given by

$$P(S_T) = (S_T - K)^+ + (K - S_T)^+ - C^E(0) - P^E(0).$$

In this case, the table of profits is given by

S_T	Profit
$S_T < K$	$K - S_T - C^E(0) - P^E(0)$
$S_T \geq K$	$S_T - K - C^E(0) - P^E(0)$

Problem 2

Let B denote the price process for the bank account. We have that $B(0) = 1$ and B(1) $= 1 + r = \frac{11}{10}$. The discounted price process for the risky asset is given by

(Continued on page 3.)

$S_1^*(0) = S_1(0)/B(0) = 5$ and $S_1^*(1) = S_1(1)/B(1) = (6,4,3)^T$. A risk neutral probability measure $Q=(Q_1,Q_2,Q_3)^T$ must satisfy the following system of equations

$$\begin{aligned} 5 &= S_1^*(0) = \mathbb{E}_Q[S_1^*(1)] = 6Q_1 + 4Q_2 + 3Q_3, \qquad (1) \\ 1 &= Q_1 + Q_2 + Q_3. \end{aligned}$$

We have that $Q_1 = 1 - Q_2 - Q_3$ and substituting this value in equation (1) we obtain

$$1 = 2Q_2 + 3Q_3 \iff Q_2 = \frac{1-3Q_3}{2}.$$

Moreover,

$$Q_1 = 1 - \frac{1-3Q_3}{2} - Q_3 = \frac{2-1+3Q_3-2Q_3}{2} = \frac{1+Q_3}{2}.$$

Hence, setting $Q_3 = \lambda$, we get $Q_\lambda = \left(\frac{1+\lambda}{2}, \frac{1-3\lambda}{2}, \lambda\right)^T$. Finally, as $Q_1 > 0, Q_2 > 0$, and $Q_3 > 0$ we have the following conditions on the parameter λ

$$\begin{aligned} Q_1 &= \frac{1+\lambda}{2} > 0 \iff \lambda > -1, \\ Q_2 &= \frac{1-3\lambda}{2} > 0 \iff \lambda < \frac{2}{3}, \\ Q_3 &= \lambda > 0, \end{aligned}$$

which yield that $\lambda \in \left(0, \frac{1}{3}\right)$. Therefore, the set of risk neutral measures $\mathbb{M}$ is given by

$$\mathbb{M} = \left\{ Q_\lambda = \left(\frac{1+\lambda}{2}, \frac{1-3\lambda}{2}, \lambda\right)^T : 0 < \lambda < \frac{1}{3} \right\}$$

By the first fundamental theorem of asset pricing we know that the market is arbitrage free because the set of risk neutral probability measures is non empty. Alternative parametrizations of $\mathbb{M}$ are

$$\begin{aligned} \mathbb{M} &= \left\{ Q_\lambda = (\lambda, 2-3\lambda, 2\lambda-1)^T, \frac{1}{2} < \lambda < \frac{2}{3} \right\} \\ &= \left\{ Q_\lambda = \left(\frac{2-\lambda}{3}, \lambda, \frac{1-2\lambda}{3}\right)^T, 0 < \lambda < \frac{1}{2} \right\}. \end{aligned}$$

a (weight 10p)

A contingent claim $X = (X_1, X_2, X_3)^T$ is attainable if there exists a portfolio $H = (H_0, H_1)^T$ such that $X = H_0 B(1) + H_1 S_1(1)$. This translates to the following system of equations

$$\begin{aligned} X_1 &= \frac{11}{10} H_0 + \frac{33}{5} H_1, \\ X_2 &= \frac{11}{10} H_0 + \frac{22}{5} H_1, \\ X_3 &= \frac{11}{10} H_0 + \frac{33}{10} H_1. \end{aligned}$$

(Continued on page 4.)

From the first equation we get that $\frac{11}{10}H_0 = X_1 - \frac{33}{5}H_1$. Substituting in the second and third equations we obtain

$$X_2 = X_1 - \frac{11}{5}H_1,$$
$$X_3 = X_1 - \frac{33}{10}H_1.$$

From the first equation we get that $H_1 = \frac{5}{11}(X_1 - X_2)$ and substituting in the third equation we finally get

$$X_3 = X_1 - \frac{33}{10}\frac{5}{11}(X_1 - X_2) \Longleftrightarrow X_1 - 3X_2 + 2X_3 = 0.$$

An alternative way of characterizing the attainable claims, when $\mathbb{M} \neq \emptyset$, is to find $X = (X_1, X_2, X_3)^T$ such that $\mathbb{E}_Q\left[\frac{X}{B(1)}\right]$ does not on $Q \in \mathbb{M}$. Hence, since $\mathbb{M} = \{Q_\lambda\}_{\lambda\in(0,1/3)}$, we have that

$$\begin{aligned}\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] &= \frac{10}{11}\left\{X_1\frac{1+\lambda}{2} + X_2\frac{1-3\lambda}{2} + X_3\lambda\right\}\\ &= \frac{5}{11}\left\{(X_1 - 3X_2 + 2X_3)\lambda + X_1 + X_2\right\},\end{aligned}$$

does not depend on λ (that is, on Q_λ) if and only if $X_1 - 3X_2 + 2X_3 = 0$.

In addition, by the second fundamental theorem of asset pricing we can conclude that the market is not complete because there are infinitely many risk neutral measures in this market.

b (weight 10p)

The contingent claim $X = (1,5,2)^T$ is not attainable because

$$X_1 - 3X_2 + 2X_3 = 1 - 15 \times 0 + 4 = -10 \neq 0.$$

Hence, there is an interval of arbitrage free prices $[V_-(X), V_+(X)]$, where $V_-(X)$ is the lower hedging price of X and $V_+(X)$ is the upper hedging price of X. Moreover, we know that

$$V_-(X) = \inf_{Q\in M}\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right]\right\} = \inf_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right]\right\},$$

and

$$V_+(X) = \sup_{Q\in M}\left\{\mathbb{E}_Q\left[\frac{X}{B(1)}\right]\right\} = \sup_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right]\right\}.$$

(Continued on page 5.)

By the computations in the previous section, we have that

$$\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] = \frac{5}{11}\{(X_1 - 3X_2 + 2X_3)\lambda + X_1 + X_2\} = \frac{5}{11}(-10\lambda + 6),$$

and

$$\begin{aligned} V_-(X) &= \inf_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\frac{5}{11}(-10\lambda+6)\right\} = \frac{5}{11}\left(6 - 10\times\frac{1}{3}\right) = \frac{5}{11}\frac{8}{3} = \frac{40}{33}, \\ V_+(X) &= \sup_{\lambda\in\left(0,\frac{1}{3}\right)}\left\{\frac{5}{11}(-10\lambda+6)\right\} = \frac{5}{11}6 = \frac{30}{11}. \end{aligned}$$

On the other hand, the contingent claim $Y = (6, 2, 0)^T$ is attainable because

$$Y_1 - 3Y_2 + 2Y_3 = 6 - 3\times 2 + 2\times 0 = 0.$$

Therefore, for this claim the upper and lower hedging price coincide and are equal to

$$\mathbb{E}_{Q_\lambda}\left[\frac{Y}{B(1)}\right] = \frac{5}{11}\{(Y_1 - 3Y_2 + 2Y_3)\lambda + Y_1 + Y_2\} = \frac{5}{11}\{Y_1 + Y_2\} = \frac{5}{11}8 = \frac{40}{11}.$$

Problem 3

a (weight 10p)

We first compute the partitions associated to $S_1(0), S_1(1)$ and $S_1(2)$. We have

$\pi_{S_1(0)} = \{S_1(0) = 3\} = \{\Omega\},$
$\pi_{S_1(1)} = \{\{S_1(1) = 2\}, \{S_1(1) = 4\}\} = \{\{\omega_2, \omega_4\}, \{\omega_1, \omega_3\}\} =: \{A_{1,1}, A_{1,2}\},$
$\pi_{S_1(2)} = \{\{S_1(2) = 1\}, \{S_1(2) = 3\}, \{S_1(2) = 7\}\} = \{\{\omega_4\}, \{\omega_2, \omega_3\}, \{\omega_1\}\} =: \{A_{2,1}, A_{2,2}, A_{2,3}\}.$

The partitions associated to $(S(0), S(1))$ and to $(S(0), S(1), S(2))$ are given by

$$\begin{aligned} \pi_{(S_1(0),S_1(1))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} = \{\Omega\cap A_{1,1}, \Omega\cap A_{1,1}\} = \{A_{1,1}, A_{1,2}\}, \\ \pi_{(S_1(0),S_1(1),S_1(2))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} \cap \pi_{S_1(2)} = \pi_{S_1(0),S_1(1)} \cap \pi_{S_1(2)} \\ &= \{A_{1,1}\cap A_{2,1}, A_{1,1}\cap A_{2,2}, A_{1,1}\cap A_{2,3}, A_{1,2}\cap A_{2,1}, A_{1,2}\cap A_{2,2}, A_{1,2}\cap A_{2,3}\} \\ &= \{\{\omega_4\}, \{\omega_2\}, \emptyset, \emptyset, \{\omega_3\}, \{\omega_1\}\} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}. \end{aligned}$$

The filtrations are given by

$$\begin{aligned} \mathcal{F}_0 &= \sigma(S_1(0)) = \sigma(\{\Omega\}) = \{\emptyset, \Omega\}, \\ \mathcal{F}_1 &= \sigma(S_1(0), S_1(1)) = \sigma(\{A_{1,1}, A_{1,2}\}) = \{\emptyset, \Omega, A_{1,1}, A_{1,2}\}, \\ \mathcal{F}_2 &= \sigma(S_1(0), S_1(1), S_1(2)) = \sigma(\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}) = \mathcal{P}(\Omega), \end{aligned}$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω.

The random variable $Y = (3, 1, 3, 2)^T$ is not measurable with respect to $\mathcal{F}_1$ because it is not constant over $A_{1,1}$, which is one of the elements in the partition generating $\mathcal{F}_1$.

(Continued on page 6.)

b (weight 20p)

Since $M = \{Q\}$ the market is arbitrage free and complete, due to the first and second fundamental theorem of asset pricing. Then, we can use the martingale method to solve the optimal portfolio problem. In this setup,$M = \{Q\}$, the martingale method consists in the following two steps:

1. We first solve the constrained optimization problem

$$\max_W \mathbb{E}\left[U(W)\right]$$
$$\text{subject to } \mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v,$$

and obtain the optimal attainable wealth $\widehat{W}$.

2. Given $\widehat{W}$, we find the optimal trading strategy $\widehat{H}$ such that its associated value process $\widehat{V}$ replicates $\widehat{W}$, that is, $\widehat{V}(2) = \widehat{W}$.

The previous constrained problem can be solved using the Lagrange multipliers method. The optimal attainable wealth $\widehat{W}$ is given by

$$\widehat{W} = I\left(\frac{\widehat{\lambda} L}{B(2)}\right),$$

where I is the inverse of $U'(u)$, L is the state-price density vector $L = \frac{Q}{P}$, $B(2)$ is the price of the risk-less asset at time 2 and $\widehat{\lambda}$ is the optimal Lagrange multiplier associated to the constraint $\mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v$. Taking into account that $r = 0, U(u) = 2u^{1/2}$, $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$ and $Q = \left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}\right)^T$,we have that

$$\begin{aligned}
i = U'(u) = u^{-1/2} &\iff I(i) = u = i^{-2}, \\
L = \left(\frac{\frac{1}{8}}{\frac{1}{4}}, \frac{\frac{1}{4}}{\frac{1}{4}}, \frac{\frac{3}{8}}{\frac{1}{4}}, \frac{\frac{1}{4}}{\frac{1}{4}}\right)^T &= \left(\frac{1}{2}, 1, \frac{3}{2}, 1\right)^T, \\
B(2) = 1,
\end{aligned}$$

which yield $\widehat{W} = \left(\widehat{\lambda} L\right)^{-2}$. The optimal Lagrange multiplier $\widehat{\lambda}$ satisfies the equation

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B(2)}\right] = \mathbb{E}_Q\left[\frac{I\left(\frac{\widehat{\lambda} L}{B(2)}\right)}{B(2)}\right] = \mathbb{E}_Q\left[\left(\widehat{\lambda} L\right)^{-2}\right] = \left(\widehat{\lambda}\right)^{-2} \mathbb{E}_Q\left[L^{-2}\right].$$

Therefore, we get

$$\widehat{\lambda} = \left(\frac{\mathbb{E}_Q\left[L^{-2}\right]}{v}\right)^{1/2}, \qquad \widehat{W} = v\frac{L^{-2}}{\mathbb{E}_Q\left[L^{-2}\right]},$$

(Continued on page 7.)

and the optimal objective value is given by

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right]=\mathbb{E}\left[2\left(v\frac{L^{-2}}{\mathbb{E}_Q\left[L^{-2}\right]}\right)^{1/2}\right]=2v^{1/2}\frac{\mathbb{E}\left[L^{-1}\right]}{\mathbb{E}_Q\left[L^{-2}\right]^{1/2}}=2v^{1/2}\mathbb{E}_Q\left[L^{-2}\right]^{1/2},$$

where we have used that $\mathbb{E}\left[L^{-1}\right]=\mathbb{E}\left[LL^{-2}\right]=\mathbb{E}_Q\left[L^{-2}\right]$. Moreover, computing $L^{-2}=\left(4,1,\frac{4}{9},1\right)^T$ and

$$\mathbb{E}_Q\left[L^{-2}\right]=4\frac{1}{8}+1\frac{1}{4}+\frac{4}{9}\frac{3}{8}+1\frac{1}{4}=\frac{7}{6},$$

we obtain $\mathbb{E}\left[U\left(\widehat{W}\right)\right]=2\left(\frac{7}{6}v\right)^{1/2}$ and $\widehat{W}=\left(\frac{24}{7}v,\frac{6}{7}v,\frac{8}{21}v,\frac{6}{7}v\right)^T$.

Finally, we have to compute the optimal trading strategy $\widehat{H}=\left\{\left(H_0\left(t\right),H_1\left(t\right)\right)^T\right\}_{t=1,2}$, that is, a self-financing and predictable process such that its asociated value process V satisfies $\mathrm{V}(2)=\widehat{W}$. We first compute the discounted increments of the risky asset

$$\Delta S_1^*\left(2\right)=\Delta S_1\left(2\right)=\left(3,1,-1,-1\right)^T,$$
$$\Delta S_1^*\left(1\right)=\Delta S_1\left(1\right)=\left(1,-1,1,-1\right)^T.$$

- For $t=2$, using that $\widehat{H}$ must be self-financing we have that $\frac{\widehat{W}}{B(2)}=\widehat{W}=\widehat{V}^*\left(1\right)+\widehat{H}_1\left(2\right)\Delta S_1^*\left(2\right)$.
  - Assuming that $\omega\in A_{1,1}=\{\omega_2,\omega_4\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned}\frac{6}{7}v&=\widehat{W}_2=\widehat{V}^*\left(1,\omega_2\right)+\widehat{H}_1\left(2,\omega_2\right)\times 1,\\ \frac{6}{7}v&=\widehat{W}_4=\widehat{V}^*\left(1,\omega_4\right)+\widehat{H}_1\left(2,\omega_4\right)\times(-1),\\ \widehat{V}^*\left(1,\omega_2\right)&=\widehat{V}^*\left(1,\omega_4\right),\\ \widehat{H}_1\left(2,\omega_2\right)&=\widehat{H}_1\left(2,\omega_4\right),\end{aligned}$$

    which, using that $r=0$, yield

$$\begin{aligned}\widehat{V}^*\left(1,\omega_2\right)=\widehat{V}^*\left(1,\omega_4\right)=V\left(1,\omega_2\right)=V\left(1,\omega_4\right)&=\frac{6}{7}v,\\ \widehat{H}_1\left(2,\omega_2\right)=\widehat{H}_1\left(2,\omega_4\right)&=0.\end{aligned}$$

  - Assuming that $\omega\in A_{1,2}=\{\omega_1,\omega_3\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned}\frac{24}{7}v&=\widehat{W}_1=\widehat{V}^*\left(1,\omega_1\right)+\widehat{H}_1\left(2,\omega_1\right)\times 3,\\ \frac{24}{63}v&=\widehat{W}_3=\widehat{V}^*\left(1,\omega_3\right)+\widehat{H}_1\left(2,\omega_3\right)\times(-1),\\ \widehat{V}^*\left(1,\omega_1\right)&=\widehat{V}^*\left(1,\omega_3\right),\\ \widehat{H}_1\left(2,\omega_1\right)&=\widehat{H}_1\left(2,\omega_3\right),\end{aligned}$$

(Continued on page 8.)

which, using that $r = 0$, yield

$$\widehat{V}^*(1,\omega_1) = \widehat{V}^*(1,\omega_3) = V(1,\omega_1) = V(1,\omega_3) = \frac{8}{7}v,$$
$$\widehat{H}_1(2,\omega_1) = \widehat{H}_1(2,\omega_3) = \frac{16}{21}v.$$

- For $t = 1$, the predictability assumption yields that $\widehat{H}_1(1)$ is constant. Moreover, using that $\widehat{H}$ must be self-financing we have that $\widehat{V}^*(1) = \widehat{V}^*(0) + \widehat{H}_1(1)\Delta S_1^*(1)$ and we get the following two equations

$$\frac{6}{7}v = \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (-1), \qquad (\text{for } \omega \in A_{1,1})$$
$$\frac{8}{7}v = \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (1), \qquad (\text{for } \omega \in A_{1,2})$$

which, using that $r = 0$, yield

$$\widehat{V}^*(0) = V(0) = v, \qquad \widehat{H}_1(1) = \frac{1}{7}v.$$

- Finally we compute $\widehat{H}_0(1)$ and $\widehat{H}_0(2)$ from the definition of value process. We have

$$\widehat{H}_0(1) = \widehat{V}^*(0) - \widehat{H}_1(1)S_1^*(0) = v - \frac{1}{7}v \times 3 = \frac{4}{7}v,$$

and

$$\widehat{H}_0(2,\omega) = \widehat{V}^*(1,\omega) - \widehat{H}_1(2,\omega)S_1^*(1,\omega)$$
$$= \begin{cases} \frac{6}{7}v - 0 \times 2 = \frac{6}{7}v & \text{if } \omega \in A_{1,1} \\ \frac{8}{7}v - \frac{16}{21}v \times 4 = -\frac{40}{21}v & \text{if } \omega \in A_{1,2} \end{cases}$$

Problem 4

a (weight 10p)

The conditional expectation of X given $\mathcal{G}$ is the unique $\mathcal{G}$- measurable random variable $\mathbb{E}[X|\mathcal{G}]$ satisfying

$$\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_B], \qquad B \in \mathcal{G}.$$

We say that a process $X = \{X(t)\}_{t=0,...,T}$ is a martingale with respect to the filtration $\mathbb{F}$ under the probability measure P if X is $\mathbb{F}$-adapted, that is, X(t) is $\mathcal{F}_t$-measurable for all $t = 0, ..., T$, and

$$\mathbb{E}[X(t+s)|\mathcal{F}_t] = X(t), \qquad t, s \geq 0.$$

Or, equivalently,

$$\mathbb{E}[X(t+1)|\mathcal{F}_t] = X(t), \qquad t \geq 0.$$

(Continued on page 9.)

b (weight 10p)

In order to prove that, if Y is $\mathcal{G}$-measurable then

$$\mathbb{E}\left[XY|\,\mathcal{G}\right] = Y\mathbb{E}\left[X|\,\mathcal{G}\right].$$

we have to prove first that $Y\mathbb{E}\left[X|\,\mathcal{G}\right]$ is $\mathcal{G}$-measurable and secondly that

$$\mathbb{E}\left[XY\mathbf{1}_B\right] = \mathbb{E}\left[Y\mathbb{E}\left[X|\,\mathcal{G}\right]\mathbf{1}_B\right], \qquad B \in \mathcal{G}. \tag{2}$$

Let $\{A_1, A_2, ..., A_m\}$be the partition that generates $\mathcal{G}$. That $Y\mathbb{E}\left[X|\,\mathcal{G}\right]$ is a $\mathcal{G}$-measurable random variable follows from the fact that the product of a $\mathcal{G}$-measurable r.v is a $\mathcal{G}$-measurable r.v., because it is constant over the subsets of the partition generating $\mathcal{G}$.

To prove (2), first note that by the linearity of the conditional expectation we can assume that $Y = \mathbf{1}_{A_i}$ for some $i \in \{1, ..., m\}$ (Recall that an arbitrary $\mathcal{G}$-measurable r.v. is of the form $\sum_{i=1}^{m} a_i \mathbf{1}_{A_i}$ with $a_i \in \mathbb{R}$). Moreover, for all $B \in \mathcal{G}$

$$\begin{aligned}\mathbb{E}\left[XY\mathbf{1}_B\right] &= \mathbb{E}\left[X\mathbf{1}_{A_i}\mathbf{1}_B\right] = \mathbb{E}\left[X\mathbf{1}_{A_i\cap B}\right] = \mathbb{E}\left[\mathbb{E}\left[X|\,\mathcal{G}\right]\mathbf{1}_{A_i\cap B}\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[X|\,\mathcal{G}\right]\mathbf{1}_{A_i}\mathbf{1}_B\right] = \mathbb{E}\left[\mathbb{E}\left[X|\,\mathcal{G}\right]Y\mathbf{1}_B\right],\end{aligned}$$

which proves (2). In the third equality we have used the definition of conditional expectation and the fact that $A_i \cap B \in \mathcal{G}$.

c (weight 10p)

1. First we prove the implication $\Rightarrow$). We have that $Z = XL$ is $\mathbb{F}$-adapted because it is the product of two $\mathbb{F}$-adapted processes. Regarding the martingale condition, we have that

$$\begin{aligned}\mathbb{E}\left[X\left(t+1\right)L\left(t+1\right)|\,\mathcal{F}_t\right] &\stackrel{(a)}{=} \mathbb{E}\left[X\left(t+1\right)\mathbb{E}\left[L\left(T\right)|\,\mathcal{F}_{t+1}\right]|\,\mathcal{F}_t\right] \\ &\stackrel{(b)}{=} \mathbb{E}\left[\mathbb{E}\left[X\left(t+1\right)L\left(T\right)|\,\mathcal{F}_{t+1}\right]|\,\mathcal{F}_t\right] \\ &\stackrel{(c)}{=} \mathbb{E}\left[X\left(t+1\right)L\left(T\right)|\,\mathcal{F}_t\right] \\ &\stackrel{(d)}{=} \frac{\mathbb{E}\left[X\left(t+1\right)L\left(T\right)|\,\mathcal{F}_t\right]}{L\left(t\right)}L\left(t\right) \\ &\stackrel{(e)}{=} \mathbb{E}_Q\left[X\left(t+1\right)|\,\mathcal{F}_t\right]L\left(t\right) \\ &\stackrel{(f)}{=} X\left(t\right)L\left(t\right),\end{aligned}$$

where we have used: (a) Definition of the process L, (b) $X\left(t+1\right)$ is $\mathcal{F}_{t+1}$-measurable and goes in $\mathbb{E}\left[\cdot|\,\mathcal{F}_{t+1}\right]$, (c) Tower law, (d) Divide and multiply by $L\left(t\right)$, (e) Formula for the conditional expectation under Q, (f) X is a martingale under Q.
Next we prove the implication$\Leftarrow$). We have that $X = Z/L$ is $\mathbb{F}$-adapted because

(Continued on page 10.)

it is the quotient of two $\mathbb{F}$-adapted processes with strictly positive denominator. Regarding the martingale condition, we have that

$$\begin{aligned}
\mathbb{E}_Q\left[X(t+1) \middle| \mathcal{F}_t \right] &\overset{(a)}{=} \frac{\mathbb{E}\left[X(t+1)\, L(T) \middle| \mathcal{F}_t \right]}{L(t)} \\
&\overset{(b)}{=} \frac{\mathbb{E}\left[X(t+1)\, \mathbb{E}\left[L(T) \middle| \mathcal{F}_{t+1} \right] \middle| \mathcal{F}_t \right]}{L(t)} \\
&\overset{(c)}{=} \frac{\mathbb{E}\left[X(t+1)\, L(t+1) \middle| \mathcal{F}_t \right]}{L(t)} \\
&\overset{(d)}{=} \frac{X(t)\, L(t)}{L(t)} = X(t)\,,
\end{aligned}$$

where we have used: (a) Formula for the conditional expectation under Q, (b) Tower law and $X(t+1)$ is $\mathcal{F}_{t+1}$-measurable and goes out $\mathbb{E}\left[\cdot \middle| \mathcal{F}_{t+1} \right]$, (c) Definition of the process L, $(d) XL$ is a martingale under P.

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — An Introduction to Mathematical Finance

Day of examination: Thursday 26. November 2020

Examination hours: 15.00 – 19.00

This problem set consists of 12 pages.

Appendices: All

Permitted aids: All

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

The price of a zero-coupon bond at time t is given by $B(t,T) = e^{-r(T-t)}$, where r is the implied annual (continuous) compounding rate. Moreover, the return of this bond bond over a period $[s,t] \subset [0,T]$ is given by

$$R(s,t) = \frac{B(t,T) - B(s,T)}{B(s,T)} = \frac{e^{-r(T-t)} - e^{-r(T-s)}}{e^{-r(T-s)}}.$$

Here, we have $s = 0, T = 1, B(0,1) = 0.93$ and $R(0,t) = 0.04$. Hence,

$$0.93 = B(0,1) = e^{-r(1-0)} \iff r = -\log(0.93) \simeq 0.0726 = 7.26\%.$$

On the other hand,

$$R(0,t) = \frac{e^{-r(1-t)} - B(0,1)}{B(0,1)},$$

which yields

$$\begin{aligned} t &= 1 + \frac{\log(B(0,1)(1 + R(0,t)))}{r} \\ &= 1 + \frac{\log(0.93(1 + 0.04))}{0.0726} \simeq 0.5406 \simeq 197.3190 \simeq 198 \text{ days}. \end{aligned}$$

(Continued on page 2.)

b (weight 10p)

Suppose that

$$V(t) < (F(t,T) - F(0,T))\, e^{-r(T-t)}. \tag{1}$$

Then, at time t:

- Borrow the amount $V(t)$.
- Pay $V(t)$ to enter a long forward position with forward price $F(0,T)$.
- Take a short forward position with forward price $F(t,T)$ (at no cost).

Next, at time T:

- Close the forward positions, getting:
- $S(T) - F(0,T)$ for the long position,
- $F(t,T) - S(T)$ for the short position.
- Pay $V(t)\, e^{r(T-t)}$ to settle the loan.

This will yield a risk free profit of

$$\begin{aligned} &S(T) - F(0,T) + F(t,T) - S(T) - V(t)\, e^{r(T-t)} \\ &= F(t,T) - F(0,T) - V(t)\, e^{r(T-t)} > 0. \end{aligned}$$

c (weight 10p)

Let $0 < K_1 < K_2 < K_3$. In this strategy you buy a call option with strike K_1 (for $C^E(0,K_1)$) and a call option with strike K_3 (for $C^E(0,K_3)$) and sell two call options with strike K_2 (for $2C^E(0,K_2)$). The profit of the strangle as a function of the final price of the stock $S(T)$ is given by

$$P(S(T)) = (S(T) - K_1)^+ + (S(T) - K_3)^+ - 2(S(T) - K_2)^+ - C,$$

where $C = C^E(0,K_1) + C^E(0,K_3) - 2C^E(0,K_2)$ is the initial cost of the strategy. In this case, the table of profits is given by

$S(T)$	Profit
$S(T) < K_1$	$-C$
$K_1 \leq S(T) < K_2$	$S(T) - K_1 - C$
$K_2 \leq S(T) < K_3$	$2K_2 - K_1 - S(T) - C$
$K_3 \leq S(T)$	$2K_2 - K_1 - K_3 - C$

(Continued on page 3.)

Problem 2

a (weight 10p)

Let B denote the price process for the bank account. We have that $B(0) = 1$ and $\mathrm{B}(1) = \frac{9}{8}$. The discounted price processes for the risky assets are given by $S_1^*(0) = S_1(0)/B(0) = 7$,$S_2^*(0) = S_2(0)/B(0) = 8$, $S_1^*(1) = S_1(1)/B(1) = (8, 10, 6, 6)^T$ and $S_2^*(1)/B(1) = (12, 6, 6, 10)^T$. A risk neutral probability measure Q=$(Q_1, Q_2, Q_3, Q_4)^T$ must satisfy the following conditions

$$\begin{aligned}\mathbb{E}_Q[S_1^*(1)] &= S_1^*(0),\\ \mathbb{E}_Q[S_2^*(1)] &= S_2^*(0),\end{aligned}$$

which are equivalent to the following equations

$$8Q_1 + 10Q_2 + 6Q_3 + 6Q_4 = 7, \tag{2}$$

$$6Q_1 + 3Q_2 + 3Q_3 + 5Q_4 = 4, \tag{3}$$

$$Q_1 + Q_2 + Q_3 + Q_4 = 1 \tag{4}$$

with the following restrictions $Q_1 > 0, Q_2 > 0, Q_3 > 0, Q_4 > 0$. From (4) we have that $Q_4 = 1 - Q_1 - Q_2 - Q_3$ and substituting this value in (2) and (3) we obtain

$$2Q_1 + 4Q_2 = 1, \tag{5}$$

$$Q_1 - 2Q_2 - 2Q_3 = -1. \tag{6}$$

From (5) we get that $Q_2 = \frac{1-2Q_1}{4}$. Substituting this value in (6) we get

$$Q_1 - 2\left(\frac{1-2Q_1}{4}\right) - 2Q_3 = -1 \Longleftrightarrow Q_3 = \frac{1+4Q_1}{4},$$

and

$$Q_4 = 1 - Q_1 - \frac{1-2Q_1}{4} - \frac{1+4Q_1}{4} = \frac{1-3Q_1}{2}.$$

Hence, setting $Q_1 = \lambda$, we get $Q_\lambda = \left(\lambda, \frac{1-2\lambda}{4}, \frac{1+4\lambda}{4}, \frac{1-3\lambda}{2}\right)^T$. Finally, using the restrictions $Q_i > 0, i = 1, ..., 4$, we have the following conditions on the parameter λ

$$\begin{aligned}Q_1 &= \lambda > 0\\ Q_2 &= \frac{1-2\lambda}{4} > 0 \Longleftrightarrow \lambda < \frac{1}{2},\\ Q_3 &= \frac{1+4\lambda}{4} > 0 \Longleftrightarrow \lambda > -\frac{1}{4},\\ Q_4 &= \frac{1-3\lambda}{2} > 0 \Longleftrightarrow \lambda < \frac{1}{3},\end{aligned}$$

(Continued on page 4.)

which yield that $\lambda \in \left(0, \frac{1}{3}\right)$. Therefore, the set of risk neutral measures $\mathbb{M}$ is given by

$$\mathbb{M} = \left\{ Q_\lambda = \left(\lambda, \frac{1-2\lambda}{4}, \frac{1+4\lambda}{4}, \frac{1-3\lambda}{2}\right)^T, 0 < \lambda < \frac{1}{3} \right\}$$

By the first fundamental theorem of asset pricing we know that the market is arbitrage free because the set of risk neutral probability measures is non empty. Alternative parametrizations of $\mathbb{M}$ are

$$\begin{aligned}
\mathbb{M} &= \left\{ Q_\lambda = \left(\frac{1-4\lambda}{2}, \lambda, \frac{3-8\lambda}{4}, \frac{12\lambda-1}{4}\right)^T, \frac{1}{12} < \lambda < \frac{3}{8} \right\} \\
&= \left\{ Q_\lambda = \left(\frac{4\lambda-1}{4}, \frac{3-4\lambda}{8}, \lambda, \frac{7-12\lambda}{8}\right)^T, \frac{1}{4} < \lambda < \frac{7}{12} \right\} \\
&= \left\{ Q_\lambda = \left(\frac{1-2\lambda}{3}, \frac{1+4\lambda}{12}, \frac{7-8\lambda}{12}, \lambda\right)^T, 0 < \lambda < \frac{1}{2} \right\}.
\end{aligned}$$

b (weight 10p)

By the second fundamental theorem of asset pricing we can conclude that the market is not complete because there are infinitely many risk neutral measures in this market. A contingent claim $X = (X_1, X_2, X_3, X_4)^T$ is attainable if there exists a portfolio $H = (H_0, H_1, H_2)^T$ such that $X = H_0 B(1) + H_1 S_1(1) + H_2 S_2(1)$. This translates to the following system of equations

$$X_1 = \frac{9}{8}H_0 + 9H_1 + \frac{27}{2}H_2, \tag{7}$$
$$X_2 = \frac{9}{8}H_0 + \frac{45}{4}H_1 + \frac{27}{4}H_2, \tag{8}$$
$$X_3 = \frac{9}{8}H_0 + \frac{27}{4}H_1 + \frac{27}{4}H_2, \tag{9}$$
$$X_4 = \frac{9}{8}H_0 + \frac{27}{4}H_1 + \frac{45}{4}H_2. \tag{10}$$

From (7) we get that $\frac{9}{8}H_0 = X_1 - 9H_1 - \frac{27}{2}H_2$. Substituting this expression for $\frac{9}{8}H_0$ in (8),(9) and (10) we obtain

$$X_2 = X_1 + \frac{9}{4}H_1 - \frac{27}{4}H_2, \tag{11}$$
$$X_3 = X_1 - \frac{9}{4}H_1 - \frac{27}{4}H_2, \tag{12}$$
$$X_4 = X_1 - \frac{9}{4}H_1 - \frac{9}{4}H_2 \tag{13}$$

(Continued on page 5.)

Solving (11) and (12) in H_1 and H_2 we get that

$$\begin{aligned} H_1 &= \frac{2}{9}\left(X_2 - X_3\right), \\ H_2 &= \frac{4X_1 - 2X_2 - 2X_3}{27}, \end{aligned}$$

and substituting these values in (13) we get

$$X_4 = X_1 - \frac{9}{4}\frac{2}{9}\left(X_2 - X_3\right) - \frac{9}{4}\frac{4X_1 - 2X_2 - 2X_3}{27} \iff 2X_1 - X_2 + 2X_3 - 3X_4 = 0.$$

Alternatively, since $\mathbb{M} \neq \emptyset$, we have that X is attainable if and only if $\mathbb{E}_{Q_\lambda}\left[X/B\left(1\right)\right]$ does not depend on λ. We have that

$$\mathbb{E}_{Q_\lambda}\left[X/B\left(1\right)\right] = \frac{1}{B\left(1\right)}\left\{\lambda\left(X_1 - \frac{X_2}{2} + X_3 - \frac{3}{2}X_4\right) + \frac{X_2 + X_3 + 2X_4}{2}\right\},$$

and the previous expectation does not depend on λ if and only if

$$X_1 - \frac{X_2}{2} + X_3 - \frac{3}{2}X_4 = 0 \iff 2X_1 - X_2 + 2X_3 - 3X_4 = 0.$$

c (weight 10p)

We have that

$$X = \begin{pmatrix} \max\left(0, S_2\left(1,\omega_1\right) - S_1\left(1,\omega_1\right) - 9/4\right) \\ \max\left(0, S_2\left(1,\omega_2\right) - S_1\left(1,\omega_2\right) - 9/4\right) \\ \max\left(0, S_2\left(1,\omega_3\right) - S_1\left(1,\omega_3\right) - 9/4\right) \\ \max\left(0, S_2\left(1,\omega_4\right) - S_1\left(1,\omega_4\right) - 9/4\right) \end{pmatrix} = \begin{pmatrix} \max\left(0, \frac{27}{2} - 9 - \frac{9}{4}\right) \\ \max\left(0, \frac{27}{4} - \frac{45}{4} - \frac{9}{4}\right) \\ \max\left(0, \frac{27}{4} - \frac{27}{4} - \frac{9}{4}\right) \\ \max\left(0, \frac{45}{4} - \frac{27}{4} - \frac{9}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{9}{4} \\ 0 \\ 0 \\ \frac{9}{4} \end{pmatrix},$$

and, therefore, it is not attainable because

$$2X_1 - X_2 + 2X_3 - 3X_4 = 2 \times \frac{9}{4} - 1 \times 0 + 2 \times 0 - 3 \times \frac{9}{4} = -\frac{9}{4} \neq 0.$$

Hence, there is an interval of arbitrage free prices $\left[V_-\left(X\right), V_+\left(X\right)\right]$, where $V_-\left(X\right)$ is the lower hedging price of X and $V_+\left(X\right)$ is the upper hedging price of X. Moreover, we know that

$$V_-\left(X\right) = \inf_{Q \in M}\left\{\mathbb{E}_Q\left[\frac{X}{B\left(1\right)}\right]\right\} = \inf_{\lambda \in \left(0,\frac{1}{3}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B\left(1\right)}\right]\right\},$$

and

$$V_+\left(X\right) = \sup_{Q \in M}\left\{\mathbb{E}_Q\left[\frac{X}{B\left(1\right)}\right]\right\} = \sup_{\lambda \in \left(0,\frac{1}{3}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B\left(1\right)}\right]\right\}.$$

(Continued on page 6.)

We have that

$$\begin{aligned}
\mathbb{E}_{Q_\lambda}\left[\frac{X}{B(1)}\right] &= \frac{8}{9}\mathbb{E}_{Q_\lambda}[X] \\
&= \frac{8}{9}\left\{\frac{9}{4}\times\lambda + 0\times\frac{1-2\lambda}{4} + 0\times\frac{1+4\lambda}{4} + \frac{9}{4}\times\frac{1-3\lambda}{2}\right\} \\
&= 2\left\{\lambda + \frac{1-3\lambda}{2}\right\} = 1-\lambda.
\end{aligned}$$

The previous computation yields

$$\begin{aligned}
V_-(X) &= \inf_{\lambda\in\left(0,\frac{1}{3}\right)}\{1-\lambda\} = \frac{2}{3}, \\
V_+(X) &= \sup_{\lambda\in\left(0,\frac{1}{3}\right)}\{1-\lambda\} = 1.
\end{aligned}$$

d (weight 5p)

The algebra of events generated by $S_1(1)$, denoted by $\mathfrak{a}(S_1(1))$, is the algebra generated by the partition

$$\begin{aligned}
\pi_1 &= \left\{\{S_1(1)=9\}, \left\{S_1(1)=\frac{45}{4}\right\}, \left\{S_1(1)=\frac{27}{4}\right\}\right\} \\
&= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3,\omega_4\}\}.
\end{aligned}$$

Moreover, a random variable (contingent claim) is measurable with respect to $\mathfrak{a}(S_1(1))$ if it is constant over the elements of π_1. In this case, since $Y_3 = Y(\omega_3) \neq Y(\omega_4) = Y_4$, Y is not constant over the elments of $\pi_1$and, hence, it is not measurable with respect to $\mathfrak{a}(S_1(1))$.

However, the algebra of events generated by $S_2(1)$, denoted by $\mathfrak{a}(S_2(1))$, is the algebra generated by the partition

$$\begin{aligned}
\pi_2 &= \left\{\left\{S_2(1)=\frac{27}{2}\right\}, \left\{S_2(1)=\frac{27}{4}\right\}, \left\{S_1(1)=\frac{45}{4}\right\}\right\} \\
&= \{\{\omega_1\}, \{\omega_2,\omega_3\}, \{\omega_4\}\},
\end{aligned}$$

and Y is constant over the elements of π_2.

Problem 3 (weight 5p)

a (weight 5p)

We first compute the partitions associated to $S_1(0), S_1(1)$ and $S_1(2)$. We have

$\pi_{S_1(0)} = \{S_1(0)=3\} = \{\Omega\},$

$\pi_{S_1(1)} = \{\{S_1(1)=2\}, \{S_1(1)=4\}\} = \{\{\omega_3,\omega_4\}, \{\omega_1,\omega_2\}\} =: \{A_{1,1}, A_{1,2}\},$

$\pi_{S_1(2)} = \{\{S_1(2)=1\}, \{S_1(2)=4\}, \{S_1(2)=6\}\} = \{\{\omega_2,\omega_4\}, \{\omega_3\}, \{\omega_1\}\} =: \{A_{2,1}, A_{2,2}, A_{2,3}\}.$

(Continued on page 7.)

The partitions associated to $(S_1(0), S_1(1))$ and to $(S_1(0), S_1(1), S_1(2))$ are given by

$$\begin{aligned}
\pi_{(S_1(0),S_1(1))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} = \{\Omega \cap A_{1,1}, \Omega \cap A_{1,1}\} = \{A_{1,1}, A_{1,2}\}, \\
\pi_{(S_1(0),S_1(1),S_1(2))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} \cap \pi_{S_1(2)} = \pi_{S_1(0),S_1(1)} \cap \pi_{S_1(2)} \\
&= \{A_{1,1} \cap A_{2,1}, A_{1,1} \cap A_{2,2}, A_{1,1} \cap A_{2,3}, A_{1,2} \cap A_{2,1}, A_{1,2} \cap A_{2,2}, A_{1,2} \cap A_{2,3}\} \\
&= \{\{\omega_4\}, \{\omega_3\}, \emptyset, \{\omega_2\}, \emptyset, \{\omega_1\}\} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}.
\end{aligned}$$

The filtrations are given by

$$\begin{aligned}
&\mathcal{F}_0 = \mathfrak{a}(S_1(0)) = \mathfrak{a}(\{\Omega\}) = \{\emptyset, \Omega\}, \\
&\mathcal{F}_1 = \mathfrak{a}(S_1(0), S_1(1)) = \mathfrak{a}(\{A_{1,1}, A_{1,2}\}) = \{\emptyset, \Omega, A_{1,1}, A_{1,2}\} = \{\emptyset, \Omega, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2\}\}, \\
&\mathcal{F}_2 = \mathfrak{a}(S_1(0), S_1(1), S_1(2)) = \mathfrak{a}(\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}) = \mathcal{P}(\Omega),
\end{aligned}$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω.

b (weight 20p)

Since $\mathbb{M} = \{Q\}$ the market is arbitrage free and complete, due to the first and second fundamental theorem of asset pricing. Then, we can use the martingale method to solve the optimal portfolio problem. In this setup, $\mathbb{M} = \{Q\}$, the martingale method consists in the following two steps:

1. We first solve the constrained optimization problem

$$\max_{W} \mathbb{E}[U(W)]$$
$$\text{subject to } \mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v,$$

   and obtain the optimal attainable wealth $\widehat{W}$.

2. Given $\widehat{W}$, we find the optimal trading strategy $\widehat{H}$ such that its associated value process $\widehat{V}$ replicates $\widehat{W}$, that is, $\widehat{V}(2) = \widehat{W}$.

The previous constrained problem can be solved using the Lagrange multipliers method. The optimal attainable wealth $\widehat{W}$ is given by

$$\widehat{W} = I\left(\frac{\widehat{\lambda} L}{B(2)}\right),$$

where I is the inverse of $U'(u)$, L is the state-price density vector $L = \frac{Q}{P}$, $B(2)$ is the price of the risk-less asset at time 2 and $\widehat{\lambda}$ is the optimal Lagrange multiplier associated to the constraint $\mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v$. Taking into account that $r = 0$,$U(u) = 2u^{1/2}$,

(Continued on page 8.)

$P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$ and $Q = \left(\frac{3}{10}, \frac{1}{5}, \frac{1}{6}, \frac{1}{3}\right)^T$,we have that

$$\begin{aligned} i = U'(u) = u^{-1/2} &\Longleftrightarrow I(i) = u = i^{-2}, \\ L &= \left(\frac{\frac{3}{10}}{\frac{1}{4}}, \frac{\frac{1}{5}}{\frac{1}{4}}, \frac{\frac{1}{6}}{\frac{1}{4}}, \frac{\frac{1}{3}}{\frac{1}{4}}\right)^T = \left(\frac{6}{5}, \frac{4}{5}, \frac{2}{3}, \frac{4}{3}\right)^T, \\ B(2) &= 1, \end{aligned}$$

which yield $\widehat{W} = \left(\widehat{\lambda} L\right)^{-2}$. The optimal Lagrange multiplier $\widehat{\lambda}$ satisfies the equation

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B(2)}\right] = \mathbb{E}_Q\left[\frac{I\left(\frac{\widehat{\lambda} L}{B(2)}\right)}{B(2)}\right] = \mathbb{E}_Q\left[\left(\widehat{\lambda} L\right)^{-2}\right] = \left(\widehat{\lambda}\right)^{-2} \mathbb{E}_Q\left[L^{-2}\right].$$

Therefore, we get

$$\widehat{\lambda} = \left(\frac{\mathbb{E}_Q\left[L^{-2}\right]}{v}\right)^{1/2}, \qquad \widehat{W} = v\frac{L^{-2}}{\mathbb{E}_Q\left[L^{-2}\right]},$$

and the optimal objective value is given by

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right] = \mathbb{E}\left[2\left(v\frac{L^{-2}}{\mathbb{E}_Q\left[L^{-2}\right]}\right)^{1/2}\right] = 2v^{1/2}\frac{\mathbb{E}\left[L^{-1}\right]}{\mathbb{E}_Q\left[L^{-2}\right]^{1/2}} = 2v^{1/2}\mathbb{E}_Q\left[L^{-2}\right]^{1/2},$$

where we have used that $\mathbb{E}\left[L^{-1}\right] = \mathbb{E}\left[LL^{-2}\right] = \mathbb{E}_Q\left[L^{-2}\right]$. Moreover,

$$\begin{aligned} \mathbb{E}_Q\left[L^{-2}\right] = \mathbb{E}\left[L^{-1}\right] &= \frac{1}{4}\left\{\left(\frac{6}{5}\right)^{-1} + \left(\frac{4}{5}\right)^{-1} + \left(\frac{2}{3}\right)^{-1} + \left(\frac{4}{3}\right)^{-1}\right\} \\ &= \frac{1}{4}\left\{\frac{5}{6} + \frac{5}{4} + \frac{3}{2} + \frac{3}{4}\right\} = \frac{13}{12}, \end{aligned}$$

and

$$L^{-2} = \left(\left(\frac{6}{5}\right)^{-2}, \left(\frac{4}{5}\right)^{-2}, \left(\frac{2}{3}\right)^{-2}, \left(\frac{4}{3}\right)^{-2}\right)^T = \left(\frac{25}{36}, \frac{25}{16}, \frac{9}{4}, \frac{9}{16}\right)^T.$$

Hence, we obtain $\mathbb{E}\left[U\left(\widehat{W}\right)\right] = 2v^{1/2}\left(\frac{13}{12}\right)^{1/2}$ and

$$\widehat{W} = \frac{v}{\frac{13}{12}}\left(\frac{25}{36}, \frac{25}{16}, \frac{9}{4}, \frac{9}{16}\right)^T = \left(\frac{25}{39}v, \frac{75}{52}v, \frac{27}{13}v, \frac{27}{52}v\right)^T.$$

Finally, we have to compute the optimal trading strategy $\widehat{H} = \left\{\left(\widehat{H}_0(t), \widehat{H}_1(t)\right)^T\right\}_{t=1,2}$, that is, a self-financing and predictable process such that its asociated value process $\widehat{V}$

(Continued on page 9.)

satisfies $\widehat{V}(2) = \widehat{W}$. We first compute, taking into account that $r = 0$, the discounted increments of the risky asset

$$\Delta S_1^*(2) = \Delta S_1(2) = (2, -3, 2, -1)^T,$$
$$\Delta S_1^*(1) = \Delta S_1(1) = (1, 1, -1, -1)^T.$$

- For $t = 2$, using that $\widehat{H}$ must be self-financing we have that $\widehat{W} = \frac{\widehat{W}}{B(2)} = \widehat{V}^*(1) + \widehat{H}_1(2)\,\Delta S_1^*(2)$.
  - Assuming that $\omega \in A_{1,1} = \{\omega_3, \omega_4\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned}\frac{27}{13}v = \widehat{W}_3 &= \widehat{V}^*(1,\omega_3) + \widehat{H}_1(2,\omega_3) \times 2,\\ \frac{27}{52}v = \widehat{W}_4 &= \widehat{V}^*(1,\omega_4) + \widehat{H}_1(2,\omega_4) \times (-1),\\ \widehat{V}^*(1,\omega_3) &= \widehat{V}^*(1,\omega_4),\\ \widehat{H}_1(2,\omega_3) &= \widehat{H}_1(2,\omega_4),\end{aligned}$$

    which, using that $r = 0$, yield

$$\begin{aligned}\widehat{V}^*(1,\omega_3) = \widehat{V}^*(1,\omega_4) = \widehat{V}(1,\omega_3) = \widehat{V}(1,\omega_4) &= \frac{27}{26}v,\\ \widehat{H}_1(2,\omega_3) = \widehat{H}_1(2,\omega_4) &= \frac{27}{52}v.\end{aligned}$$

  - Assuming that $\omega \in A_{1,2} = \{\omega_1, \omega_2\}$ and the predictability of $\widehat{H}$ we get the equations

$$\begin{aligned}\frac{25}{39}v = \widehat{W}_1 &= \widehat{V}^*(1,\omega_1) + \widehat{H}_1(2,\omega_1) \times 2,\\ \frac{75}{52}v = \widehat{W}_2 &= \widehat{V}^*(1,\omega_2) + \widehat{H}_1(2,\omega_2) \times (-3),\\ \widehat{V}^*(1,\omega_1) &= \widehat{V}^*(1,\omega_2),\\ \widehat{H}_1(2,\omega_1) &= \widehat{H}_1(2,\omega_2),\end{aligned}$$

    which, using that $r = 0$, yield

$$\begin{aligned}\widehat{V}^*(1,\omega_1) = \widehat{V}^*(1,\omega_2) = \widehat{V}(1,\omega_1) = \widehat{V}(1,\omega_2) &= \frac{25}{26}v,\\ \widehat{H}_1(2,\omega_1) = \widehat{H}_1(2,\omega_2) &= -\frac{25}{156}v.\end{aligned}$$

- For $t = 1$, the predictability assumption yields that $\widehat{H}_1(1)$ is constant. Moreover, using that $\widehat{H}$ must be self-financing we have that $\widehat{V}^*(1) = \widehat{V}^*(0) + \widehat{H}_1(1)\,\Delta S_1^*(1)$

(Continued on page 10.)

and we get the following two equations

$$\frac{27}{26}v = \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (-1), \qquad (\text{for } \omega \in A_{1,1})$$
$$\frac{25}{26}v = \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (1), \qquad (\text{for } \omega \in A_{1,2})$$

which, using that $r = 0$, yield

$$\widehat{V}^*(0) = \widehat{V}(0) = v, \qquad \widehat{H}_1(1) = -\frac{1}{26}v.$$

- Finally we compute $H_0(1)$ and $H_0(2)$ from the definition of value process. We have

$$\widehat{H}_0(1) = \widehat{V}^*(0) - \widehat{H}_1(1) S_1^*(0) = v + \frac{1}{26}v \times 3 = \frac{29}{26}v,$$

and

$$\begin{aligned}\widehat{H}_0(2,\omega) &= \widehat{V}^*(1,\omega) - \widehat{H}_1(2,\omega) S_1^*(1,\omega) \\ &= \begin{cases} \frac{27}{26}v - \frac{27}{52}v \times 2 = 0 & \text{if } \omega \in A_{1,1} \\ \frac{25}{26}v + \frac{25}{156}v \times 4 = \frac{125}{78}v & \text{if } \omega \in A_{1,2} \end{cases}\end{aligned}$$

Problem 4

a (weight 10p)

The conditional expectation of X given $\mathcal{G}$ is the unique random variable $\mathbb{E}[X|\mathcal{G}]$ satisfying:

1. $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$- measurable.
2. $\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbf{1}_B], \qquad B \in \mathcal{G}.$

$1 \Rightarrow 2$) That Z is $\mathcal{G}$-measurable follows from the $\mathcal{G}$-measurability of $\mathbb{E}[X|\mathcal{G}]$. Moreover, we can reason as follows

$$\begin{aligned}\mathbb{E}[(X-Z)Y] &\overset{(a)}{=} \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])Y] \overset{(b)}{=} \mathbb{E}[\mathbb{E}[(X-\mathbb{E}[X|\mathcal{G}])Y|\mathcal{G}]], \\ &\overset{(c)}{=} \mathbb{E}[\mathbb{E}[(X-\mathbb{E}[X|\mathcal{G}])|\mathcal{G}]Y] \overset{(d)}{=} \mathbb{E}[\mathbb{E}[(\mathbb{E}[X|\mathcal{G}]-\mathbb{E}[X|\mathcal{G}])|\mathcal{G}]Y] = 0,\end{aligned}$$

where we have used that: (a)By assumption, (b)Law of total expectation, (c) what is $\mathcal{G}$-measurable goes out, (d)Linearity of conditional expectation and what is $\mathcal{G}$-measurable goes out again.

$2 \Rightarrow 1$) Property 1. of conditional expectation follows by assumption. Property 2. follows by taking $Y = \mathbf{1}_B, B \in \mathcal{G}$. Then,

$$0 = \mathbb{E}[(X-Z)Y] = \mathbb{E}[(X-Z)\mathbf{1}_B] \Longleftrightarrow \mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[Z\mathbf{1}_B].$$

Since B is an arbitrary set in $\mathcal{G}$, the previous equality shows that Z satisfies property 2. of the conditional expectation.

(Continued on page 11.)

b (weight 10p)

A process M is an $\mathbb{F}$-martingale if M is $\mathbb{F}$-adapted and satisfies

$$\mathbb{E}\left[M(t+1)|\mathcal{F}_t\right] = M(t), \qquad t = 0, ..., T-1.$$

Let M be a $\mathbb{G}$-adapted process that is an $\mathbb{F}$-martingale, with $\mathcal{G}_t \subseteq \mathcal{F}_t$. To prove that M is also a $\mathbb{G}$-martingale we only need to prove the martingale property because by assumption is $\mathbb{G}$-adapted. Then,

$$\mathbb{E}\left[M(t+1)|\mathcal{G}_t\right] \overset{(a)}{=} \mathbb{E}\left[\mathbb{E}\left[M(t+1)|\mathcal{F}_t\right]|\mathcal{G}_t\right] \overset{(b)}{=} \mathbb{E}\left[M(t)|\mathcal{G}_t\right] \overset{(c)}{=} M(t), \qquad t = 0, ..., T-1,$$

where we have used that: (a)The tower property of conditional expectation and $\mathcal{G}_t \subseteq \mathcal{F}_t$, $(b)M$ is an $\mathbb{F}$-martingale, (c) M is $\mathbb{G}$-adapted ($M(t)$ is $\mathcal{G}_t$-measurable) and the property that if Z is a $\mathcal{G}$-measurable random variable then $Z = \mathbb{E}\left[Z|\mathcal{G}\right]$.

c (weight 10p)

Note that

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \qquad \mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2\}\}, \qquad \mathcal{F}_2 = \mathcal{P}(\Omega).$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω. Therefore, the predictability constraint on the process A implies that $A(1) = a_{1,1}$ (a constant) and $A(2,\omega) = a_{2,1}\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + a_{2,2}\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega)$. The square of the price process is given by

$$\begin{aligned} S_1^2(0) &= 9 \\ S_1^2(1,\omega) &= 16\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + 4\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega), \\ S_1^2(2,\omega) &= 36\mathbf{1}_{\{\omega_1\}}(\omega) + 16\mathbf{1}_{\{\omega_3\}}(\omega) + 1\mathbf{1}_{\{\omega_2,\omega_4\}}(\omega). \end{aligned}$$

The process A is $\mathbb{F}$-adapted because it is $\mathbb{F}$-predictable. This yields that M_t is $\mathcal{F}_t$-measurable because it is a function of the two $\mathcal{F}_t$-measurable random variables S_t and A_t. Hence, M is $\mathbb{F}$-adapted. Now we only need to prove the martingale property, which boils down to check

$$M(0) = \mathbb{E}\left[M(1)|\mathcal{F}_0\right] = \mathbb{E}\left[M(1)\right] \tag{14}$$

and

$$M(1) = \mathbb{E}\left[M(2)|\mathcal{F}_1\right]. \tag{15}$$

Since $A(0) = 0$, we have that $M(0) = S_1^2(0) - A(0) = S_1^2(0) = 9$ and

$$\mathbb{E}\left[M(1)\right] = \mathbb{E}\left[S_1^2(1)\right] - A(1) = 16(P_1 + P_2) + 4(P_3 + P_4) - a_{1,1}.$$

Hence, using equation (14), we get

$$A(1) = a_{1,1} = 16(P_1 + P_2) + 4(P_3 + P_4) - 9.$$

(Continued on page 12.)

On the other hand,

$$\begin{aligned}
M(1) &= S_1^2(1) - A(1) \\
&= (16 - a_{1,1})\,\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + (4 - a_{1,1})\,\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega), \\
M(2) &= S_1^2(2) - A(2) \\
&= (36 - a_{2,1})\,\mathbf{1}_{\{\omega_1\}}(\omega) + (1 - a_{2,1})\,\mathbf{1}_{\{\omega_2\}}(\omega) \\
&\quad + (16 - a_{2,2})\,\mathbf{1}_{\{\omega_3\}}(\omega) + (1 - a_{2,2})\,\mathbf{1}_{\{\omega_4\}}(\omega)
\end{aligned}$$

and

$$\mathbb{E}\left[M(2)\,|\,\mathcal{F}_1\right] = \mathbb{E}\left[M(2)\,|\,\{\omega_1,\omega_2\}\right]\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + \mathbb{E}\left[M(2)\,|\,\{\omega_3,\omega_4\}\right]\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega).$$

Moreover,

$$\begin{aligned}
\mathbb{E}\left[M(2)\,|\,\{\omega_1,\omega_2\}\right] = \mathbb{E}\left[M(2)\,|\,\{\omega_1,\omega_2\}\right] &= \frac{(36 - a_{2,1})\,P_1 + (1 - a_{2,1})\,P_2}{P_1 + P_2} \\
&= \frac{36P_1 + P_2}{P_1 + P_2} - a_{2,1}, \\
\mathbb{E}\left[M(2)\,|\,\{\omega_3,\omega_4\}\right] = \mathbb{E}\left[M(2)\,|\,\{\omega_3,\omega_4\}\right] &= \frac{(16 - a_{2,2})\,P_3 + (1 - a_{2,2})\,P_4}{P_3 + P_4} \\
&= \frac{16P_3 + P_4}{P_3 + P_4} - a_{2,2},
\end{aligned}$$

and, therefore,

$$\mathbb{E}\left[M(2)\,|\,\mathcal{F}_1\right] = \left(\frac{36P_1 + P_2}{P_1 + P_2} - a_{2,1}\right)\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + \left(\frac{16P_3 + P_4}{P_3 + P_4} - a_{2,2}\right)\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega).$$

Finally, using equation (15), we get

$$\begin{aligned}
a_{2,1} &= \frac{36P_1 + P_2}{P_1 + P_2} - 16 + a_{1,1}, \\
a_{2,2} &= \frac{16P_3 + P_4}{P_3 + P_4} - 4 + a_{1,1}.
\end{aligned}$$

If we take $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$ we get

$$\begin{aligned}
a_{1,1} &= 16 \times \frac{1}{2} + 4 \times \frac{1}{2} - 9 = 1, \\
a_{2,1} &= \frac{36 \times \frac{1}{4} + \frac{1}{4}}{\frac{1}{2}} - 16 + 1 = \frac{37}{2} - 15 = \frac{7}{2}, \\
a_{2,2} &= \frac{16 \times \frac{1}{4} + \frac{1}{4}}{\frac{1}{2}} - 4 + 1 = \frac{17}{2} - 3 = \frac{11}{2}.
\end{aligned}$$

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: 20 January 2023

Examination hours: 09.00 – 13.00

This problem set consists of 13subsection.4.2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Solution Compute the expected return of the whole portfolio. The total portfolio value is

$$V = V_1 + V_2 + V_3 = 10000 \text{ NOK}.$$

The portfolio weights are

$$w_1 = \frac{V_1}{V} = \frac{1000}{10000} = 0.10, \qquad w_2 = \frac{V_2}{V} = \frac{3500}{10000} = 0.35, \qquad w_3 = \frac{V_3}{V} = \frac{5500}{10000} = 0.55.$$

The expected portfolio return is defined as the weighted average of the expected returns:

$$\mathbb{E}[R_p] = w_1\mathbb{E}[R_1] + w_2\mathbb{E}[R_2] + w_3\mathbb{E}[R_3].$$

Substituting the values,

$$\mathbb{E}[R_p] = 0.10 \cdot 0.10 + 0.35 \cdot 0.15 + 0.55 \cdot 0.05.$$

Compute each term:

$$0.10 \cdot 0.10 = 0.01, \qquad 0.35 \cdot 0.15 = 0.0525, \qquad 0.55 \cdot 0.05 = 0.0275.$$

Thus,

$$\mathbb{E}[R_p] = 0.01 + 0.0525 + 0.0275 = 0.09.$$

$$\boxed{\mathbb{E}[R_p] = 9\%.}$$

(Continued on page 2.)

b (weight 10p)

Solution

The variance of a two-asset portfolio is given by

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2.$$

First note that

$$\sigma_1 = \sqrt{0.0004} = 0.02, \qquad \sigma_2 = \sqrt{0.0009} = 0.03.$$

(a) Portfolio variance

We compute

$$\begin{aligned}
\sigma_p^2 &= 0.4^2 \cdot 0.0004 + 0.6^2 \cdot 0.0009 + 2 \cdot 0.4 \cdot 0.6 \cdot 0.5 \cdot 0.02 \cdot 0.03 \\
&= 0.16 \cdot 0.0004 + 0.36 \cdot 0.0009 + 0.48 \cdot 0.5 \cdot 0.02 \cdot 0.03 \\
&= 0.000064 + 0.000324 + 0.000144 \\
&= 0.000532.
\end{aligned}$$

Thus, the portfolio variance is

$$\boxed{\sigma_p^2 = 0.000532.}$$

c (weight 10p)

Strip: long 1 call + long 2 puts with strike K

The call and put payoffs are

$$C(S) = (S-K)^+ = \max(S-K, 0), \qquad P(S) = (K-S)^+ = \max(K-S, 0).$$

The payoff of the strip is

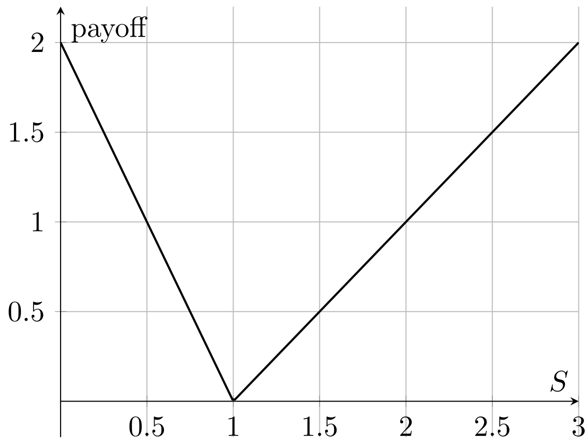
$$\boxed{\pi_{\text{strip}}(S) = (S-K)^+ + 2(K-S)^+}.$$

Equivalently, piecewise:

$$\pi_{\text{strip}}(S) = \begin{cases} 2(K-S), & S < K, \\ S-K, & S \geq K. \end{cases}$$

(Continued on page 3.)

Payoff diagram. Below is a payoff diagram for a strip with strike $K = 1$ (for illustration):



Problem 2

a (weight 10p)

Solution We assume $0 < d < 1 < u$, $S_0 = 1$, and the bond satisfies $B_t \equiv 1$.

Since $d^2 < K < ud$, the payoff of the call at time $t = 2$ is

$$C_2 = \begin{cases} u^2 - K, & S_2 = u^2, \\ ud - K, & S_2 = ud, \\ 0, & S_2 = d^2. \end{cases}$$

Step 1: Replication at time $t = 1$

At time 1 we consider a one-period portfolio consisting of

$$H_1^{(1)} \quad \text{(shares)}, \qquad H_0^{(1)} \quad \text{(bond)}.$$

We choose $(H_1^{(1)}, H_0^{(1)})$ separately in each node so that

$$H_1^{(1)} S_2 + H_0^{(1)} = C_2.$$

Node $S_1 = u$. Possible prices at $t = 2$ are u^2 and ud. We solve

$$\begin{cases} H_1^{(1,u)} u^2 + H_0^{(1,u)} = u^2 - K, \\ H_1^{(1,u)} ud + H_0^{(1,u)} = ud - K. \end{cases}$$

(Continued on page 4.)

Subtracting the second equation from the first gives

$$H_1^{(1,u)}(u^2 - ud) = u^2 - ud \quad \Rightarrow \quad H_1^{(1,u)} = 1.$$

Then

$$H_0^{(1,u)} = -K.$$

Hence the value of the replicating portfolio in this node is

$$V_1^{(u)} = H_1^{(1,u)}u + H_0^{(1,u)} = u - K.$$

Node $S_1 = d$. Possible prices at $t = 2$ are ud and d^2. We solve

$$\begin{cases} H_1^{(1,d)}ud + H_0^{(1,d)} = ud - K, \\ H_1^{(1,d)}d^2 + H_0^{(1,d)} = 0. \end{cases}$$

Subtracting,

$$H_1^{(1,d)}(ud - d^2) = ud - K \quad \Rightarrow \quad H_1^{(1,d)} = \frac{ud - K}{d(u-d)}.$$

From the second equation,

$$H_0^{(1,d)} = -H_1^{(1,d)}d^2 = -\frac{d(ud-K)}{u-d}.$$

Thus the portfolio value in the down node is

$$\begin{aligned} V_1^{(d)} &= H_1^{(1,d)}d + H_0^{(1,d)} \\ &= \frac{ud-K}{d(u-d)}\,d - \frac{d(ud-K)}{u-d} \\ &= \frac{ud-K}{u-d} - \frac{d(ud-K)}{u-d} \\ &= \frac{(1-d)(ud-K)}{u-d}. \end{aligned}$$

Step 2: Replication at time $t = 0$

At time 0 we hold

$$H_1^{(0)} \quad \text{(shares)}, \qquad H_0^{(0)} \quad \text{(bond)},$$

and require the value of this portfolio at time 1 to match the values $V_1^{(u)}$ and $V_1^{(d)}$ obtained above:

$$\begin{cases} H_1^{(0)}u + H_0^{(0)} = V_1^{(u)} = u - K, \\ H_1^{(0)}d + H_0^{(0)} = V_1^{(d)} = \dfrac{(1-d)(ud-K)}{u-d}. \end{cases}$$

(Continued on page 5.)

Subtracting the second equation from the first:

$$H_1^{(0)}(u-d) = (u-K) - \frac{(1-d)(ud-K)}{u-d},$$

so

$$H_1^{(0)} = \frac{u-K}{u-d} - \frac{(1-d)(ud-K)}{(u-d)^2}.$$

Then

$$H_0^{(0)} = u - K - H_1^{(0)}u.$$

Step 3: Option price at time 0

Since $S_0 = 1$ and the bond is worth 1, the time–0 value of the replicating portfolio is

$$V_0 = H_1^{(0)}S_0 + H_0^{(0)} = H_1^{(0)} + H_0^{(0)}.$$

If the market price C_0 of the call differs from V_0, an arbitrage is possible:

- If $C_0 > V_0$, short the option and buy the replicating portfolio, locking in $C_0 - V_0 > 0$.
- If $C_0 < V_0$, buy the option and short the replicating portfolio, locking in $V_0 - C_0 > 0$.

Therefore, by the absence of arbitrage,

$$\boxed{C_0 = V_0 = H_1^{(0)} + H_0^{(0)}}.$$

b (weight 10p)

Solution In the risk-neutral measure Q, the discounted stock price must be a martingale. Since $r = 0$, this means

$$S_0 = \mathbb{E}^Q[S_1].$$

Let q be the risk-neutral probability of an up-move in each period:

$$\mathbb{P}^Q(S_1 = u) = q, \qquad \mathbb{P}^Q(S_1 = d) = 1 - q.$$

Then

$$1 = S_0 = \mathbb{E}^Q[S_1] = qu + (1-q)d.$$

Solving for q gives

$$\boxed{q = \frac{1-d}{u-d}}.$$

(Continued on page 6.)

Risk-neutral valuation

Under Q the probabilities of the terminal stock prices are

$$\mathbb{P}^Q(S_2 = u^2) = q^2, \quad \mathbb{P}^Q(S_2 = ud) = 2q(1-q), \quad \mathbb{P}^Q(S_2 = d^2) = (1-q)^2.$$

The payoff of the call is

$$C_2 = \begin{cases} u^2 - K, & S_2 = u^2, \\ ud - K, & S_2 = ud, \\ 0, & S_2 = d^2. \end{cases}$$

Therefore the risk-neutral expected payoff is

$$\mathbb{E}^Q[C_2] = q^2(u^2 - K) + 2q(1-q)(ud - K).$$

Since $r = 0$, the present value is just this expectation, so the risk-neutral pricing formula gives

$$C_0 = \mathbb{E}^Q[C_2].$$

A straightforward algebraic simplification, using $q = \frac{1-d}{u-d}$, shows that this C_0 coincides with $V_0 = H_1^{(0)} + H_0^{(0)}$ computed in part (a). Hence the option price equals the present risk-neutral expectation of its payoff.

c (weight 10p)

Solution At time $t = 1$ we can also price the option under the risk-neutral measure Q.

Node $S_1 = u$

Conditional on $S_1 = u$, the stock at time 2 is either u^2 (up) or ud (down), with probabilities q and $1 - q$, respectively. Thus

$$\mathbb{E}^Q[S_2 \mid S_1 = u] = qu^2 + (1-q)ud = u\,(qu + (1-q)d) = u \cdot 1 = u,$$

since $qu + (1-q)d = 1$ from part (b). Because in this node the payoff is always positive, we have

$$C_2 = S_2 - K \quad \text{when } S_1 = u,$$

so

$$C_1(u) = \mathbb{E}^Q[C_2 \mid S_1 = u] = \mathbb{E}^Q[S_2 \mid S_1 = u] - K = u - K.$$

But this is exactly the value of the replicating portfolio found in part (a):

$$C_1(u) = u - K = V_1^{(u)}.$$

(Continued on page 7.)

Node $S_1 = d$

Conditional on $S_1 = d$, the stock at time 2 is either ud (up) or d^2 (down). The payoff of the call is positive only in the up-state:

$$C_2 = \begin{cases} ud - K, & \text{with prob. } q, \\ 0, & \text{with prob. } 1 - q. \end{cases}$$

Therefore

$$C_1(d) = \mathbb{E}^Q[C_2 \mid S_1 = d] = q(ud - K) = \frac{1-d}{u-d}(ud - K) = \frac{(1-d)(ud-K)}{u-d}.$$

But this is exactly the value of the replicating portfolio in the down-node:

$$C_1(d) = V_1^{(d)}.$$

Conclusion

In each node at time 1 the option price obtained from the risk-neutral expectation coincides with the value of the replicating portfolio:

$$C_1(u) = V_1^{(u)}, \qquad C_1(d) = V_1^{(d)}.$$

Hence the price of the option at time 1 equals the value of its replicating portfolio at time 1.

Problem 3

a (weight 10p)

The filtration generated by S_1 is

$$\mathcal{F}_0 = \sigma(S_1(0)) = \sigma(\{\Omega\}) = \{\emptyset, \Omega\},$$

$$\mathcal{F}_1 = \sigma(S_1(0), S_1(1)) = \sigma(\{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}) = \{\emptyset, \Omega, \{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\},$$

$$\mathcal{F}_2 = \sigma(S_1(0), S_1(1), S_1(2)) = \sigma(\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}) = \mathcal{P}(\Omega),$$

where $\mathcal{P}(\Omega)$ is the power set of Ω.

To compute $\mathbb{E}[S_1(2,\omega) \mid \mathcal{F}_1]$ we condition on the atoms of $\mathcal{F}_1$:

$$\mathbb{E}[S_1(2,\omega) \mid \mathcal{F}_1] = \mathbb{E}[S_1(2,\omega) \mid \{\omega_1, \omega_3\}]\, 1_{\{\omega_1,\omega_3\}} + \mathbb{E}[S_1(2,\omega) \mid \{\omega_2, \omega_4\}]\, 1_{\{\omega_2,\omega_4\}}.$$

Since $P(\omega_i) = \frac{1}{4}$ for all i,

$$\mathbb{E}[S_1(2,\omega) \mid \{\omega_1, \omega_3\}] = \frac{S_1(2,\omega_1)P(\omega_1) + S_1(2,\omega_3)P(\omega_3)}{P(\omega_1) + P(\omega_3)} = \frac{7 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4}}{\frac{1}{4} + \frac{1}{4}} = 5,$$

(Continued on page 8.)

$$\mathbb{E}[S_1(2,\omega) \mid \{\omega_2,\omega_4\}] = \frac{S_1(2,\omega_2)P(\omega_2) + S_1(2,\omega_4)P(\omega_4)}{P(\omega_2)+P(\omega_4)} = \frac{3\cdot\frac{1}{4} + 1\cdot\frac{1}{4}}{\frac{1}{4}+\frac{1}{4}} = 2.$$

Hence

$$\boxed{\mathbb{E}[S_1(2,\omega) \mid \mathcal{F}_1] = 5\,1_{\{\omega_1,\omega_3\}} + 2\,1_{\{\omega_2,\omega_4\}}.}$$

b (weight 20p)

Let $Q = \left(\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}\right)^T$ be the unique martingale measure on this market (you do not have to prove this). Consider the following optimal portfolio problem

$$\max_{H \in \mathbb{H}} \mathbb{E}\left[U\left(V\left(2\right)\right)\right]$$
$$\text{subject to } V\left(0\right) = v,$$

where v is a given strictly positive real number, $\mathbb{H}$ is the set of all self-financing and predictable trading strategies and $U\left(u\right) = 2u^{1/2}$. Compute the optimal attainable wealth, the optimal objective value and the optimal trading strategy.

(b) Optimal portfolio problem

Let $Q = (\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4})^T$ be the unique martingale measure. We consider

$$\max_{H\in\mathbb{H}} \ \mathbb{E}\big[U(V(2))\big] \quad \text{s.t.} \quad V(0) = v > 0,$$

with $U(u) = 2u^{1/2}$ and $\mathbb{H}$ the set of predictable self-financing strategies.

Since the market is arbitrage free and complete (unique martingale measure), we can use the martingale method.

Step 1: Optimal attainable wealth. Let L be the state price density

$$L = \frac{Q}{P} = \begin{pmatrix} \frac{1/8}{1/4} \\ \frac{1/4}{1/4} \\ \frac{3/8}{1/4} \\ \frac{1/4}{1/4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \\ \frac{3}{2} \\ 1 \end{pmatrix}, \qquad B(2) = 1.$$

We solve the static problem

$$\max_{W} \ \mathbb{E}[U(W)] \quad \text{subject to} \quad \mathbb{E}^Q\Big[\frac{W}{B(2)}\Big] = v.$$

Here

$$U(u) = 2u^{1/2} \quad \Rightarrow \quad U'(u) = u^{-1/2}, \qquad I(y) = (U')^{-1}(y) = y^{-2}.$$

(Continued on page 9.)

Hence the optimal attainable wealth is

$$\widehat{W} = I\Big(\hat{\lambda}\frac{L}{B(2)}\Big) = I(\hat{\lambda}L) = (\hat{\lambda}L)^{-2},$$

for some Lagrange multiplier $\hat{\lambda} > 0$.

Componentwise,

$$\widehat{W}(\omega_i) = (\hat{\lambda}L(\omega_i))^{-2} = \hat{\lambda}^{-2}L(\omega_i)^{-2}.$$

Since

$$L^{-2} = \begin{pmatrix} (1/2)^{-2} \\ 1^{-2} \\ (3/2)^{-2} \\ 1^{-2} \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ \frac{4}{9} \\ 1 \end{pmatrix},$$

we have

$$\widehat{W} = \hat{\lambda}^{-2}\begin{pmatrix} 4 \\ 1 \\ \frac{4}{9} \\ 1 \end{pmatrix}.$$

The constraint $\mathbb{E}^Q[\widehat{W}] = v$ gives

$$v = \hat{\lambda}^{-2}\sum_{i=1}^{4} Q_i L(\omega_i)^{-2} = \hat{\lambda}^{-2}\Big(\tfrac{1}{8}\cdot 4 + \tfrac{1}{4}\cdot 1 + \tfrac{3}{8}\cdot\tfrac{4}{9} + \tfrac{1}{4}\cdot 1\Big) = \hat{\lambda}^{-2}\cdot\frac{7}{6}.$$

Thus

$$\hat{\lambda}^{-2} = v\cdot\frac{6}{7},$$

and the optimal attainable wealth is

$$\widehat{W} = \frac{6v}{7}\begin{pmatrix} 4 \\ 1 \\ \frac{4}{9} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{24}{7}v \\ \frac{6}{7}v \\ \frac{8}{21}v \\ \frac{6}{7}v \end{pmatrix}.$$

Step 2: Optimal objective value. The optimal value of the objective function is

$$\begin{aligned}\mathbb{E}[U(\widehat{W})] = \mathbb{E}\big[2\widehat{W}^{1/2}\big] &= 2\sum_{i=1}^{4} P(\omega_i)\sqrt{\widehat{W}(\omega_i)} \\ &= 2\cdot\frac{1}{4}\Big(\sqrt{\tfrac{24}{7}v} + \sqrt{\tfrac{6}{7}v} + \sqrt{\tfrac{8}{21}v} + \sqrt{\tfrac{6}{7}v}\Big).\end{aligned}$$

(Continued on page 10.)

Step 3: Optimal trading strategy $\widehat{H}$. We now find a predictable self-financing strategy

$$\widehat{H} = \{(H_0(t), H_1(t))^T\}_{t=1,2}$$

such that its value process V satisfies $V(2) = \widehat{W}$.

Since $r = 0$, the discounted and non-discounted prices coincide. The increments of the risky asset are

$$\Delta S_1(1) = S_1(1) - S_1(0) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \qquad \Delta S_1(2) = S_1(2) - S_1(1) = \begin{pmatrix} 3 \\ 1 \\ -1 \\ -1 \end{pmatrix}.$$

Time $t = 2$. Self-financing gives

$$\widehat{W} = V(2) = V(1) + H_1(2)\Delta S_1(2).$$

Because of predictability,

$$H_1(2,\omega_1) = H_1(2,\omega_3) =: h_u, \qquad H_1(2,\omega_2) = H_1(2,\omega_4) =: h_d,$$

and

$$V(1,\omega_1) = V(1,\omega_3) =: x_u, \qquad V(1,\omega_2) = V(1,\omega_4) =: x_d.$$

From $\widehat{W}$ we obtain the system

$$\begin{cases} \dfrac{24}{7}v = x_u + 3h_u & (\omega_1), \\ \dfrac{8}{21}v = x_u - h_u & (\omega_3), \end{cases} \qquad \begin{cases} \dfrac{6}{7}v = x_d + h_d & (\omega_2), \\ \dfrac{6}{7}v = x_d - h_d & (\omega_4). \end{cases}$$

From the down-branch equations we get $h_d = 0$ and $x_d = \frac{6}{7}v$.

From the up-branch equations,

$$\Big(\frac{24}{7} - \frac{8}{21}\Big)v = 4h_u \quad \Rightarrow \quad h_u = \frac{16}{21}v,$$

and then $x_u = \frac{8}{7}v$.

Thus

$$V(1) = \begin{pmatrix} \frac{8}{7}v \\ \frac{6}{7}v \\ \frac{8}{7}v \\ \frac{6}{7}v \end{pmatrix}, \qquad H_1(2) = \begin{pmatrix} \frac{16}{21}v \\ 0 \\ \frac{16}{21}v \\ 0 \end{pmatrix}.$$

Time $t = 1$. Predictability implies that $H_1(1)$ is constant (same for all ω). Self-financing gives

$$V(1) = V(0) + H_1(1)\Delta S_1(1),$$

(Continued on page 11.)

so for $\omega \in \{\omega_1, \omega_3\}$,

$$\frac{8}{7}v = v + H_1(1) \cdot 1,$$

and for $\omega \in \{\omega_2, \omega_4\}$,

$$\frac{6}{7}v = v + H_1(1) \cdot (-1).$$

Both equations yield

$$H_1(1) = \frac{1}{7}v, \qquad V(0) = v.$$

Bond holdings. Finally, we obtain $H_0(1)$ and $H_0(2)$ from

$$V(t) = H_0(t)B(t) + H_1(t)S_1(t), \qquad B(t) = 1.$$

At time 1:

$$H_0(1) = V(0) - H_1(1)S_1(0) = v - \frac{1}{7}v \cdot 3 = \frac{4}{7}v.$$

At time 2:

$$H_0(2,\omega) = V(1,\omega) - H_1(2,\omega)S_1(1,\omega) = \begin{cases} \frac{8}{7}v - \frac{16}{21}v \cdot 4 = -\frac{40}{21}v, & \omega \in \{\omega_1, \omega_3\}, \\ \frac{6}{7}v - 0 \cdot 2 = \frac{6}{7}v, & \omega \in \{\omega_2, \omega_4\}. \end{cases}$$

Thus the *optimal trading strategy* $\widehat{H}$ is given by

$$\boxed{\begin{aligned} &H_1(1) = \frac{1}{7}v, \qquad H_0(1) = \frac{4}{7}v, \\ &H_1(2,\omega) = \begin{cases} \frac{16}{21}v, & \omega \in \{\omega_1, \omega_3\}, \\ 0, & \omega \in \{\omega_2, \omega_4\}, \end{cases} \\ &H_0(2,\omega) = \begin{cases} -\frac{40}{21}v, & \omega \in \{\omega_1, \omega_3\}, \\ \frac{6}{7}v, & \omega \in \{\omega_2, \omega_4\}. \end{cases} \end{aligned}}$$

and it replicates the optimal attainable wealth $\widehat{W}$ found above.

Problem 4

Let $(\Omega, \mathcal{F}, P)$ be a finite probability space.

a (weight 10p)

Let $\{X_n,\ n \geq 1\}$ be independent identically distributed random variables such that $\mathbb{E}[X_i] = a$, $i \geq 1$. Set $\mathcal{F}_n = \sigma(X_1, X_2, ..., X_n), n \geq 1$. Find the conditional expectations:

$$\mathbb{E}[X_1 \cdot X_2 \cdot ... \cdot X_n | \mathcal{F}_k],\ n > k.$$

(Continued on page 12.)

b (weight 10p)

Solution.

To prove that G is a martingale we must show:
1. $G(t)$ is $\mathcal{F}_t$âmeasurable for each t (i.e. G is $\mathbb{F}$-adapted);
2. for all $t = 0, \dots, T-1$,

$$\mathbb{E}[\,G(t+1) \mid \mathcal{F}_t\,] = G(t) \quad \text{(martingale property).}$$

(We also tacitly assume integrability of all terms so that all expectations exist.)

1. Adaptedness.

First note that if X and Y are $\mathcal{G}$âmeasurable, then $X + Y$ and XY are also $\mathcal{G}$-measurable for any σ-algebra $\mathcal{G}$.

The process H is predictable, hence in particular adapted: $H(t)$ is $\mathcal{F}_{t-1}$-measurable. The process Z is adapted because it is a martingale, so $Z(u)$ is $\mathcal{F}_u$-measurable for every u. Moreover, $Z(u-1)$ is $\mathcal{F}_{u-1}$-measurable and therefore also $\mathcal{F}_u$-measurable, since $\mathbb{F}$ is a filtration and $\mathcal{F}_{u-1} \subseteq \mathcal{F}_u$.

Thus, for each fixed $u \geq 1$,

$$Z(u) - Z(u-1) \quad \text{and} \quad H(u)$$

are $\mathcal{F}_u$-measurable, and hence their product

$$H(u)\big(Z(u) - Z(u-1)\big)$$

is $\mathcal{F}_u$-measurable.

For $t \geq 1$ we have

$$G(t) = \sum_{u=1}^{t} H(u)\big(Z(u) - Z(u-1)\big).$$

Each summand is $\mathcal{F}_u$-measurable, and since $u \leq t$ we have $\mathcal{F}_u \subseteq \mathcal{F}_t$. Therefore each summand is $\mathcal{F}_t$-measurable, and so is their finite sum $G(t)$. Also $G(0) = 0$ is trivially $\mathcal{F}_0$-measurable. Hence G is $\mathbb{F}$-adapted.

2. Martingale property.

For $t = 0, \dots, T-1$ we can write

$$G(t+1) = \sum_{u=1}^{t+1} H(u)\big(Z(u) - Z(u-1)\big) = G(t) + H(t+1)\big(Z(t+1) - Z(t)\big).$$

Taking conditional expectation with respect to $\mathcal{F}_t$ gives

$$\begin{aligned}
\mathbb{E}[G(t+1) \mid \mathcal{F}_t] &= \mathbb{E}\big[G(t) + H(t+1)(Z(t+1) - Z(t)) \,\big|\, \mathcal{F}_t\big] \\
&= \mathbb{E}[G(t) \mid \mathcal{F}_t] + \mathbb{E}\big[H(t+1)(Z(t+1) - Z(t)) \mid \mathcal{F}_t\big] \\
&= G(t) + H(t+1)\,\mathbb{E}\big[Z(t+1) - Z(t) \mid \mathcal{F}_t\big],
\end{aligned}$$

(Continued on page 13.)

where we used:

- linearity of conditional expectation in the second line;
- the fact that $G(t)$ is $\mathcal{F}_t$-measurable in the third line;
- that $H(t+1)$ is $\mathcal{F}_t$-measurable (predictability) so it can be taken out of the conditional expectation.

Since Z is a martingale,

$$\mathbb{E}[Z(t+1) \mid \mathcal{F}_t] = Z(t),$$

and hence

$$\mathbb{E}[Z(t+1) - Z(t) \mid \mathcal{F}_t] = 0.$$

Therefore,

$$\mathbb{E}[G(t+1) \mid \mathcal{F}_t] = G(t) + H(t+1) \cdot 0 = G(t).$$

Thus G satisfies the martingale property.

Since G is adapted and $\mathbb{E}[G(t+1) \mid \mathcal{F}_t] = G(t)$ for all $t = 0, \ldots, T-1$, we conclude that

$$\boxed{G \text{ is a martingale with respect to } \mathbb{F}.}$$

(Continued on page 14.)