STK-MAT3710/4710: Probability Theory
Mandatory assignment Fall 2025

Submission deadline

Thursday 30th October 2025, 14:30 in Canvas (`canvas.uio.no`).

Instructions

If you are taking STK-MAT3710, you can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). If you are taking STK-MAT4710, you must write your solution in LaTeX. The assignment must be submitted as a single pdf file. Scanned pages must be clearly legible. The submission must contain your name and course code.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. You only have **one attempt** at the assignment, and you need to have the assignment approved in order to take the exam. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

Answer all problems or, at least, try. To pass the assignment we expect that you answer 2 out of the 3 exercises correctly.

Complete guidelines about delivery of mandatory assignments:

`uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html`

Problem 1

Write down and explain with **at least one** example the concepts presented below. It is okay to use examples from the lecture notes, but it will be **highly** appreciated if you build your own examples!

(a) A σ-algebra on a set E.

(b) A measurable space.

(c) A measure μ on a measurable space and a measure space.

(d) A complete measure (with an example of complete and non-complete measure).

(e) A measurable function between measurable spaces.

(f) The concept of almost everywhere (a.s) or almost surely (a.s) in the context of probability. Give an example of a property that holds a.e. or a.s. but not on the entire space.

(g) The concept of push-forward measure.

(h) A simple function.

(i) The integral of a simple function with respect to a measure.

(j) The integral of a measurable function with respect to a measure.

(k) The integral of a function with respect to the push-forward measure.

(l) Concept of probability, probability space and random variable.

(m) Formula for transformation of random variables with an example.

(n) Expectation of a random variable.

(o) Formula for the expectation with respect to the distribution function.

(p) Concept of independent random variables.

Problem 2

Let $(\Omega, \mathcal{A}, \mathbb{P})$ the probability space with $\Omega = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$ and $\mathbb{P}$ is the probability measure defined as

$$\mathbb{P}[B] = \int_B \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz, \quad B \in \mathcal{B}(\mathbb{R}),$$

where $\mu, \sigma \in \mathbb{R}$, $\sigma > 0$ are two fixed parameters.

Suppose that the outcomes ω are states of an economy, being negative values of ω seen as *negative states* and positive values as *positive states*.

You run a business in this economy that invests some initial capital C_0. This business has an aleatory rate of return, denoted by ρ. We assume that $\rho : \Omega \to \mathbb{R}$ is a mapping that interconnects the states of the economy with the rate of return of your investment. Your own analysis on the business indicated that $\rho(\omega) = \mu + \sigma\omega$, $\omega \in \Omega$ for fixed parameters $\mu, \sigma \in \mathbb{R}$, $\sigma > 0$.

(a) Prove that $\mathbb{P}$ is indeed a probability measure on the measurable space $(\Omega, \mathcal{A})$.

(b) Prove that ρ is a random variable.

Assume that this investment runs for t years and that returns are compounded continuously. This essentially means that after t years, your capital will have increased (or decreased) to

$$C(\omega) = C_0 e^{\rho(\omega)t}, \quad \omega \in \Omega.$$

Note that $C \in (0, \infty)$.

(c) Prove that C, the capital gains from investing C_0 is indeed a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$.

(d) Find the law of C and compute the expected gains, i.e. $\mathbb{E}[C]$.

(e) You are only interested in running this business if your gains C exceed a given threshold $K > 0$, i.e. $C > K$. Find the distribution of the random variable $Y = (C - K)\mathbb{I}_{\{C>K\}}$ and the expectation of this new random variable Y. What type of random variable is Y? Note: you will not be able to find an analytical closed expression, but you can leave the expression in terms of integrals.

Problem 3 (Hölder's inequality)

The proof of this exercise can be found in may places in the internet, but try to do it yourself and understand the proof.

(a) **Young's inequality**: Let $a, b \in [0, \infty)$ and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(b) **Hölder's inequality**: Let $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g : \mathbb{R} \to \mathbb{R}$ two integrable functions. Show that

$$\int |fg| \leq \left(\int |f|^p \right)^{\frac{1}{p}} \left(\int |g|^q \right)^{\frac{1}{q}}.$$

Hint: Use Young's inequality.

(c) Let X, Y two random variables and p, q as in the previous items. Prove that

$$\mathbb{E}[|XY|] \leq \mathbb{E}[|X|^p]^{\frac{1}{p}} \mathbb{E}[|Y|^q]^{\frac{1}{q}}.$$