UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3710/4710 — Probability Theory

Day of examination: 16th December 2024

Examination hours: 3:00 pm – 7:00 pm

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature and the three remaining ones are more applied. All items within a problem are worth the same unless otherwise specified. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (2 points)

Let E be an uncountable set. Define the class of subsets of E given by

$$\mathcal{E} = \{A \subseteq E : \text{ either } A \text{ is countable, or } A^c \text{ countable}\}.$$

(a) Show that if $A, B \subseteq E$ are uncountable such that $A \cap B = \emptyset$, then either $A \notin \mathcal{E}$ or $B \notin \mathcal{E}$.

(b) Show that $\mathcal{E}$ is a σ-algebra.

(c) Define the mapping $\mu : \mathcal{E} \to [0, \infty]$ by $\mu(X) = \mathbb{I}_{\{X^c \text{ countable}\}}$. Show that μ is a probability measure.

Problem 2 Random variables (3 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\{X_n\}_{n=1}^{\infty}$ a sequence of independent identically distributed random variables with uniform distribution on $[0, 1]$, i.e. with densities $f_{X_n}(x) = 1$, $x \in [0, 1]$, $n \geq 1$.

Define the random variables

$$L_{(1:n)} = \min\{X_1, \ldots, X_n\}, \quad H_{(1:n)} = \max\{X_1, \ldots, X_n\}, \quad n \geq 2.$$

It may not be surprising that $L_{(1:n)}$ and $H_{(1:n)}$ are not independent and, in fact, it can be shown that their joint density is given by

$$f_{(L_{(1:n)}, H_{(1:n)})}(l, h) = n(n-1)(h-l)^{n-2}, \quad 0 \leq l < h \leq 1.$$

(Continued on page 2.)

(a) Prove that the density function of the range, denoted by R_n, of n independent uniform observations is given by

$$f_{R_n}(r) = n(n-1)r^{n-2}(1-r), \quad 0 < r < 1.$$

(b) Study the convergence of the sequence $\{R_n\}_{n=1}^{\infty}$ in distribution, probability, almost surely and in L^p, $p \geq 1$. Note: $\sum_{n=1}^{\infty} nr^n < \infty$ for every real r such that $|r| < 1$.

(c) Consider independent observations $X_1, X_2, \ldots$ of the uniform distribution. What is the probability that we observe k consecutive observations that are, at most, within distance $\delta > 0$?

Problem 3 Conditional expectation (2 points)

Let X and Y be two independent Bernoulli distributed random variables in some probability space, i.e. $X, Y \in \{0, 1\}$ and $\mathbb{P}[X = 1] = \mathbb{P}[Y = 1] = p$ for some $p \in [0, 1]$. Define the random variable $Z = \mathbb{I}_{\{X+Y=0\}}$.

(a) Compute $\mathbb{E}[Z]$, $\mathbb{E}[X|Z]$ and verify that $\mathbb{E}[\mathbb{E}[X|Z]] = \mathbb{E}[X]$.

(b) Compute $\mathbb{E}[Z|X]$. Is it true that $\mathbb{E}[\mathbb{E}[X|Z]|X] = \mathbb{E}[\mathbb{E}[Z|X]|Z]$? What about $\mathbb{E}[\mathbb{E}[X|Z]|X, Z] = \mathbb{E}[X|Z]$? Justify your answer.

Problem 4 Martingales (3 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X, Y two independent random variables in it. Assume that X is uniformly distributed on $[0, 1]$, i.e. $f(x) = 1$ for $x \in [0, 1]$ and $\mathbb{P}[Y = -1] = \mathbb{P}[Y = 1] = \frac{1}{2}$. Consider the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(X)$ and $\mathcal{F}_n = \sigma(X, Y)$ for all $n \geq 2$. Define for each $n \geq 1$ the processes

$$M_n = \frac{Y}{X}\mathbb{I}_{\{n \geq 2\}}, \quad \tau_n = \begin{cases} 1, & n < \frac{1}{X}, \\ \infty, & n \geq \frac{1}{X} \end{cases}.$$

(a) Show that $\{M_n\}_{n \geq 1}$ is not a martingale.

(b) Show that $\{\tau_n\}_{n \geq 1}$ is a monotone sequence of stopping times which converges almost surely to infinity. Note: we have defined stopping times as being finite. Here, we allow infinity to be an outcome. The definition remains the same with the additional requirement that $\{\tau_n = \infty\}$ also needs to be measurable.

(c) Prove that $\{M_{\tau_k \wedge n}\}_{n \geq 1}$ is a martingale for every $k \geq 1$. Here, $x \wedge y \triangleq \min\{x, y\}$. A process like $\{M_n\}_{n \geq 1}$ is known as local martingale.

GOOD LUCK!

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Trial exam in: STK3710/4710 — Probability Theory

Day of examination: 20th November 2024

Examination hours: 2:00 pm – 6:00 pm

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (3 points)

Let $\{X_n\}_{n=1}^{\infty}$, X be random variables defined in a probability space.

(a) (1 point) Define the concept of convergence in probability and convergence almost surely.

(b) (2 points) Prove that $X_n \xrightarrow{n\to\infty} X$ in probability if, and only if

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{|X_n - X|}{1 + |X_n - X|}\right] = 0.$$

Hint: Note that $\frac{|x|}{1+|x|} < 1$ for all x and that $f(x) = \frac{x}{1+x}$ is an increasing function for $x \geq 0$. It may also be useful to split the sample space into the events $\{|X_n - X| > \varepsilon\}$ and $\{|X_n - X| \leq \varepsilon\}$.

Problem 2 Random variables (2 points)

The Pareto distribution is an absolutely continuous probability distribution characterized by having the following density function,

$$f(x) = \frac{\alpha\lambda^{\alpha}}{x^{\alpha+1}}, \quad x \geq \lambda,$$

for positive real parameters α, λ and $\alpha > 1$ to guarantee finite expectation. Note that the distribution function is given by $F(x) = 1 - \left(\frac{\lambda}{x}\right)^{\alpha}$, $x \geq \lambda$.

(Continued on page 2.)

Consider $n \geq 1$ independent random variables $X_1, \dots, X_n$ where X_i is Pareto-distributed with parameters $\lambda > 0$ and $\alpha_i > 1$, $i = 1, \dots, n$.

Define the the random variables $Y_n = \min\{X_1, \dots, X_n\}$, $n \geq 1$.

(a) Find the distribution of Y_n and its expectation.

(b) Give a sufficient condition on the sequence of parameters $\{\alpha_i\}_{i=1}^n$ to guarantee that $\{Y_n\}_{n=1}^\infty$ converges almost surely to λ. Does the sequence converge in L^1 under your condition? Why or why not.

Problem 3 Conditional expectation (3 points)

We consider an insurance portfolio in which there is a large number n of policyholders. We model the number of the insurance claims of the policyholders by i.i.d. random variables $X_1, \dots, X_n$. Thus X_i is the number of claims of policyholder i. Let $p_k \triangleq \mathbb{P}[X_i = k]$ for all $i = 1, \dots, n$ and denote by $m = \mathbb{E}[X_i] = \sum_{k \geq 1} k p_k < \infty$ and $s^2 = \mathbb{E}[X_i^2] - m^2 = \sum_{k \geq 1} k^2 p_k - m^2 < \infty$ the expectation and variance.

We pick an insurance claim uniformly at random. This claim belongs to some policyholder among these n. Denote by Z_n be the number of claims of such policyholder.

(a) Justify that for every integer $k \geq 1$ we have

$$\mathbb{P}[Z_n = k | X_1, \dots, X_n] = \frac{k \sum_{i=1}^n \mathbb{I}_{\{X_i = k\}}}{\sum_{i=1}^n X_i}.$$

(b) Show that $\{Z_n\}_{n=1}^\infty$ converges in distribution to a random variable Z_∞ with probability mass function given by $\mathbb{P}[Z_\infty = k] = \frac{k p_k}{m}$, $k \geq 1$.

(c) Show that

$$\mathbb{E}[Z_n] \xrightarrow{n \to \infty} \frac{s^2 + m^2}{m}.$$

Problem 4 Martingales (2 points)

Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent and identically distributed random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = \sigma^2 < \infty$. Show that the sequence

$$M_n \triangleq \left(\sum_{i=1}^n X_i \right)^2 - \sigma^2 n, \quad n \geq 1,$$

is a martingale with respect to the filtration generated by $X_1, \dots, X_n$, $n \geq 1$. Does $\left\{ \frac{M_n}{n^2} \right\}_{n=1}^\infty$ converge in L^1 and almost surely? To what limit and why.

GOOD LUCK!

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Deferred exam in: STK-MAT3710/4710 — Probability Theory

Day of examination: 18th of January 2024

Examination hours: 15:00 – 19:00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (2 points)

Let X be an integrable random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F}, \mathcal{G}$ sub-σ-algebras such that $\mathcal{G} \subseteq \mathcal{F} \subseteq \mathcal{A}$.

(a) Prove that if $\mathcal{F}$ is independent of $\sigma(X)$, then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

(b) Prove that $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}]$.

Problem 2 Random variables (3 points)

Consider the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ and Y on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ given by

$$X_n = \frac{(-1)^n Y}{n}, \quad n \geq 1,$$

where Y is exponentially distributed with parameter $\lambda > 0$, i.e. Y is absolutely continuous with density function $f_Y(y) = \lambda e^{-\lambda y}$, $y > 0$.

(a) Show that X_n, $n \geq 1$ are absolutely continuous with density functions given by

$$f_{X_n}(x) = \lambda n e^{-\lambda n x}, \quad x > 0,$$

if n is even and

$$f_{X_n}(x) = \lambda n e^{\lambda n x}, \quad x < 0,$$

if n is odd.

(Continued on page 2.)

(b) Discuss the convergence of the sequence $\{X_n\}_{n=1}^{\infty}$ almost surely, in L^p for any $p \geq 1$, in probability and in distribution.

(c) Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} a_n < \infty$. We say that X_n and X_{n+1} are *friends* if their distance is at most a_n. Prove that the infinite sequence $X_1(\omega), X_2(\omega), \ldots$ has a finite number of friends.

Problem 3 Conditional expectation (3 points)

Let X be uniformly distributed on $[0,1]$, i.e. X is absolutely continuous with density $f_X(x) = 1$, $x \in [0,1]$. Moreover, let N be geometrically distributed on the set of natural numbers $\{1,2,\ldots\}$ with parameter $p \in (0,1)$, i.e. N has probability mass function $\mathbb{P}[N = k] = (1-p)^{k-1}p$, $k = 1,2,\ldots$. Assume that N and X are independent and define the random variable $Y = X^{1/N}$.

(a) Show that Y is absolutely continuous with density function given by

$$f_Y(y) = \frac{p}{(1-(1-p)y)^2}, \quad y \in [0,1]$$

and that

$$\mathbb{E}[Y] = \frac{1}{1-p} + \frac{p}{(1-p)^2} \log p.$$

(b) Compute $\mathbb{E}[Y|\sigma(N)]$ and use it to show that

$$\sum_{k=1}^{\infty} \frac{k}{k+1} \frac{1}{2^k} = 2\,(1 - \log 2)\,.$$

(c) Find the distribution of (N, Y).

Problem 4 Martingales (2 points)

Let $\{X_n\}_{n=0}^{\infty}$ a time-homogeneous Markov chain on a finite state space S, i.e. the Markov property holds a.s. $\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] = \mathbb{E}[f(X_{n+1})|\sigma(X_n)]$ for all $n \geq 0$ and the transition probabilities $p_{ij} \triangleq \mathbb{P}[X_{n+1} = j|X_n = i]$, $i, j \in S$ do not depend on n. Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $n \geq 0$.

For any function $f : S \to \mathbb{R}$ such that $\mathbb{E}[|f(X_n)|] < \infty$, define the function $Pf : S \to \mathbb{R}$

$$Pf(i) = \mathbb{E}[f(X_{n+1})|X_n = i], \quad i \in S.$$

(a) Prove that $M_n \triangleq f(X_n) - \sum_{k=0}^{n-1} (Pf(X_k) - f(X_k))$, $n \geq 1$ is a martingale with respect to the given filtration.

(b) Show that for every $\alpha > 0$, $\mathbb{P}[M_n^* \geq \alpha] \leq \frac{(2n+1)c}{\alpha}$ where $M_n^* = \sup_{k=1,\ldots,n} |M_k|$ and $c = \sup_k \mathbb{E}[|f(X_k)|]$.

GOOD LUCK!

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in: STK-MAT3710/4710 — Probability Theory

Day of examination: 8th December 2023

Examination hours: 9:00 am – 1:00 pm

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (2 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, X a random variable on it and $\mathcal{F} \subseteq \mathcal{A}$ a sub-σ-algebra.

(a) Define the concept of expectation of a random variable X from the measure-theoretical definition and outline the steps to construct it.

(b) Assuming that X is integrable, define the concept of conditional expectation of X given $\mathcal{F}$ through the theoretical definition.

Problem 2 Random variables (3 points)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent, identically distributed random variables with exponential distribution with parameter 1, i.e. each X_n is absolutely continuous with density function $f_{X_n}(x) = e^{-x}$, $x > 0$.

Define, for each $n \geq 1$, the events

$$A_n = \left\{ |X_n - X_{n-1}| \leq \frac{1}{n^2} \right\},$$

where $X_0 = 0$ by convention.

We call outcomes in A_n *friends*. In other words, we say that $X_{n-1}(\omega)$ and $X_n(\omega)$ are friends if $\omega \in A_n$.

(Continued on page 2.)

(a) Show that the random variables $Y_n = |X_n - X_{n-1}|$, $n \geq 1$ are again exponentially distributed with parameter 1. Hint: You may want to use $\mathbb{P}[A] = \int_{-\infty}^{\infty} \mathbb{P}[A|X = x] f_X(x) dx$ for an absolutely continuous random variable X with density f_X and any event A.

(b) Let I_n be the random variable that is one if X_{n-1} and X_n are friends and 0 otherwise. Show that $\{I_n\}_{n=1}^{\infty}$ converges in probability to 0.

(c) Will you observe an infinite number of friends in the infinite sequence $X_1, X_2, \ldots$? Justify your answer.

Problem 3 Conditional expectation (3 points)

Suppose that $a, b > 0$ real and X and Y are two random variables with values in $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{N}_0 = \{0, 1, \ldots\}$, respectively, whose distribution function is given by

$$\mathbb{P}[X \leq x, Y = n] = b \int_0^x \frac{(az)^n}{n!} e^{-(a+b)z} dz, \quad (x, n) \in \mathbb{R}_+ \times \mathbb{N}_0.$$

You may need the fact that $\int_0^{\infty} \frac{b^{n+1}}{n!} x^n e^{-bx} dx = 1$ and the Taylor expansion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{R}$. Also $\sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}$ for $|r| < 1$.

(a) Prove that the marginal distributions of X and Y are given by

$$\mathbb{P}[X \leq x] = 1 - e^{-bx}, x \in \mathbb{R}_+, \quad \mathbb{P}[Y = n] = \frac{b}{a+b} \left(\frac{a}{a+b} \right)^n, n \in \mathbb{N}_0.$$

(b) Show that $\mathbb{E}[h(X)|Y = n] = \frac{(a+b)^{n+1}}{n!} \int_0^{\infty} h(x) x^n e^{-(a+b)x} dx$ for a Borel measurable function h.

(c) Compute $\mathbb{E}[X|\sigma(Y)]$ and verify that $\mathbb{E}[\mathbb{E}[X|\sigma(Y)]] = \mathbb{E}[X]$.

Problem 4 Martingales (2 points)

Let $\{M_n\}_{n=0}^{\infty}$ be a martingale relative to a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$ and $\{X_n\}_{n=0}^{\infty}$ a bounded (i.e. $|X_n| \leq C$ for some constant $C > 0$ for all $n \geq 0$) stochastic process adapted to $\{\mathcal{F}_n\}_{n=0}^{\infty}$. Define the stochastic process Y_n, $n \geq 0$ by

$$Y_0 = 0, \quad Y_n = \sum_{k=1}^{n} X_{k-1}(M_k - M_{k-1}), \quad n \geq 1.$$

The process $\{Y_n\}_{n=0}^{\infty}$ is known as *discrete stochastic integral of* $\{X_n\}_{n=0}^{\infty}$ *with respect to the martingale* $\{M_n\}_{n=0}^{\infty}$.

(a) Show that $\{Y_n\}_{n=0}^{\infty}$ is a martingale relative to $\{\mathcal{F}_n\}_{n=0}^{\infty}$.

(b) Show that if, in addition, $\mathbb{E}[|M_n|^2] < \infty$ for every $n \geq 1$ then $\mathbb{E}[|Y_n^*|^2] < \infty$ for every $n \geq 1$ as well, where $Y_n^* = \sup_{k=0,\ldots,n} |Y_k|$.

GOOD LUCK!

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Trial exam in: STK3710/4710 — Probability Theory

Day of examination: 20th November 2023

Examination hours: 2:00 pm – 6:00 pm

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more oriented to solving through reasoning. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (3 points)

Let E be a set and $(F, \mathcal{F})$ a measurable space. Consider a transformation $f : E \to F$ between E and F. It is implicitly assumed that $f^{-1}(F) = E$.

(a) Given subsets $A, B \in \mathcal{F}$ show that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

and

$$f^{-1}(A^c) = f^{-1}(A)^c,$$

where $A^c = F \setminus A$ stands for the *complement of* A.

(b) Define a class of subsets of E as

$$\mathcal{E} = \{A \subseteq E : \text{ there exists } B \in \mathcal{F} \text{ with } A = f^{-1}(B)\}.$$

Show that $\mathcal{E}$ is a σ-algebra on E and that f is an $(\mathcal{E}, \mathcal{F})$-measurable transformation.

(c) Let μ be a measure on $(E, \mathcal{E})$ where $\mathcal{E}$ is given as in item (b). Define the push-forward measure of μ via f and show that it is a measure on $(F, \mathcal{F})$.

(Continued on page 2.)

Problem 2 Random variables (2 points)

(a) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with the following distribution

$$\mathbb{P}[X_n = -n] = \mathbb{P}[X_n = n] = \frac{1}{2\sqrt{n}}, \quad \mathbb{P}[X_n = 0] = 1 - \frac{1}{\sqrt{n}}, \quad n \geq 1.$$

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Discuss the convergence of the sequence $\{X_n\}_{n=1}^{\infty}$ in L^p for any $p \geq 1$, almost surely, in probability and in distribution.

(b) Consider the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$ the Borel σ-algebra on $[0,1]$ and $\mathbb{P}$ the Lebesgue measure, i.e. $\mathbb{P}[(a,b]) = b - a$ for all intervals $(a,b] \subset [0,1]$.

Consider on this probability space the random variables

$$X_n(\omega) = \frac{n+1}{n}\omega + (1-\omega)^n, \quad n \geq 1$$

and

$$X(\omega) = \omega.$$

Show that $\{X_n\}_{n=1}^{\infty}$ converges, almost surely, to X and discuss whether it also converges in L^p, for $p \geq 1$. You may find useful the following inequality: for real numbers a, b and $p \geq 1$, $(|a| + |b|)^p \leq 2^{p-1}\left(|a|^p + |b|^p\right)$.

Problem 3 Conditional expectation (3 points)

Let $n \geq 1$ be a fixed natural number and $\{X_i\}_{i=1}^{n}$ a family of independent and identically distributed random variables. Let $S_n = \sum_{i=1}^{n} X_i$ denote their sum. Furthermore, consider $\mathcal{F} = \sigma(S_n)$ the σ-algebra generated by the sum of the n random variables.

(a) Compute $\mathbb{E}[X_1|\mathcal{F}]$.

(b) Let Y, Z be two independent standard normally distributed random variables. Let $a, b \in \mathbb{R}$. Find the conditions on a and b for the random variables $(1-a)Y + bZ$ and $aY - bZ$ to be independent.

(c) Assume now, in addition to independence, that the distribution of the X_i, $i = 1, \ldots, n$ is the standard normal distribution, that is $X_i \sim N(0,1)$ for al $i = 1, \ldots, n$. Compute $\mathbb{E}[X_1^2|\mathcal{F}]$.

(Continued on page 3.)

Problem 4 Martingales (2 points)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables all with distribution $N(-\mu, 1)$, $\mu > 0$ real.

(a) Find the values of $a \in \mathbb{R}$ for which the stochastic process $Y_n = e^{a \sum_{i=1}^{n} X_i}$, $n \geq 1$, is a submartingale, a martingale and a supermartingale with respect to the natural filtration of $\{X_n\}_{n=1}^{\infty}$.

(b) Assume that $a = 2\mu$ (then $\{Y_n\}_{n=1}^{\infty}$ is martingale) and let $x > 0$ real. Define $S_n = \sum_{i=1}^{n} X_i$. Show that

$$\mathbb{P}\left[\sup_{n\geq 1} S_n > x\right] \leq e^{-2\mu x}.$$

GOOD LUCK!

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK-MAT3710/4710 — Probability Theory.
Day of examination:	Monday, December 5th, 2022.
Examination hours:	15.00 – 19.00.

This problem set consists of 3 pages.

Appendices:	Formula sheet.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1a, 1b, 2 etc.) count equally. If there is a problem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated.

We use $\mathbb{N}_0$ to denote the natural numbers with 0 included, i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$.

Problem 1 (30 points)

a) Let a be a real number and assume that Y_a is a random variable such that

$$Y_a = \begin{cases} 1 + a & \text{with probability } \frac{1}{2} \\ -1 + a & \text{with probability } \frac{1}{2} \end{cases}$$

Show that the characteristic function of Y_a is $\phi_a(t) = e^{ita} \cos t$.

b) For $n \in \mathbb{N}$, let $X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}$ be n independent copies of $Y_{\frac{1}{\sqrt{n}}}$, and put

$$S_n = \frac{X_1^{(n)} + X_2^{(n)} + \cdots + X_n^{(n)}}{\sqrt{n}}$$

Find the characteristic function ϕ_{S_n} of S_n.

c) Show that $\{S_n\}$ converges in distribution to a normal distribution. What is its mean and variance?

Problem 2 (10 points) A radio station is running a lottery every day. The first day the chance of winning is $\frac{1}{100}$, the second day it is $\frac{1}{101}$, the third day $\frac{1}{102}$ and so on. What is the probability of winning infinitely many times if you live forever and play the lottery every day?

(Continued on page 2.)

Problem 3 (10 points) Assume that X and Y are two integrable random variables. Show that the family of all sets Λ such that

$$\int_\Lambda X\,dP = \int_\Lambda Y\,dP$$

is a monotone class.

Problem 4 (10 points) Assume that $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of independent events. Let

$$P_n = P(A_1) + P(A_2) + \cdots + P(A_n)$$

and let $X_n(\omega)$ be the number of sets among $A_1, A_2, \ldots, A_n$ that ω belongs to, i.e., $X_n(\omega) = |\{i \le n \,|\, \omega \in A_i\}|$. Show that the sequence $\left\{\frac{X_n - P_n}{n}\right\}$ converges to 0 a.s.

Problem 5 (50 points)

We assume that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of nonnegative, integrable random variables with finite mean $E[X_n] = \mu_n > 0$. We also assume that Y is a random variable which takes values in $\mathbb{N}_0$ and is independent of $\{X_n\}_{n\in\mathbb{N}}$. Define a random variable Z by

$$Z(\omega) = \begin{cases} X_1(\omega) + X_2(\omega) + \cdots + X_{Y(\omega)}(\omega) & \text{if } Y(\omega) > 0 \\ \\ 0 & \text{if } Y(\omega) = 0 \end{cases}.$$

a) Show that

$$Z(\omega) = \sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(\omega)(X_1(\omega) + X_2(\omega) + \cdots + X_n(\omega))$$

and that

$$E[Z] = \sum_{n=1}^{\infty} (\mu_1 + \mu_2 + \cdots + \mu_n) P[Y = n]$$

b) Assume that $\mathcal{G}$ is a σ-algebra such that Y is $\mathcal{G}$-measurable and the $\{X_n\}$'s are independent of $\mathcal{G}$. Show that if Z is integrable, then

$$E[Z|\mathcal{G}](\omega) = \sum_{n=1}^{\infty} (\mu_1 + \cdots + \mu_n) \mathbf{1}_{[Y=n]}(\omega)$$

In the rest of the problem, we assume that X is a random variable with mean $\mu > 0$ taking values in $\mathbb{N}_0$. We shall study a probabilistic model for how a population of animals develops from generation to generation. If generation k consists of $Z_k(\omega) \in \mathbb{N}_0$ individuals, we assume that each of these gives rise to a random number of offsprings distributed according to X. We also assume that the number of offsprings is independent from individual to individual and from generation to generation.

To model this mathematically, we assume that $\{X_i^{(k)}\}_{k,i\in\mathbb{N}}$ are independent copies of X. We let $Z(0) \in \mathbb{N}$ and define inductively

$$Z_{k+1}(\omega) = \begin{cases} X_1^{(k+1)}(\omega) + X_2^{(k+1)}(\omega) + \cdots + X_{Z_k(\omega)}^{(k+1)}(\omega) & \text{if } Z_k(\omega) > 0 \\ \\ 0 & \text{if } Z_k(\omega) = 0 \end{cases}$$

We also assume that X is chosen such that Z_k is integrable for all k.

(Continued on page 3.)

c) Let $\mathcal{F}_k$ be the σ-algebra generated by all $\{X_i^{(j)}\}$ with $j \leq k$. Show that

$$E[Z_{k+1}|\mathcal{F}_k] = \mu Z_k$$

For which values of μ is $\{Z_k\}$ an $\{\mathcal{F}_k\}$-submartingale, an $\{\mathcal{F}_k\}$-martingale, or an $\{\mathcal{F}_k\}$-supermartingale, respectively?

d) Show that $Y_k = \frac{Z_k}{\mu^k}$ is an $\{\mathcal{F}_k\}$-martingale.

One can prove that if X has finite variance σ^2, then there is a $K \in \mathbb{R}$ such that $E[Y_k^2] \leq K$ for all k (you can use this without proof), and hence $\{Y_k\}$ is uniformly integrable.

e) Assume that X has finite variance and that $\mu > 1$. Explain that $\{Y_k\}$ converges almost surely and in L^1 to an integrable random variable Y_∞. Show that there must be a set of positive probability where Y_∞ is strictly positive, and conclude that Z_k goes to infinity with positive probability.

GOOD LUCK

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Examination in: STK-MAT3710/4710 - Probability Theory.
Day of examination: Friday, December 3, 2021.

Examination hours: 15:00 - 19:00.

This problem set consists of 2 pages and 4 problems.

Appendices: Formula sheet.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

If there is a problem you can not solve, you may still use the result in the sequel. All answers have to be substantiated. For all problems, at each step, please explain the mathematical facts, which let you move on. Lack of explanation may cause a partial credit or no credits. Please write down your solutions completely in English.

[**Problem 1**] (9 points)

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{X_n : n \geq 1\}$ and $\{Y_n : n \geq 1\}$ be sequences of random variables such that $\sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty$. Show that $\sum_{n=1}^{\infty} X_n$ and $\sum_{n=1}^{\infty} Y_n$ converge or diverge together.

[**Problem 2**]

Let $(\Omega, \mathcal{F}, P)$ be a probability space and the random variable X is given by: $P(X = -5) = \frac{1}{4}$, $P(X = 3) = \frac{1}{4}$, and $P(X = 1) = \frac{1}{2}$.

a) (9 points) Find the Taylor expansion of the characteristic function of X.

b) (9 points) Assume that $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of independent random variables with the same distribution as X. Find the characteristic function of:

$$S_n = \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}.$$

c) (9 points) Use the result in **(b)** to show directly that S_n converges in distribution to a normal distribution (you are not allow to use a version of the Central Limit Theorem).

[Problem 3]

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n : n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a sub-sigma field of $\mathcal{F}$. Suppose that $\{X_n, \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ be a martingale and $E[X_n^2] < \infty$ for all $n \in \mathbb{N}_0$. Moreover, assume that $n < r < m$ for all $m, n, r \in \mathbb{N}_0$.

a) (9 points) Show that for all $j > n$ ($j \in \mathbb{N}_0$), $E[X_j X_n] = E[X_n^2]$.

b) (5 points) Show that $E[(X_m - X_r)X_n] = 0$.

c) (9 points) Show that $E[(X_m - X_r)^2|\mathcal{F}_n] = E[X_m^2|F_n] - E[X_r^2|F_n]$.

d) (9 points) Suppose that there exists a constant K such that $E[X_n^2] \leq K$ for all $n \in \mathbb{N}_0$. Show that $\{X_n\}_{n\in\mathbb{N}_0}$ is mean-square Cauchy convergent.

[Problem 4]

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_n : n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ be a sub-sigma field of $\mathcal{F}$. Suppose that $\{X_n, \mathcal{F}_n, n = 0, 1, 2, \ldots\}$ be a martingale and let K be a constant. Define T and S with $\lambda_2 < 0 < \lambda_1$ as follows:

$$T(\omega) = \begin{cases} \inf\{n \in \mathbb{N}_0 : X_n \geq \lambda_1\}, & \text{if such n exists,} \\ N, & \text{if no such } n \text{ exists,} \end{cases}$$

and

$$S(\omega) = \begin{cases} \inf\{n \in \mathbb{N}_0 : X_n \leq \lambda_2\}, & \text{if such } n \text{ exists,} \\ N, & \text{if no such } n \text{ exists.} \end{cases}$$

a) (9 points) Prove that $\gamma = \max(T, S)$ is a *bounded* stopping time.

b) (10 points) Let us define $X_n^+ = \max(X_n, K)$ and assume that $K < \lambda_2$. Let $E[X_S^+] = L$ be (L is a constant), then, is $E[X_\gamma^+]$ less than L, equal to L or greater than L? Justify your reasoning.
Hint: $X^+ = \max(X, K)$ is a nondecreasing convex function of X.

c) (13 points) Let assume $K = 0$. Suppose that $M_n = \lim_{m\to\infty} E[X_{m+n}^+|\mathcal{F}_n]$, ($m \geq 0$), exists almost surely, moreover, M_n is integrable and $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}_0$. Prove that M_n is a martingale with respect to $\mathcal{F}_n$.

GOOD LUCK!

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3710/4710 — Probability Theory.

Day of examination: Friday, December 11th, 2020.

Examination hours: 15.00 – 19.00.

This problem set consists of 2 pages.

Appendices: Formula sheet.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1a, 1b etc.) count equally. If there is a problem you cannot solve, you may still use the result in the sequel. All answers have to be substantiated.

We use $\mathbb{N}_0$ to denote the natural numbers with 0 included, i.e. $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$.

Problem 1 (50 points)

In this problem λ is a positive real number.

a) Let Y be a Poisson random variable with intensity λ, i.e. Y is taking values in the nonnegative integers $\mathbb{N}_0$, and

$$P[Y = n] = \frac{\lambda^n}{n!} e^{-\lambda}$$

for each $n \in \mathbb{N}_0$. Show that the characteristic function of Y is

$$\phi_Y(t) = e^{\lambda(e^{it}-1)}.$$

b) Find $E[Y]$, e.g. by differentiating ϕ_Y.

c) Let $n \in \mathbb{N}$ and assume that $n > \lambda$. Let X_n be a random variable taking only two values 0 and 1 with probabilities $P[X_n = 1] = \frac{\lambda}{n}$ and $P[X_n = 0] = 1 - \frac{\lambda}{n}$, respectively. Show that the characteristic function of X_n is

$$\phi_{X_n}(t) = 1 + \frac{\lambda}{n}(e^{it} - 1).$$

d) Let $S_n = X_n^{(1)} + X_n^{(2)} + \ldots + X_n^{(n)}$ be the sum of n independent copies of X_n. Find the characteristic function of S_n.

e) Show that the sequence $\{S_n\}_{n \in \mathbb{N}}$ converges to Y in distribution (here Y is the random variable in question a)).

(Continued on page 2.)

Problem 2 (20 points)

In this problem a and b are real numbers such that $b < 0 < a$. Assume that p is a real number between 0 and 1, and let X be a random variable taking the values a and b with probabilities $P[X = a] = p$, $P[X = b] = 1 - p$, respectively.

Assume that $\{X_j\}_{j\in\mathbb{N}}$ is an independent sequence of copies of X, and define a process $M = \{M_n\}_{n\in\mathbb{N}_0}$ by putting $M_0 = 0$ and $M_n = \sum_{j=1}^{n} X_j$ for $n > 0$. Define the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ by letting $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(X_1, X_2, \ldots, X_n)$ for $n > 0$.

a) For which value of p is M a martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$? For which values of p is it a submartingale and for which is it a supermartingale?

b) Assume from now on that $a = 1$ and $b = -1$. Pick two integers k, m such that $k < 0 < m$ and define

$$T(\omega) = \inf\{n \in \mathbb{N}_0 \colon M_n(\omega) = k \text{ or } M_n(\omega) = m\}.$$

Define $M_T(\omega) = M_{T(\omega)}(\omega)$ (you may assume without proof that T is finite a.s.). For which values of p is $E[M_T] > 0$ and for which values of p is $E[M_T] < 0$? Explain your reasoning.

Problem 3 (40 points)

In this problem you may use that a finite sum of gaussian random variables is gaussian.

a) Let $\{X_n\}_{n\in\mathbb{N}}$ be an independent sequence of gaussian random variables with mean zero and variance 1. Put $Y_N = X_1 + X_2 + \cdots + X_N$. Find the mean and variance of Y_N.

b) Explain that if $a > 0$, then

$$\int_a^\infty e^{-\frac{x^2}{2}}\,dx \le \int_a^\infty \frac{x}{a} e^{-\frac{x^2}{2}}\,dx$$

and use this to prove that

$$\int_a^\infty e^{-\frac{x^2}{2}}\,dx \le \frac{1}{a} e^{-\frac{a^2}{2}}.$$

c) Let $\epsilon > 0$. Show that for all $N \in \mathbb{N}$,

$$P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] \le \frac{e^{-\frac{N^\epsilon}{2}}}{\sqrt{2\pi}}.$$

d) Prove that

$$P\left[Y_N > \sqrt{N^{1+\epsilon}} \text{ for infinitely many } N\right] = 0.$$

You may use without proof that $\lim_{N\to\infty} \frac{N^p}{e^{\frac{N^\epsilon}{2}}} = 0$ for all $p > 0$.

THE END

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK-MAT3710/4710 — Probability Theory.

Day of examination: Wednesday, December 11th, 2019.

Examination hours: 14.30 – 18.30.

This problem set consists of 2 pages.

Appendices: Formula sheet.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All items (Problems 1, 2a, 2b etc.) count equally. If there is a problem you can not solve, you may still use the result in the sequel. All answers have to be substantiated.

Problem 1 (10 points)

Let $\{X_n\}_{n\in\mathbb{N}}$ be independent and identically distributed random variables taking values in the set $\{1,2,3,4,5,6\}$. Assume that $P([X_n = i]) = \frac{1}{6}$ for $i = 1,2,3,4,5,6$. Show that for almost all ω, the sequence $X_1(\omega), X_2(\omega), X_3(\omega), \ldots$ contains infinitely many occurences of 17 consecutive 6's.

Problem 2

A distribution function is given by

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2}(x+1) & \text{for } -1 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

a) (10 points) Find a real expression for the characteristic function of F.

b) (10 points) Assume that $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of independent random variables with distribution function F. Find the characteristic function of

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}\ .$$

c) (10 points) Use the result in b) to "show directly" that S_n converges in distribution to a normal distribution ("show directly" means that you are not allowed to use a version of the Central Limit Theorem).

(Continued on page 2.)

Problem 3

In this problem, $(\Omega, \mathcal{F}, P)$ is a probability space; $\mathbb{N}$ is the timeline; and $\{M_n\}_{n\in\mathbb{N}}$ is a martingale with respect to a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ on $(\Omega, \mathcal{F}, P)$. We assume that all M_n have finite second moments, and for each $n \in \mathbb{N}$, we let $\Delta M_n = M_{n+1} - M_n$ be the forward increment of M.

a) (10 points) Show that if $n \leq m$, then $E[\Delta M_m|\mathcal{F}_n] = 0$ and $E[\Delta M_m M_n|\mathcal{F}_n] = 0$.

b) (10 points) Show that if $n < m$, then $E[\Delta M_m \Delta M_n|\mathcal{F}_n] = 0$.

c) (10 points) Show that if $n < m$, then

$$E[(M_m - M_n)^2|\mathcal{F}_n] = \sum_{k=n}^{m-1} E[\Delta M_k^2|\mathcal{F}_n]$$

Problem 4

Recall that a function $F\colon \mathbb{R} \to \mathbb{R}$ is a distribution function if

(i) F is right continuous and increasing

(ii) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

a) (10 points) Show that if F is a distribution function and Y is a random variable, then

$$K(x) = E[F(x - Y)]$$

is also a distribution function. (To save time, you need only check one of the two conditions $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.)

In the rest of the problem, $X, Y\colon \Omega \to \mathbb{R}$ are two independent random variables with distribution functions F and G, respectively. Our aim is to show that the distribution function H of $X + Y$ is given by

$$H(x) = E[F(x - Y)]\ . \tag{1}$$

b) (10 points) Assume first that Y is of the form $Y = \sum_{n=1}^{\infty} a_n \mathbf{1}_{A_n}$, where the a_n's are distinct real numbers and $\{A_n\}_{n\in\mathbb{N}}$ is a partition of Ω; i.e. the sets are mutually disjoint and their union is all of Ω. Show that (1) holds in this case. Specify where you use the independence of X and Y.

c) (10 points) If Y is a general random variable, let

$$\underline{Y}_n = \sum_{k=-\infty}^{\infty} \frac{k}{2^n} \mathbf{1}_{(k2^{-n},(k+1)2^{-n}]}$$

be the usual lower approximation of Y. Show that

$$E[F(x - Y)] = \lim_{n\to\infty} E[F(x - \underline{Y}_n)]\ .$$

d) (10 points) Show that equation (1) holds for all random variables Y that are independent of X. Conclude that

$$H(x) = \int_{-\infty}^{\infty} F(x - y)\, dG(y)$$

THE END

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	STK.MAT3710/4710 — Probability Theory
Day of examination:	16th December 2024
Examination hours:	3:00 pm – 7:00 pm
This problem set consists of 9 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature and the three remaining ones are more applied. All items within a problem are worth the same unless otherwise specified. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Grading: The total score is 10 points. The grading scale is F [0,4), E [4,5), D [5, 6), C [6, 7), B [7, 8.5), A [8.5,10].

Problem 1 Theory (2 points)

Let E be an uncountable set. Define the class of subsets of E given by

$$\mathcal{E} = \{A \subseteq E : \text{ either } A \text{ is countable, or } A^c \text{ countable}\}.$$

(a) Show that if $A, B \subseteq E$ are uncountable such that $A \cap B = \emptyset$, then either $A \notin \mathcal{E}$ or $B \notin \mathcal{E}$.

Solution: Assume that A and B are uncountable *and* both belong to $\mathcal{E}$ then it must be the case that A^c and B^c are countable. But then we would have $A^c \cup B^c = (A \cap B)^c = \emptyset^c = E$ and we would have that the union of two countable sets, i.e. A^c and B^c, gives an uncountable set, i.e. E, which is not possible. Hence A and B cannot both belong to $\mathcal{E}$. **Points:** Give 2/3p for this item. Subtract 0.2 for mistakes.

(b) Show that $\mathcal{E}$ is a σ-algebra.

Solution: We need to verify three conditions:

1. $\emptyset \in \mathcal{E}$ since the empty set is countable.
2. let $\{A_i\}_{i=1}^\infty \subset \mathcal{E}$ be disjoint, then if all the sets are countable we have $\cup_{i=1}^\infty A_i$ is countable and hence in $\mathcal{E}$. If one of them is uncountable, say A_j, then the union is uncountable. Note that

(Continued on page 2.)

we can only allow one of them being uncountable since they are disjoint, in virtue of item (a). However,

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

is countable since it is the intersection of sets, where A_j^c is countable. If the sets are not disjoint, we can select a family of disjoint sets such that their unions are equal.

3. Finally, $A \in \mathcal{E}$ and if A is countable then A^c is uncountable with countable complement and vice versa. Hence $A^c \in \mathcal{E}$.

Points: Give 2/3p for this item. Give 0.33p for a properly written definition of σ-algebra. Give 0.33 for verifying the definition. Subtract 0.2 for mistakes.

(c) Define the mapping $\mu : \mathcal{E} \to [0, \infty]$ by $\mu(X) = \mathbb{I}_{\{X^c \text{ countable}\}}$. Show that μ is a probability measure.

Solution: We need to verify three conditions:

1. $\mu() = 0$ since $\emptyset \in \mathcal{E}$ is countable.
2. let $\{A_i\}_{i=1}^{\infty} \subset \mathcal{E}$ be disjoint, then if all A_i's are countable we have that the union is countable and hence

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0.$$

But if one A_j is uncountable then the union is uncountable and hence

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 1.$$

On the other hand, if all A_i are countable then $\mu(A_i) = 0$ for all i and hence

$$\sum_{i=1}^{\infty} \mu(A_i) = 0.$$

But if only one, say A_j is uncountable then $\mu(A_i) = 0$ for all $i \neq j$ and $\mu(A_j) = 1$ and hence

$$\sum_{i=1}^{\infty} \mu(A_i) = 1.$$

As a result,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i).$$

(Continued on page 3.)

3. Finally, $\mu(E) = 1$ since E is uncountable.

Points: Give 2/3p for this item. Give 0.33p for a properly written definition of measure. Give 0.33 for verifying the definition. Subtract 0.2 for mistakes.

Problem 2 Random variables (3 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\{X_n\}_{n=1}^{\infty}$ a sequence of independent identically distributed random variables with uniform distribution on $[0,1]$, i.e. with densities $f_{X_n}(x) = 1$, $x \in [0,1]$, $n \geq 1$.

Define the random variables

$$L_{(1:n)} = \min\{X_1, \ldots, X_n\}, \quad H_{(1:n)} = \max\{X_1, \ldots, X_n\}, \quad n \geq 2.$$

It may not be surprising that $L_{(1:n)}$ and $H_{(1:n)}$ are not independent and, in fact, it can be shown that their joint distribution is given by

$$f_{(L_{(1:n)}, H_{(1:n)})}(l, h) = n(n-1)(h-l)^{n-2}, \quad 0 \leq l < h \leq 1.$$

(a) Prove that the distribution of the range, denoted by R_n, of n independent uniform observations is given by

$$f_{R_n}(r) = n(n-1)r^{n-2}(1-r), \quad 0 < r < 1.$$

Solution: The range is $R_n = H_{(1:n)} - L_{(1:n)}$. Let $(u, v) = g(l, h) = (l, h-l)$ then $g^{-1}(u, v) = (u, u+v)$. Then $|Dg^{-1}(u, v)| = 1$ and $0 \leq l < h \leq 1$, $0 \leq u + v \leq 1$ and $0 \leq u, v \leq 1$. Then

$$f_{(U,V)}(u, v) = f_{(L,H)}(g^{-1}(u, v)) = n(n-1)v^n - 2, \quad 0 \leq u + v \leq 1.$$

Now to get R we need to integrate away U, i.e.

$$f_V(v) = \int_0^1 n(n-1)v^{n-2}\mathbb{I}_{\{0 \leq u+v \leq 1\}} du = n(n-1)v^{n-2}(1-v), \quad 0 \leq v \leq 1,$$

which is exactly the density given in the exercise.

Points: Give 0.25p for a properly written formula of change of variables. Give 0.5 for finding the joint transformation but not the marginal. Subtract 0.2 for mistakes.

(b) Study the convergence of the sequence $\{R_n\}_{n=1}^{\infty}$ in distribution, probability, almost surely and in L^p, $p \geq 1$. Note: $\sum_{n=1}^{\infty} nr^n < \infty$ for every real r such that $|r| < 1$.

Solution: We have that

$$F_{R_n}(r) = \int_0^r f_{R_n}(u) du = nr^n \left[\frac{1}{r} - \frac{n-1}{n} \right],$$

(Continued on page 4.)

which tends to 0 as n tends to infinity as long as $r < 1$, when $r \geq 1$ we have that the limit is 1. Hence $R_n \to 1$ in distribution.

Moreover,

$$\mathbb{P}[|R_n - 1| > \varepsilon] = \mathbb{P}[R_n \leq 1 - \varepsilon] = n(1-\varepsilon)^n \left[\frac{1}{1-\varepsilon} - \frac{n-1}{n}\right]$$

which converges to zero by the previous argument since $1 - \varepsilon < 1$. Hence, R_n converges to 1 in probability, as well. Note that we knew this already, since 1 is a constant and convergence in distribution to a constant implies convergence in probability.

For a.s. convergence we aim to apply Borel-Cantelli lemma. Define $A_n = \{|R_n - 1| > \varepsilon\}$. We have already computes $\mathbb{P}[A_n]$. Namely,

$$\mathbb{P}[A_n] = nr^n \left[\frac{1}{1-\varepsilon} - \frac{n-1}{n}\right] \leq \frac{n(1-\varepsilon)^n}{1-\varepsilon}$$

and for every $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] \leq \sum_{n=1}^{\infty} \frac{n(1-\varepsilon)^n}{1-\varepsilon} < \infty,$$

and by Borel-Cantelli's first lemma we have

$$\mathbb{P}\left[\limsup_n \{|R_n - 1| > \varepsilon\}\right] = 0$$

and a.s. convergence follows.

Now, since $|R_n - 1|^p \leq C$ is bounded for every $p \geq 1$, we have by Lebesgue's dominated convergence theorem that

$$\lim_n \mathbb{E}[|R_n - 1|^p] = \mathbb{E}[\lim_n |R_n - 1|^p] = 0,$$

where we used continuity of the function $|\cdot|^p$ and almost sure convergence.

Points: Give 0.25p per convergence type (when justified). Give 0.25 for Borel-Cantelli (written theorem, not just name). Subtract 0.2 for mistakes.

(c) Consider independent observations $X_1, X_2, \ldots$ of the uniform distribution. What is the probability that we observe k consecutive observations that are, at most, within distance $\delta > 0$?

Solution: We can divide the observations in string of length k, i.e. $X_1, \ldots, X_k$ then $X_{k+1}, \ldots, X_{2k}$ and so on. Hence, we look at strings

$$\{X_{nk+1}, \ldots, X_{(n+1)k}\}_{n=1}^{\infty}.$$

Any of these string will consist of all observations being at most δ away from each other if their range is less than δ, i.e. $H_{(nk+1:(n+1)k)} -$

(Continued on page 5.)

$L_{(nk+1:(n+1)k)} < \delta$. Note that since the observations are i.i.d. this is the same as looking at the event $R_k < \delta$. We are asking whether the event $\{R_k < \delta\}$ for fixed k will occurs infinitely many times or not. But

$$\sum_{n=1}^{\infty} \mathbb{P}[R_k < \delta] = \infty$$

and hence by Borel-Cantelli's second lemma, since R_k are independent, we conclude that there will be infinitely many strings of length k that are at most a distance of δ from each other, for any arbitrary k.

Points: Give 0.25p if there is an attempt to apply Borel-Cantelli. Give 0.25 for Borel-Cantelli (written theorem, not just name). Subtract 0.2 for mistakes.

Problem 3 Conditional expectation (2 points)

Let X and Y be two independent Bernoulli distributed random variables in some probability space, i.e. $X, Y \in \{0, 1\}$ and $\mathbb{P}[X = 1] = \mathbb{P}[Y = 1] = p$ for some $p \in [0, 1]$. Define the random variable $Z = \mathbb{I}_{\{X+Y=0\}}$.

(a) Compute $\mathbb{E}[Z]$, $\mathbb{E}[X|Z]$ and verify that $\mathbb{E}[\mathbb{E}[X|Z]] = \mathbb{E}[X]$.

Solution:

$\mathbb{E}[Z] = \sum_{x,y\in\{0,1\}} \mathbb{I}_{\{X+Y=0\}}\mathbb{P}[X = x]\mathbb{P}[Y = y] = \mathbb{P}[X = 0]\mathbb{P}[Y = 0] = (1 - p)^2$.

To compute $\mathbb{E}[X|Z]$ note that the σ-algebra generated by Z is finite, i.e. $\{Z = 0\}$ and $\{Z = 1\}$ then let us compute the conditional expectation on these two events:Note that Z is also Bernoully with probability $\mathbb{P}[Z = 1] = (1 - p)^2$. Hence,

$$\begin{aligned}
\mathbb{E}[X|Z = 0] &= \frac{\mathbb{E}[X\mathbb{I}_{\{Z=0\}}]}{\mathbb{P}[Z = 0]} \\
&= \frac{\mathbb{P}[X = 1]\mathbb{P}[Y = 1] + \mathbb{P}[X = 1]\mathbb{P}[Y = 0]}{1 - (1 - p)^2} \\
&= \frac{p^2 + p(1 - p)}{1 - (1 - p)^2} \\
&= \frac{1}{2 - p}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}[X|Z = 1] &= \frac{\mathbb{E}[X\mathbb{I}_{\{Z=1\}}]}{\mathbb{P}[Z = 1]} \\
&= \frac{X\mathbb{I}_{\{X+Y=0\}}}{(1 - p)^2} = 0.
\end{aligned}$$

(Continued on page 6.)

In a summary,

$$\mathbb{E}[X|Z=z] = \frac{1}{2-p}\mathbb{I}_{\{z=0\}}$$

and hence

$$\mathbb{E}[X|Z] = \frac{1-Z}{2-p}.$$

Now, we have by linearity of expectation,

$$\mathbb{E}[\mathbb{E}[X|Z]] = \frac{1-\mathbb{E}[Z]}{2-p} = \frac{1-(1-p)^2}{2-p} = p$$

which also coincides with $\mathbb{E}[X] = p$.

Points: Give 0.5p for $\mathbb{E}[X|Z]$ and 0.25p for the two other. Give 0.25p if someone writes down the definition of conditional expectation perfectly. Subtract 0.2 for mistakes. In particular, subtract 0.2 if candidate does not express $\mathbb{E}[X|Z]$ as a Borel-measurable function of Z.

(b) Compute $\mathbb{E}[Z|X]$. Is it true that $\mathbb{E}[\mathbb{E}[X|Z]|X] = \mathbb{E}[\mathbb{E}[Z|X]|Z]$? What about $\mathbb{E}[\mathbb{E}[X|Z]|X,Z] = \mathbb{E}[X|Z]$? Justify your answer.

Solution: We have,

$$\begin{aligned}
\mathbb{E}[Z|X=x] &= \sum_{z\in\{0,1\}} z\mathbb{P}[Z=z|X=x] \\
&= \mathbb{P}[Z=1|X=x] \\
&= \mathbb{P}[X+Y=0|X=x] \\
&= \mathbb{P}[x+Y=0] \\
&= (1-p)(1-X).
\end{aligned}$$

hence,

$$\mathbb{E}[Z|X] = (1-p)(1-X).$$

Now, we have on the one hand,

$$\mathbb{E}[\mathbb{E}[X|Z]|X] = \mathbb{E}\left[\frac{1-Z}{2-p}\middle|X\right] = \frac{1-\mathbb{E}[Z|X]}{2-p} = \frac{1-(1-p)(1-X)}{2-p}$$

and on the other hand,

$$\mathbb{E}[\mathbb{E}[Z|X]|Z] = \mathbb{E}[(1-p)(1-X)|Z] = (1-p)(1-\mathbb{E}[X|Z]) = (1-p)\left(1-\frac{1-Z}{2-p}\right).$$

These random quantities are always different for all $p\in[0,1]$.

It is true that $\mathbb{E}[\mathbb{E}[X|Z]|X,Z] = \mathbb{E}[X|Z]$ since $\sigma(Z)\subseteq\sigma(X,Z)$ and by the stability property we get the result.

Points: Give 0.5p for $\mathbb{E}[Z|X]$ and 0.25p for the questions. Give 0.25p if someone writes down the definition of conditional expectation perfectly, but only once (not both in (a) and (b)). Subtract 0.2 for mistakes. In particular, subtract 0.2 if candidate does not express conditional expectations as Borel-measurable functions.

(Continued on page 7.)

Problem 4 Martingales (3 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X, Y two independent random variables in it. Assume that X is uniformly distributed on $[0,1]$, i.e. $f(x) = 1$ for $x \in [0,1]$ and $\mathbb{P}[Y = -1] = \mathbb{P}[Y = 1] = \frac{1}{2}$. Consider the filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \sigma(X)$ and $\mathcal{F}_n = \sigma(X, Y)$ for al $n \geq 2$. Define for each $n \geq 1$ the processes

$$M_n = \frac{Y}{X}\mathbb{I}_{\{n\geq 2\}}$$

and

$$\tau_n = \begin{cases} 1, & n < \frac{1}{X}, \\ \infty, & n \geq \frac{1}{X} \end{cases} .$$

(a) Show that $\{M_n\}_{n\geq 1}$ is not a martingale.

Solution: A martingale needs to satisfy three conditions: adaptedness, integrability and the martingale property. In this case, we see that M_n is adapted to the given filtration. However, the second property breaks down for $n \geq 2$. Indeed, take $n \geq 2$,

$$\mathbb{E}[|M_n|] = \mathbb{E}\left[\frac{|Y|}{|X|}\right] = \mathbb{E}[1/X] = \infty,$$

since $|Y| = 1$ and the inverse of the uniform has infinite expectation.

Points: Give 1p for this item. Give 0.5p if candidate writes down definition of martingale perfectly. Subtract 0.2 for mistakes.

(b) Show that $\{\tau_n\}_{n\geq 1}$ is a monotone sequence of stopping times which converges almost surely to infinity. Note: we have defined stopping times as being finite. Here, we allow infinity to be an outcome. The definition remains the same with the additional requirement that $\{\tau_n = \infty\}$ also needs to be measurable.

Solution: For each $n \geq 1$ we have that for every $k \in \mathbb{N}$,

$$\{\tau_n \leq k\} = \{\tau_n = 1\} = \left\{X < \frac{1}{n}\right\} \in \mathcal{F}_n$$

for every n. Also,

$$\{\tau_n = \infty\} = \{X \geq 1/n\} \in \mathcal{F}_n$$

hence τ_n is a stopping time for every $n \geq 1$.

It is monotone non-decreasing,

$$\begin{aligned}
\mathbb{P}\left[\tau_n \leq \tau_{n+1}\right] &= \mathbb{P}\left[\tau_n = 1, \tau_{n+1} = 1\right] + \mathbb{P}\left[\tau_n = 1, \tau_{n+1} = \infty\right] + \mathbb{P}\left[\tau_n = \infty, \tau_{n+1} = \infty\right] \\
&= \mathbb{P}\left[X < \frac{1}{n}, X < \frac{1}{n+1}\right] + \mathbb{P}\left[X < \frac{1}{n}, X \geq \frac{1}{n+1}\right] + \mathbb{P}\left[X \geq \frac{1}{n}, X \geq \frac{1}{n+1}\right] \\
&= \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+1} + 1 - \frac{1}{n} = 1.
\end{aligned}$$

(Continued on page 8.)

Finally, $\frac{1}{\tau_n} = \mathbb{I}_{\{n<\frac{1}{X}\}}$ which clearly converges to 0 almost surely (actually surely). Hence $\tau_n \to \infty$ almost surely (actually surely).

Points: Give 1p for this item. Give 0.5 for stopping time and 0.25 for monotonicity and 0.25 for limit Give 0.25p if candidate writes down definition of stopping time perfectly. Subtract 0.2 for mistakes.

(c) Prove that $\{M_{\tau_k \wedge n}\}_{n\geq 1}$ is a martingale for every $k \geq 1$. Here, $x \wedge y \triangleq \min\{x, y\}$. A process like $\{M_n\}_{n\geq 1}$ is known as local martingale.

Solution: We need to check the three properties for martingale, but first let us find a simpler expression for $M_{\tau_k \wedge n}$. Since τ_k is either 1 of ∞ then $\tau_k \wedge n$ is either 1 or n for $n \geq 1$. We have $M_{\tau_k \wedge 0} = M_0 = 0$ and $M_{\tau_k \wedge 1} = M_1 = 0$ as well. Now, for every $n \geq 2$,

$$M_{\tau_k \wedge n} = \begin{cases} M_1 & \text{if} \quad \tau_k = 1, \\ M_n & \text{if} \quad \tau_k = \infty. \end{cases} = \begin{cases} 0 & \text{if} \quad X < \frac{1}{k}, \\ M_n & \text{if} \quad X \geq \frac{1}{k}. \end{cases} ,$$

since $M_1 = 0$.

In conclusion,

$$M_{\tau_k \wedge n} = M_n \mathbb{I}_{\{X\geq \frac{1}{k}\}} = \frac{Y}{X} \mathbb{I}_{\{X\geq \frac{1}{k}\}},$$

for $n \geq 2$ which is clearly adapted since $\frac{Y}{X}$ and $\mathbb{I}_{\{X\geq\frac{1}{k}\}}$ are $\mathcal{F}_n = \sigma(X, Y)$-measurable for $n \geq 2$, i.e. $M_{\tau_k \wedge n}$ is $\mathcal{F}_n$-measurable for each $n \geq 1$.

Moreover, for $n = 1$, $M_{\tau_k \wedge n} = 0$ and hence integrable and for $n \geq 2$,

$$\mathbb{E}[|M_{\tau_k \wedge n}|] = \mathbb{E}\left[\frac{|Y|}{X}\mathbb{I}_{\{X\geq\frac{1}{k}\}}\right] = \mathbb{E}\left[\frac{1}{X}\mathbb{I}_{\{X\geq\frac{1}{k}\}}\right] = \int_{1/k}^{1} \frac{1}{x} dx = \log k < \infty$$

for every $k \geq 1$ and $n \geq 2$.

Finally, the martingale property. We have $M_{\tau_k \wedge 1} = M_1 = 0$ so we start for $n \geq 1$,

$$\mathbb{E}[M_{\tau_k \wedge (n+1)} | \mathcal{F}_n] = \mathbb{E}[M_{n+1} \mathbb{I}_{\{X\geq\frac{1}{k}\}} | \mathcal{F}_n].$$

For $n = 1$ we have

$$\mathbb{E}[M_2 \mathbb{I}_{\{X\geq\frac{1}{k}\}} | \sigma(X)] = \mathbb{E}\left[\frac{Y}{X}\mathbb{I}_{\{X\geq\frac{1}{k}\}} | \sigma(X)\right].$$

Since $\frac{1}{X}\mathbb{I}_{\{X\geq\frac{1}{k}\}}$ is $\sigma(X)$-measurable and Y is independent of X we have

$$\mathbb{E}[M_2 \mathbb{I}_{\{X\geq\frac{1}{k}\}} | \sigma(X)] = \frac{1}{X}\mathbb{I}_{\{X\geq\frac{1}{k}\}} \mathbb{E}[Y] = 0 = M_{\tau_k \wedge 1}.$$

(Continued on page 9.)

Now, for $n \geq 2$,

$$\mathbb{E}[M_{\tau_k \wedge (n+1)} | \mathcal{F}_n] = \mathbb{E}\left[\frac{Y}{X}\mathbb{I}_{\{X \geq \frac{1}{k}\}} | \sigma(X, Y)\right] = \underbrace{\frac{Y}{X}}_{=M_n} \mathbb{I}_{\{X \geq \frac{1}{k}\}} = M_{\tau_k \wedge n}$$

and hence $M_{\tau_k \wedge n}$, $n \geq 1$ is a martingale w.r.t. the given filtration.

Points: Give 1p for this item. Give 0.5p if candidate writes down definition of martingale perfectly (if not done in item (a)). Give 0.25 points for adaptedness, 0.25 for integrability and 0.5 for martingale property. Subtract 0.2 for mistakes.

GOOD LUCK!

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Trial exam in: STK3710/4710 — Probability Theory

Day of examination: 20th November 2024

Examination hours: 2:00 pm – 6:00 pm

This problem set consists of 9 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (3 points)

Let $\{X_n\}_{n=1}^{\infty}$, X be random variables defined in a probability space.

(a) (1 point) Define the concept of convergence in probability and convergence almost surely.

Solution: We say that a sequence of random variables $\{X_n\}_{n\geq 1}$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ converges *in probability* to a random variable X on the same probability space if, and only if for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left[|X_n - X| > \varepsilon\right] = 0.$$

We say that a sequence of random variables $\{X_n\}_{n\geq 1}$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ converges *almost surely* to a random variable X on the same probability space if, and only if

$$\mathbb{P}\left[\lim_{n\to\infty} X_n = X\right] = 1.$$

Points: Give 0.5p per definition. Give 0.5 for equivalent definitions. Subtract 0.2 for mistakes.

(b) (2 points) Prove that $X_n \xrightarrow{n\to\infty} X$ in probability if, and only if

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\right] = 0.$$

(Continued on page 2.)

Hint: Note that $\frac{|x|}{1+|x|} < 1$ for all x and that $f(x) = \frac{x}{1+x}$ is an increasing function for $x \geq 0$. It may also be useful to split the sample space into the events $\{|X_n - X| > \varepsilon\}$ and $\{|X_n - X| \leq \varepsilon\}$.

Solution: Note that X does not need to be integrable since $\frac{1}{1+|x|} \leq 1$ for any x and hence the expectations are always finite.

Let us prove that convergence in probability implies the given limit. We assume that for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \mathbb{P}\left[|X_n - X| > \varepsilon\right] = 0.$$

Then

$$\begin{aligned}
\mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\right] &= \mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\mathbb{I}_{\{|X_n-X|\leq\varepsilon\}}\right] + \mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\mathbb{I}_{\{|X_n-X|>\varepsilon\}}\right] \\
&\leq \varepsilon + \mathbb{E}\left[\mathbb{I}_{\{|X_n-X|>\varepsilon\}}\right] \\
&= \varepsilon + \mathbb{P}\left[|X_n - X| > \varepsilon\right],
\end{aligned}$$

where we used that $\frac{1}{1+|x|} \leq 1$ on the first term and that $\frac{|x|}{1+|x|} \leq 1$ on the second. Since the probability converges to zero by assumption it means that for every $\varepsilon > 0$ there is an n_0 such that for all $n \geq n_0$ $\mathbb{P}[|X_n - X| > \varepsilon] \leq \varepsilon$ and in particular for all $n \geq n_0$ we have

$$\mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\right] \leq 2\varepsilon,$$

hence the limit follows.

On the contrary, we have

$$\mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\right] \geq \mathbb{E}\left[\frac{|X_n - X|}{1+|X_n - X|}\mathbb{I}_{\{|X_n-X|>\varepsilon\}}\right] \geq \frac{\varepsilon}{1+\varepsilon}\mathbb{P}\left[|X_n - X| > \varepsilon\right],$$

where we used that $f(x) = \frac{x}{1+x}$ is increasing for $x \geq 0$ and hence $f(|X_n - X|) \geq f(\varepsilon)$. Then since the left-hand side converges to zero and the right-hand side is positive. We have convergence to zero and hence convergence in probability.

Points: Give 1p for each implication. Within each implication, give 0.5 if there is a meaningful attempt and 0.25 if Markov inequality is mentioned. Subtract 0.2 for mistakes.

Problem 2 Random variables (2 points)

The Pareto distribution is an absolutely continuous probability distribution characterized by having the following density function,

$$f(x) = \frac{\alpha\lambda^\alpha}{x^{\alpha+1}}, \quad x \geq \lambda,$$

(Continued on page 3.)

for positive real parameters α, λ and $\alpha > 1$ to guarantee finite expectation. Note that the distribution function is given by $F(x) = 1 - \left(\frac{\lambda}{x}\right)^{\alpha}, x \geq \lambda$.

Consider $n \geq 1$ independent random variables $X_1, \ldots, X_n$ where X_i is Pareto-distributed with parameters $\lambda > 0$ and $\alpha_i > 1, i = 1, \ldots, n$.

Define the the random variables $Y_n = \min\{X_1, \ldots, X_n\}, n \geq 1$.

(a) Find the distribution of Y_n and its expectation.

Solution: Note that $Y_{(n)}$ takes values in $[\lambda, \infty)$. Let $y \geq \lambda$, then

$$\mathbb{P}[Y_n > y] = \mathbb{P}[Y_1 > y, \ldots, Y_n] = \prod_{i=1}^{n} \mathbb{P}[Y_i > y] = \prod_{i=1}^{n} \left(\frac{\lambda}{y}\right)^{\alpha_i} = \left(\frac{\lambda}{y}\right)^{\sum_{i=1}^{n} \alpha_i}$$

hence $Y_{(n)}$ is Pareto-distributed with parameters λ and $\sum_{i=1}^{n} \alpha_i$.
The mean is given by

$$\mathbb{E}[Y_{(n)}] = \int_{\lambda}^{\infty} x d f_n(x),$$

where $f_n(x) \triangleq \frac{a_n \lambda^{a_n}}{x^{a_n+1}}$ and $a_n = \sum_{i=1}^{n} \alpha_i$. Thus we have

$$\mathbb{E}[Y_{(n)}] = a_n \lambda^{a_n} \int_{\lambda}^{\infty} x^{-a_n} dx = \frac{\lambda a_n}{a_n - 1}.$$

Points: Give 0.5 for the distribution and 0.5 for the mean. Give 0.25 for finding the tail distribution. Give 0.1 for writing down an expression for the expectation. Subtract 0.2 for mistakes.

(b) Give a sufficient condition on the sequence of parameters $\{\alpha_i\}_{i=1}^{n}$ to guarantee that $\{Y_n\}_{n=1}^{\infty}$ converges almost surely to λ. Does the sequence converge in L^1 under your condition? Why or why not.

Solution: We try to apply Borel-Cantelli's first lemma to show that the sequence converges almost surely to λ. Define the events $A_n = \{|Y_{(n)} - \lambda| > \varepsilon\}$ for an arbitrarily small $\varepsilon > 0$. Denote by $a_n = \sum_{i=1}^{n} \alpha_i$. Then

$$\begin{aligned}
\mathbb{P}[A_n] &= \mathbb{P}[|Y_{(n)} - \lambda| > \varepsilon] \\
&= \mathbb{P}[Y_{(n)} > \lambda + \varepsilon] \\
&= \left(\frac{\lambda}{\lambda + \varepsilon}\right)^{a_n}.
\end{aligned}$$

A general sufficient condition would be

$$\sum_{n=1}^{\infty} r^{a_n} < \infty \tag{1}$$

(Continued on page 4.)

for $|r| < 1$ so that we have

$$\mathbb{P}[\limsup_n A_n] = 0$$

and hence $\{Y_{(n)}\}_{n=1}^{\infty}$ converges to λ a.s. An example is simply taking $\alpha_i = \alpha$ constant, then $a_n = \alpha n$.

Yes, the sequence converges in L^1 trivially since $Y_{(n)} \geq \lambda$ hence

$$\lim_n \mathbb{E}[|Y_{(n)} - \lambda|] = \lim_n \mathbb{E}[Y_{(n)}] - \lambda = \lim_n \frac{\lambda a_n}{a_n - 1} - \lambda = 0$$

provided $\lim_n a_n = \infty$, which is the case if condition (1) holds.

Points: Give 0.5 for mentioning and trying to apply the right Borel-Cantelli lemma. Give 0.5 for the L^1-convergence. Subtract 0.2 for mistakes.

Problem 3 Conditional expectation (3 points)

We consider an insurance portfolio in which there is a large number n of policyholders. We model the number of the insurance claims of the policyholders by i.i.d. random variables $X_1, \dots, X_n$. Thus X_i is the number of claims of policyholder i. Let $p_k \triangleq \mathbb{P}[X_i = k]$ for all $i = 1, \dots, n$ and denote by $m = \mathbb{E}[X_i] = \sum_{k\geq 1} k p_k < \infty$ and $s^2 = \mathbb{E}[X_i^2] - m^2 = \sum_{k\geq 1} k^2 p_k - m^2 < \infty$ the expectation and variance.

We pick an insurance claim uniformly at random. This claim belongs to some policyholder among these n. Denote by Z_n be the number of claims of such policyholder.

(a) Justify that for every integer $k \geq 1$ we have

$$\mathbb{P}[Z_n = k | X_1, \dots, X_n] = \frac{k \sum_{i=1}^n \mathbb{I}_{\{X_i = k\}}}{\sum_{i=1}^n X_i}.$$

Solution: The portfolio has $X_1 + \cdots + X_n$ claims in total. Given a total number of claims $X_1 + \cdots + X_n$ we choose at random one. Then we want to see how many among all the n have exactly k claims. This number is $\mathbb{I}_{\{X_1 = k\}} + \cdots + \mathbb{I}_{\{X_n = k\}}$. Then the total number of claims among all policyholders with exactly k claims is $k \sum_{i=1} \mathbb{I}_{\{X_i = k\}}$. Hence, the probability that we observe k claims in a randomly chosen claim from a policyholder among the n is given by

$$\mathbb{P}[Z_n = k | X_1, \dots, X_n] = \frac{k \sum_{i=1}^n \mathbb{I}_{\{X_i = k\}}}{\sum_{i=1}^n X_i}.$$

Points: Give 1p for this item. Give 0.5 for some meaningful justification. Subtract 0.2 for mistakes.

(Continued on page 5.)

(b) Show that $\{Z_n\}_{n=1}^{\infty}$ converges in distribution to a random variable Z_∞ with probability mass function given by $\mathbb{P}[Z_\infty = k] = \frac{kp_k}{m}$, $k \geq 1$.

Solution: By dividing by n we have

$$\mathbb{P}[Z_n = k | X_1, \ldots, X_n] = \frac{k \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{X_i = k\}}}{\frac{1}{n} \sum_{i=1}^{n} X_i}.$$

Since X_i are independent and identically distributed, we have, by the strong law of large numbers, that

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{I}_{\{X_i = k\}} \to \mathbb{E}[\mathbb{I}_{\{X_1 = k\}}] = \mathbb{P}[X_1 = k] = p_k$$

and

$$\frac{1}{n} \sum_{i=1}^{n} X_i \to \mathbb{E}[X_1] = m.$$

Hence,

$$\mathbb{P}[Z_n = k | X_1, \ldots, X_n] \to \frac{kp_k}{m}.$$

Now,

$$\mathbb{P}[Z_n = k] = \mathbb{E}[\mathbb{P}[Z_n = k | X_1, \ldots, X_n]].$$

Since $\mathbb{P}[Z_n = k | X_1, \ldots, X_n]$ is a.s. bounded (by one) we have by Lebesgue's dominated convergence theorem that

$$\mathbb{P}[Z_n = k] = \mathbb{E}[\mathbb{P}[Z_n = k | X_1, \ldots, X_n]] \to \frac{kp_k}{m}$$

almost surely.

Points: Give 0.5 for detecting strong law of large numbers. Give 0.25 for mentioning dominated convergence (or subtract for forgetting). Subtract 0.2 for mistakes.

(c) Show that

$$\mathbb{E}[Z_n] \xrightarrow{n \to \infty} \frac{s^2 + m^2}{m}.$$

Solution:

Z_n converges in distribution to Z_∞ with the given distribution. Convergence in distribution does not imply convergence of the means to the limiting mean. In other words, $Z_n \to Z_\infty$ in distribution does not imply $\mathbb{E}[Z_n] \to \mathbb{E}[Z_\infty]$. Example: $\mathbb{P}[X_n = n] = \frac{1}{n}$ and $\mathbb{P}[X_n = 0] = 1 - \frac{1}{n}$. Then $X_n \to 0$ in distribution but $\mathbb{E}[X_n] = 1$ which is not $\mathbb{E}[X_\infty] = 0$. But in this case it is true! There are two ways we can expect convergence of the means when having «only» convergence in distribution. One is having convergence in mean, i.e. $\mathbb{E}[|Z_n - Z_\infty] \to 0$ which is stronger than convergence in probability

(Continued on page 6.)

and distribution and another one is having uniform integrability. Indeed,

$$\lim_{C\to\infty} \sup_n \mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}] = 0.$$

Note that

$$\begin{aligned}\mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}] &= \mathbb{E}\left[k\mathbb{I}_{\{k\geq C\}}\frac{k\sum_{i=1}^n \mathbb{I}_{\{X_i=k\}}}{\sum_{i=1}^n X_i}\right] \\ &= \mathbb{E}\left[\frac{\sum_{i=1}^n X_i^2\mathbb{I}_{\{X_i\geq C\}}}{\sum_{i=1}^n X_i}\right].\end{aligned}$$

By the strong law of large numbers we have

$$\frac{\frac{1}{n}\sum_{i=1}^n X_i^2\mathbb{I}_{\{X_i\geq C\}}}{\frac{1}{n}\sum_{i=1}^n X_i} \to \frac{\mathbb{E}[X_1^2\mathbb{I}_{\{X_1\geq C\}}]}{m}.$$

Since $\mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}]$ is a convergent sequence (strong law of large numbers) then it is bounded. Thus

$$\sup_n \mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}] < \infty.$$

Now, without loss of generality assume that C is integer, then

$$\mathbb{E}[X_1^2\mathbb{I}_{\{X_1\geq C\}}] = \sum_{k=C}^{\infty} k^2 p_k.$$

The last term is the tail of a convergent series, so it must converge to 0. In a summary,

$$\lim_{C\to\infty} \sup_n \mathbb{E}[|Z_n|\mathbb{I}_{\{|Z_n|\geq C\}}] = 0.$$

Therefore,

$$\mathbb{E}[Z_n] \to \mathbb{E}[Z_\infty] = \sum_k k\frac{kp_k}{m} = \frac{s^2+m^2}{m}.$$

Observation: Note that a sufficient condition for $\{Z_n\}_n$ to be uniformly integrable is by Theorem 8.7.3 from the lecture notes, that $\sup_n \mathbb{E}[|Z_n|^p] < \infty$ for some $p > 1$. But

$$\mathbb{E}[|Z_n|^p] = \mathbb{E}\left[\frac{\sum_{i=1}^n X_i^{p+1}}{\sum_{i=1}^n X_i}\right]$$

and even for p very close to 1, we have $1+p > 2$ and we have not assumed that X_i has higher order moments than 2. Hence, this

(Continued on page 7.)

theorem fails and one needs to prove uniform integrability from the very definition! (Definition 8.7.1).

Finally, if you want, you can try to show that Z_n actually also converges to the limit $L = \frac{s^2+m^2}{m}$ in L^1. It is a nice exercise. Hint: you need to combine the law of total expectation as we have done several times, then get rid of the absolute value $|Z_n - L|$, by considering the cases that $Z_n > L$ and $Z_n < L$ and finally apply the strong law of large numbers.

Points: Give 0.2 for noting the law of total expectation. Give 0.5 for computing the conditional expectation. Give 0.25 for the law of large numbers and 0.25 for Lebesgue's dominated convergence theorem. Give consideration or 0.25 if the candidate notes that uniform integrability is useful in this context. Subtract 0.2 for mistakes.

Problem 4 Martingales (2 points)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent and identically distributed random variables with $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[X_1^2] = \sigma^2 < \infty$. Show that the sequence

$$M_n \triangleq \left(\sum_{i=1}^{n} X_i\right)^2 - \sigma^2 n, \quad n \geq 1,$$

is a martingale with respect to the filtration generated by $X_1, \dots, X_n$, $n \geq 1$. Does $\left\{\frac{M_n}{n^2}\right\}_{n=1}^{\infty}$ converge in L^1 and almost surely? To what limit and why.

Solution: To show that $\{M_n\}_{n=1}^{\infty}$ is a martingale, we need to verify the three conditions:

1. $\{M_n\}_{n=1}^{\infty}$ is adapted.
2. M_n is integrable for all n.
3. the martingale property: for all $m \geq n \geq 1$ we have

$$\mathbb{E}[M_m|\mathcal{F}_n] = M_n.$$

Property one is trivial in this exercise since M_n is constructed as sums and powers of $X_1, \dots, X_n$ and hence M_n is measurable w.r.t. $X_1, \dots, X_n$ for each $n \geq 1$ and hence adapted.

Property two:

$$\mathbb{E}\left[\left|\left(\sum_{i=1}^{n} X_i\right)^2 - \sigma^2 n\right|\right] \leq C \sum_{i=1}^{n} \mathbb{E}\left[|X_i|^2\right] + C\sigma^2 n = 2Cn\sigma^2 < \infty$$

(Continued on page 8.)

for some constant C (independent of n but it is not so important).

Finally, the martingale property. It is enough to check for $m = n+1$ since we have $\mathcal{F}_n \subseteq \mathcal{F}_{m-1} \subseteq \mathcal{F}_m$ for $m \geq n+1$ and by the tower property we have

$$\mathbb{E}[M_m|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[M_m|\mathcal{F}_{m-1}]|\mathcal{F}_n] = \mathbb{E}[M_{m-1}|\mathcal{F}_n].$$

Now,

$$\begin{aligned}\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n+1} X_i\right)^2 - \sigma^2(n+1)|\mathcal{F}_n\right] \\ &= \mathbb{E}\left[\left(\sum_{i=1}^{n} X_i + X_{n+1}\right)^2 |\mathcal{F}_n\right] - \sigma^2(n+1).\end{aligned}$$

Now, $\sum_{i=1}^n X_i$ is $\mathcal{F}_n$-measurable and X_{n+1} is independent of $\mathcal{F}_n$. Hence, we can treat $\sum_{i=1}^n X_i$ as a constant and apply expectation on X_{n+1}. To do so and not get confused by what is the operator $\mathbb{E}$ applied on, we can write:

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i + X_{n+1}\right)^2 |\mathcal{F}_n\right] = \mathbb{E}\left[(y + X_{n+1})^2\right]\Big|_{y=\sum_{i=1}^n X_i}.$$

Now,

$$\mathbb{E}\left[(y + X_{n+1})^2\right] = y^2 + 2y\mathbb{E}[X_{n+1}] + \mathbb{E}[X_{n+1}^2] = y^2 + \sigma^2.$$

Back to the main equation, we arrive to

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} X_i + X_{n+1}\right)^2 |\mathcal{F}_n\right] - \sigma^2(n+1) = \left(\sum_{i=1}^{n} X_i\right)^2 + \sigma^2 - \sigma^2(n+1) = M_n.$$

as we wanted to show.

We have,

$$\frac{M_n}{n^2} = \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 - \frac{\sigma^2}{n}.$$

Since by the strong law of large numbers $\frac{1}{n}\sum_{i=1}^n X_i \to \mathbb{E}[X_1]$ and $\mathbb{E}[X_1] = 0$ then we have

$$\frac{M_n}{n^2} \xrightarrow{a.s.} 0$$

as n goes to infinity.

Almost sure convergence does not imply L^1-convergence, but we may try to check L^1-convergence directly. Indeed,

$$\mathbb{E}\left[\left|\frac{M_n}{n^2}\right|\right] = \mathbb{E}\left[\left|\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 - \frac{\sigma^2}{n}\right|\right].$$

(Continued on page 9.)

Furthermore, we have

$$\mathbb{E}\left[\left|\frac{M_n}{n^2}\right|\right] \leq \mathbb{E}\left[\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i\right)^2\right] + \frac{\sigma^2}{n} = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}[X_i^2] + \frac{\sigma^2}{n},$$

since $\mathbb{E}[X_i X_j] = 0$, $i \neq j$, since they are independent with 0 mean. Hence,

$$\mathbb{E}\left[\left|\frac{M_n}{n^2}\right|\right] \leq \frac{\mathbb{E}[X_1^2]}{n} + \frac{\sigma^2}{n},$$

which goes to 0 as $n \to \infty$.

One can also prove the L^1-convergence here by using the second martingale convergence theorem on the submartingale M_n/n^2, since $\frac{1}{n}\sum_{i=1}^{n} X_i$ is uniformly integrable.

Points: Give 1 for showing the martingale part and 1p for the rest. Give 0.5 for a.s. convergence and 0.5 for L^1-convergence. Give 0.25 for mentioning strong law of large numbers. Give 0.25 for dominated convergence. Subtract 0.2 for mistakes.

GOOD LUCK!

(Continued on page 10.)

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Deferred exam in:	STK-MAT3710/4710 — Probability Theory
Day of examination:	18th of January 2024
Examination hours:	15:00 – 19:00
This problem set consists of 8 pages.	
Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Grading: The total score is 10 points. The grading scale is F [0,4), E [4,5), D [5, 6), C [6, 7), B [7, 8.5), A [8.5,10].

Problem 1 Theory (2 points)

Let X be an integrable random variable defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F}, \mathcal{G}$ sub-σ-algebras such that $\mathcal{G} \subseteq \mathcal{F} \subseteq \mathcal{A}$.

(a) Prove that if $\mathcal{F}$ is independent of $\sigma(X)$, then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

Solution: Let $Z = \mathbb{E}[X|\mathcal{F}]$, then Z is the $\mathbb{P}$-a.s. unique $\mathcal{F}$-measurable and integrable random variable such that

$$\int_F Z d\mathbb{P} = \int_F X d\mathbb{P}, \quad F \in \mathcal{F}.$$

Observe that $Z = \mathbb{E}[X]$ satisfies the requirement. It is a constant, so it is integrable and obviously $\mathcal{F}$-measurable. Finally,

$$\int_F \mathbb{E}[X] d\mathbb{P} = \mathbb{E}[X]\mathbb{P}[F] = \mathbb{E}[\mathbb{I}_F X] = \int_F X d\mathbb{P}$$

for all $F \in \mathcal{F}$. Hence, $Z = \mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

Points: Give 0.5 if the rigorous definition of conditional expectation is given. Give 0.75 if the exercise is solved for random variables. Subtract 0.2 per mistake and 0.5 per serious mistake.

(Continued on page 2.)

(b) Prove that $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}]$.

Solution: Let $Z = \mathbb{E}[X|\mathcal{F}]$, then $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]|\mathcal{G}] = \mathbb{E}[Z|\mathcal{G}]$ is the object such that

$$\int_G \mathbb{E}[Z|\mathcal{G}]d\mathbb{P} = \int_G Zd\mathbb{P}$$

for all $G \in \mathcal{G}$. But, now $Z = \mathbb{E}[X|\mathcal{F}]$ is the object that satisfies

$$\int_F Zd\mathbb{P} = \int_F Xd\mathbb{P}$$

for al $F \in \mathcal{F}$, in particular for $F \in \mathcal{G}$ since $\mathcal{G} \subseteq \mathcal{F}$. Hence,

$$\int_G Zd\mathbb{P} = \int_G Xd\mathbb{P}$$

for all $G \in \mathcal{G}$. In a summary, we have proven

$$\int_G \mathbb{E}[Z|\mathcal{G}]d\mathbb{P} = \int_G Xd\mathbb{P}$$

for all $G \in \mathcal{G}$, so

$$\mathbb{E}[Z|\mathcal{G}] = \mathbb{E}[X|\mathcal{G}]$$

as we wanted to show.

Points: Give 0.5 if the first step is conducted. Give 0.5 for the rest. Subtract 0.2 per mistake and 0.5 per serious mistake.

Problem 2 Random variables (3 points)

Consider the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ and Y on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ given by

$$X_n = \frac{(-1)^n Y}{n}, \quad n \geq 1,$$

where Y is exponentially distributed with parameter $\lambda > 0$, i.e. Y is absolutely continuous with density function $f_Y(y) = \lambda e^{-\lambda y}, y > 0$.

(a) Show that X_n, $n \geq 1$ are absolutely continuous with density functions given by

$$f_{X_n}(x) = \lambda n e^{-\lambda n x}, \quad x > 0,$$

if n is even and

$$f_{X_n}(x) = \lambda n e^{\lambda n x}, \quad x < 0,$$

if n is odd.

(Continued on page 3.)

Solution: Let us assume that n is even, then $X_n = \frac{Y}{n}$. Since $Y \in (0, \infty)$ so does $X_n = \frac{Y}{n}$ as well. Let $x \in (0, \infty)$, the distribution function of X_n is given by

$$\mathbb{P}[X_n \leq x] = \mathbb{P}\left[\frac{Y}{n} \leq x\right] = \mathbb{P}[Y \leq nx] = 1 - e^{-\lambda nx}.$$

We see that the distribution function is differentiable, hence the density is given by $f_{X_n}(x) = \lambda n e^{-\lambda nx}$, $x > 0$.
Now if n is odd then $X_n = \frac{-Y}{n}$. Since $Y \in (0, \infty)$ then $X_n = \frac{-Y}{n} \in (-\infty, 0)$. Let $x \in (-\infty, 0)$, the distribution function of X_n now is given by

$$\mathbb{P}[X_n \leq x] = \mathbb{P}\left[\frac{-Y}{n} \leq x\right] = \mathbb{P}[Y \geq -nx] = e^{\lambda nx}.$$

We see that the distribution function is differentiable, hence the density is given by $f_{X_n}(x) = \lambda n e^{\lambda nx}$, $x < 0$.

Points: Give 0.5 per each and 0.2 for an attempt to compute the distribution. Subtract 0.2 per mistake and 0.5 per serious mistake.

(b) Discuss the convergence of the sequence $\{X_n\}_{n=1}^{\infty}$ almost surely, in L^p for any $p \geq 1$, in probability and in distribution.

Solution: Let us try to show a.s. convergence, then we will automatically get convergence in probability and thereby convergence in distribution. From the distribution functions above we can directly see convergence to 0, in distribution. Hence, if there is a limit, it will be 0. Let us define the events $A_n = \{|X_n| > \varepsilon\}$ for an arbitrarily small $\varepsilon > 0$. Then

$$\mathbb{P}[A_n] = \mathbb{P}[|X_n| > \varepsilon] = \mathbb{P}[Y > n\varepsilon] = e^{-\lambda n\varepsilon}.$$

Now,

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \sum_{n=1}^{\infty} e^{-\lambda n\varepsilon} = \sum_{n=1}^{\infty} \left(e^{-\lambda\varepsilon}\right)^n < \infty$$

since $e^{-\lambda\varepsilon} < 1$ and we have a geometric series. Hence, by Borel-Cantelli's first lemma

$$\mathbb{P}[\limsup_n A_n] = 0$$

meaning that the events $\{|X_n| > \varepsilon\}$ do not happen infinitely often for any arbitrarily small $\varepsilon > 0$ and hence X_n converges almost surely to 0.
Another way to prove a.s. convergence is by direct definition:

$$\mathbb{P}\left[\lim_n X_n(\omega) = 0\right] = \mathbb{P}\left[Y(\omega) \lim_n \frac{(-1)^n}{n} = 0\right] = \mathbb{P}\left[\lim_n \frac{(-1)^n}{n} = 0\right] = 1.$$

(Continued on page 4.)

Almost sure convergence implies convergence in probability (which we also see in the fact that $\lim_n \mathbb{P}[A_n] = 0$) and convergence in distribution.
Finally, since

$$|X_n|^p \leq \frac{Y^p}{n^p} \leq Y^p$$

and $\mathbb{E}[Y^p] < \infty$. By Lebesgue's dominated convergence theorem

$$\lim_n \mathbb{E}[|X_n|^p] = \mathbb{E}[|\lim_n X_n|^p] = 0,$$

where we used the continuity of the function $|\cdot|^p$ and that $X_n \to 0$ almost surely.

Points: Give 0.2 per each type of convergence and 0.1 per just defining each type. Give 0.2 if the right Borel-Cantelli lemma is stated correctly. Subtract 0.2 per mistake and 0.5 per serious mistake.

(c) Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} a_n < \infty$. We say that X_n and X_{n+1} are *friends* if their distance is at most a_n. Prove that the infinite sequence $X_1(\omega), X_2(\omega), \ldots$ has a finite number of friends.

Solution: We need to look at the occurrence of the events $B_n = \{|X_{n+1} - X_n| \leq a_n\}$. Since B_n, $n \geq 1$ are not independent we can again hope for Borel-Cantelli's first lemma. Observe that

$$|X_{n+1} - X_n| = \frac{2n+1}{n(n+1)} Y.$$

$$\mathbb{P}[A_n] = \mathbb{P}\left[\frac{2n+1}{n(n+1)} Y \leq a_n\right] = 1 - e^{-\lambda \frac{n(n+1)a_n}{2n+1}}.$$

Now, using that $1 - e^{-x} \leq x$ we get

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] \leq \lambda \sum_{n=1}^{\infty} \frac{n(n+1)a_n}{2n+1} < \infty,$$

by assumption of a_n. Then, by the first Borel-Cantelli lemma the events A_n do not happen infinitely often. Hence, we conclude that we will only observe a finite number of friends.

Points: Give 0.2 if the first Borel-Cantelli lemma is stated correctly. Subtract 0.2 per mistake and 0.5 per serious mistake.

Problem 3 Conditional expectation (3 points)

Let X be uniformly distributed on $[0,1]$, i.e. X is absolutely continuous with density $f_X(x) = 1$, $x \in [0,1]$. Moreover, let N be geometrically

(Continued on page 5.)

distributed on the set of natural numbers $\{1, 2, \dots\}$ with parameter $p \in (0, 1)$, i.e. N has probability mass function $\mathbb{P}[N = k] = (1-p)^{k-1}p$, $k = 1, 2, \dots$. Assume that N and X are independent and define the random variable $Y = X^{1/N}$.

(a) Show that Y is absolutely continuous with density function given by

$$f_Y(y) = \frac{p}{(1-(1-p)y)^2}, \quad y \in [0,1]$$

and that

$$\mathbb{E}[Y] = \frac{1}{1-p} + \frac{p}{(1-p)^2}\log p.$$

Solution: Since X is uniform then $X \in (0,1)$ and since $N \geq 1$, then $X^{1/N} \in (0,1)$ as well. Let $y \in (0,1)$, conditioning on N, by the law of total probability, we have

$$\begin{aligned}
F_Y(y) &= \sum_{k=1}^{\infty} \mathbb{P}[X^{1/N} \leq y | N = k]\mathbb{P}[N = k] \\
&= \sum_{k=1}^{\infty} \mathbb{P}[X^{1/k} \leq y](1-p)^{k-1}p \\
&= p\sum_{k=1}^{\infty} \mathbb{P}[X \leq y^k](1-p)^{k-1} \\
&= \frac{p}{1-p}\sum_{k=1}^{\infty}((1-p)y)^k \\
&= p\frac{y}{1-(1-p)y}.
\end{aligned}$$

We can see that the distribution function is differentiable in $y \in (0,1)$ hence the density function is given by

$$f_Y(y) = \frac{d}{dy}\mathbb{P}[Y \leq y] = \frac{p}{(1-(1-p)y)^2}, \quad y \in (0,1).$$

Now, the expectation can be computed by using the density function

$$\mathbb{E}[Y] = \int_0^1 \frac{py}{(1-(1-p)y)^2}dy.$$

We use integration-by-parts, by choosing $u = py$ and $dv = (1-(1-$

(Continued on page 6.)

$p)y)^{-2}dy$ then $du = pdy$ and $v = \frac{1}{1-p}\frac{1}{1-(1-p)y}$. Therefore,

$$\begin{aligned}
\mathbb{E}[Y] &= \int_0^1 \frac{py}{(1-(1-p)y)^2}dy \\
&= \frac{py}{1-p}\frac{1}{1-(1-p)y}\Big|_0^1 - \frac{p}{1-p}\int_0^1 \frac{dy}{1-(1-p)y} \\
&= \frac{1}{1-p} + \frac{p}{(1-p)^2}\log(1-(1-p)y)\Big|_0^1 \\
&= \frac{1}{1-p} + \frac{p}{(1-p)^2}\log p.
\end{aligned}$$

Points: Give 0.5 for density and 0.5 for expectation. Give 0.2 for noticing that one has to condition on N. Subtract 0.2 per mistake and 0.5 per serious mistake.

(b) Compute $\mathbb{E}[Y|\sigma(N)]$ and use it to show that

$$\sum_{k=1}^{\infty} \frac{k}{k+1}\frac{1}{2^k} = 2\,(1-\log 2)\,.$$

Solution: Since N is discrete, let us just fix an event $N = k$ and then we leave it as a function of N. We have

$$\mathbb{E}[Y|N=k] = \mathbb{E}[X^{1/N}|N=k] = \mathbb{E}[X^{1/k}] = \int_0^1 x^{1/k}dx = \frac{k}{k+1}.$$

Hence,

$$\mathbb{E}[Y|\sigma(N)] = \frac{N}{N+1}.$$

Note that it is not easy to compute $\mathbb{E}\left[\frac{N}{N+1}\right]$ since, by the law of the unconscious statistician, it is given by

$$\mathbb{E}\left[\frac{N}{N+1}\right] = \sum_{k=1}^{\infty}\frac{k}{k+1}(1-p)^{k-1}p.$$

But, we know that

$$\mathbb{E}\left[\frac{N+1}{N}\right] = \mathbb{E}\left[\mathbb{E}[Y|\sigma(N)]\right] = \mathbb{E}[Y],$$

by the tower property of the conditional expectation and we have computed $\mathbb{E}[Y]$. Hence,

$$\mathbb{E}\left[\frac{N}{N+1}\right] = \sum_{k=1}^{\infty}\frac{k}{k+1}(1-p)^{k-1}p = \frac{1}{1-p} + \frac{p}{(1-p)^2}\log p.$$

So,

$$\sum_{k=1}^{\infty}\frac{k}{k+1}(1-p)^{k-1} = \frac{1}{p(1-p)} + \frac{1}{(1-p)^2}\log p.$$

(Continued on page 7.)

Finally,

$$\sum_{k=1}^{\infty} \frac{k}{k+1}(1-p)^k = \frac{1}{p} + \frac{1}{1-p}\log p.$$

Plugging $p = \frac{1}{2}$ leads to the desired result.

Points: Give 0.5 for computing the conditional expectation and 0.25 for mentioning or using the tower property. Give 0.2 for connecting that to the given formula. Subtract 0.2 per mistake and 0.5 per serious mistake.

(c) Find the distribution of (N, Y).

Solution: We need to find $\mathbb{P}[N = k, Y \leq y]$ for $k = 1, 2, \ldots$ and $y \in (0, 1)$. It is obvious that N and Y are not independent.

$$\begin{aligned}
\mathbb{P}[N = k, Y \leq y] &= \mathbb{P}[N = k, X^{1/N} \leq y] \\
&= \mathbb{P}[N = k, X \leq y^N] \\
&= \mathbb{P}[X \leq y^N | N = k]\mathbb{P}[N = k] \\
&= \mathbb{P}[X \leq y^k]\mathbb{P}[N = k] \\
&= y^k(1-p)^{k-1}p.
\end{aligned}$$

Points: Give 0.5 for computing the conditional distribution. Subtract 0.2 per mistake and 0.5 per serious mistake.

Problem 4 Martingales (2 points)

Let $\{X_n\}_{n=0}^{\infty}$ a time-homogeneous Markov chain on a finite state space S, i.e. the Markov property holds a.s. $\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] = \mathbb{E}[f(X_{n+1})|\sigma(X_n)]$ for all $n \geq 0$ and the transition probabilities $p_{ij} \triangleq \mathbb{P}[X_{n+1} = j | X_n = i]$, $i, j \in S$ do not depend on n. Let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, $n \geq 0$.

For any function $f : S \to \mathbb{R}$ such that $\mathbb{E}[|f(X_n)|] < \infty$, define the function $Pf : S \to \mathbb{R}$

$$Pf(i) = \mathbb{E}[f(X_{n+1})|X_n = i], \quad i \in S.$$

(a) Prove that $M_n \triangleq f(X_n) - \sum_{k=0}^{n-1}(Pf(X_k) - f(X_k))$, $n \geq 1$ is a martingale with respect to the given filtration.

Solution: We need to check for the three conditions: adaptedness, integrability and the martingale property.
Adaptedness: M_n is a function of $X_0, \ldots, X_n$ hence M_n is $\mathcal{F}_n$-measurable.
Integrability: $\mathbb{E}[|M_n|] \leq C\mathbb{E}[|f(X_0)|] < \infty$ where we used the tower property and the fact that $\mathbb{E}[|f(X_n)|] = \mathbb{E}[|f(X_0)|]$ since the chain is

(Continued on page 8.)

homogeneous.
Martingale property: First observe that

$$
\begin{aligned}
M_{n+1} &= f(X_{n+1}) - Pf(X_n) + f(X_n) - \sum_{k=0}^{n-1} (Pf(X_k) - f(X_k)) \\
&= f(X_{n+1}) - Pf(X_n) + M_n.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[f(X_{n+1}) - Pf(X_n) + M_n|\mathcal{F}_n\right] \\
&= \mathbb{E}\left[f(X_{n+1}) - Pf(X_n)|\mathcal{F}_n\right] + M_n \\
&= \mathbb{E}\left[f(X_{n+1})|\mathcal{F}_n\right] - Pf(X_n) + M_n
\end{aligned}
$$

Now, we need to show that $\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] - Pf(X_n) = 0$. Indeed, by the Markov property we have

$$
\mathbb{E}[f(X_{n+1})|\mathcal{F}_n] = \mathbb{E}[f(X_{n+1})|X_n] = Pf(X_n)
$$

and the result follows.

Points: Give 0.2 for adaptedness, 0.2 for integrability and 0.6 for the martingale property. Give 0.2 for defining only the concept of martingale. Subtract 0.2 per mistake and 0.5 per serious mistake.

(b) Show that for every $\alpha > 0$, $\mathbb{P}[M_n^* \geq \alpha] \leq \frac{(2n+1)c}{\alpha}$ where $M_n^* = \sup_{k=1,\ldots,n} |M_k|$ and $c = \mathbb{E}[|f(X_0)|]$.

Solution: We simply use Doob's maximal inequality:

$$
\mathbb{P}[M_n^* \geq \alpha] \leq \frac{\mathbb{E}[|M_n|]}{\alpha}.
$$

Now, it remains to estimate $\mathbb{E}[|M_n|]$. Using the triangle inequality, Jensen's inequality (for conditional expectations) and the tower property we have

$$
\mathbb{E}[|M_n|] \leq \mathbb{E}[|f(X_n)|] + \sum_{k=0}^{n-1} \mathbb{E}[|f(X_k)|] + \sum_{k=0}^{n-1} \mathbb{E}[|f(X_k)|]
$$

and since $\mathbb{E}[|f(X_k)|] \leq \sup_k \mathbb{E}[|f(X_k)|] = c$ for all k we have

$$
\mathbb{E}[|M_n|] \leq c + nc + nc = (2n+1)c
$$

and the result follows.

Points: Give 0.5 for stating Doob's maximal inequality. Give 0.5 for estimating the expectation. Give 0.2 for mentioning Jensen and 0.2 for tower property. Subtract 0.2 per mistake and 0.5 per serious mistake.

GOOD LUCK!

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Constituent exam in:	STK-MAT3710/4710 — Probability Theory
Day of examination:	8th December 2023
Examination hours:	9:00 am – 1:00 pm

This problem set consists of 9 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more applied. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Grading: The total score is 10 points. The grading scale is F [0,4), E [4,5), D [5, 6), C [6, 7), B [7, 8.5), A [8.5,10].

Problem 1 Theory (2 points)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, X a random variable on it and $\mathcal{F} \subseteq \mathcal{A}$ a sub-σ-algebra.

(a) Define the concept of expectation of a random variable X from the measure-theoretical definition and outline the steps to construct it.

Solution: The expectation of a random variable X is defined as the integral of the measurable function X w.r.t. the probability measure $\mathbb{P}$, that is

$$\mathbb{E}[X] = \int_\Omega X(\omega) d\mathbb{P}(\omega).$$

The above quantity is constructed as follows: first you assume that X is a simple random variable, i.e. for events in the σ-algebra $\{A_i\}_{i=1}^n \subset \mathcal{A}$ and non-negative real numbers $\{a_i\}_{i=1}^n \subset [0, \infty)$ we have

$$X(\omega) = \sum_{i=1}^n a_i \mathbb{I}_{A_i}(\omega).$$

Then we define the integral as

$$\mathbb{E}[X] = \sum_{i=1}^n a_i \mathbb{P}[A_i].$$

(Continued on page 2.)

For any general non-negative random variable X, we have a result saying that there exists a increasing sequence of simple random variables $\{X_n\}_{n=1}^{\infty}$ such that $X_n \to X$ a.s. (or pointwise), then we define the integral of X as

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n].$$

Finally, for any general random variable X, we split X as $X = X^+ - X^-$ where $X^+ = \max\{X, 0\}$ and $X^- = -\min\{X, 0\}$ the positive and negative parts which are non-negative and define

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-].$$

We say that X is integrable if, and only if

$$\mathbb{E}[|X|] = \mathbb{E}[X^+] + \mathbb{E}[X^-] < \infty.$$

Points: Give 0.25p if the expectation is defined for simple r.v.'s and 0.5p for non-negative r.v.'s via limit. Give the left 0.25p for a general r.v. splitting into positive and negative part. Give consideration if integrability of X is discussed or if, in addition, the connection to the push-forward measure is discussed. Give only 0.5p if the expectation is defined via the distribution/density functions. Subtract 0.2 per mistake and 0.5 per serious mistake.

(b) Assuming that X is integrable, define the concept of conditional expectation of X given $\mathcal{F}$ through the theoretical definition.

Solution: We define the *conditional expectation* of X, given $\mathcal{F}$ as the $\mathbb{P}$-a.s. unique $\mathbb{P}$-integrable and $\mathcal{F}$-measurable random variable Z such that

$$\int_F Z(\omega) d\mathbb{P}(\omega) = \int_F X(\omega) d\mathbb{P}(\omega)$$

for all events $F \in \mathcal{F}$. In such case, we denote Z by $\mathbb{E}[X|\mathcal{F}]$.
Points: Give 1p for everything. Subtract 0.5 if $\mathcal{F}$-measurability of Z is forgotten in the definition. Give max 0.25 if the candidate only provides the definition of $\mathbb{E}[X|F]$ for an event $F \in \mathcal{F}$. Subtract 0.2 per mistake and 0.5 per serious mistake.

Problem 2 Random variables (3 points)

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent, identically distributed random variables with exponential distribution with parameter 1, i.e. each X_n is absolutely continuous with density function $f_{X_n}(x) = e^{-x}$, $x > 0$.

(Continued on page 3.)

Define, for each $n \geq 1$, the events

$$A_n = \left\{ |X_n - X_{n-1}| \leq \frac{1}{n^2} \right\},$$

where $X_0 = 0$ by convention.

We call outcomes in A_n *friends*. In other words, we say that $X_{n-1}(\omega)$ and $X_n(\omega)$ are friends if $\omega \in A_n$.

(a) Show that the random variables $Y_n = |X_n - X_{n-1}|$, $n \geq 1$ are again exponentially distributed with parameter 1. Hint: You may use the law of total probability for absolutely continuous random variables, that is $\mathbb{P}[A] = \int_{-\infty}^{\infty} \mathbb{P}[A|X = x] f_X(x) dx$ for an absolutely continuous random variable X with density f_X and any event A.

Solution: We apply the law of total expectation: let $x > 0$, then

$$\begin{aligned}
\mathbb{P}[|X_n - X_{n-1}| \leq x] &= \int_{-\infty}^{\infty} \mathbb{P}[|X_n - X_{n-1}| \leq x | X_{n-1} = y] f_{X_{n-1}}(y) dy \\
&= \int_0^{\infty} \mathbb{P}[|X_n - y| \leq x] e^{-y} dy \\
&= \int_0^{\infty} \mathbb{P}[-x \leq X_n - y \leq x] e^{-y} dy \\
&= \int_0^{\infty} \mathbb{P}[y - x \leq X_n \leq y + x] e^{-y} dy \\
&= \int_0^{\infty} \left(\mathbb{P}[X_n \leq y + x] - \mathbb{P}[X_n \leq y - x]\right) e^{-y} dy \\
&= \int_0^{\infty} \mathbb{P}[X_n \leq y + x] e^{-y} dy - \int_0^{\infty} \mathbb{P}[X_n \leq y - x] e^{-y} dy
\end{aligned}$$

Now, observe that the probability $\mathbb{P}[X_n \leq y - x]$ is 0 for $y < x$, so it makes only sense to integrate for $y > x$. Thus,

$$\begin{aligned}
&\mathbb{P}[|X_n - X_{n-1}| \leq x] = \\
&\qquad = \int_0^{\infty} \mathbb{P}[X_n \leq y + x] e^{-y} dy - \int_x^{\infty} \mathbb{P}[X_n \leq y - x] e^{-y} dy \\
&\qquad = \int_0^{\infty} \left(1 - e^{-(y+x)}\right) e^{-y} dy - \int_x^{\infty} \left(1 - e^{-(y-x)}\right) e^{-y} dy \\
&\qquad = 1 - \frac{e^{-x}}{2} - e^{-x} + \frac{e^{-x}}{2} \\
&\qquad = 1 - e^{-x}.
\end{aligned}$$

Hence, $\mathbb{P}[|X_n - X_{n-1}| \leq x]$, $x > 0$ which is the distribution function of an exponential random variable with parameter 1.

Points: Give full score if the candidate uses the transformation of variables formula or any other method (e.g. characteristic functions is possible too). Give 0.5 if there is a meaningful attempt, e.g. if the first display in the solution is written down. Subtract 0.2 per mistake and 0.5 per serious mistake.

(Continued on page 4.)

(b) Let I_n be the random variable that is one if X_{n-1} and X_n are friends and 0 otherwise. Show that $\{I_n\}_{n=1}^{\infty}$ converges in probability to 0.

Solution: The random variable I_n is defined as

$$I_n(\omega) = \begin{cases} 1, \text{ if } |X_n(\omega) - X_{n-1}(\omega)| \leq \frac{1}{n^2}, \\ 0, \text{ otherwise} \end{cases} = \begin{cases} 1, \text{ if } \omega \in A_n, \\ 0, \text{ otherwise} \end{cases}.$$

That is $I_n = \mathbb{I}_{A_n}$.

$\{I_n\}_{n\geq 1}$ will converge to 0 in probability if, and only if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}[|I_n| > \varepsilon] = 0.$$

Now, take $\varepsilon \in (0,1)$, then the event $\{|I_n| > \varepsilon\} = \{I_n = 1\}$. Therefore,

$$\mathbb{P}[|I_n| > \varepsilon] = \mathbb{P}[I_n = 1] = \mathbb{P}[A_n] = \mathbb{P}[|X_n - X_{n-1}| \leq \frac{1}{n^2}] = 1 - e^{-\frac{1}{n^2}},$$

where the last follows from item (a). Then, since $\lim_n e^{-\frac{1}{n^2}} = 1$ it follows that

$$\lim_n \mathbb{P}[|I_n| > \varepsilon] = \lim_n \left(1 - e^{-\frac{1}{n^2}}\right) = 0$$

and convergence in probability follows.

Another way of proving convergence is by simply observing that

$$\mathbb{P}[I_n = 1] = \mathbb{P}[A_n], \quad \mathbb{P}[I_n = 0] = 1 - \mathbb{P}[A_n]$$

and showing that $\lim_n \mathbb{P}[A_n] = 0$. This implies that I_n converges to 0 in distribution and since 0 is a constant then convergence in probability follows, as well.

Points: Give 0.25 for just writing the definition of convergence in probability. Give 0.25 extra for a meaningful attempt. Subtract 0.2 per mistake and 0.5 per serious mistake.

(c) Will you observe an infinite number of friends in the infinite sequence $X_1, X_2, \ldots$? Justify your answer.

Solution: The event that we observe infinitely many friends is given by $\limsup_n A_n$. Hence, we will observe infinitely many friends if $\mathbb{P}[\limsup_n A_n] = 1$ or not if $\mathbb{P}[\limsup_n A_n] = 0$. We may attempt to apply Borel-Cantelli's first lemma, since we do not need to assume that A_n are independent, which they are not. Then we have

$$\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \sum_{n=1}^{\infty} \left(1 - e^{-\frac{1}{n^2}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

where we used the elementary estimate $1 - e^{-x} \leq x$ for $x > 0$. By Borel-Cantelli's first lemma we conclude that

$$\mathbb{P}[\limsup_n A_n] = 0$$

(Continued on page 5.)

and hence we will not observe an infinite number of friends.

Points: Give 0.5 for just writing Borel-Cantelli's lemma correct. Give 0.25 for mentioning which Borel-Cantelli lemma applies here. Subtract 0.2 per mistake and 0.5 per serious mistake.

Problem 3 Conditional expectation (3 points)

Suppose that $a, b > 0$ real and X and Y are two random variables with values in $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{N}_0 = \{0, 1, \dots\}$, respectively, whose distribution function is given by

$$\mathbb{P}[X \leq x, Y = n] = b \int_0^x \frac{(az)^n}{n!} e^{-(a+b)z} dz, \quad (x, n) \in \mathbb{R}_+ \times \mathbb{N}_0.$$

You may need the fact that $\int_0^\infty \frac{b^{n+1}}{n!} x^n e^{-bx} dx = 1$ and the Taylor expansion $e^z = \sum_{n=0}^\infty \frac{z^n}{n!}$, $z \in \mathbb{R}$. Also $\sum_{n=0}^\infty n r^n = \frac{r}{(1-r)^2}$ for $|r| < 1$.

(a) Prove that the marginal distributions of X and Y are given by

$$\mathbb{P}[X \leq x] = 1 - e^{-bx}, x \in \mathbb{R}_+, \quad \mathbb{P}[Y = n] = \frac{b}{a+b} \left(\frac{a}{a+b} \right)^n, n \in \mathbb{N}_0.$$

Solution: The marginal distributions are found by *integrating away* the other variable. Since Y takes values on $\mathbb{N}_+$ then

$$\mathbb{P}[X \leq x] = \sum_{n=0}^\infty \mathbb{P}[X \leq x, Y = n].$$

Then

$$\sum_{n=0}^\infty b \int_0^x \frac{(az)^n}{n!} e^{-(a+b)z} dz = b \lim_N \int_0^x \sum_{n=0}^N \frac{(az)^n}{n!} e^{-(a+b)z} dz.$$

In order to pull the limit inside, we note that the sequence of functions

$$f_N(z) = \sum_{n=0}^N \frac{(az)^n}{n!} e^{-(a+b)z}, \quad N \geq 0$$

is strictly increasing and converges pointwise in z towards

$$\lim_N f_N(z) = e^{-bz},$$

which is integrable. Hence, by Lebesgue's monotone convergence theorem, we have

$$\mathbb{P}[X \leq x] = b \int_0^x \sum_{n=0}^\infty \frac{(az)^n}{n!} e^{-(a+b)z} dz.$$

(Continued on page 6.)

and since $\sum_{n=0}^{\infty} \frac{(az)^n}{n!} = e^{az}$ we conclude that

$$\mathbb{P}[X \leq x] = b \int_0^x e^{-bz} dz = 1 - e^{-bz},$$

which corresponds to an exponential distribution with parameter $b > 0$.

Similarly,

$$\begin{aligned}\mathbb{P}[Y = n] &= \lim_{x \to \infty} \mathbb{P}[X \leq x, Y = n] \\ &= b \int_0^{\infty} \frac{(az)^n}{n!} e^{-(a+b)z} dz = \frac{b}{a+b} \left(\frac{a}{a+b}\right)^n,\end{aligned}$$

where we used that $\int_0^{\infty} \frac{z^n}{n!} e^{-(a+b)z} dz = \frac{1}{(a+b)^n}$ as suggested by the exercise.

Points: Give 0.5 for each distribution. Subtract 0.2 if monotone convergence theorem is not mentioned. Give 0.2 per each, i.e. 0.4 if the candidate mentions how to compute the marginals. Subtract 0.2 per mistake and 0.5 per serious mistake.

(b) Show that $\mathbb{E}[h(X)|Y = n] = \frac{(a+b)^{n+1}}{n!} \int_0^{\infty} h(x) x^n e^{-(a+b)x} dx$ for a Borel measurable function h.

Solution: We simply need the conditional density of $X|Y = n$ which we almost have, since

$$\mathbb{P}[X \leq x | Y = n] = \frac{\mathbb{P}[X \leq x, Y = n]}{\mathbb{P}[Y = n]}.$$

Hence, the density of X, given $Y = n$ is

$$f_{X|Y=n}(x|n) = \frac{d}{dx} \mathbb{P}[X \leq x | Y = n] = \frac{\frac{d}{dx} \mathbb{P}[X \leq x, Y = n]}{\mathbb{P}[Y = n]}.$$

By the fundamental theorem of calculus we have

$$\frac{d}{dx} \mathbb{P}[X \leq x, Y = n] = b \frac{d}{dx} \int_0^x \frac{(az)^n}{n!} e^{-(a+b)z} dz = b \frac{(ax)^n}{n!} e^{-(a+b)x}.$$

Thus,

$$f_{X|Y=n}(x|n) = b \frac{(ax)^n}{n!} e^{-(a+b)x} \frac{a+b}{b} \left(\frac{a+b}{a}\right)^n = \frac{(a+b)^{n+1}}{n!} x^n e^{-(a+b)x}.$$

Finally,

$$\begin{aligned}\mathbb{E}[h(X)|Y = n] &= \int_0^{\infty} h(x) f_{X|Y=n}(x|n) dx \\ &= \frac{(a+b)^{n+1}}{n!} \int_0^{\infty} h(x) x^n e^{-(a+b)x} dx,\end{aligned}$$

and the formula follows.

Points: Give 0.5 for a general formula without completing the computation. Subtract 0.2 per mistake and 0.5 per serious mistake.

(Continued on page 7.)

(c) Compute $\mathbb{E}[X|\sigma(Y)]$ and verify that $\mathbb{E}[\mathbb{E}[X|\sigma(Y)]] = \mathbb{E}[X]$.

Solution: To compute $\mathbb{E}[X|\sigma(Y)]$ we simply acknowledge that this is an $\sigma(Y)$-measurable random variable, i.e. a (Borel) measurable function of Y. We may just "pretend" that $Y = n$ and use the formula from (b) with $h(x) = x$. Hence,

$$\mathbb{E}[X|Y = n] = \frac{(a+b)^n}{n!} \int_0^\infty x^{n+1} e^{-(a+b)x} dx.$$

Then we use the formula suggested in the exercise to conclude that

$$\mathbb{E}[X|Y = n] = \frac{(a+b)^{n+1}}{n!} \int_0^\infty x^{n+1} e^{-(a+b)x} dx = \frac{n+1}{a+b}.$$

As a result,

$$\mathbb{E}[X|\sigma(Y)] = \frac{Y+1}{a+b},$$

which we can confirm is a Borel measurable function of Y. Now, we can compute $\mathbb{E}[X]$ by the tower property:

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[\mathbb{E}[X|\sigma(Y)]] \\ &= \mathbb{E}\left[\frac{Y+1}{a+b}\right] \\ &= \frac{\mathbb{E}[Y]+1}{a+b}.\end{aligned}$$

Now,

$$\mathbb{E}[Y] = \sum_{n=0}^\infty n\mathbb{P}[Y = n] = \frac{b}{a+b} \sum_{n=0}^\infty n \left(\frac{a}{a+b}\right)^n = \frac{a}{b},$$

by applying the formula $\sum_{n=0}^\infty nr^n = \frac{r}{(1-r)^2}$ with $r = a/(a+b)$. We conclude that

$$\mathbb{E}[X] = \frac{1}{b}.$$

If one recognizes in (a) that $X \sim \exp(b)$ then one can confirm that the result is correct and that the tower property obviously holds.

Points: Give 0.5 for each expectation, but only give points for $\mathbb{E}[X]$ if computed via tower property. Subtract 0.2 if $\mathbb{E}[X|\sigma(Y)]$ is not recognized as a measurable function of Y. Subtract 0.2 per mistake and 0.5 per serious mistake.

Problem 4 Martingales (2 points)

Let $\{M_n\}_{n=0}^\infty$ be a martingale relative to a filtration $\{\mathcal{F}_n\}_{n=0}^\infty$ and $\{X_n\}_{n=0}^\infty$ a bounded (i.e. $|X_n| \le C$ for some constant $C > 0$ for all $n \ge 0$) stochastic process adapted to $\{\mathcal{F}_n\}_{n=0}^\infty$. Define the stochastic process Y_n, $n \ge 0$ by

$$Y_0 = 0, \quad Y_n = \sum_{k=1}^n X_{k-1}(M_k - M_{k-1}), \quad n \ge 1.$$

(Continued on page 8.)

The process $\{Y_n\}_{n=0}^{\infty}$ is known as *discrete stochastic integral of* $\{X_n\}_{n=0}^{\infty}$ *with respect to the martingale* $\{M_n\}_{n=0}^{\infty}$.

(a) Show that $\{Y_n\}_{n=0}^{\infty}$ is a martingale relative to $\{\mathcal{F}_n\}_{n=0}^{\infty}$.

Solution: We need to check the three conditions for martingale.

1. (Adaptedness): Y_n needs to be $\mathcal{F}_n$-measurable. Indeed,

$$Y_n = \sum_{k=1}^{n} X_{k-1}(M_k - M_{k-1}).$$

We have that X_{k-1} is $\mathcal{F}_{k-1}$-measurable and since $\mathcal{F}_{k-1} \subset \mathcal{F}_n$ for all $k = 1, \ldots, n$ then X_{k-1} is also $\mathcal{F}_n$-measurable. Also, $M_k - M_{k-1}$ are all $\mathcal{F}_k$-measurable for $k = 1, \ldots, n$ since M_k is a martingale. Since $\mathcal{F}_k \subset \mathcal{F}_n$ for all $k = 0, \ldots, n$. Hence, $M_k - M_{k-1}$ is also $\mathcal{F}_n$-measurable. Then $X_{k-1}(M_k - M_{k-1})$ is $\mathcal{F}_n$-measurable for all $k = 1, \ldots, n$ and hence the sum is also $\mathcal{F}_n$-measurable.

2. (Integrability): We need to check that $\mathbb{E}[|Y_n|] < \infty$ for all $n \geq 1$. By the triangle inequality we have

$$\mathbb{E}[|Y_n|] \leq \sum_{k=1}^{n} \mathbb{E}[|X_{k-1}||M_k - M_{k-1}|].$$

Since $|X_{k-1}| \leq C$ then

$$\mathbb{E}[|Y_n|] \leq C \sum_{k=1}^{n} \mathbb{E}[|M_k - M_{k-1}|].$$

Finally, since $\{M_n\}_{n=0}^{\infty}$ is a martingale we have $\mathbb{E}[|M_n|] < \infty$ for all $n \geq 1$, hence $\mathbb{E}[|M_k - M_{k-1}|] \leq \mathbb{E}[|M_k|] + \mathbb{E}[|M_{k-1}|] < \infty$ for all $k = 1, \ldots, n$.

3. (Martingale property): It suffices to check

$$\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n, \quad n \geq 0,$$

since by the tower property it holds for $m \geq n$,

$$\mathbb{E}[Y_m|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Y_m|\mathcal{F}_{m-1}]|\mathcal{F}_n] = \cdots = \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n.$$

Then,

$$\begin{aligned}\mathbb{E}[Y_{n+1}|\mathcal{F}_n] &= \mathbb{E}[Y_n + X_n(M_{n+1} - M_n)|\mathcal{F}_n] \\ &= Y_n + \mathbb{E}[X_n(M_{n+1} - M_n)|\mathcal{F}_n],\end{aligned}$$

where we used that Y_n is $\mathcal{F}_n$-measurable. Now, since $\{X_n\}_{n=1}^{\infty}$ is adapted, we have that X_n is $\mathcal{F}_n$-measurable, hence we can take

(Continued on page 9.)

it out of the conditional expectation. Finally, since $\{M_n\}_{n=0}^{\infty}$ is a martingale, we have

$$\mathbb{E}[M_{n+1} - M_n | \mathcal{F}_n] = \mathbb{E}[M_{n+1} | \mathcal{F}_n] - \mathbb{E}[M_n | \mathcal{F}_n] = M_n - M_n = 0$$

and hence,

$$\mathbb{E}[Y_{n+1} | \mathcal{F}_n] = Y_n,$$

for every $n \geq 0$ and the result follows.

Points: Give 0.5 for checking adaptedness and integrability and 0.5 for the martingale property. For not mentioning properties of conditional expectation, subtract 0.2 per each (e.g. measurability, towerproperty, etc.). Subtract 0.2 per mistake and 0.5 per serious mistake.

(b) Show that if, in addition, $\mathbb{E}[|M_n|^2] < \infty$ for every $n \geq 1$ then $\mathbb{E}[|Y_n^*|^2] < \infty$ for every $n \geq 1$ as well, where $Y_n^* = \sup_{k=0,\dots,n} |Y_k|$.

Solution: Since $\{Y_n\}_{n=1}^{\infty}$ is a martingale by item (i), then by Doob's L^p martingale inequality with $p = 2$ we have

$$\mathbb{E}[|Y_n^*|^2] \leq K\mathbb{E}[|Y_n|^2]$$

for a constant $K > 0$ that is not relevant (actually $K = 4$). Observe that

$$|Y_n|^2 \leq n \sum_{k=1}^{n} |X_{k-1}|^2 |M_k - M_{k-1}|^2$$

and since $|X_{k-1}| \leq C$ so $|X_{k-1}|^2 \leq C^2$. Furthermore,

$$\mathbb{E}[|Y_n|^2] \leq C^2 n \sum_{k=1}^{n} \mathbb{E}[|M_k - M_{k-1}|^2] < \infty$$

for each $n \geq 1$ since $\mathbb{E}[|M_k|^2] < \infty$ for each $k = 0, \dots, n$. Altogether,

$$\mathbb{E}[|Y_n^*|^2] \leq KC^2 n \sum_{k=1}^{n} \mathbb{E}[|M_k - M_{k-1}|^2] < \infty$$

for each $n \geq 1$.

Points: Give 0.25 for only mentioning Doob's L^p martingale inequality. Give 0.5 if it is in addition applied and give 0.75 if the candidate uses the boundedness of X_k but it remains to conclude. Subtract 0.2 per mistake and 0.5 per serious mistake.

GOOD LUCK!

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Trial exam in: STK3710/4710 — Probability Theory

Day of examination: 20th November 2023

Examination hours: 2:00 pm – 6:00 pm

This problem set consists of 11 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

This exam consists of four problems. The first one is of theoretical nature, while the three remaining problems are more oriented to solving through reasoning. Make sure to be **precise** and **rigorous** when stating mathematical results and formulae. Please, **write clearly**, **orderly** and avoid scribbling.

Problem 1 Theory (3 points)

Let E be a set and $(F, \mathcal{F})$ a measurable space. Consider a transformation $f : E \to F$ between E and F. It is implicitly assumed that $f^{-1}(F) = E$.

(a) Given subsets $A, B \in \mathcal{F}$ show that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

and

$$f^{-1}(A^c) = f^{-1}(A)^c,$$

where $A^c = F \setminus A$ stands for the *complement of* A.

Now we show that $f^{-1}(A \cup B) \supseteq f^{-1}(A) \cup f^{-1}(B)$: let $x \in f^{-1}(A) \cup f^{-1}(B)$, then $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. In particular, we can say that $x \in f^{-1}(A \cup B)$ or $x \in f^{-1}(A \cup B)$ and hence $x \in f^{-1}(A \cup B)$. Finally, observe that

$$E = f^{-1}(F) = f^{-1}(A \cup A^c) = f^{-1}(A) \cup f^{-1}(A^c),$$

where the last equality follows from the previous assertion. Then

$$f^{-1}(A)^c = E \setminus f^{-1}(A) = f^{-1}(A^c).$$

Points: Give 0.5 per each statement. Give 0.1 if the candidate tries to solve the exercise elementwise, but fails. Subtract 0.2 per mistake and 0.5 per serious mistakes.

(Continued on page 2.)

(b) Define a class of subsets of E as

$$\mathcal{E} = \{A \subseteq E : \text{ there exists } B \in \mathcal{F} \text{ with } A = f^{-1}(B)\}.$$

Show that $\mathcal{E}$ is a σ-algebra on E and that f is an $(\mathcal{E}, \mathcal{F})$-measurable transformation.

Solution: To show that $\mathcal{E}$ is a σ-algebra we need to show that the three axioms of σ-algebra are fulfilled. Namely,

1. $\emptyset \in \mathcal{E}$.
2. For a countable family of measurable sets $\{A_i\}_{i=1}^{\infty} \subset \mathcal{E}$ we have that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$.
3. For every $A \in \mathcal{E}$ we have $A^c = E \setminus A \in \mathcal{E}$.

1. We have that $f^{-1}(\emptyset) = \{x \in E : f(x) \in \emptyset\} = \emptyset$ so, $\emptyset \in \mathcal{E}$.
2. Let now $\{A_i\}_{i=1}^{\infty} \subset \mathcal{E}$ be a countable family of measurable sets in $\mathcal{E}$. Since $A_i \in \mathcal{E}$ it means that there is an $B_i \in \mathcal{F}$ such that $A_i = f^{-1}(B_i)$. Hence,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} f^{-1}(B_i) = f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right).$$

Now, since $\mathcal{F}$ is a σ-algebra then $\bigcup_{i=1}^{\infty} B_i \in \mathcal{F}$. So, for $\bigcup_{i=1}^{\infty} A_i$ there is a set in $\mathcal{F}$, namely $\bigcup_{i=1}^{\infty} B_i$ such that $\bigcup_{i=1}^{\infty} A_i = f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)$, hence $\bigcup_{i=1}^{\infty} A_i \in \mathcal{E}$.
3. Finally, if $A \in \mathcal{E}$ then there is $B \in \mathcal{F}$ such that $A = f^{-1}(B)$. Then $A^c = f^{-1}(B)^c = f^{-1}(B^c)$ and since $B \in \mathcal{F}$ so is $B^c \in \mathcal{F}$ too. Hence, for A^c there is an B^c such that $A^c = f^{-1}(B^c)$ and hence $A^c \in \mathcal{E}$.
We need to prove that $f : (E, \mathcal{E}) \to (F, \mathcal{F})$ is measurable, that is for all $B \in \mathcal{F}$ we have $f^{-1}(B) \in \mathcal{E}$. Indeed, $\mathcal{E}$ is defined such way, that is $\mathcal{E} = \{f^{-1}(B), B \in \mathcal{F}\}$. Actually, $\mathcal{E}$ is the minimal σ-algebra that makes f measurable.

Points: Give 0.75 for the σ-algebra and 0.25 for the measurability of f. Give 0.25 only for writing down what is a σ-algebra and a measurable function but nothing else. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(c) Let μ be a measure on $(E, \mathcal{E})$ where $\mathcal{E}$ is given as in item (b). Define the push-forward measure of μ via f and show that it is a measure on $(F, \mathcal{F})$.

Solution: The push-forward measure is defined as

$$\begin{aligned} mu_f :& \mathcal{F} \to [0, \infty] \\ & B \mapsto \mu_f(B) = \mu[f^{-1}(B)], \end{aligned}$$

where $\mu : \mathcal{E} \to [0, \infty]$ is a measure on $(E, \mathcal{E})$. We see that f^{-1} maps sets from $\mathcal{F}$ into sets from $\mathcal{E}$ and then μ maps sets from $\mathcal{E}$ into a value

(Continued on page 3.)

$[0, \infty]$.

We need to check that μ_f satisfies the axioms for measure which are:

1. $\mu_f[\emptyset] = 0$.
2. For a countable family of disjoint measurable sets $\{B_i\}_{i=1}^{\infty} \subset \mathcal{F}$ we have that
$$\mu_f\left[\bigcup_{i=1}^{\infty} B_i\right] = \sum_{i=1}^{\infty} \mu_f[B_i]$$
.

1. We have $\mu_f[\emptyset] = \mu[f^{-1}(\emptyset)] = \mu[\emptyset] = 0$, where the last step follows since μ *is* a measure.

2. Let $\{B_i\}_{i=1}^{\infty} \subset \mathcal{F}$ disjoint. Then

$$\begin{aligned}\mu_f\left[\bigcup_{i=1}^{\infty} B_i\right] &= \mu\left[f^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right)\right] \\ &= \mu\left[\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right].\end{aligned}$$

Now, we claim that if C, D are two disjoint sets, then $f^{-1}(C), f^{-1}(D)$ are also disjoint. Indeed, assume not. Then there is $x \in f^{-1}(C) \cap f^{-1}(D)$ which means that $x \in f^{-1}(C)$ *and* $x \in f^{-1}(D)$. Therefore, it means that there is an $y \in C$ such that $y = f(x)$ and there is an $z \in D$ such that $z = f(x)$ which implies $y = z$, in particular, $y \in C$ and $y \in D$ contradicting that C and D are disjoint.

To finish we simply use the fact that μ is a measure:

$$\begin{aligned}\mu_f\left[\bigcup_{i=1}^{\infty} B_i\right] &= \mu\left[\bigcup_{i=1}^{\infty} f^{-1}(B_i)\right] \\ &= \sum_{i=1}^{\infty} \mu[f^{-1}(B_i)] \\ &= \sum_{i=1}^{\infty} \mu_f[B_i].\end{aligned}$$

Points: Give 0.5 for the definition of push-forward measure and 0.5 for the measurability. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

Problem 2 Random variables (2 points)

(a) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent random variables with the following distribution

$$\mathbb{P}[X_n = -n] = \mathbb{P}[X_n = n] = \frac{1}{2\sqrt{n}}, \quad \mathbb{P}[X_n = 0] = 1 - \frac{1}{\sqrt{n}}, \quad n \geq 1.$$

(Continued on page 4.)

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Discuss the convergence of the sequence $\{X_n\}_{n=1}^{\infty}$ in L^p for any $p \geq 1$, almost surely, in probability and in distribution.

Solution: We can see that the probabilities for $-n$ and n vanish and for 0 prevails. Hence, we have see that $\{X_n\}_{n=1}^{\infty}$ converges, in distribution, to the constant 0, i.e.

$$X_n \xrightarrow{d} 0.$$

Let us check convergence in probability. We need to check whether for every $\varepsilon > 0$ we have

$$\lim_n \mathbb{P}[|X_n| > \varepsilon] = 0.$$

Let $\varepsilon > 0$ be small enough, say $\varepsilon < 1$. Then the event $\{|X_n| > \varepsilon\}$ is the event that $\{X_n = -n\}$ or $\{X_n = n\}$ that is $\{|X_n| = n\}$. Thus

$$\mathbb{P}[|X_n| > \varepsilon] = \mathbb{P}[|X_n| = n] = \frac{1}{\sqrt{n}} \xrightarrow{n\to\infty} 0.$$

Hence,

$$X_n \xrightarrow{P} 0,$$

as well.

For almost sure convergence, we aim to use the equivalent definition that for all $\varepsilon > 0$

$$\mathbb{P}[\limsup_n \{|X_n| > \varepsilon\}] = 0.$$

We have some work done. We know that

$$\mathbb{P}[\{|X_n| > \varepsilon\}] = \frac{1}{\sqrt{n}}.$$

We have

$$\sum_{n=1}^{\infty} \mathbb{P}[|X_n| > \varepsilon] = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty,$$

which implies, since $\{X_n\}_{n=1}^{\infty}$ are independent, by the second Borel-Cantelli lemma that

$$\mathbb{P}[\limsup_n \{|X_n| > \varepsilon\}] = 1$$

and hence convergence almost surely does not hold.

Now, we need to discuss the limit of $\mathbb{E}[|X_n|^p]$ for any $p \geq 1$. We have

$$\mathbb{E}[|X_n|^p] = |n|^p \frac{1}{2\sqrt{n}} + |n|^p \frac{1}{2\sqrt{n}} = n^{p-\frac{1}{2}}$$

(Continued on page 5.)

which does not converge to 0 for any $p \geq 1$. Hence, X_n does not converge in any L^p sense for any $p \geq 1$.

Points: Give 0.25 per each type of convergence. Give 0.25 for only writing down all the definitions of convergence without being used. Consider positively mentioning the use of Borel-Cantelli lemma for the almost sure convergence. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(b) Consider the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}([0,1])$ the Borel σ-algebra on $[0,1]$ and $\mathbb{P}$ the Lebesgue measure, i.e. $\mathbb{P}[(a,b]) = b - a$ for all intervals $(a,b] \subset [0,1]$.

Consider on this probability space the random variables

$$X_n(\omega) = \frac{n+1}{n}\omega + (1-\omega)^n, \quad n \geq 1$$

and

$$X(\omega) = \omega.$$

Show that $\{X_n\}_{n=1}^{\infty}$ converges, almost surely, to X and discuss whether it also converges in L^p, for $p \geq 1$. You may find useful the following inequality: for real numbers a, b and $p \geq 1$, $(|a| + |b|)^p \leq 2^{p-1}(|a|^p + |b|^p)$.

Solution: We need to prove that

$$\mathbb{P}[\lim_n X_n = X] = 1.$$

Let $\omega \in (0,1]$, then

$$\lim_n X_n(\omega) = \lim_n \frac{n+1}{n}\omega + \lim_n (1-\omega)^n = \omega,$$

where we used that $0 \leq 1 - \omega < 1$ for $\omega \in (0,1]$ hence $\lim_n (1-\omega)^n = 0$. Hence,

$$\lim_n X_n(\omega) = \omega$$

for $\omega \in (0,1]$. Since $\mathbb{P}[(0,1]] = 1$ we have that $X_n \to X$ almost surely. You may notice that for $\omega = 0$ we have $\lim_n X_n(0) = 1$ and hence $X_n(0)$ does not converge to $X(0)$.

Now, let us study convergence in L^p, $p \geq 1$. We have

$$\begin{aligned}
\mathbb{E}[|X_n - X|^p] &= \int_0^1 \left| \frac{n+1}{n}\omega + (1-\omega)^n - \omega \right|^p d\omega \\
&= \int_0^1 \left| \frac{1}{n}\omega + (1-\omega)^n \right|^p d\omega.
\end{aligned}$$

(Continued on page 6.)

Using now the proposed inequality we have

$$\begin{aligned}\mathbb{E}[|X_n - X|^p] &\leq \int_0^1 \frac{1}{n}\omega^p + (1-\omega)^{np} d\omega \\ &= \frac{2^{p-1}}{p+1}\frac{1}{n} + 2^{p-1}\frac{1}{np+1}\end{aligned}$$

which converges to 0 as $n \to \infty$ for all $p \geq 1$. Hence, $\{X_n\}$ converges to X in L^p for all $p \geq 1$.

Points: Give 0.5 per each type of convergence. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

Problem 3 Conditional expectation (3 points)

Let $n \geq 1$ be a fixed natural number and $\{X_i\}_{i=1}^n$ a family of independent and identically distributed random variables. Let $S_n = \sum_{i=1}^n X_i$ denote their sum. Furthermore, consider $\mathcal{F} = \sigma(S_n)$ the σ-algebra generated by the sum of the n random variables.

(a) Compute $\mathbb{E}[X_1|\mathcal{F}]$.

Solution: One has the following intuitive fact:

$$\mathbb{E}[X_1|\mathcal{F}] = \cdots = \mathbb{E}[X_n|\mathcal{F}].$$

Recall the following result (Exercise from Chapter 7):

If X and Y are two random variables, then there is a measurable function f such that $\mathbb{E}[X|Y] = f(Y)$ a.s. Moreover, if (X', Y') is identically distributed to (X, Y) then $\mathbb{E}[X'|Y'] = f(Y')$ for the same function f.

Proving the above result is not difficult: the first part is a Theorem 7.3.1 from the lecture notes. The second part is reduced to showing that $f(Y')$ is indeed the same as $\mathbb{E}[X'|Y']$ which is implied by showing that $\mathbb{E}[\mathbb{I}_A f(Y')] = \mathbb{E}[\mathbb{I}_A X']$ for all $A \in \sigma(Y')$.
In our case we have (X_1, S_n) is identically distributed as (X_2, S_n) and (X_3, S_n) and so on. So the result above applies and we are, now certainly, able to say that

$$\mathbb{E}[X_1|\sigma(S_n)] = \mathbb{E}[X_2|\sigma(S_n)] = \cdots = \mathbb{E}[X_n|\sigma(S_n)].$$

Now, we can see that

$$\sum_{i=1}^n \mathbb{E}[X_i|\mathcal{F}] = n\mathbb{E}[X_1|\mathcal{F}].$$

But on the other hand, it is also true that

$$\sum_{i=1}^n \mathbb{E}[X_i|\mathcal{F}] = \mathbb{E}[\sum_{i=1}^n X_i|\mathcal{F}] = \mathbb{E}[S_n|\sigma(S_n)] = S_n.$$

(Continued on page 7.)

In conclusion,

$$\mathbb{E}[X_1|\mathcal{F}] = \frac{S_n}{n},$$

meaning that the best guess for a given random observation X_i, knowing the sum of all, is the sample mean.

Points: Give 0.25 for writing down the definition of conditional expectation. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(b) Let Y, Z be two independent standard normally distributed random variables. Let $a, b \in \mathbb{R}$. Find the conditions on a and b for the random variables $(1-a)Y + bZ$ and $aY - bZ$ to be independent.

Solution: Let $(U, V) = g(Y, Z)$, where $g(y, z) = ((1-a)y + bz, ay - bz)$, then $g^{-1}(u, v) = \left(u + v, \frac{a}{b}u - \frac{1-a}{b}v\right)$ and

$$|Dg^{-1}(u,v)| = |\begin{pmatrix} 1 & 1 \\ \frac{a}{b} & -\frac{1-a}{b} \end{pmatrix}| = |-\frac{1-a}{b} - \frac{a}{b}| = \frac{1}{|b|}.$$

The density of the transformed vector is

$$\begin{aligned} f_{(U,V)}(u,v) &= f_{(Y,Z)}\left(g^{-1}(u,v)\right)|Dg^{-1}(u,v)| \\ &= f_Y(u+v) f_Z\left(\frac{a}{b}u - \frac{1-a}{b}v\right)\frac{1}{|b|}, \end{aligned}$$

where f_Y and f_Z are the density functions of a standard normal distribution.

Now, the function $f_{(U,V)}(u,v)$ will factor in u and v if the cross term in the exponent of the normal densities cancel out. The exponent we get is

$$-\frac{1}{2}\left[(u+v)^2 + \left(\frac{a}{b}u - \frac{1-a}{b}v\right)^2\right].$$

The cross term is:

$$2uv - 2\frac{a}{b}u\frac{1-a}{b}v.$$

So we need to impose that

$$2uv - 2\frac{a}{b}u\frac{1-a}{b}v = 0$$

which yields

$$(1-a)a = b^2.$$

An alternative way is to resort to the fact that if (X, Y) is a bivariate normal distribution and X and Y are uncorrelated, then X and Y are independent. This property is far from being true in general, but it holds for the particular case of normal distributions. Hence, it is enough to fins a and b such that $\mathbb{E}[((1-a)Y + Z)(aY - bZ)] = 0$.

(Continued on page 8.)

Points: Give 0.1 for writing down the formula for transformation. Give 0.25 for transforming the random vector correctly. Give 0.25 for stating independence correctly. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(c) Assume now, in addition to independence, that the distribution of the X_i, $i = 1, \dots, n$ is the standard normal distribution, that is $X_i \sim N(0,1)$ for al $i = 1, \dots, n$. Compute $\mathbb{E}[X_1^2|\mathcal{F}]$.

Solution: To compute $\mathbb{E}[X_1^2|\mathcal{F}]$, we can not proceed as for $\mathbb{E}[X_1|\mathcal{F}]$ from (a). But, we may note that $X_1 - \frac{1}{n}S_n$ and $\frac{1}{n}S_n$ are independent. Indeed,

$$X_1 - \frac{1}{n}S_n = \left(1 - \frac{1}{n}\right) X_1 - \frac{1}{n}\sum_{i=2}^{n} X_i$$

and

$$\frac{1}{n}S_n = \frac{1}{n}X_1 + \frac{1}{n}\sum_{i=2}^{n} X_i.$$

We have $X_1 \sim N(0,1)$ and $\frac{1}{n}\sum_{i=2}^{n} X_i \sim N(0, \frac{n-1}{n^2}) \sim \frac{\sqrt{n-1}}{n}Z$ with $Z \sim N(0,1)$. Choose $Y = X_1$ and $a = \frac{1}{n}$ and $b = \frac{\sqrt{n-1}}{n}$ which satisfies the condition of item (a). Therefore,

$$X_1 - \frac{1}{n}S_n = (1 - \frac{1}{n})Y - \frac{\sqrt{n-1}}{n}Z$$

and

$$\frac{1}{n}S_n = \frac{1}{n}Y + \frac{\sqrt{n-1}}{n}Z$$

are independent.
In virtue of that, we can proceed as follows:

$$\mathbb{E}[X_1^2|\mathcal{F}] = \mathbb{E}[\left(X_1 - \frac{1}{n}S_n + \frac{1}{n}S_n\right)^2 |\mathcal{F}].$$

Now, since $\frac{1}{n}S_n$ is $\sigma(S_n)$-measurable, we can *freeze* $\frac{1}{n}S_n$ and use the fact that $X_1 - \frac{1}{n}S_n$ is independent of $\frac{1}{n}S_n$. Then,

$$\begin{aligned}
\mathbb{E}[X_1^2|\mathcal{F}] &= \mathbb{E}[\left(X_1 - \frac{1}{n}S_n + y\right)^2 |\mathcal{F}]\Big|_{y=\frac{1}{n}S_n} \\
&= \mathbb{E}[\left(X_1 - \frac{1}{n}S_n + y\right)^2]\Big|_{y=\frac{1}{n}S_n} \\
&= \mathbb{E}[(X_1 - \overline{X}_n + y)^2]\Big|_{y=\overline{X}_n} \\
&= \mathbb{E}[(X_1 - \overline{X}_n)^2] + 2\overline{X}_n\mathbb{E}[X_1 - \overline{X}_n] + \overline{X}_n^2,
\end{aligned}$$

(Continued on page 9.)

where $\overline{X}_n = \frac{1}{n}S_n$ is the sample mean.
Note that $\mathbb{E}[X_1 - \overline{X}_n] = 0$ and that

$$X_1 - \overline{X}_n \sim N(0, \frac{n-1}{n}).$$

Therefore,

$$\mathbb{E}[X_1^2|\mathcal{F}] = \frac{n-1}{n} + \overline{X}_n^2.$$

Points: Give 0.5 for applying the previous item correctly and observing the independence of $X_1 - \overline{X}_n$ and $\overline{X}_n$. Have consideration in trying to compute the conditional expectation using its properties. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

Problem 4 Martingales (2 points)

Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent and identically distributed random variables all with distribution $N(-\mu, 1)$, $\mu > 0$ real.

(a) Find the values of $a \in \mathbb{R}$ for which the stochastic process $Y_n = e^{a\sum_{i=1}^n X_i}$, $n \geq 1$, is a submartingale, a martingale and a supermartingale with respect to the natural filtration of $\{X_n\}_{n=1}^\infty$.

Solution: For martingality we need to check the three conditions:

1. (adaptedness) Y_n is $\mathcal{F}_n$-measurable for all $n \geq 1$.
2. (integrability) $\mathbb{E}[|Y_n|] < \infty$ for all $n \geq 1$.
3. (martingale property) For $n, m \geq 1$ with $m \geq n$ we have

$$\mathbb{E}[Y_m|\mathcal{F}_n] = Y_n.$$

1. The first condition is easy to verify since

$$Y_n = f(X_1, \dots, X_n)$$

for a measurable function f, namely $f(x_1, \dots, x_n) = e^{a\sum_{i=1}^n x_i}$, actually f is continuous, so also measurable, for any finite value of a.
2. $\mathbb{E}[|Y_n|] = \mathbb{E}[Y_n]$ since $Y_n > 0$ a.s. ans furthermore, since X_i are i.i.d. normally distributed then $a\sum_{i=1}^n X_i \sim N(-an\mu, a^2 n)$. Hence, $e^{a\sum_{i=1}^n X_i}$ is log-normally distributed with mean $-an\mu$ and variance $a^2 n$. Hence,

$$\mathbb{E}[Y_n] = e^{-an\mu + \frac{1}{2}a^2 n} < \infty$$

for every $n \geq 1$ and any finite value of a.
3. Finally, let us check the martingale condition for $m = n + 1$,

(Continued on page 10.)

since by the tower property it will hold for any $m \geq n$. Indeed, if $\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n$ then

$$\mathbb{E}[Y_m|\mathcal{F}_n] = \mathbb{E}[\mathbb{E}[Y_m|\mathcal{F}_{m-1}]|\mathcal{F}_n] = \mathbb{E}[Y_{m-1}|\mathcal{F}_n] = \cdots = \mathbb{E}[Y_{n+1}|\mathcal{F}_n] = Y_n,$$

where we used the tower property and $\mathcal{F}_n \subset \mathcal{F}_{m-1}$ for $m \geq n+1$. So,

$$\begin{aligned}\mathbb{E}[Y_{n+1}|\mathcal{F}_n] = \mathbb{E}[e^{a\sum_{i=1}^{n+1} X_i}|\mathcal{F}_n]& \\ &= \mathbb{E}[e^{a\sum_{i=1}^{n} X_i} e^{aX_{n+1}}|\mathcal{F}_n] \\ &= \mathbb{E}[Y_n e^{aX_{n+1}}|\mathcal{F}_n] \\ &= Y_n \mathbb{E}[e^{aX_{n+1}}|\mathcal{F}_n] \\ &= Y_n e^{-a\mu+\frac{1}{2}a^2}.\end{aligned}$$

Then $\{Y_n\}_{n=1}^{\infty}$ will be a submartingale if, and only if

$$e^{-a\mu+\frac{1}{2}a^2} > 1,$$

a martingale if, and only if

$$e^{-a\mu+\frac{1}{2}a^2} = 1,$$

and a supermartingale if, and ony if

$$e^{-a\mu+\frac{1}{2}a^2} < 1.$$

The roots of $-a\mu + \frac{1}{2}a^2$ are $a = 0$ and $a = 2\mu$. Then $-a\mu + \frac{1}{2}a^2 < 0$ for $a < 0$ and $a > 2\mu$ and $-a\mu + \frac{1}{2}a^2 > 0$ for $0 < a < 2\mu$. So, $\{Y_n\}_{n=1}^{\infty}$ is submartingale if $a < 0$ or $a > 2\mu$, a martingale if $a = 0$ or $a = 2\mu$ and a supermartingale if $0 < a < 2\mu$.

Points: Give 0.5 for verifying martingality and 0.5 for sub- and supermartingality. Give 0.25 for writing down the definition of conditional expectation and nothing else. Give 0.2 for verifying the two first properties and 0.6 for the martingale property. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

(b) Assume that $a = 2\mu$ (then $\{Y_n\}_{n=1}^{\infty}$ is martingale) and let $x > 0$ real. Define $S_n = \sum_{i=1}^{n} X_i$. Show that

$$\mathbb{P}\left[\sup_{n\geq 1} S_n > x\right] \leq e^{-2\mu x}.$$

Solution: Let $Y_n = e^{2\mu S_n}$, $n \geq 1$. Then by (a) $\{Y_n\}_{n=1}^{\infty}$ is a martingale. This means that, by Doob's first martingale inequality we have

$$\mathbb{P}[Y_n^* > \alpha] \leq \frac{\mathbb{E}[Y_n]}{\alpha} = \frac{1}{\alpha},$$

(Continued on page 11.)

where $Y_n^* \triangleq \sup_{k=1,\dots,n} |Y_k|$.
However, this is not quite what we want, we wish an estimate for $\sup_n S_n$. Observe that

$$Y_n^* = \sup_{k=1,\dots,n} |Y_k| = \sup_{k=1,\dots,n} e^{2\mu S_k} = e^{2\mu \sup_{k=1,\dots,n} S_k}.$$

Therefore,

$$\log(Y_n^*) = 2\mu \sup_{k=1,\dots,n} S_k.$$

So,

$$\sup_n S_n = \frac{1}{2\mu} \lim_n \log(Y_n^*).$$

We turn now to estimating the probability. We have,

$$\begin{aligned}
\mathbb{P}\left[\sup_n S_n > x\right] &= \mathbb{P}\left[\frac{1}{2\mu} \lim_n \log(Y_n^*) > x\right] \\
&= \mathbb{P}\left[\log(\lim_n Y_n^*) > 2\mu x\right] \\
&= \mathbb{P}\left[\lim_n Y_n^* > e^{2\mu x}\right] \\
&\leq \limsup_n \mathbb{P}\left[Y_n^* > e^{2\mu x}\right] \\
&\leq \limsup_n \frac{\mathbb{E}[Y_n]}{e^{2\mu x}} \\
&= \limsup_n \frac{1}{e^{2\mu x}} \\
&= e^{-2\mu x},
\end{aligned}$$

where we used Doob's first martingale inequality from item (a) in the fifth equality.

Points: Give 0.2 for stating Doob's first martingale inequality. Give 0.2 for mentioning Fatou or subtract 0.2 if it is not mentioned. Subtract 0.2 for mistakes and 0.5 for serious mistakes.

GOOD LUCK!

(Continued on page 12.)

STK-MAT3710/4710: Solution to Exam 2022

Problem 1 a) We have

$$\phi_a(t) = E[e^{itY_a}] = e^{it(1+a)} \cdot \frac{1}{2} + e^{it(-1+a)} \cdot \frac{1}{2}$$

$$= e^{iat}\frac{e^{it} + e^{-it}}{2} = e^{ita}\cos t$$

b) Using the independence of $X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}$, we get

$$\phi_{S_n}(t) = E[e^{itS_n}] = E\left[e^{it\frac{X_1^{(n)}+X_2^{(n)}+\cdots+X_n^{(n)}}{\sqrt{n}}}\right]$$

$$= E\left[e^{i\frac{t}{\sqrt{n}}X_1^{(n)}}\right] \cdot E\left[e^{i\frac{t}{\sqrt{n}}X_2^{(n)}}\right] \cdot \ \ldots \cdot E\left[e^{i\frac{t}{\sqrt{n}}X_n^{(n)}}\right]$$

$$= \left[\phi_{\frac{1}{\sqrt{n}}}\left(\frac{t}{\sqrt{n}}\right)\right]^n = \left[e^{i\frac{t}{\sqrt{n}}\cdot\frac{1}{\sqrt{n}}}\cos\frac{t}{\sqrt{n}}\right]^n = e^{it}\left[\cos\frac{t}{\sqrt{n}}\right]^n$$

c) Using the Taylor expansion $\cos x = 1 - \frac{x^2}{2} + o(x^2)$, we get

$$\lim_{n\to\infty}\left[\cos\frac{t}{\sqrt{n}}\right]^n = \lim_{n\to\infty}\left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n = e^{-\frac{t^2}{2}}$$

by Lemma 6.34. This means that

$$\lim_{n\to\infty}\phi_{S_n}(t) = e^{it-\frac{t^2}{2}}$$

which is the characteristic function of an $\mathcal{N}(1,1)$ random variable. By Lévy's Continuity Theorem, S_n converges in distribution to a normal distribution with mean 1 and variance 1.

Problem 2 Let A_n be the event that you win on day n. Then the A_n's are independent (we must assume) and $P(A_n) = \frac{1}{99+n}$. The series

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty}\frac{1}{99+n}$$

diverges (it is the tail of the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$), and thus the probability of winning infinitely many times is 1 by the Borel-Cantelli Lemma.

Problem 3 Let $\mathcal{M} = \{\Lambda : \int_\Lambda X\,dP = \int_\Lambda Y\,dP\}$; we must prove that $\mathcal{M}$ is closed under increasing unions and decreasing intersections.

Assume first that $\{\Lambda_n\}$ is an increasing sequence of sets in $\mathcal{M}$ with union Λ. Since the sequence $\{\mathbf{1}_{\Lambda_n}X\}$ is dominated by the integrable function $|X|$, the Dominated Convergence Theorem tells us that

$$\int_\Lambda X\,dP = \int \mathbf{1}_\Lambda X\,dP = \int \lim_{n\to\infty}\mathbf{1}_{\Lambda_n}X\,dP \overset{DCT}{=} \lim_{n\to\infty}\int \mathbf{1}_{\Lambda_n}X\,dP = \lim_{n\to\infty}\int_{\Lambda_n} X\,dP$$

and similarly,

$$\int_\Lambda Y\,dP = \lim_{n\to\infty} \int_{\Lambda_n} Y\,dP.$$

This shows that $\int_\Lambda X\,dP = \int_\Lambda Y\,dP$, and hence $\Lambda \in \mathcal{M}$.

Assume now that $\{\Lambda_n\}$ is a decreasing sequence of sets in $\mathcal{M}$ with intersection Λ. As above, the sequence $\{\mathbf{1}_{\Lambda_n} X\}$ is dominated by the integrable function $|X|$, and the Dominated Convergence Theorem gives

$$\int_\Lambda X\,dP = \int \mathbf{1}_\Lambda X\,dP = \int \lim_{n\to\infty} \mathbf{1}_{\Lambda_n} X\,dP \overset{DCT}{=} \lim_{n\to\infty} \int \mathbf{1}_{\Lambda_n} X\,dP = \lim_{n\to\infty} \int_{\Lambda_n} X\,dP$$

and similarly,

$$\int_\Lambda Y\,dP = \lim_{n\to\infty} \int_{\Lambda_n} Y\,dP.$$

This shows that $\int_\Lambda X\,dP = \int_\Lambda Y\,dP$, and hence $\Lambda \in \mathcal{M}$.

Problem 4 Let $Y_n = \mathbf{1}_{A_n}$. Then $E[Y_n] = P(A_n)$, and the random variables $Z_n = Y_n - P(A_n)$ have mean 0. Since the Z_n's are independent and obviously have bounded fourth moments, Cantelli's Strong Law of Large Numbers tells us that $\frac{Z_1+Z_2+\cdots+Z_n}{n}$ converges to 0 a.s. But $X_n = Y_1 + Y_2 + \ldots + Y_n$ and $P_n = P(A_1) + P(A_2) + \cdots + P(A_n)$, and thus

$$\frac{X_n - P_n}{n} = \frac{Z_1 + Z_2 + \cdots + Z_n}{n} \to 0 \qquad \text{a.s.}$$

Problem 5

a) To show that

$$Z(\omega) = \sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(\omega)(X_1(\omega) + X_2(\omega) + \cdots + X_n(\omega)),$$

we first observe that if $Y(\omega) = 0$, then both sides are 0. Next we observe that if $Y(\omega) = n$ for $n > 0$, then both sides equal $X_1(\omega) + X_2(\omega) + \cdots + X_n(\omega)$, and hence the expressions are equal for all ω.

Taking expectations on both sides of the equation above, we get

$$E[Z] = E\left[\sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)\right]$$

Since all terms are positive, the Monotone Convergence Theorem gives

$$E[Z] = E\left[\lim_{m\to\infty} \sum_{n=1}^{m} \mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)\right]$$

$$= \lim_{m\to\infty} E\left[\sum_{n=1}^{m} \mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)\right]$$

Using the independence of Y and the X_i's, we see that this equals

$$\lim_{m\to\infty} \left[\sum_{n=1}^{m} E[\mathbf{1}_{[Y=n]}]E[X_1 + X_2 + \cdots + X_n]\right]$$

$$= \lim_{m\to\infty} \left[\sum_{n=1}^{m} P[Y = n](\mu_1 + \mu_2 + \cdots + \mu_n) \right]$$

$$= \sum_{n=1}^{\infty} P[Y = n](\mu_1 + \mu_2 + \cdots + \mu_n)$$

and hence

$$E[Z] = \sum_{n=1}^{\infty} (\mu_1 + \mu_2 + \cdots + \mu_n) P[Y = n]$$

b) Using the Monotone Convergence Theorem for conditional expextations, we get

$$E[Z|\mathcal{G}] = E\left[\sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)|\mathcal{G} \right]$$

$$= \lim_{m\to\infty} E\left[\sum_{n=1}^{m} \mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)|\mathcal{G} \right]$$

$$= \lim_{m\to\infty} \sum_{n=1}^{m} E[\mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)|\mathcal{G}]$$

Since Y is $\mathcal{G}$-measurable and the $\{X_n\}$'s are independent of $\mathcal{G}$, we have

$$E[\mathbf{1}_{[Y=n]}(X_1 + X_2 + \cdots + X_n)|\mathcal{G}] = \mathbf{1}_{[Y=n]} E[X_1 + X_2 + \cdots + X_n|\mathcal{G}]$$

$$= \mathbf{1}_{[Y=n]}(\mu_1 + \mu_2 + \cdots + \mu_n)$$

and hence

$$E[Z|\mathcal{G}] = \lim_{m\to\infty} \sum_{n=1}^{m} \mathbf{1}_{[Y=n]}(\mu_1 + \mu_2 + \cdots + \mu_n)$$

$$= \sum_{n=1}^{\infty} \mathbf{1}_{[Y=n]}(\mu_1 + \mu_2 + \cdots + \mu_n).$$

c) Since Z_k is $\mathcal{F}_k$-measurable by induction, and $X_i^{(k+1)}$ is independent of $\mathcal{F}_k$ by assumption, we get from b) that

$$E[Z_{k+1}|\mathcal{F}_k] = \sum_{n=1}^{\infty} \mathbf{1}_{[Z_k=n]}(\mu + \mu + \cdots + \mu) = \mu \sum_{n=1}^{\infty} n\mathbf{1}_{[Z_k=n]} = \mu Z_k.$$

Since $Z_k \geq 0$, we see that $E[Z_{k+1}|\mathcal{F}_k] \geq Z_k$ when $\mu > 1$, $E[Z_{k+1}|\mathcal{F}_k] = Z_k$ for $\mu = 1$, and $E[Z_{k+1}|\mathcal{F}_k] \leq Z_k$ when $\mu < 1$. This means that Z_k is a submartingale, a martingale, and a supermartingale according to whether $\mu \geq 1$, $\mu = 1$, or $\mu \leq 1$.

d) Using c), we see that

$$E[Y_{k+1}|\mathcal{F}_k] = \frac{1}{\mu^{k+1}} E[Z_{k+1}|\mathcal{F}_k] = \frac{\mu}{\mu^{k+1}} Z_k = Y_k$$

which shows that $Y_k = \frac{Z_k}{\mu^k}$ is a martingale.

e) We use Theorem 6.31: Since $\{Y_k\}$ is a uniformly intergrable martingale, Y_k converges a.s. and in L^1 to an integrable random variable Y_∞ such that the augmented process $Y_0, Y_1, \ldots, Y_\infty$ is also a martingale. Hence $E[Y_\infty] = E[Y_0] = E[Z_0] = Z_0 > 0$, which proves that Y_∞ is positive on a set Ω_0 of positive measure. Since $Z_k = \mu^k Y_k$, it follows that Z_k goes to infinity on Ω_0.

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Soltions of the Examination, STK-MAT3710/4710 - Probability Theory.

[**Problem 1**] (9 points) **Solution:** By the first Borel-Cantelli lemma, $P(X_n \neq Y_n, i.o.) = 0$. Hence, $X_n = Y_n$ for all, just not for finitely many values of n, almost surely. Off an event of probability zero, the sequences are identical for all large n. Hence, both converge or diverge together.

[**Problem 2**]

a) (9 points) **Solution:** $E[X] = -5 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} = 0$.
$E[X^2] = 25 \cdot \frac{1}{4} + 9 \cdot \frac{1}{4} + 1 \cdot \frac{2}{4} = 9$.
Then, by Taylor expansion formula of the characteristic function:
$\phi_X(t) = 1 - \frac{9}{2}t^2 + o(t^2)$.
If you have written :

$$\phi_X(t) = \sum_{k=0}^{n} \frac{1}{k!} E[X^k](it)^k + o(t^n) = \sum_{k=0}^{n} \frac{(-5)^k + 3^k + 2}{4k!}(it)^k + o(t^n)$$

is also accepted.

b) (9 points) **Solution:**

$$\begin{aligned}
\phi_{S_n}(t) &= E[e^{itS_n}] = E[e^{it(\frac{X_1+X_2+\ldots+X_n}{\sqrt{n}})}] \\
&= E[e^{\frac{it}{\sqrt{n}}X_1} \cdot e^{\frac{it}{\sqrt{n}}X_2} \cdots e^{\frac{it}{\sqrt{n}}X_n}] \\
&= E[e^{\frac{it}{\sqrt{n}}X_1}] \cdot E[e^{\frac{it}{\sqrt{n}}X_2}] \cdots E[e^{\frac{it}{\sqrt{n}}X_n}], \quad (X_n\text{'s are independent}) \\
&= (E[e^{\frac{it}{\sqrt{n}}X_1}])^n \quad (X_n\text{'s are identical distributed}) \\
&= (\phi_{X_1}(\frac{t}{\sqrt{n}}))^n = (\phi_X(\frac{t}{\sqrt{n}}))^n \\
&= (1 - \frac{9}{2}\frac{t^2}{n} + o(\frac{t^2}{n}))^n.
\end{aligned}$$

Last step runs since X_n's and X have the same distribution (so they have same characteristic function).

c) (9 points) **Solution:** Let us define $z_n = \frac{-9}{2}t^2 + o(t^2)$, hence by the Lemma 6.34,

$$\lim_{n\to\infty}(1 + \frac{z_n}{n})^n = e^{\frac{-9}{2}t^2}, \quad (z_n \to \frac{-9}{2}t^2, \ (n \to \infty)).$$

Since $e^{\frac{-9}{2}t^2}$ is continuous at $t = 0$ and $\phi_{S_n}(t) \to e^{\frac{-9}{2}t^2}$ $(n \to \infty)$, which is the characteristic function of a random variable with Normal distribution, by Levy continuity theorem, S_n converges in distribution to a normal distribution with $N(0, 9)$.

[Problem 3]
a) (9 points) **Solution:**

$$\begin{aligned} E[X_j X_n] &= E[E[X_j X_n|\mathcal{F}_n]] \quad (n < j,\ X_n \text{ is } \mathcal{F}_n\text{-measurable}) \\ &= E[X_n E[X_j|\mathcal{F}_n]] = E[X_n^2] \quad (X_n \text{ is a martingale wrt } \mathcal{F}_n). \end{aligned}$$

First step runs via the rule of conditional expectation of a random variable is equal to its expectation.

b) (5 points) **Solution:**

$$E[(X_m - X_r)X_n] = E[X_m X_n] - E[X_r X_n] = E[X_n^2] - E[X_n^2] = 0 \quad \text{by (a) and expectation is linear}$$

c) (9 points) **Solution:**

$$\begin{aligned} E[(X_m - X_r)^2|\mathcal{F}_n] &= E[X_m^2|\mathcal{F}_n] - 2E[X_m X_r|\mathcal{F}_n] + E[X_r^2|\mathcal{F}_n] \quad \text{conditional expectation is linear} \\ &= E[X_m^2|\mathcal{F}_n] - E[X_r^2|\mathcal{F}_n], \quad \text{(by 1).} \end{aligned}$$

and

$$\begin{aligned} E[X_m X_r|\mathcal{F}_n] &= E[E[X_m X_r|\mathcal{F}_r]|\mathcal{F}_n] \quad \text{by tower property, } n < r, \text{ and } r < m \\ &= E[X_r E[X_m|\mathcal{F}_r]|\mathcal{F}_n] \quad (X_r \text{ is } \mathcal{F}_r\text{-measurable}) \\ &= E[X_r X_r|\mathcal{F}_n] = E[X_r^2|\mathcal{F}_n] \quad (X_n \text{ is a martingale wrt } \mathcal{F}_n) \quad (1) \end{aligned}$$

d) (9 points) **Solution:**
By taking expectation of both sides of the equation obtained in part (c) and by the linearity of conditional expectation, we get:

$$\begin{aligned} &0 \le E[E[(X_m - X_r)^2|\mathcal{F}_n]] = E[E[X_m^2|\mathcal{F}_n] - E[X_r^2|\mathcal{F}_n]] = E[E[(X_m^2 - X_r^2)|\mathcal{F}_n]] \\ &0 \le E[(X_m - X_r)^2] = E[X_m^2 - X_r^2], \quad r < m. \end{aligned}$$

Observe that since $E[X_m^2 - X_r^2] \ge 0$ for $r < m$, $E[X_m^2] \ge E[X_r^2]$, for all $r < m$. Hence, $\left\{E(X_n^2) : n \ge 1\right\}$ is nondecreasing and by hypothesis it is bounded. Therefore, $\left\{E(X_n^2) : n \ge 1\right\}$ is convergent. Then,

$$\lim_{m\to\infty} E[X_m^2] = \lim_{r\to\infty} E[X_r^2], \quad \text{(If limit exists, it is unique.)}$$

By Squeeze Theorem, $\{X_n\}_{n\in\mathbb{N}_0}$ is Cauchy convergent.

[Problem 4]

a) (9 points) **Solution:**

Both S and T are bounded First Hitting Times, hence they are stopping times. The natural filtration generated by X_n can be considered as a sub-filtration of $\mathcal{F}_n$, hence both S and T are $\mathcal{F}_n$-measurable and both $\{T \leq n\} \in \mathcal{F}_n$ and $\{S \leq n\} \in \mathcal{F}_n$. Then,

$$\{\gamma = \max(T, S) \leq n\} = \{T \leq n\} \cap \{S \leq n\} \in \mathcal{F}_n,$$

σ-algebras are closed under finite intersection. Since, both S and T are bounded, maximum of their upper bound is an upper bound for γ.

b) (10 points) **Solution:**

Let us call $\max(X_n, K) = X_n^+ = \Phi(X_n)$, since Φ is a convex function and X_n is an $\mathcal{F}_n$-martingale; hence by Jensen's inequality:

$$E[X_{n+1}^+|\mathcal{F}_n] = E[\Phi(X_{n+1})|\mathcal{F}_n] \geq \Phi(E[X_{n+1}|\mathcal{F}_n]) = \Phi(X_n) = X_n^+.$$

Hence, X_n^+ is an $\mathcal{F}_n$-submartingale.
Observe that since $K < \lambda_2$, replacing X_n^+ with X_n in the definitions of S or T does not matter. By hypothesis, $\gamma \geq S$ and both are bounded stopping-times, hence, by Theorem 9.9, since X_n^+ is an $\mathcal{F}_n$-submartingale, (X_γ^+, X_S^+) is an $(\mathcal{F}_S, \mathcal{F}_\gamma)$ submartingale. Then, by applying expectation to both sides of the inequality:

$$\begin{aligned}
&E[X_\gamma^+|\mathcal{F}_S] \geq X_S^+ \\
&E[E[X_\gamma^+|\mathcal{F}_S]] \geq E[X_S^+] \\
&E[X_\gamma^+] \geq E[X_S^+] = L.
\end{aligned}$$

c) (13 points) **Solution:**

Since $K = 0$, $U_m = E[X_{m+n}^+|\mathcal{F}_n]$, $m \geq 0$ is a nonnegative sequence of random variables. (Remember that conditional expectation of a random variable is a random variable as well.) Moreover, by Tower property:

$$U_{m+1} = E[X_{m+n+1}^+|\mathcal{F}_n] = E[E[X_{m+n+1}^+|\mathcal{F}_{n+m}]|\mathcal{F}_n] \geq E[X_{n+m}^+|\mathcal{F}_n] = U_m,$$

since X_n^+ is an $\mathcal{F}_n$-submartingale. We see that for all $m \geq 0$, $U_{m+1} \geq U_m$, then U_m is nondecreasing (or increasing). So we are allowed to use Monotone Convergence Theorem for conditional expectation. Then,

$$\begin{aligned}
E[M_{n+1}|\mathcal{F}_n] &= E[\lim_{m\to\infty} E[X_{m+n+1}^+|\mathcal{F}_{n+1}]|\mathcal{F}_n] \\
&= \lim_{m\to\infty} E[E[X_{m+n+1}^+|\mathcal{F}_{n+1}]|\mathcal{F}_n] \\
&= \lim_{m\to\infty} E[X_{m+n+1}^+|\mathcal{F}_n] = M_n, \quad \text{by Tower property.}
\end{aligned}$$

Note that last step runs also by the fact that if limit exists, it has to be unique and all subsequences approach to same point.

STK-MAT3710: Solution to Exam 2020

Problem 1 a) We have

$$\phi_Y(t) = E[e^{itY}] = \sum_{n=0}^{\infty} e^{itn}\frac{\lambda^n}{n!}e^{-\lambda} = e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda e^{it})^n}{n!} = e^{-\lambda}e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}$$

where we have used the Taylor series for the exponential function.

b) Note first that since the series

$$\sum_{n=0} n\frac{\lambda^n}{n!}e^{-\lambda} = \sum_{n=1}\frac{\lambda^n}{(n-1)!}e^{-\lambda}$$

converges, Y is integrable, and hence ϕ_Y is differentiable with $\phi_Y'(0) = iE(Y)$. Differentiating we get

$$\phi_Y'(t) = e^{\lambda(e^{it}-1)}\lambda i e^{it} = i\lambda e^{it}e^{\lambda(e^{it}-1)},$$

which means that $\phi_Y'(0) = i\lambda$. Thus

$$E[Y] = \frac{\phi_Y'(0)}{i} = \frac{i\lambda}{i} = \lambda.$$

c) We have

$$\phi_X(t) = E[e^{itX}] = e^{it0}(1-\frac{\lambda}{n}) + e^{it1}\frac{\lambda}{n} = 1 + \left(e^{it}-1\right)\frac{\lambda}{n}.$$

d) By independence, we have

$$\begin{aligned}\phi_{S_n} = E[e^{itS_n}] &= E[e^{it(X_n^{(1)}+\cdots+X_n^{(n)})}] = E[e^{itX_n^{(1)}}]\cdot\ldots\cdot E[e^{itX_n^{(n)}}] \\ &= \phi_X(t)^n = \left(1+\left(e^{it}-1\right)\frac{\lambda}{n}\right)^n.\end{aligned}$$

e) By the definition of e (or a formula on the formula sheet),

$$\lim_{n\to\infty}\phi_{S_n}(t) = \lim_{n\to\infty}\left(1+\left(e^{it}-1\right)\frac{\lambda}{n}\right)^n = e^{\lambda(e^{it}-1)}$$

which is the characteristic function of Y, and hence S_n converges to Y in distribution by Lévy's Continuity Theorem.

Problem 2 a) M_n is obviously $\mathcal{F}_n$-measurable and integrable. Putting $\Delta M_n = M_{n+1} - M_n = X_{n+1}$ and using that X_{n+1} is independent of $\mathcal{F}_n$, we have

$$E[\Delta M_n \,|\, \mathcal{F}_n] = E[X_{n+1}\,|\,\mathcal{F}_n] = E[X_{n+1}] = ap + b(1-p) = (a-b)p + b$$

This quantity is 0 when $p = \frac{-b}{a-b}$ (which is between 0 and 1 since a is positive and b is negative), it is positive when $p \geq \frac{-b}{a-b}$, and it is negative when $p \leq \frac{-b}{a-b}$. Hence M is a martingale when $p = \frac{-b}{a-b}$, a submartingale when $p \geq \frac{-b}{a-b}$, and a supermartingale when $p \leq \frac{-b}{a-b}$.

b) Observe first that with $a = 1$, $b = -1$, the critical value in a) becomes $p = \frac{-b}{a-b} = \frac{1}{2}$. Note also that since T is a first hitting time, it is a stopping time. Let $Y_n = M_{n \wedge T}$. Then Y is bounded and $Y_T = M_T$. Moreover, Y is a martingale/submartingale/supermartingale iff M is one. Applying Theorem 9.11 (optional stopping for bounded processes) to the stopping times 1 and T, we get $E[Y_T] = E[Y_1]$ if Y is a martingale, $E[Y_T] \geq E[Y_1]$ if Y is a submartingale, and $E[Y_T] \leq E[Y_1]$ if Y is a supermartingale. As $E[Y_1] = E[M_1]$ is zero, greater than zero, or less than 0 according to whether $p = \frac{1}{2}$, $p > \frac{1}{2}$, or $p < \frac{1}{2}$, we see that $E[M_T] = E[M_1] = 0$ if $p = \frac{1}{2}$, $E[M_T] \geq E[M_1] > 0$ if $p > \frac{1}{2}$, and $E[M_T] \leq E[M_1] < 0$ if $p < \frac{1}{2}$.

Problem 3 a) We have

$$E[Y_N] = E[X_1 + X_2 + \cdots + X_N] = E[X_1] + E[X_2] + \cdots + E[X_N] = 0.$$

As the X_n are independent,

$$\mathrm{var}(Y_n) = \mathrm{var}(X_1 + X_2 + \cdots + X_N) = \mathrm{var}(X_1) + \mathrm{var}(X_2) + \cdots + \mathrm{var}(X_N) = N.$$

b) We have

$$\int_a^\infty e^{-\frac{x^2}{2}}\, dx \leq \int_a^\infty \frac{x}{a} e^{-\frac{x^2}{2}}\, dx$$

since $e^{-\frac{x^2}{2}} \leq \frac{x}{a} e^{-\frac{x^2}{2}}$ on the interval $[a, \infty)$ that we are integrating over. Hence using the substitution $u = \frac{x^2}{2}$, $du = x\, dx$, we get

$$\int_a^\infty e^{-\frac{x^2}{2}}\, dx \leq \int_a^\infty \frac{x}{a} e^{-\frac{x^2}{2}}\, dx = \int_{\frac{a^2}{2}}^\infty \frac{1}{a} e^{-u}\, du = \left[-\frac{1}{a} e^{-u}\right]_{\frac{a^2}{2}}^\infty = \frac{1}{a} e^{-\frac{a^2}{2}}.$$

c) Since Y_N is gaussian with mean 0 and variance N, we have

$$P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] = \int_{\sqrt{N^{1+\epsilon}}}^\infty \frac{1}{\sqrt{2\pi N}} e^{-\frac{u^2}{2N}}\, du.$$

Substituting $x = \frac{u}{\sqrt{N}}$, $dx = \frac{du}{\sqrt{N}}$ and using the inequality above, we get

$$P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] = \int_{N^{\frac{\epsilon}{2}}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, du \leq \frac{e^{-\frac{N^\epsilon}{2}}}{\sqrt{2\pi} N^{\frac{\epsilon}{2}}} \leq \frac{e^{-\frac{N^\epsilon}{2}}}{\sqrt{2\pi}}.$$

d) By the first half of Borel-Cantelli's lemma,

$$P\left[Y_N > \sqrt{N^{1+\epsilon}} \text{ for infinitely many } N\right] = 0$$

if

$$\sum_{N=1}^\infty P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] < \infty.$$

By c)

$$\sum_{N=1}^\infty P\left[Y_N > \sqrt{N^{1+\epsilon}}\right] \leq \frac{1}{\sqrt{2\pi}} \sum_{N=1}^\infty e^{-\frac{N^\epsilon}{2}},$$

and hence it suffices to show that the series $\sum_{N=1}^{\infty} e^{-\frac{N^\epsilon}{2}}$ converges. Using the hint with $p = 2$, we see that

$$\lim_{N\to\infty} \frac{e^{-\frac{N^\epsilon}{2}}}{\frac{1}{N^2}} = \lim_{N\to\infty} \frac{N^2}{e^{\frac{N^\epsilon}{2}}} = 0.$$

Since $\sum_{N=1}^{\infty} \frac{1}{N^2}$ converges, this means that $\sum_{N=1}^{\infty} e^{-\frac{N^\epsilon}{2}}$ converges by the Limit Comparison Test, and we are done.

(To see that the hint is correct, choose $m \in \mathbb{N}$ so large that $m\epsilon > p$ and note that by Taylor's Formula

$$e^{\frac{N^\epsilon}{2}} > \sum_{k=0}^{m} \frac{(\frac{N^\epsilon}{2})^k}{k!} > \frac{N^{m\epsilon}}{2^m m!}.$$

Hence

$$\lim_{N\to\infty} \frac{N^p}{e^{\frac{N^\epsilon}{2}}} \leq \lim_{N\to\infty} \frac{N^p}{\frac{N^{m\epsilon}}{2^m m!}} = 0$$

as $m\epsilon > p$.)

Exam in STK-MAT3710/4710, Fall 2019. Solutions

Problem 1

Cut $\mathbb{N}$ into sequences of length 17:

$I_0 = \{1, 2, \ldots, 17\}, I_1 = \{18, 19, \ldots, 34\}, \ldots, I_k = \{17k+1, 17k+2, \ldots, 18k\}, \ldots$.

The sets

$$A_k = \{\omega : X_j(\omega) = 6 \text{ for all } j \in I_k\}$$

are independent and have probability $P(A_k) = \frac{1}{6^{17}} > 0$. Hence $\sum_{k=0}^{\infty} P(A_k) = \infty$. According to the Converse Borel-Cantelli Lemma, $\limsup A_k$ has probability 1, and each ω in $\limsup A_k$ clearly contains infinitely many occurences of 17 consecutive 6's.

Problem 2

a) Differentiating we see that the distribution has the density

$$f(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{1}{2} & \text{for} -1 \leq x \leq 1 \\ 0 & \text{for } x > 1 \end{cases}$$

The characteristic function is

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \frac{1}{2}\int_{-1}^{1} e^{itx}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{itx}}{it}\right]_{x=-1}^{x=1} = \frac{e^{it} - e^{-it}}{2it} = \frac{\sin t}{t}$$

b) We have

$$\phi_{S_n}(t) = E\left[e^{itS_n}\right] = E\left[e^{i\frac{t}{\sqrt{n}}X_1} e^{i\frac{t}{\sqrt{n}}X_2} \cdot \ldots \cdot e^{i\frac{t}{\sqrt{n}}X_n}\right]$$

$$= E\left[e^{i\frac{t}{\sqrt{n}}X_1}\right] E\left[e^{i\frac{t}{\sqrt{n}}X_2}\right] \cdot \ldots \cdot E\left[e^{i\frac{t}{\sqrt{n}}X_n}\right] = \left(\phi(\frac{t}{\sqrt{n}})\right)^n = \left(\frac{\sin\frac{t}{\sqrt{n}}}{\frac{t}{\sqrt{n}}}\right)^n$$

c) The Taylor expansion of the sine function is

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

and hence

$$\frac{\sin\frac{t}{\sqrt{n}}}{\frac{t}{\sqrt{n}}} = 1 - \frac{t^2}{6n} + o\left(\frac{t^2}{n}\right)$$

Consequently (e.g. by Lemma 6.34)

$$\phi_{S_n}(t) = \left(1 - \frac{t^2}{6n} + o\left(\frac{t^2}{n}\right)\right)^n \to e^{-\frac{t^2}{6}}.$$

As $e^{-\frac{t^2}{6}}$ is the characteristic function of a normal distribution with mean 0 and variance $\sigma^2 = \frac{1}{3}$, the result follows from Lévy's Continuity Theorem.

Problem 3

a) Note first that

$$E[\Delta M_m|\mathcal{F}_n] = E[M_{m+1}|\mathcal{F}_n] - E[M_m|\mathcal{F}_n] = M_n - M_n = 0$$

by the martingale property.

For the second part, observe that since M_n is $\mathcal{F}_n$-measurable, we have

$$E[\Delta M_m M_n|\mathcal{F}_n] = M_n E[\Delta M_m|\mathcal{F}_n] = 0$$

where the last step uses what we just proved above.

b) As ΔM_n is $\mathcal{F}_m$-measurable, we have by the tower property

$$E[\Delta M_m \Delta M_n|\mathcal{F}_n] = E\big[E[\Delta M_n \Delta M_m|\mathcal{F}_m]|\mathcal{F}_n\big]$$

$$= E\big[\Delta M_n E[\Delta M_m|\mathcal{F}_m]|\mathcal{F}_n\big] = E\big[\Delta M_n \cdot 0|\mathcal{F}_n\big] = 0$$

c) Since $M_m - M_n = \sum_{k=n}^{m-1} \Delta M_k$, we have

$$E\left[(M_m - M_n)^2|\mathcal{F}_n\right] = E\left[\left(\sum_{k=n}^{m-1} \Delta M_k\right)^2 \middle| \mathcal{F}_n\right] = \sum_{i,j=n}^{m-1} E\left[\Delta M_i \Delta M_j|\mathcal{F}_n\right]$$

$$= 2 \sum_{n \le i < j \le m-1} E[\Delta M_i \Delta M_j|\mathcal{F}_n] + \sum_{k=n}^{m-1} E[\Delta M_k^2|\mathcal{F}_n]$$

All terms in the first sum are zero since by b) and the tower property, we have:

$$E[\Delta M_i \Delta M_j|\mathcal{F}_n] = E\left[E[\Delta M_i \Delta M_j|\mathcal{F}_i]|\mathcal{F}_n\right] = E[0|\mathcal{F}_n] = 0$$

Hence

$$E[(M_m^2 - M_n)^2|\mathcal{F}_n] = \sum_{k=n}^{m-1} E[\Delta M_k^2|\mathcal{F}_n]$$

Problem 4

a) Assume $x_1 < x_2$. As F is increasing, we have

$$K(x_1) = E[F(x_1 - Y)] \le E[F(x_2 - Y)] = K(x_2)$$

which shows that K is increasing. To prove right continuity, note that if $\{x_n\}$ is a sequence decreasing to x, the Bounded Convergence Theorem tells us that (note that the integrand is bounded by 1):

$$\lim_{n\to\infty} K(x_n) = \lim_{n\to\infty} E[F(x_n - Y)] = E[\lim_{n\to\infty} F(x_n - Y)] = E[F(x-Y)] = K(x)$$

since F is right continuous. It follows that K is right continuous.

To check the limit conditions, first note that if $x \to -\infty$, then $x - Y \to -\infty$, and hence $\lim_{x\to-\infty} F(x-Y) = 0$. Applying the Bounded Convergence Theorem again, we see that if $x_n \to -\infty$, then

$$\lim_{n\to\infty} K(x_n) = \lim_{n\to\infty} E[F(x_n - Y)] = E[\lim_{n\to\infty} F(x_n - Y)] = E[0] = 0$$

The other limit is similar: If $x_n \to \infty$, then

$$\lim_{n\to\infty} K(x_n) = \lim_{n\to\infty} E[F(x_n - Y)] = E[\lim_{n\to\infty} F(x_n - Y)] = E[1] = 1$$

as $\lim_{x\to\infty} F(x) = 1$.

b) We have

$$H(x) = P[X + Y \le x] = P[X + \sum_{n\in\mathbb{N}} a_n \mathbf{1}_{A_n} \le x] = \sum_{n\in\mathbb{N}} P([X \le x - a_n] \cap A_n)$$

$$= \sum_{n\in\mathbb{N}} P([X \le x - a_n] \cap [Y = a_n]) = \sum_{n\in\mathbb{N}} P[X \le x - a_n] P[Y = a_n]$$

$$= \sum_{n\in\mathbb{N}} P[X \le x - a_n] P[A_n] = \sum_{n\in\mathbb{N}} F(x - a_n) P[A_n] = E[F(x-Y)]$$

where we have used the independence to get the equality in the middle of the second line.

c) As $\underline{Y}_n$ increases to Y, we see that $x - \underline{Y}_n$ approaches $x - Y$ from the right, and since F is right continuous, it follows that $F(x - Y_n) \to F(x - Y)$. Using the Bounded Convergence Theorem again, we get

$$\lim_{n\to\infty} E[F(x - \underline{Y}_n)] = E[\lim_{n\to\infty} F(x - \underline{Y}_n)] = E[F(x-Y)] \ .$$

d) If H_n is the distribution function of $X + Y_n$, we know from b) that

$$H_n(x) = E[F(x - Y_n)] \ ,$$

and we have already seen that the right hand side converges pointwise to $K(x) = E[F(x-Y)]$ which is a distribution function by a). On the other hand, H_n is the distribution function of $X + \underline{Y}_n$ and since $X + \underline{Y}_n$ converges pointwise (and hence in distribution) to $X + Y$, the distribution functions H_n converge to the distribution function H of $X + Y$ at all continuity points of H. This means that the two distribution functions H and K coincide at all continuity points of H, and hence they have to be equal for all x.

The last formula,

$$H(x) = \int_{-\infty,\infty} F(x-y)\,dG(y)$$

follows immediately from $H(x) = E[F(x-Y)]$ as we have $E[f(Y)] = \int_{-\infty}^{\infty} f(y)\,dG(y)$ for all Borel functions f.