STK-MAT3710
Probability Theory

OBLIG 1

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Problem 1

A σ-**algebra** $\mathcal{F}$ on a set E is a collection of subsets of E such that:

1. $E \in \mathcal{F}$,
2. if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$,
3. if $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

With this, $\mathcal{F}$ is closed under complements and countable unions.

If we let $E = \{1, 2, 3\}$. Then a σ-algebra on E is:

$$\mathcal{F} = \{\varnothing, \{1, 2\}, \{3\}, E\}$$

b)

A measurable space is a pair $(E, \mathcal{F})$ where $\mathcal{F}$ is a σ-algebra on E. For example, $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra.

c)

A measure μ on a measurable space $(E, \mathcal{F})$ is a function $\mu : \mathcal{F} \to [0, \infty]$ such that

1. $\mu(\varnothing) = 0$
2. $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ for disjoint $A_n \in \mathcal{F}$

The triple $(E, \mathcal{F}, \mu)$ is a measure space. For example, Lebesgue measure λ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

d)

A measure μ on $(E, \mathcal{F})$ is complete if every subset of a μ-null set is measurable and has measure zero; that is, if $N \in \mathcal{F}$ with $\mu(N) = 0$ and $A \subseteq N$, then $A \in \mathcal{F}$ and $\mu(A) = 0$.

Example: The Lebesgue measure on $\mathbb{R}$ is complete.

Non-complete example: Let $\mu(A) = \lambda(A \cap \mathbb{Q})$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where λ denotes Lebesgue measure. Then $\mathbb{R} \setminus \mathbb{Q}$ has μ-measure zero, but some of its subsets are not Borel-measurable.

e)

A function $f : (E, \mathcal{E}) \to (F, \mathcal{F})$ between measurable spaces is measurable if $f^{-1}(B) \in \mathcal{E}$ for all $B \in \mathcal{F}$.

For example, $f(x) = x^2 : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ is measurable.

f)

A property $P(\omega)$ is said to hold *almost everywhere* (a.e.) with respect to a measure μ — or *almost surely* (a.s.) in the probabilistic setting — if the set where P fails has measure zero.

Example: On $([0, 1], \mathcal{B}([0, 1]), \lambda)$, define

$$f(x) = \begin{cases} 0, & x \notin \mathbb{Q}, \\ 1, & x \in \mathbb{Q}. \end{cases}$$

Then $f(x) = 0$ almost everywhere, since the rationals $\mathbb{Q} \cap [0, 1]$ form a Lebesgue null set, but $f(x) \neq 0$ everywhere.

g)

Let $f : (E, \mathcal{E}) \to (F, \mathcal{F})$ be a measurable map, and let μ be a measure on $(E, \mathcal{E})$. The *push-forward measure* $f_*\mu$ on $(F, \mathcal{F})$ is defined by

$$f_*\mu(B) = \mu\big(f^{-1}(B)\big), \qquad B \in \mathcal{F}.$$

Example: For $f(x) = x^2$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and μ the standard normal measure, $f_*\mu$ is the distribution of X^2 where $X \sim \mathcal{N}(0, 1)$.

h)

A *simple function* on a measurable space $(E, \mathcal{E})$ is a measurable function that takes only finitely many real values, i.e.

$$\varphi = \sum_{k=1}^{n} c_k \mathbf{1}_{A_k}, \qquad c_k \in \mathbb{R},\ A_k \in \mathcal{E}.$$

Example: $\varphi(x) = \mathbf{1}_{[0,1]}(x) + 2\,\mathbf{1}_{(1,2]}(x)$ on $\mathbb{R}$.

i)

Let $\varphi = \sum_{k=1}^{n} c_k \mathbf{1}_{A_k}$ be a simple function on $(E, \mathcal{E})$, where $c_k \in \mathbb{R}$ and $A_k \in \mathcal{E}$ are disjoint. The integral of φ with respect to a measure μ is defined as

$$\int_E \varphi \, d\mu = \sum_{k=1}^{n} c_k \, \mu(A_k),$$

with the convention that $0 \cdot \infty = 0$.

j)

Let $f : E \to \mathbb{R}$ be a measurable function. If $f \geq 0$, its integral with respect to a measure μ is defined by

$$\int_E f \, d\mu = \sup \left\{ \int_E \varphi \, d\mu \ : \ 0 \leq \varphi \leq f, \ \varphi \text{ simple} \right\}.$$

For a general measurable function f, write $f = f^+ - f^-$ where

$$f^+ = \max(f, 0), \qquad f^- = \max(-f, 0).$$

If both $\int f^+ \, d\mu$ and $\int f^- \, d\mu$ are finite, f is said to be μ-integrable, and

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

k)

Let $(E, \mathcal{E}, \mu)$ be a measure space, $(F, \mathcal{F})$ a measurable space, and $f : E \to F$ a measurable map. If $\nu = f_* \mu$ denotes the push-forward measure defined by $\nu(B) = \mu(f^{-1}(B))$ for $B \in \mathcal{F}$, then for any measurable function $g : F \to \mathbb{R}$,

$$\int_F g \, d\nu = \int_E (g \circ f) \, d\mu,$$

whenever the integrals exist.

l)

A *probability measure* is a measure P on a measurable space $(\Omega, \mathcal{A})$ such that $P(\Omega) = 1$. The triplet $(\Omega, \mathcal{A}, P)$ is called a *probability space*.

A *random variable* is a measurable map

$$X : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})).$$

Example: Let $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}([0, 1])$, and let P be the Lebesgue measure. Then $X(\omega) = \omega$ is a random variable on $(\Omega, \mathcal{A}, P)$.

m)

Let X be a random variable with distribution function F_X, and let $Y = g(X)$ where $g : \mathbb{R} \to \mathbb{R}$ is a strictly increasing and continuous function. Then the distribution function of Y is given by

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P\big(X \leq g^{-1}(y)\big) = F_X(g^{-1}(y)).$$

Example: If $X \sim \mathcal{N}(0,1)$ and $Y = e^X$, then

$$F_Y(y) = P(e^X \leq y) = P(X \leq \log y) = \Phi(\log y), \qquad y > 0,$$

where Φ denotes the standard normal distribution function.

n)

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable. If $X \geq 0$, the expectation of X is defined as

$$\mathbb{E}[X] = \int_\Omega X \, dP = \int_0^\infty P(X > t) \, dt.$$

For a general random variable X, write $X = X^+ - X^-$ where $X^+ = \max(X, 0)$ and $X^- = \max(-X, 0)$. If at least one of $\mathbb{E}[X^+]$ or $\mathbb{E}[X^-]$ is finite, then X is integrable and

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-].$$

o)

Let X be a real-valued random variable with distribution function F_X. Then the expectation of X can be expressed as the Lebesgue–Stieltjes integral

$$\mathbb{E}[X] = \int_{-\infty}^\infty x \, dF_X(x).$$

Equivalently, it can be written in terms of the tail probabilities as

$$\mathbb{E}[X] = \int_0^\infty \big(1 - F_X(x)\big) \, dx - \int_{-\infty}^0 F_X(x) \, dx.$$

p)

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $X_1, \ldots, X_n$ random variables defined on it. They are said to be *independent* if, for all Borel sets $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$,

$$P(X_1 \in B_1, \ldots, X_n \in B_n) = \prod_{i=1}^n P(X_i \in B_i).$$

Equivalently, the σ-algebras generated by the random variables,

$$\sigma(X_i) = \{X_i^{-1}(B) : B \in \mathcal{B}(\mathbb{R})\}, \qquad i = 1, \ldots, n,$$

form an independent family of sub-σ-algebras of $\mathcal{A}$.

Problem 2

a)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the measurable space with

$$\Omega = \mathbb{R}, \quad \mathcal{A} = \mathcal{B}(\mathbb{R}), \quad \mathbb{P}[B] = \int_B \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz, \qquad B \in \mathcal{B}(\mathbb{R}).$$

To verify that $\mathbb{P}$ is a probability measure:

- $\mathbb{P}[B] \geq 0$ for all $B \in \mathcal{B}(\mathbb{R})$, since the integrand is non-negative.
- $\mathbb{P}[\mathbb{R}] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = 1$ (Gaussian integral).
- For any countable collection of pairwise disjoint sets $(B_n)_{n\geq 1} \subset \mathcal{B}(\mathbb{R})$,

  $$\mathbb{P}\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \int_{\cup_n B_n} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = \sum_{n=1}^{\infty} \int_{B_n} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz = \sum_{n=1}^{\infty} \mathbb{P}[B_n],$$

  by countable additivity of the Lebesgue integral.

Hence, $\mathbb{P}$ satisfies non-negativity, normalization, and countable additivity, so it is a probability measure on $(\Omega, \mathcal{A})$.

b)

We have $\rho : \Omega \to \mathbb{R}$ given by $\rho(\omega) = \mu + \sigma\omega$. The map $\omega \mapsto \mu + \sigma\omega$ is affine and hence continuous, so it is Borel–measurable. Equivalently, for any $B \in \mathcal{B}(\mathbb{R})$,

$$\rho^{-1}(B) = \{\omega \in \Omega : \mu + \sigma\omega \in B\} = \{\omega \in \Omega : \omega \in (B - \mu)/\sigma\} = \frac{B - \mu}{\sigma} \in \mathcal{B}(\mathbb{R}).$$

Thus ρ is $\mathcal{A}/\mathcal{B}(\mathbb{R})$–measurable, i.e. a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$.

(c)

Define $C : \Omega \to (0, \infty)$ by

$$C(\omega) = C_0\, e^{\rho(\omega)\, t} = C_0\, e^{(\mu + \sigma\omega)t}.$$

The functions $\omega \mapsto (\mu + \sigma\omega)t,\ \ x \mapsto e^x$, and $x \mapsto C_0 x$ are continuous, hence measurable. Therefore their composition is measurable:

$$C = \big(\, x \mapsto C_0 x \,\big) \circ \big(\, x \mapsto e^x \,\big) \circ \big(\, \omega \mapsto (\mu + \sigma\omega)t \,\big).$$

Hence C is $\mathcal{A}/\mathcal{B}((0, \infty))$–measurable and thus a random variable on $(\Omega, \mathcal{A}, \mathbb{P})$.

d)

Let $Z(\omega) = \omega$ under $\mathbb{P}$; then $Z \sim \mathcal{N}(0,1)$. Hence

$$\rho = \mu + \sigma Z, \qquad C = C_0 \exp\big((\mu + \sigma Z)t\big).$$

Therefore

$$\log C \sim \mathcal{N}\big(\log C_0 + \mu t,\ (\sigma t)^2\big)\,,$$

so C is log-normal:

$$C \sim \text{LogNormal}\big(\log C_0 + \mu t,\ (\sigma t)^2\big)\,,$$

with density (for $c > 0$)

$$f_C(c) = \frac{1}{c\,\sigma t\sqrt{2\pi}} \exp\left(-\frac{\big(\log(c/C_0) - \mu t\big)^2}{2(\sigma t)^2}\right).$$

Its expectation is

$$\mathbb{E}[C] = C_0 \exp\big(\mu t + \tfrac{1}{2}(\sigma t)^2\big)\,.$$

(e)

Let $Y = (C - K)\,\mathbf{1}_{\{C>K\}} = (C-K)^+$ with $K > 0$.

Distribution of Y. $Y \geq 0$ a.s. and has an atom at 0:

$$\mathbb{P}(Y = 0) = \mathbb{P}(C \leq K) = \Phi\left(\frac{\log(K/C_0) - \mu t}{\sigma t}\right).$$

For $y < 0$, $F_Y(y) = 0$. For $y \geq 0$,

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(C \leq K + y) = \Phi\left(\frac{\log\big((K+y)/C_0\big) - \mu t}{\sigma t}\right).$$

Hence Y is a *mixed* random variable: a point mass at 0 plus an absolutely continuous part on $(0, \infty)$ with density

$$f_Y(y) = f_C(K + y), \qquad y > 0.$$

Expectation of Y. Several equivalent integral forms:

$$\mathbb{E}[Y] = \int_0^\infty \mathbb{P}(Y > y)\,dy = \int_0^\infty \mathbb{P}(C > K{+}y)\,dy = \int_K^\infty \big(1{-}F_C(c)\big)\,dc = \int_K^\infty (c{-}K)\,f_C(c)\,dc.$$

Optional closed form (in terms of Φ). Let $m = \log C_0 + \mu t$ and $s = \sigma t$. Then

$$\mathbb{E}[Y] = e^{m + \frac{1}{2}s^2}\,\Phi\left(\frac{m + s^2 - \log K}{s}\right) - K\,\Phi\left(\frac{m - \log K}{s}\right).$$

Problem 3

(a)

Let $a, b \in [0, \infty)$ and $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Consider the convex functions $\phi(u) = \frac{u^p}{p}$ and $\psi(v) = \frac{v^q}{q}$ on $[0, \infty)$. By convexity (Fenchel–Young),

$$ab \leq \phi(a) + \psi(b) = \frac{a^p}{p} + \frac{b^q}{q}.$$

Equality holds iff $a^{p-1} = b^{q-1}$, i.e. $a^p = b^q$.

(b)

Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g : \mathbb{R} \to \mathbb{R}$ integrable with $\int |f|^p < \infty$ and $\int |g|^q < \infty$ (Lebesgue measure). If either norm is zero the claim is trivial. Otherwise set

$$F = \frac{|f|}{\|f\|_p}, \qquad G = \frac{|g|}{\|g\|_q}.$$

Then $F, G \geq 0$ and $\int F^p = \int G^q = 1$. By Young's inequality applied pointwise, $FG \leq \frac{F^p}{p} + \frac{G^q}{q}$. Integrating,

$$\int |fg| = \|f\|_p \, \|g\|_q \int FG \leq \|f\|_p \, \|g\|_q \left(\frac{1}{p} + \frac{1}{q} \right) = \|f\|_p \, \|g\|_q.$$

Hence

$$\int |fg| \leq \left(\int |f|^p \right)^{1/p} \left(\int |g|^q \right)^{1/q}.$$

(c)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and X, Y random variables with $\mathbb{E}[|X|^p] < \infty, \mathbb{E}[|Y|^q] < \infty$. Applying part (b) with $\mu = \mathbb{P}$, $f = |X|$, $g = |Y|$ gives

$$\mathbb{E}[\,|XY|\,] \leq \left(\mathbb{E}[|X|^p] \right)^{1/p} \left(\mathbb{E}[|Y|^q] \right)^{1/q}.$$