UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4060/STK9060 — Time series

Day of examination: Tuesday May 29th 2018

Examination hours: 2.30 p.m. – 6.30 p.m.

This problem set consists of 2 pages.

Appendices: Note on state space model and Kalman filter

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let $\{x_t\}_{t=-\infty}^{\infty}$ be a stationary time series with $E[x_t] = 0$ and $cov(x_{t+h}, x_t) = \gamma_x(h) < \infty$ for $h = 0, \pm 1, \ldots$.

a) When does the spectral density, $f_x(\omega)$, of the time series $\{x_t\}$ exist and how is it defined?

b) Let $\{a_j\}_{j=-\infty}^{\infty}$ be a sequence of real numbers such that $\Sigma_{j=-\infty}^{\infty}|a_j| < \infty$. Define a new time series $\{y_t\}_{t=-\infty}^{\infty}$ by $y_t = \Sigma_{j=-\infty}^{\infty} a_j x_{t-j}$. Explain how the spectral density, $f_y(\omega)$, of $\{y_t\}$ can be expressed by the spectral density $f_x(\omega)$.

c) Let $y_t = x_t - x_{t-4}$ where $\{x_t\}$ is a moving average time series of order 1, i.e. a MA(1) time series. What is the spectral density, $f_y(\omega)$, of $\{y_t\}$?

Problem 2

Consider the linear process

$$x_t = \Sigma_{j=-\infty}^{\infty} \psi_j w_{t-j} \tag{1}$$

where w_t is white noise $w_t \sim wn(0, \sigma_w^2)$ and $\Sigma_{j=-\infty}^{\infty}|\psi_j| < \infty$.

a) Show that the covariance of x_t is given by $cov(x_{t+h}, x_t) = \gamma_x(h) = \sigma_w^2 \Sigma_{j=-\infty}^{\infty} \psi_{j+h}\psi_j$.

b) Explain what an ARMA(p,q) time series is and what it means that the time series is causal and invertible. What are the conditions an ARMA(p,q) time series must satisfy in order to be causal and invertible?

c) Express a causal and invertible ARMA(1,1) time series on the form (1).

(Continued on page 2.)

d) Use the results from part a) and c) to find an expression for the covariance of a causal and invertible ARMA(1,1) time series.

Problem 3

Let $\{w_t\}$ and $\{v_t\}$ be to sequences of independent normally distributed random variables where $w_t \sim N(0, \sigma_w^2)$ and $v_t \sim N(0, \sigma_v^2)$.

a) Show that the two recursive equations

$$\bigtriangledown^2 \mu_t = w_t \text{ and } y_t = \mu_t + v_t \tag{2}$$

and

$$\mu_t = \mu_{t-1} + \eta_t,\ \eta_t = \eta_{t-1} + w_t \text{ and } y_t = \mu_t + v_t \tag{3}$$

both can be expressed as state space models by defining the states as $x_t = (\mu_t, \mu_{t-1})'$ and $x_t = (\mu_t, \eta_t)'$. Explain what the state and measurement equations are.

b) Show that the normal initial distributions $x_0^0 = (0,0)'$ and $P_0^0 = \begin{pmatrix} \sigma_w^2 & 0 \\ 0 & \sigma_w^2 \end{pmatrix}$ for (2) and $x_0^0 = (0,0)'$ and $p_0^0 = \begin{pmatrix} \sigma_w^2 & \sigma_w^2 \\ \sigma_w^2 & 2\sigma_w^2 \end{pmatrix}$ for (3) will imply the same distributions for observations from the recursions in (2) and (3) .

c) Using the initial distributions from part b) find the Kalman gain at time t=1, i.e. K_1, for the two models.

END

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4060 — Time series

Day of examination: Monday June 6th 2016

Examination hours: 14.30 – 18.30

This problem set consists of 2 pages.

Appendices: Note on state space model and Kalman filter

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let x_t be a stationary time series with $E[x_t] = 0$ and $var(x_t) < \infty$.

a) Explain what an ARMA(p,q) time series is and what it means that is causal and invertible.

b) Is the time series x_t satisfying

$$x_t = -0.1x_{t-1} + 0.12x_{t-2} + w_t$$

causal?

c) Explain what the partial autocorrelation coefficient is and how it is used to determine the order of an autoregressive time series.

Problem 2

The bivariate AR(1) process

$$\begin{pmatrix} \psi_t \\ \psi_t^* \end{pmatrix} = 0.5 \begin{pmatrix} \cos\lambda & \sin\lambda \\ -\sin\lambda & \cos\lambda \end{pmatrix} \begin{pmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{pmatrix} + \begin{pmatrix} \kappa_t \\ \kappa_t^* \end{pmatrix}$$

where κ_t and κ_t^* are independent $N(0, \sigma_\kappa^2)$ and $N(0, \sigma_{\kappa^*}^2)$ respectively can be used for describing cyclical patterns. Consider first λ as known. Then two ways to formulate the model as a part of a larger model is as

(i) the *trend and cycle* model where

$$\begin{aligned} y_t &= \mu_t + \psi_t + \epsilon_t \\ \mu_t &= \mu_{t-1} + \beta_{t-1} + \eta_t \\ \beta_t &= \beta_{t-1} + \zeta_t \end{aligned}$$

(Continued on page 2.)

and as

(ii) the *cyclical trend* model where

$$\begin{aligned} y_t &= \mu_t + \epsilon_t \\ \mu_t &= \mu_{t-1} + \beta_{t-1} + \psi_{t-1} + \eta_t \\ \beta_t &= \beta_{t-1} + \zeta_t \end{aligned}$$

where ϵ_t, η_t and ζ_t are independent and mutual independent $N(0, \sigma_\epsilon^2)$, $N(0, \sigma_\eta^2)$ and $N(0, \sigma_\zeta^2)$ variables, which also are independent of the κ_t's and κ_t^*'s.

a) Formulate both models as state-space models.

b) Assume that the initial state has expectation zero and covariance matrix equal to the identity matrix so $\mathbf{x}_0^0 = 0$ and $P_0^0 = I$. Describe what the Kalman gain at time 1, K_1, is for the *trend and cycle* model.

c) If now the parameter λ is not assumed to be known, how would you estimate it based on observations $y_1, \ldots, y_n$?

Problem 3

Consider observations $z_1, \ldots, z_n$ from a time series with covariance function $\gamma(h)$ satisfying $\sum_{-\infty}^{\infty} |\gamma(h)| < \infty$ so that the spectral density $f_z(\omega)$ exists. Recall that the discrete Fourier transform is given as $d_z(j/n) = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} z_t e^{-2\pi i (j/n) t}$, $j = 0, \ldots, n-1$.

a) Define the periodogram and state its main properties. Explain how the periodogram can be used for estimating the spectral density.

b) Let the time series y_t be defined as

$$y_t = \mu + x_t, \ t = 1, \ldots, n$$

where x_t is a stationary time series with discrete Fourier transform d_x. The expectation $E[x_t] = 0$ and μ is a known constant. Express the discrete Fourier transform of y_t in terms of the discrete Fourier transform of x_t, i.e. d_x, and μ.

c) Let y_t be as in part b) and let

$$y'_t = \begin{cases} y_t, \ t = 1, \ldots, n \\ 0, \ t = n+1, \ldots, n' \end{cases}$$

i.e. y_t padded. Find the discrete Fourier transform of y'_t. Compare the result to what you found in part b).

d) Discuss whether it is preferable to subtract μ from the series y_t or from the padded version y'_t when the focus of interest is the periodic behavior of y_t.

END

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK9060 — Time Series

Day of examination: Wednesday May 28th 2014

Examination hours: 09.00 – 13.00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

In this problem you are asked to consider autoregressive time series.

a) How is an autoregressive time series of order p defined? Explain what it means that it is causal.

Consider the autoregressive time series of order two

$$x_t = x_{t-1}/3 + 2x_{t-2}/9 + w_t$$

where w_t is a time series of white noise, i.e. $w_t \sim wn(0, \sigma^2)$.

b) Show that x_t defined above is causal and find the representation $x_t = \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$ where $\psi_0 = 1$.

c) Explain that the auto correlation function of x_t satisfies a difference equation. Find the the auto correlation function of x_t.

d) What is the partial autocorrelation function of x_t?

Problem 2

Consider observations $y_1, \ldots, y_n$ described by the equations

$$\begin{aligned} x_t &= \mu + a x_{t-1} + w_t, \ |a| < 1 \\ y_t &= b x_t + v_t \end{aligned}$$

where $w_t, t = 1, \ldots, n$ and $v_t, t = 1, \ldots, n$ are independent random variables where $w_t \sim iidN(0, \sigma_w^2)$ and $v_t \sim iidN(0, \sigma_v^2)$. The variable x_0, $x_0 \sim N(0, \sigma_0^2)$, is independent of w_t and v_t

a) Interpret the equations as a state-space model. Explain what the state equation and the observation equation are.

(Continued on page 2.)

b) What are the conditions for the bivariate series $(x_t, y_t)'$ to be stationary? Find the expectation and covariance of the stationary series $(x_t, y_t)'$ satisfying the equations above?

c) What is the stationary distribution of $(x_t, y_t)'$ described in part b)?

Problem 3

Let $x_t, t = 0, \pm 1, \pm 2, \ldots$ be a stationary time series such that $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$ where $\gamma(h)$ is the covariance between x_{t+h} and x_t.

a) Define the spectral density $f_X(\omega)$ of x_t. Explain why $f_X(\omega) = f_X(-\omega)$ and $f_X(\omega) = f_X(1-\omega)$.

Let $a_j, j = 0, \pm 1, \pm 2, \ldots$ be a sequence of scalars so that $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Let $y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}$.

b) Show that the spectral density of y_t has the form

$$f_Y(\omega) = |A(\omega)|^2 f_X(\omega)$$

where $A(\omega) = \sum_{j=-\infty}^{\infty} a_j \mathrm{e}^{-2\pi i \omega j}$.

c) Let w_t be a white noise series, $w_t \sim wn(0, \sigma_w^2)$, and y_t be the ARMA(p,q) process defined by $\phi(B)y_t = \theta(B)w_t$ where the roots of the polynomials $\phi(z) = 0$ and $\theta(z) = 0$ are outside the unit circle. Use the result from part b) to explain that

$$f_Y(\omega) = \sigma_w^2 \frac{|\theta(\mathrm{e}^{-2\pi i \omega})|^2}{|\phi(\mathrm{e}^{-2\pi i \omega})|^2}.$$

END

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4060 — Time Series

Day of examination: Monday June 4'th 2012

Examination hours: 09.00 – 13.00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider a linear time series $\{x_t\}_{t=-\infty}^{\infty}$ of the form

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$$

where $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$ and $\{w_t\}_{t=-\infty}^{\infty}$ is a sequence of white noise, i.e a sequence of random variables where $E[w_t] = 0$, $E[w_t^2] = \sigma_w^2$ for all t and $E[w_s w_t] = 0$ for all $s \neq t$.

a) Explain why $\{x_t\}_{t=-\infty}^{\infty}$ is weakly stationary. Define the autocovariances $\gamma(h)$ of $\{x_t\}_{t=-\infty}^{\infty}$ and show that

$$\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h},\ h = 0, \pm 1, \pm 2, \ldots.$$

b) Explain why a weakly stationary autoregressive time series of order 1 having finite variance and satisfying the stochastic difference equation

$$x_t = \phi x_{t-1} + w_t,\ t = 0, \pm 1, \pm 2, \ldots$$

where $|\phi| < 1$, is a linear process. What does it mean that it is causal?

c) What is meant by a causal, invertible ARMA-process? How can properties of their autocorrelation (ACF) and partial autocorrelaton functions (PACF) be used to identify such processes?

d) Define the Yule-Walker equations, and explain how they can be used to estimate the coefficients in a stationary, causal autoregressive process of order p, i.e. an AR(p) process.

(Continued on page 2.)

e) Consider the stationary, causal autoregressive process of order 2

$$x_t = (5/6)x_{t-1} - (1/6)x_{t-2} + w_t.$$

Find the autocorrelation (ACF) and partial autocorrelation function (PACF) of this process.

Problem 2

Consider a stationary time series $\{x_t\}_{t=-\infty}^{\infty}$ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$.

a) Explain how the spectral density is defined and how it should be interpreted.

b) What is the spectral density in the autoregressive time series

$$x_t = 0.5x_{t-1} + w_t, t = 0, \pm 1, \pm 2, \ldots$$

where $\{w_t\}_{t=-\infty}^{\infty}$ is a sequence of white noise, $wn(0, \sigma_w^2)$.

c) If $x_1, \ldots, x_n$ are observations from the time series $\{x_t\}_{t=-\infty}^{\infty}$ defined in the beginning of the problem, how are the periodogram $I(j/n), j = 0, 1, \ldots, n-1$ defined? Show that the expectation satisfies

$$E[I(j/n)] = \sum_{h=-(n-1)}^{(n-1)} (1 - \frac{|h|}{n})\gamma(h)e^{-2\pi i(j/n)h},\ j = 1, \ldots, n-1$$

d) Describe the main properties of the distribution of the periodogram, and explain how they can used to estimate the spectral density.

END

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i: STK4060/STK9060 — Tidsrekker

Eksamensdag: Torsdag 3. juni 2010

Tid for eksamen: 14.30 – 17.30

Oppgavesettet er på 2 sider.

Vedlegg: Ingen

Tillatte hjelpemidler: Godkjent kalkulator

Kontroller at oppgavesettet er komplett før
du begynner å besvare spørsmålene.

Oppgave 1

a) Hva er en tidsrekke? Når er en tidsrekke

(i) strengt stasjonær?

(ii) (svakt (weakly)) stasjonær?

b) Definer hva du mener med en ARMA(p,q)-prosess. Hva mener en når en sier at en prosess er kausal (causal)? Formuler en betingelse på modellkoeffisientene som er ekvivalent med kravet om kausalitet. Spesialiser til AR(1)-tilfellet.

c) Betrakt AR(2)-prosessen definert ved

$$x_t = 0.25x_{t-2} + w_t,$$

hvor w_t er hvit støy med varians σ^2. Finn røttene av det autoregressive polynomet for x_t, og vis at prosessen er kausal. Vis at $x_t = \sum_{j=0}^{\infty} 0.25^j w_{t-2j}$, og at $Var(x_t) = \sigma^2/0.9375$.

d) Formuler et sett med homogene lineære differensligninger som autokorrelasjonen tilfredsstiller, og skriv opp fra dette den generelle formen som autokorrelasjonen må ha.

e) Finn autokorrelasjonsfunksjonen når prosessen er stasjonær. (Hint: Multipliser modell-ligningen med x_{t-1} og ta forventning. Man kan vise at $E(w_t x_{t-1}) = 0$. Prøv å vise dette.)

f) For det stasjonære tilfellet, finn krysskorrelasjonsfunksjonen mellom x_t og w_t ved først å finne et sett med homogene lineære differensligninger som krysskovariansen må tilfredsstille. Se på både positive og negative tidsdifferanser, og gi en alternativ tolkning av krysskorrelasjonen for negative tidsdifferanser.

(Fortsettes på side 2.)

Oppgave 2

Definer spektraltettheten (the spectral density) av en stasjonær tidsrekke x_t uttrykt ved autokovariansfunksjonen, og formuler den motsatte relasjonen. Hvilken betingelse må være tilfredsstilt for at disse to relasjonene skal holde? Finn autokovariansen og spektraltettheten for tidsrekken

$$x_t = w_t - 0.5 w_{t-1},$$

der w_t er hvit støy med varians 1.

SLUTT

Solutions STK4060/STK9060-sp18

Problem 1

a) See S&S page 173.

b) See S&S Property 4.3 page 175.

c) Let $y_t = x_t - x_{t-4}$. Then $y_t = (1 - B^4)x_t$ so by part b) $f_y(\omega) = |(1-e^{-2\pi i\omega 4})|^2 f_x(\omega)$. Since $\{x_t\}$ is a MA(1) time series, $x_t = w_t + \theta w_{t-1}$ where $w_t \sim wn(0, \sigma_w^2$. Then $f_x(\omega) = |(1 + \theta e^{-2\pi i\omega})|^2\sigma_w^2$ also fram part b). Then $f_y(\omega) = |(1 - e^{-2\pi i\omega 4})|^2 f_x(\omega) = |(1 - e^{-2\pi i\omega 4})|^2|(1 + \theta e^{-2\pi i\omega})|^2\sigma_w^2 = [2 - 2\cos(8\pi\omega)][(1 + \theta^2) + 2\theta\cos(2\pi\omega)]$

Problem 2

a) For a linear process $x_t = \Sigma_{j=-\infty}^{\infty}\psi_j w_{t-j}$ where w_t is white noise $w_t \sim wn(0, \sigma_w^2)$ and $\Sigma_{j=-\infty}^{\infty}|\psi_j| < \infty$ $\gamma_x(h) = \sigma_w^2\Sigma_{j=-\infty}^{\infty}\psi_{j+h}\psi_j$ is equation (1.32) page 25 in S&S. It is Problem 1.11 in S&S. $E(x_t) = 0$ so

$$\begin{aligned}
\gamma(h) &= cov(x_{t+h}, x_t)E(x_{t+h}x_t) \\
&= E[(\sum_{j=-\infty}^{\infty}\psi_j w_{t+h-j})(\sum_{j=-\infty}^{\infty}\psi_j w_{t-j})] \\
&= E[(\sum_{j=-\infty}^{\infty}\psi_{j+h} w_{t-j})(\sum_{j=-\infty}^{\infty}\psi_j w_{t-j})] \\
&= \sum_{j=-\infty}^{\infty}\psi_{j+h}\psi_j\sigma_w^2.
\end{aligned}$$

b) See definitions 3.5, 3.6, 3.7 and 3.8 and properties 3.1 and 3.2, pages 83-86 in S&S.

c) A causal and invertible ARMA(1,1) time series satisfies $|\phi| < 1$ and $|\theta| < 1$ From property 3.1 in S&S the coefficients in the linear process $x_t = \Sigma_{j=-\infty}^{\infty}\psi_j w_{t-j}$ satisfying $(1 - \phi B)x_t = (1 + \theta B)w_t$ is given by the coefficients of the power series $(1+\theta z)/(1-\phi z)$. Since $|\phi|<1$, $1/(1-\phi z) = 1+\phi z+\phi^2 z^2+\cdots$ and $(1+\theta z)(1+\phi z+\phi^2 z^2+\cdots) = 1+(\theta+\phi)z+(\phi^2+\phi\theta)z^2+\cdots+(\phi^j+\phi^{j-1}\theta)z^j+\cdots$, so and $\psi_0 = 1$ and $\psi_j = (\phi+\theta)\phi^{j-1},\ j = 1,2,\ldots$.

d) From parts a) and c), when $h = 0$

$$\begin{aligned}
\gamma_x(0) &= \sigma_w^2[1+(\theta+\phi)^2+(\theta+\phi)^2\phi^2+\cdots] \\
&= \sigma_w^2(1+\frac{(\theta+\phi)^2}{1-\phi^2}) \\
&= \sigma_w^2\frac{1-\phi^2+\theta^2+2\theta\phi+\phi^2}{1-\phi^2} \\
&= \sigma_w^2\frac{1+2\theta\phi+\theta^2}{1-\phi^2}
\end{aligned}$$

and when $h \ge 1$

$$\begin{aligned}
\gamma_x(h) &= \sigma_w^2\Sigma_{j=-\infty}^{\infty}\psi_{j+h}\psi_j \\
&= \sigma_w^2(\psi_0\psi_{0+h}+\psi_1\psi_{1+h}+\psi_2\psi_{2+h}+\cdots) \\
&= \sigma_w^2[(\phi+\theta)\phi^{h-1}+(\phi+\theta)(\phi+\theta)\phi^h+(\phi+\theta)(\phi+\theta)\phi^{h+1}+\cdots] \\
&= \sigma_w^2[(\phi+\theta)\phi^{h-1}+(\phi+\theta)^2\phi^h(1+\phi^2+\phi^4+\cdots] \\
&= \sigma_w^2(\phi+\theta)\phi^{h-1}(1+\frac{\phi(\theta+\phi)}{1-\phi^2}) \\
&= \sigma_w^2(\phi+\theta)\phi^{h-1}\frac{(1-\phi^2+\theta\phi+\phi^2)}{1-\phi^2} \\
&= \sigma_w^2\frac{(1+\theta\phi)(\theta+\phi}{1-\phi^2}\phi^{h-1}.
\end{aligned}$$

Problem 3

a) First model. Since $\bigtriangledown^2\mu_t = (\mu_t-\mu_{t-1})-(\mu_{t-1}-\mu_{t-2}) = \mu_t-2\mu_{t-1}+\mu_{t-2}$, one gets

$$\begin{aligned}
\text{State equation: } \begin{pmatrix}\mu_t\\ \mu_{t-1}\end{pmatrix} &= \begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\mu_{t-1}\\ \mu_{t-2}\end{pmatrix}+\begin{pmatrix}w_t\\ 0\end{pmatrix} \\
\text{Measurement equation: } y_t &= (1,0)\begin{pmatrix}\mu_t\\ \mu_{t-1}\end{pmatrix}+v_t
\end{aligned}$$

Second model

$$
\begin{aligned}
\text{State equation:}\ \begin{pmatrix} \mu_t \\ \eta_t \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ \eta_{t-1} \end{pmatrix} + \begin{pmatrix} w_t \\ w_t \end{pmatrix} \\
\text{Measurement equation:}\ y_t &= (1,0) \begin{pmatrix} \mu_t \\ \eta_t \end{pmatrix} + v_t
\end{aligned}
$$

b) Since $\mu_t = \mu_{t-1} + \eta_t = \mu_{t-1} + \eta_{t-1} + w_t = \mu_{t-1} + (\mu_{t-1} - \mu_{t-2}) + w_t$ it follows that if $\begin{pmatrix} \mu_t \\ \mu_{t-1} \end{pmatrix} \sim N(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_w^2 & 0 \\ 0 & \sigma_w^2 \end{pmatrix})$ then $\begin{pmatrix} \mu_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} \mu_t \\ \mu_t - \mu_{t-1} \end{pmatrix} \sim N(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_w^2 & \sigma_w^2 \\ \sigma_w^2 & 2\sigma_w^2 \end{pmatrix})$. Thus choosing the first of this two distributions as initial distribution in the first model, and the other distribution as the initial distribution for the second model, the observations $y_1, \dots, y_n$ will have the same distribution in the two cases.

c) First model

$$
\begin{aligned}
x_1^0 &= \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} x_0^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
P_1^0 &= [\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]\sigma_w^2 = \begin{pmatrix} 6 & 2 \\ 2 & 0 \end{pmatrix} \sigma_w^2 \\
x_1^1 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + K_1(y_1 - (1,0)K_1) \\
K_1 &= \begin{pmatrix} 6 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma_w^2/[(1,0) \begin{pmatrix} 6 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma_w^2 + \sigma_v^2] \\
&= \begin{pmatrix} 6 \\ 2 \end{pmatrix} \sigma_w^2/(6\sigma_w^2 + \sigma_v^2)
\end{aligned}
$$

Second model

$$\begin{aligned}
x_1^0 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x_0^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
P_1^0 &= [\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}]\sigma_w^2 = \begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix} \sigma_w^2 \\
x_1^1 &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + K_1(y_1 - (1,0)K_1) \\
K_1 &= \begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma_w^2 / [(1,0) \begin{pmatrix} 6 & 4 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sigma_w^2 + \sigma_v^2] \\
&= \begin{pmatrix} 6 \\ 4 \end{pmatrix} \sigma_w^2 / (6\sigma_w^2 + \sigma_v^2)
\end{aligned}$$

Problem 1

a) See S&S page 92 and S&S page 94 and 95.

b) The characteristic polynomials equals

$$\phi(z) = 0.12z^2 - 0.1 - 1$$

which has roots

$$\frac{0.1 \pm \sqrt{0.01 + 4 \times 0.12}}{2 \times 0.12} = \begin{cases} 0.8/0.24 = 1/0.3 \\ -0.6/0.24 = -1/0.4 \end{cases}$$

Both are outside the unit circle so the time series is causal.

c) For definition of partial correlation function see S&S page 106. In a AR(p) model it has the property of being equal to zero for arguments larger than p. Hence from plotting the partial autocorrelation function one can determine the order from observing when the partial autocorrelation function is equal to zero.

Problem 2

(i) For the *trend and cycle* model let the state be

$$\mathbf{x_t} = \begin{pmatrix} \mu_t \\ \beta_t \\ \psi_t \\ \psi_t^* \end{pmatrix}$$

Then the transition or state equation is given by

$$\begin{aligned} \mathbf{x_t} &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5\cos\lambda & 0.5\sin\lambda \\ 0 & 0 & -0.5\sin\lambda & 0.5\cos\lambda \end{pmatrix} \begin{pmatrix} \mu_{t-1} \\ \beta_{t-1} \\ \psi_{t-1} \\ \psi_{t-1}^* \end{pmatrix} + \begin{pmatrix} \eta_t \\ \zeta_t \\ \psi_t \\ \psi_t^* \end{pmatrix} \\ &= \Phi \mathbf{x_{t-1}} + \begin{pmatrix} \eta_t \\ \zeta_t \\ \psi_t \\ \psi_t^* \end{pmatrix} \end{aligned}$$

and the measurement or observation equation

$$y_t = (1,0,1,0)\begin{pmatrix}\mu_t\\ \beta_t\\ \psi_t\\ \psi_t^*\end{pmatrix} + \epsilon_t = \mu_t + \psi_t + \epsilon_t.$$

For the *cyclical trend* model let the state be the same. Now the state equation is

$$\begin{aligned}\mathbf{x_t} &= \begin{pmatrix}1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0.5\cos\lambda & 0.5\sin\lambda\\ 0 & 0 & -0.5\sin\lambda & 0.5\cos\lambda\end{pmatrix} + \begin{pmatrix}\mu_{t-1}\\ \beta_{t-1}\\ \psi_{t-1}\\ \psi_{t-1}^*\end{pmatrix} + \begin{pmatrix}\eta_t\\ \zeta_t\\ \psi_t\\ \psi_t^*\end{pmatrix}\\ &= \Phi\mathbf{x_{t-1}} + \begin{pmatrix}\eta_t\\ \zeta_t\\ \psi_t\\ \psi_t^*\end{pmatrix}\end{aligned}$$

and the measurement or observation equation

$$y_t = (1,0,0,0)\begin{pmatrix}\mu_t\\ \beta_t\\ \psi_t\\ \psi_t^*\end{pmatrix} + \epsilon_t = \mu_t + \epsilon_t.$$

b) For *trend and cycle* model

$$\begin{aligned}\Phi P_0^0\Phi' &= \Phi\Phi' = \begin{pmatrix}1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0.5\cos\lambda & 0.5\sin\lambda\\ 0 & 0 & -0.5\sin\lambda & 0.5\cos\lambda\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0.5\cos\lambda & -0.5\sin\lambda\\ 0 & 0 & 0.5\sin\lambda & 0.5\cos\lambda\end{pmatrix}\\ &= \begin{pmatrix}2 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0.25(\cos^2\lambda+\sin^2\lambda) & 0.25(-\cos\lambda\sin\lambda+\cos\lambda\sin\lambda)\\ 0 & 0 & 0.25(-\cos\lambda\sin\lambda+\cos\lambda\sin\lambda) & 0.25(\cos^2\lambda+\sin^2\lambda)\end{pmatrix}\\ &= \begin{pmatrix}2 & 1 & 0 & 0\\ 1 & 1 & 0 & 0\\ 0 & 0 & 0.25 & 0\\ 0 & 0 & 0 & 0.25\end{pmatrix}\end{aligned}$$

so

$$P_1^0 = \Phi P_0^0 \Phi' + Q = + \begin{pmatrix} 2+\sigma_\eta^2 & 1 & 0 & 0 \\ 1 & 1+\sigma_\zeta^2 & 0 & 0 \\ 0 & 0 & 0.25+\sigma_\psi^2 & 0 \\ 0 & 0 & 0 & 0.25+\sigma_{\psi^*}^2 \end{pmatrix}$$

so $A\Phi P_1^0 A' + R = (1,0,1,0)(\Phi P_0^0 \Phi' + Q)(1,0,1,0)' + \sigma_\epsilon^2) = 2.25 + \sigma_\eta^2 + \sigma_\psi^2 + \sigma_\epsilon^2$. Since $P_1^0 A' = (2+\sigma_\eta^2, 1, 0.25+\sigma_\psi^2, 0)'$, $K_1 = P_1^0 A'(A\Phi P_1^0 A' + R)^{-1} = (2+\sigma_\eta^2, 1, 0.25+\sigma_\psi^2, 0)'/((2.25+\sigma_\eta^2+\sigma_\psi^2+\sigma_\epsilon^2)$

c) Using the Kalman filter y_t^{t-1} can be calculated for each $t = 1, \dots, n$. But $y_t - y_t^{t-1}$ are independent $y_t - y_t^{t-1} \sim N(0, AP_t^{t-1}A' + \sigma_\epsilon^2)$. Hence the likelihood can be found as a function of σ_ϵ^2, σ_η^2,σ_ζ^2,σ_ψ^2 and $\sigma_{\psi^*}^2$. Using an optimization algorithm the maximal value can be found for each value of λ. Plotting the maxima as a function of λ, allows the maximum likelihood estimate for λ to be read off as the value that corresponds to the maximum value of the plot.

Problem 3

a) The periodogarm is the square of the modulus of the discrete Fourier transform, i.e. $I(j/n) = |d(j/n)|$, $j = 0, \dots, n-1$. For large n $2I(j/n)/f_x(j/n)$, $j = 1, \dots, [n/2]$ are approximately independent χ_2^2 variables. For parts of the specter where the spectral densities are not varying too much smoothed versions of the periodogram of the form $\frac{1}{(2m+1)} \sum_{k=-m}^{m} I((j+k)/n)$ will therefore have approximate expectation $f_x(j/n)$ and variance $f_x(j/n)^2/(2m+1)$.

b)

$$d_y(j/n) = \sum_{t=1}^{n} (\mu + x_t) e^{-2\pi i (j/n)t}/\sqrt{n} = \mu \sum_{t=1}^{n} e^{-2\pi i (j/n)t}/\sqrt{n} + \sum_{t=1}^{n} x_t e^{-2\pi i (j/n)t}/\sqrt{n}.$$

But

$$\begin{aligned} \sum_{t=1}^{n} e^{-2\pi i (j/n)t} &= e^{-2\pi i (j/n)} \sum_{t=0}^{n-1} e^{-2\pi i (j/n)t} \\ &= \begin{cases} n & \text{for } j = 0 \\ e^{-2\pi i (j/n)} \frac{1 - e^{-2\pi i (j/n)n}}{1 - e^{-2\pi i (j/n)}} = 0 & \text{for } j = 1, \dots, n-1 \text{ and } j \neq n/2,\ n \text{ even} \\ -n & \text{for } j = n/2,\ n \text{ even.} \end{cases} \end{aligned}$$

Hence

$$d_y(j/n) = \begin{cases} \sqrt{n}(\mu + \bar{x}) & \text{for } j = 0 \\ d_x(j/n) & \text{for } j = 1, \ldots, n-1 \text{ and } j \neq n/2,\ n \text{ even} \\ -\sqrt{n}(\mu + \bar{x}) & \text{for } j = n/2,\ n \text{ even.} \end{cases}$$

cfr. S&S page 188

c) For $j = 1, \ldots, n'$ and $j \neq n'/2,\ n'$ even

$$\begin{aligned} d_{y'}(j/n') &= \frac{1}{\sqrt{n'}} \sum_{t=1}^{n} \mu e^{-2\pi i (j/n')t} + \frac{1}{\sqrt{n'}} \sum_{t=1}^{n} x_t e^{-2\pi i (j/n')t} \\ &= \frac{\mu}{\sqrt{n'}} e^{-2\pi i (j/n')} \sum_{t=0}^{n-1} e^{-2\pi i (j/n')(t-1)} + \frac{1}{\sqrt{n'}} \sum_{t=1}^{n} x_t e^{-2\pi i (j/n')t} \\ &= \frac{\mu}{\sqrt{n'}} e^{-2\pi i (j/n')} \frac{1 - e^{-2\pi i (j/n')n}}{1 - e^{-2\pi i (j/n')}} + \frac{1}{\sqrt{n'}} \sum_{t=1}^{n} x_t e^{-2\pi i (j/n')t}, \end{aligned}$$

for $j = 0$

$$d_{y'}(0) = (n\mu + n\bar{x})/\sqrt{n'}.$$

and for $j = n'/2, n' even$

$$d_{y'}(j/n') = -(n\mu + n\bar{x})/\sqrt{n'}.$$

d) The variables of interest are the observations $y_t,\ t = 1, \ldots, n$ whose Fourier transform only depends on μ for frequency 0. In the padded version the dependency of μ shows up in the discrete Fourier transform at all frequencies $j/n',\ j = 0, \ldots, n - 1'$. Since padding is an auxiliary technique to facilitate the use of the Fast Fourier Transform, FFT, it seems reasonably to prefer a method where the dependency on μ is small.

END

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4060 — Time Series

Day of examination: Wednesday May 28th 2014

Examination hours: 09.00 – 13.00

This problem set consists of 2 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

In this problem you are asked to consider autoregressive time series.

a) How is an autoregressive time series of order p defined? Explain what it means that it is causal.

Consider the autoregressive time series of order two

$$x_t = x_{t-1}/3 + 2x_{t-2}/9 + w_t$$

where w_t is a time series of white noise, i.e. $w_t \sim wn(0, \sigma^2)$.

b) Show that x_t defined above is causal.

c) Explain that the auto correlation function of x_t satisfies a difference equation. Find the the auto correlation function of x_t.

d) What is the partial autocorrelation function of x_t?

Problem 2

Consider observations $y_1, \ldots, y_n$ described by the equations

$$\begin{aligned} x_t &= \mu + a x_{t-1} + w_t, \ |a| < 1 \\ y_t &= b x_t + v_t \end{aligned}$$

where $w_t, t = 1, \ldots, n$ and $v_t, t = 1, \ldots, n$ are independent random variables where $w_t \sim iidN(0, \sigma_w^2)$ and $v_t \sim iidN(0, \sigma_v^2)$. The variable x_0, $x_0 \sim N(0, \sigma_0^2)$, is independent of w_t and v_t

a) Interpret the equations as a state-space model. Explain what the state equation and the observation equation are.

(Continued on page 2.)

b) What are the conditions for the bivariate series $(x_t, y_t)'$ to be stationary? Find the expectation and covariance of the stationary series $(x_t, y_t)'$ satisfying the equations above?

c) What is the stationary distribution of $(x_t, y_t)'$ described in part b)?

Problem 3

Let $x_t, t = 0, \pm 1, \pm 2, \ldots$ be a stationary time series such that $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$ where $\gamma(h)$ is the covariance between x_{t+h} and x_t.

a) Define the spectral density $f_X(\omega)$ of x_t. Explain why $f_X(\omega) = f_X(-\omega)$ and $f_X(\omega) = f_X(1-\omega)$.

Let $a_j, j = 0, \pm 1, \pm 2, \ldots$ be a sequence of scalars so that $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Let $y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}$.

b) Show that the spectral density of y_t has the form

$$f_Y(\omega) = |A(\omega)|^2 f_X(\omega)$$

where $A(\omega) = \sum_{j=-\infty}^{\infty} a_j \mathrm{e}^{-2\pi i \omega j}$.

c) Let w_t be a white noise series, $w_t \sim wn(0, \sigma_w^2)$, and y_t be the ARMA(p,q) process defined by $\phi(B) y_t = \theta(B) w_t$ where the roots of the polynomials $\phi(z) = 0$ and $\theta(z) = 0$ are outside the unit circle. Use the result from part b) to explain that

$$f_Y(\omega) = \sigma_w^2 \frac{|\theta(\mathrm{e}^{-2\pi i \omega})|^2}{|\phi(\mathrm{e}^{-2\pi i \omega})|^2}.$$

END

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: STK4060 — Time Series

Day of examination: Monday June 4'th 2012

Examination hours: 09.00 – 13.00

This problem set consists of 4 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Solution proposal

Problem 1

a) Weak stationarity: $E(X_t) = \mu$, $Var(X_t) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j^2$ independent og t, and autocovarinces $\gamma(h) = Cov(x_{t+h}, x_t)$ only dependent on h.

For all integer h

$$\begin{aligned}
\gamma(h) &= E[(\sum_{j=-\infty}^{\infty} \psi_j w_{t+h-j})(\sum_{j=-\infty}^{\infty} \psi_j w_{t-j})] \\
&= E[(\sum_{j=-\infty}^{\infty} \psi_{j+h} w_{t-j})(\sum_{j=-\infty}^{\infty} \psi_j w_{t-j})] \\
&= \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}
\end{aligned}$$

because $E(w_s w_t) = \sigma_w^2,\ s = t$ and $E(w_s w_t) = 0\ s \neq t$.

b) Solving backwards

$$x_t = \phi^k x_{t-k} + w_t + \cdots + \phi^k w_{t-k}.$$

Because $|\phi| < 1$,

$$E[x_t - w_t + \cdots + \phi^k w_{t-k}]^2 = \phi^{2k} E[X_{t-k}^2] \to 0$$

since $E[x_{t-k}^2] < \infty$ by assumption. Hence $x_t = \sum_{k=0}^{\infty} \phi^k w_{t-k}$, which is linear with

$$\psi_j = \begin{cases} \theta^j\ j \geq 0 \\ 0\ j < 0 \end{cases}.$$

Since $\psi_j = 0\ j < 0$, the time series is causal.

c) An ARMA(p,q) process is a weak stationary time series satisfing the stochastic difference equation

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t + \theta_1 e_{t-1} + \cdots + \theta_q w_{t-q}$$

where $w_t,\ t = 0, \pm 1, \pm 2, \ldots$ is a sequence of white noise.

The time series x_t is causal if the solutiuons of $1 - \phi_1 z - \cdots - \phi_p z^p = 0$ are larger than 1 in absolute value.

The time series x_t is invertible if the solutiuons of $1 + \theta_1 z - \cdots - \theta_q z^q = 0$ are larger than 1 in absolute value.

For stationary AR(p) time series the autocorrelation function (ACF), $\rho(h)$, is exponentially decreasing and the partial autocorrelation function (PACF), ϕ_{hh}, equals zero if $h > p$.

For MA(q) time series the autocorrelation function, $\rho(h)$ equals zero if $h > q$ and the partial autocorrelation function, ϕ_{hh}, is exponentially decreasing.

For ARMA(p,q), where $p, q > 0$ processes both ACF and PACF are decreasing.

By plotting ACF and PACF, pure AR(p) and MA(q) processes can be identified. If neither is appropriate, this suggests an ARMA(p,q).

d) For a stationary process the Yule-Walker equations are in matrix notation defined as

$$\Gamma_n \phi_n = \gamma_n \text{ and } \sigma_w^2 = \gamma(0) - \phi_n' \gamma_n$$

where Γ_n is the covariance matrix

$$\Gamma_n = \begin{pmatrix} \gamma(0) & \cdots & \gamma(n-1) \\ \vdots & & \vdots \\ \gamma(n-1) & \cdots & \gamma(0) \end{pmatrix} \text{ and } \gamma_n = \begin{pmatrix} \gamma(1) \\ \vdots \\ \gamma(n) \end{pmatrix}.$$

For an causal AR(p) time series

$$x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p} = w_t$$

one get by sucessively multiplying both sides by $x_t, x_{t-1}, \cdots, x_{t-p}$, using the assumed causality and taking expectations

$$\begin{aligned} \phi_1 \gamma(1) + \cdots + \phi_p \gamma(p) &= \gamma(0) - \sigma_w^2 \\ \phi_1 \gamma(0) + \cdots + \phi_p \gamma(p-1) &= \gamma(1) \\ \phi_1 \gamma(p-1) + \cdots \phi_p \gamma(0) &= \gamma(p) \end{aligned}$$

The first equation is $\sigma_w^2 = \gamma(0) - \phi_n' \gamma_n$ and the p last equations $\Gamma_n \phi_n = \gamma_n$.

Estimating the theoretical autocovariances, $\gamma(h)$ by the corresponding empirical, $\hat{\gamma}(h)$, one obtains estimators for the parameters $\phi_1, \cdots, \phi_p$ and σ_w^2.

e) The polynomial equation $1-(5/6)z+(1/6)z^2=0$ has solutions 3 and 2, and hence the AR(2) time series defined by

$$x_t-(5/6)x_{t-1}+(1/6)x_{t-2}=w_t$$

is causal. Using this one gets by multiplying by $x_t, x_{t-1}, \ldots$ and taking expectations

$$\begin{aligned}\gamma(0)-(5/6)\gamma(1)+(1/6)\gamma(2) &= E(x_t w_t)=\sigma_w^2\\ \gamma(1)-(5/6)\gamma(0)+(1/6)\gamma(1) &= E(x_{t-1}w_t)=0\\ \gamma(k)-(5/6)\gamma(k-1)+(1/6)\gamma(k-2) &= E(x_{t-k}w_t)=0\end{aligned}$$

for $k=2,3,\ldots$.

The covariances sayisfy the difference equation

$$\gamma(k)-(5/6)\gamma(k-1)+(1/6)\gamma(k-2)=0,\ k=1,\ldots.$$

Dividing by $\gamma(0)$ yields the diffence equation

$$\rho(k)-(5/6)\rho(k-1)+(1/6)\rho(k-2)=0,\ k=1,\ldots$$

for the autocovariance function. This difference equation has the general solution

$$\gamma(k)=c_1(1/2)^k+c_2(1/3)^k,\ k=2,\ldots$$

and initial conditions $\rho(0)=1$ and $\rho(1)-(5/6)\rho(0)+(1/6)\rho(1)=0$, i.e $\rho(0)=1$ and $\rho(1)=5/7$. Thus c_1 and c_2 are determined by

$$\begin{aligned}c_1+c_2=\rho(0)=1\\ c_1(1/2)+c_2(1/3)=\rho(1)=5/7\end{aligned}$$

which have solutions $c_1=(16/7)$ and $c_2=-(9/7)$.

Therefore the autocorrelation function is $\rho(k)=[(16/7)(1/2)^k-(9/7)(1/3)^k]/2,\ k=0,1,2,\ldots$.

The PACF is found by solving recursively the Yule-Walker equations,i.e. for h=1

$$\gamma(0)\phi_{11}=\gamma(1) \text{ or } \phi_{11}=\rho(1)=10/7.$$

Since the process is AR(2) we know that the solution $\phi_{22}=\phi_2=1/4$ and that $\phi_{kk}=0, k=3,4,\ldots$.

Problem 2

a) The spectral density $f_x(\omega)$ is defined as

$$f_x(\omega)=\sum_{h=-\infty}^{\infty}\gamma(h)e^{-2\pi\omega h}$$

where γ is the autocovariance function.

Large values of $f_x(\omega)$ indicates that the corresponding frequencies contribute much to the periodic variation in the time series $\{x_t\}$.

b) Using that the auto covariances are $\gamma(h) = \phi^{|h|}\sigma_w^2$ in a stationary causal AR(1) process, the spectral density can be found directly from the definition.

Alternatively, in an ARMA(p,q) process with autoregressive polynomial $\phi(z)$ and moving average polynomial $\theta(z)$ the spectral density is

$$f_x(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2}$$

which in this case means

$$f_x(\omega) = \sigma_w^2 \frac{1}{|1 - 0.5e^{-2\pi i\omega})|^2} = \frac{\sigma_w^2}{1.25 - \cos(2\pi\omega)}.$$

c) The periodogram is defined as

$$I(j/n) = |\frac{1}{\sqrt{n}}\sum_{t=1}^{n} x_t e^{-2\pi i(j/n)t}|^2,\ j = 0, 1, \ldots, n-1.$$

For $j = 1, \ldots, n-1$

$$\sum_{t=1}^{n} \mu e^{-2\pi i(j/n)t} = \mu e^{-2\pi i(j/n)} \sum_{t=0}^{n-1}(e^{-2\pi i(j/n)})^t = \mu e^{-2\pi i(j/n)} \frac{1 - e^{-2\pi i(j/n)n}}{1 - e^{-2\pi i(j/n)}} = 0.$$

Therefore, for $j = 1, \ldots, n-1$

$$I(j/n) = |\frac{1}{\sqrt{n}}\sum_{t=1}^{n}(x_t - \mu)e^{-2\pi i(j/n)t}|^2 = \frac{1}{n}\sum_{s,t=1}^{n}(x_s - \mu)(x_t - \mu)(e^{-2\pi i(j/n)(s-t)}.$$

Hence,

$$E[I(j/n)] = \frac{1}{n}\sum_{s,t=1}^{n}\gamma(s-t)e^{-2\pi i(j/n)(s-t)} = \frac{1}{n}\sum_{h=-(n-1)}^{n-1}(n-|h|)\gamma(h)e^{-2\pi i(j/n)h}.$$

by changing summation variable $h = s - t$.

d) For $j \neq k$ $I(j/n)$ and $I(k/n)$ are for large values of n under the regularity conditions described in Shumway and Stoffer, in particular $\sum_{h=-\infty}^{\infty}|h|\gamma(h) < \infty$, approximately uncorrelated. For linear processes $(\frac{2I(j_1/n)}{f_x(j_1/n)}, \ldots, \frac{2I(j_k/n)}{f_x(j_k/n)})'$ is moreover approximately distributed as $(X_1, \ldots, X_k)'$ where $X_1, \ldots, X_k$ are independently χ^2-distributed with 2 degrees of fredom.

The spectral density f_x is continous and if it is not varying too much, gain can be obtained by smoothing/averaging over adjacent frequencies. But there is a tradeoff, if the smooting is too coarse, bias will result when f_x is not constant, and if not enough frequencies are included in the smooth the variance will increase.

SLUTT

Løsningsforslag, eksamen STK 4060 våren 2010.

Oppgave 1.

a) En tidsrekke er en stokastisk prosess $\{x_t; t = 1, 2, 3, \ldots\}$ eller $\{x_t; t = 0, \pm 1, \pm 2, \ldots\}$, dvs. en samling av tilfeldige variable indeksert ved en diskret tidsparameter. (i) Tidsrekken er strengt stasjonær hvis $\{x_{t_1}, x_{t_2} \ldots, x_{t_r}\}$ har samme fordeling som $\{x_{t_1+h}, x_{t_2+h} \ldots, x_{t_r+h}\}$ for alle r, alle h og alle valg av $t_1, ..., t_r$. (ii) Tidsrekken er (svakt) stasjonær hvis forventningen μ_t av x_t er konstant i t og autokorrelasjonsfunksjonen $\gamma(s,t) = E((x_s - \mu_s)(x_t - \mu_t))$ bare avhenger av $|s - t|$.

b) En ARMA(p,q)-prosess er en stasjonær tidsrekke bestemt av sammenhengen $x_t = \phi_1 x_{t-1} + ... + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + ... + \theta_q w_{t-q}$ der $\phi_p \neq 0$, $\theta_q \neq 0$ og w_t er hvit støy med $\sigma_w^2 > 0$. Prosessen er kausal hvis den kan betraktes som bestemt av fortiden, det vil si at vi har en konvergent rekke av formen $x_t = \sum_{i=0}^{\infty} \psi_i w_{t-i}$. En nødvendig og tilstrekkelig betingelse for at en ARMA(p,q)-prosess skal være kausal, er at røttene i det autoregressive polynomet $\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p$ alle ligger utenfor enhetssirkelen. I AR(1)-tilfellet $x_t = \phi x_{t-1} + w_t$ svarer dette til $|\phi| < 1$.

c) Autoregressivt polynom: $1 - 0.25z^2$ har røtter ± 2, altså utenfor enhetssirkelen, derfor er prosessen kausal. Modell-ligningen kan skrives:

$$(1 - 0.25B^2)x_t = w_t,$$

så

$$x_t = (1 - 0.25B^2)^{-1} w_t = \sum_{j=0}^{\infty} (0.25B^2)^j w_t = \sum_{j=0}^{\infty} 0.25^j w_{t-2j}.$$

Dette gir

$$Var(x_t) = E(x_t^2) = E\{\sum_{j=0}^{\infty} 0.25^j w_{t-2j} \sum_{k=0}^{\infty} 0.25^k w_{t-2k}\} = \sum_j \sum_k 0.25^{j+k} E(w_{t-2j} w_{t-2k})$$

$$= \sum_{j=0}^{\infty} 0.25^{2j} \sigma^2 = \frac{\sigma^2}{1 - 0.25^2} = \sigma^2 / 0.9375.$$

d) De lineære ligningene som autokorrelasjonen (og autokovariansen) må tilfredsstille, er

$$\rho(h) - 0.25\rho(h-2) = 0, \;\; h \geq 2.$$

Det tilhørende polynomet $1 - 0.25z^2$ har igjen røttene ± 2. Fra dette blir den generelle formen for autokorrelasjonsfunksjonen av formen

$$\rho(h) = c_1 z_1^{-h} + c_2 z_2^{-h} = c_1 2^{-h} + c_2 (-1)^h 2^{-h}.$$

e) Bruker initialbetingelsene. Først:

$$\rho(0) = c_1 + c_2 = 1.$$

Så multipliserer vi modell-ligningen med x_{t-1} og tar forventning:

$$E(x_t x_{t-1}) = 0.25 E(x_{t-2} x_{t-1}) + E(w_t x_{t-1}).$$

Det siste leddet er 0 siden $x_{t-1} = \sum_{j=0}^{\infty} \psi_j w_{t-1-j}$. Altså fra stasjonaritet:

$$\rho(1) = 0.25\rho(1),$$

så $\rho(1) = c_1 2^{-1} - c_2 2^{-1} = 0$ og $c_1 = c_2$. De to ligningene gir $c_1 = c_2 = \frac{1}{2}$, og

$$\rho(h) = 2^{-h-1} + (-1)^h 2^{-h-1},$$

eller: $\rho(h) = 2^{-h}$ for h partall; $\rho(h) = 0$ for odde h.

f) Skriv opp modell-ligningen for h erstattet med $t+h$, multipliser med w_t og ta forventning:

$$E(x_{t+h}w_t) = 0.25E(x_{t+h-2}w_t) + E(w_{t+h}w_t).$$

Dividert med $\sqrt{\gamma_x(0)\gamma_w(0)}$ gir dette den homogene ligningen:

$$\rho_{xw}(h) - 0.25\rho_{xw}(h-2) = 0, \quad h = 1,2,3,... \; og \; h = -1,-2,-3,....$$

Ligningen for positive h har som før den generelle løsningen

$$\rho_{xw}(h) = c_1 2^{-h} + c_2(-1)^h 2^{-h}.$$

For negative h er $\rho_{xw}(h) = \rho_{wx}(-h)$. Ligningen for negative h kan da skrives, om h erstattes med $-h$

$$\rho_{wx}(h) - 0.25\rho_{wx}(h+2) = 0.$$

Denne har polynom $z^2 - 0.25$ med røtter ± 0.5 og den generelle løsningen

$$\rho_{wx}(h) = d_1 2^h + d_2(-1)^h 2^h.$$

Siden $\rho_{wx}(h)$ må være begrenset når $h \to \infty$, krever dette $d_1 = d_2 = 0$ og $\rho_{wx}(h) = 0$ for $h = 1,2,3,....$ Dette er konsistent med at $\gamma_{wx}(h) = E(w(t+h)x(t))$ og $x(t) = \sum_{j=0}^{\infty} \psi_j w_{t-j}$, og kunne også ha vært utledet fra det.

Går nå tilbake til ligningen og løsningen for $\rho_{xw}(h)$ for positiv h, og finner først $\rho_{xw}(1) = 0.25\rho_{xw}(-1) = 0$. Dette gir $\rho_{xw}(h) = 0$ for odde h og $c_1 = c_2 = c$. Settes $h = 0$, finner vi at $\gamma_{xw}(0) = E(x_t w_t) = 0.25\gamma_{xw}(-2) + E(w_t^2) = \sigma^2$. Siden $\gamma_x(0) = \sigma^2/0.9375$ og $\gamma_w(0) = \sigma^2$, blir $\rho_{xh}(0) = \sqrt{0.9375} = 0.968$ og

$$\rho_{xw}(h) = 2c2^{-h} = 0.968 \cdot 2^{-h}$$

når h er et ikke-negativt partall.

Oppgave 2.

La $\gamma(h)$ være autokovariansfunksjonen til x_t. Da er spektraltettheten definert ved

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i \omega h}, \;\; -1/2 \leq \omega \leq 1/2.$$

Den inverse relasjonen er

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i \omega h} f(\omega) d\omega, \;\; h = 0, \pm 1, \pm 2, ...$$

Betingelsen for at disse relasjonene skal holde, er

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty.$$

For $x_t = w_t - 0.5 w_{t-1}$ har vi

$$\gamma(h) = E(x_{t+h} x_t) = E[(w_{t+h} - 0.5 w_{t+h-1})(w_t - 0.5 w_{t-1})],$$

som gir:

$$\gamma(0) = 1 + 0.5^2 = 1.25,$$

$$\gamma(\pm 1) = -0.5,$$

$$\gamma(h) = 0 \;\; for |h| \geq 2.$$

Spektraltettheten blir

$$f(\omega) = 1.25 - 0.5(e^{-2\pi i\omega} + e^{2\pi i\omega}) = 1.25 - 0.25cos(2\pi\omega).$$