FYS3150
Project 2

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Abstract

In this project, we have developed a model for the solar system using object-oriented programming. Object-orientation eased the transition from initial two- and three-body problems to the final ten-body problem, involving the planets, the Sun and Pluto.

We have implemented two famous methods for solving ordinary differential equations: The basic forward Euler method, and the more sophisticated velocity Verlet method. The second one is by far superior in regards to accuracy, and our timing of the algorithms show an increase in CPU time of only a factor of 2 to 3.

In addition to looking at the stability of the orbits and escape velocities, we have reproduced the observed perihelion precession angle per century of Mercury (subtracting classical effects) using general relativity.

1 Introduction

Modelling the solar system can be quite intricate, with all the different gravitational interactions between the sun and the planets giving rice to a bunch of coupled ordinary differential equations. Except for the various physical constants, the equations are very similar. Typing in all the equations manually would be an error-prone nightmare, but luckily, object-orientation makes our lives easier. In this project, we will use object-orientated programming to simplify the extension of our program from a simple two- and three-body problem (Earth-Sun and Earth-Sun-Jupiter), to a model of the whole solar system (including the eight planets and Pluto, but not including moons, asteroids, comets etc.).

This "write once, run many times" philosophy also extends on our experiences from the previous project. In Project 2, we developed an eigenvalue solver that could be applied to diverse physical problems. This time around, we have developed a coupled ordinary differential equation solver which can also be used in Molecular Dynamics simulations (Project 5).

There are numerous different methods for solving ordinary differential equations. In this project, we will implement both the basic forward Euler method and the more sophisticated velocity Verlet method, and look at the differences in precision and performance.

By modelling different two- and three-body systems, we can look at how the planets affect each other's motion, determine escape velocities, and see what happens if we alter the gravitational force. We will for instance try to reproduce the observed deviation from the classical prediction of the perihelion precession of Mercury using general relativity. Historically, this was an important test of Einstein's theory of general relativity.

2 Theory

Both the forward Euler method and the velocity Verlet method is based on Taylor expansions. But before we look at these methods, we need to look at the equations governing the motion of the celestial bodies of the solar system.

2.1 The Solar System

2.1.1 Laws of motion

When we work with the solar system and the motion of the planets, the only force that is acting is the gravitational force. Starting with the Earth-Sun-system, the gravitational force is given by Newton's law of gravity:

$$F_G = \frac{GM_\odot M_E}{r^2}$$

where G is the gravitational constant, $M_\odot$ and M_E is the mass of the Sun and Earth, respectively, and r is the distance between the Sun and Earth. We would like to rewrite this equation into units of AU and years, which is convenient when looking at the solar system. If we assume that Earth's orbit around the sun is circular, we know that

$$F_G = \frac{M_E v_E^2}{r} = \frac{GM_\odot M_E}{r^2}$$

where v_E^2/r is the centripetal acceleration. Since $r = 1$ AU and $v_E = 2\pi r$ AU/year for this orbit, we get that

$$GM_\odot = v_E^2 r = 4\pi^2 \text{ AU}^3/\text{year}^2$$

Using Newton's second law of motion, we get the following differential equations for the motion of Earth:

$$\frac{d^2x}{dt^2} = \frac{F_{G,x}}{M_E} = -\frac{GM_\odot}{r^3}x \tag{1}$$

and similar equations for the y- and z-direction.

Equation (1) can be written into two first-order differential equations:

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x$$
$$\frac{dx}{dt} = v_x$$

The same goes for the y- and z-direction. We've now got six coupled differential equations that we have to solve.

If we add another planet, say Jupiter, we have to add the gravitational force from Jupiter on Earth. This gives us the following equations for the motion of Earth:

$$\frac{dv_x}{dt} = -\frac{GM_\odot}{r^3}x - \frac{GM_\odot M_J/M_\odot}{r_{EJ}^3}(x_E - x_J)$$
$$\frac{dx}{dt} = v_x$$

We can keep adding planets in this way. We only need to find the force between two objects once, since Newton's third law of motion gives us the opposite force.

We will also look at the perihelion precession of Mercury. The observed precession could not be fully explained with Newtonian gravity. The relativistic correction of the classical gravitational force is given by

$$F_G = \frac{GM_\odot M_{\text{Mercury}}}{r^2}\left[1 + \frac{3l^2}{r^2c^2}\right]$$

where l is Mercury's angular momentum per unit mass, and c is the speed of light. The perihelion angle is given by

$$\tan\theta_p = \frac{y_p}{x_p}$$

where (x_p, y_p) is the position of Mercury at perihelion.

The masses, positions and velocities for the planets was obtained from NASA [1].

2.1.2 Escape velocity

We consider a single planet in a circular orbit 1 AU from the sun. To find the escape velocity of the planet, we can use energy considerations. Since the energy is conserved, we get that

$$\begin{gathered} K_0 + V_0 = K_1 + V_1 \\ \frac{1}{2}mv_{\text{escape}}^2 - \frac{GM_\odot m}{r_0} = 0 + 0 = 0 \end{gathered}$$

(where $K_1 = 0$ since we're looking for the lowest velocity needed to escape the gravitational field.) This gives us the following escape velocity:

$$v_{\text{escape}} = \sqrt{\frac{2GM_\odot}{r_0}} = \sqrt{8\pi^2} = 2\pi\sqrt{2}\ \text{AU/year}$$

We see that the escape velocity differ from the circular orbit velocity by a factor of $\sqrt{2}$.

What happens to the escape velocity if we alter the gravitational force? We can try to replace the newtonian gravitational force with

$$F_G = \frac{GM_\odot m}{r^\beta}$$

where $\beta \in [2, 3]$. The potential energy is now given by

$$V(r) = \int_r^\infty F_G\, dr = -\frac{GM_\odot m}{(\beta - 1)r^{\beta-1}}$$

Repeating the above calculations of the escape velocity, we obtain

$$v_{\text{escape}} = 2\pi\sqrt{\frac{2}{\beta - 1}}\ \text{AU/year}$$

We see that that when $\beta \to 3$ the escape velocity approaches the circular orbit speed. The circular orbit is no longer stable.

2.2 Forward Euler

We want to solve the following differential equation:

$$y'(t) = f(t_i, y_i)$$

This equation can be discretised, with the initial value $y_0 = y(t_0)$ and a step size $h = \frac{b-a}{n}$. For a value $y(t_i)$ and t_i, the next value $y(t_{i+1}) = y(t_i + h)$ is given by the Taylor expansion

$$y(t_{i+1}) = y(t_i) + h \cdot \Delta(t_i, y_i) + O(h^{p+1})$$

where $\Delta(t_i, y_i) = y'(t_i) + \cdots + y^{(p)}(t_i)\frac{h^{p-1}}{p!}$

If we truncate Δ at the first derivative, we get the steps of the forward Euler method:

$$y(t_{i+1}) = y(t_i) + h \cdot y'(t_i) + O(h^2)$$

We see that the local truncation error is of the order $O(h^2)$, which gives a global truncation error of $n \cdot O(h^2) \approx O(h)$ after n steps.

Applied to Newton's second law of motion, we get that

$$\begin{aligned} x_{i+1} &\approx x_i + h \cdot v_i \\ v_{i+1} &\approx v_i + h \cdot a_i \end{aligned}$$

If we calculate a_i as $\frac{F_i}{m}$, this algorithm requires $5n$ FLOPs.

2.3 Velocity Verlet

Now, we start of with the two coupled differential equations derived from Newton's second law:

$$\begin{aligned} \frac{dx}{dt} &= v(x,t) \\ \frac{dv}{dt} &= a(x,t) \end{aligned}$$

The Taylor expansions of x and v are given by

$$\begin{aligned} x(t_{i+1}) &= x(t_i) + hx^{(1)}(t_i) + \frac{h^2}{2}x^{(2)}(t_i) + O(h^3) \\ v(t_{i+1}) &= v(t_i) + hv^{(1)}(t_i) + \frac{h^2}{2}v^{(2)}(t_i) + O(h^3) \end{aligned} \tag{2}$$

We need to approximate $v^{(2)}(t)$, which is not known. We use the forward Euler method:

$$v^{(1)}(t_{i+1}) = v^{(1)}(t_i) + hv^{(2)}(t_i) + O(h^2) \quad \Rightarrow \quad v^{(2)}(t_i) \approx \frac{v^{(1)}(t_{i+1}) - v^{(1)}(t_i)}{h}$$

Inserting this into equation (2), we get that

$$v(t_{i+1}) = v(t_i) + \frac{h}{2}\left(v^{(1)}(t_{i+1}) + v^{(1)}(t_i)\right) + O(h^3)$$

Our equations for both position and velocity is then

$$x(t_{i+1}) = x(t_i) + hx^{(1)}(t_i) + \frac{h^2}{2}x^{(2)}(t_i) + O(h^3)$$
$$v(t_{i+1}) = v(t_i) + \frac{h}{2}\left(v^{(1)}(t_{i+1}) + v^{(1)}(t_i)\right) + O(h^3)$$

We see that the global error is of the order $\approx O(h^2)$. We also see that when the acceleration $v^{(1)}$ depends on the position x, we need to calculate the new position before the new velocity.

If we calculate $a_i = x^{(2)}(t_i) = v^{(1)}(t_i)$ as $\frac{F_i}{m}$, we get the final equations

$$x_{i+1} \approx x_i + h \cdot v_i + \frac{h^2}{2m}F_i$$
$$v_{i+1} \approx v_i + \frac{h}{2m}(F_{i+1} + F_i)$$

The constant $\frac{h}{2m}$ only needs to be calculated once, and we end up with $8n$ FLOPs for this algorithm.

An important feature of the velocity Verlet method is the fact that it is symplectic, which means that it conserves energy (this is not the case for the forward Euler method).

3 Implementation

The programs used in this project can be found on https://github.com/egilsk/fys3150x/tree/master/project3. Inspiration is taken from the programs in the course github repository [2].

I planned to make a unit test based on conservation of energy and angular momentum, but unfortunately I ran out of time.

4 Results

(In the plots given, the size of the sun is exaggerated to improve its visibility)

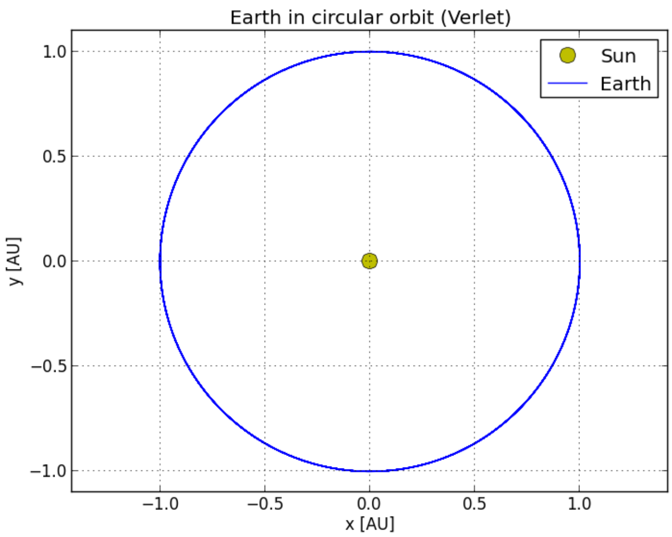

Figure 1: Earth's circular orbit around a fixed sun, using the velocity Verlet method (10 years, $n = 10^3$).

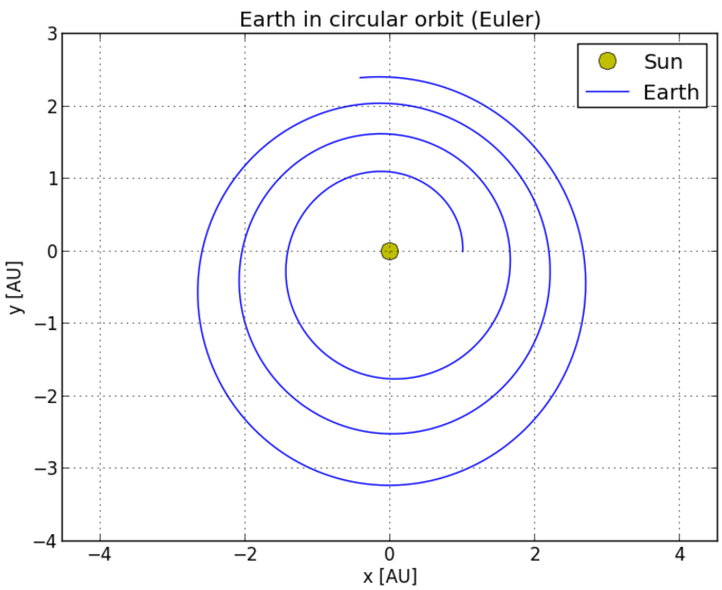

Figure 2: Earth's circular orbit around a fixed sun, using the forward Euler method (10 years, $n = 10^3$).

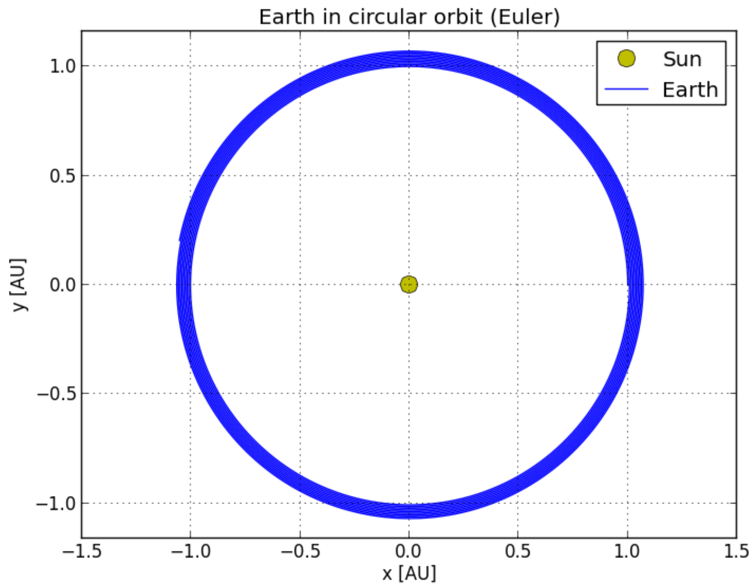

Figure 3: Earth's circular orbit around a fixed sun, using the forward Euler method (10 years, $n = 10^5$).

n	Time used [s]
$1 \cdot 10^3$	$1.4 \cdot 10^{-3}$
$1 \cdot 10^4$	$1.18 \cdot 10^{-2}$
$1 \cdot 10^5$	$7.96 \cdot 10^{-2}$
$1 \cdot 10^6$	$7.51 \cdot 10^{-1}$

(a) Timing of Forward Euler

n	Time used [s]
$1 \cdot 10^3$	$3.67 \cdot 10^{-3}$
$1 \cdot 10^4$	$2.64 \cdot 10^{-2}$
$1 \cdot 10^5$	$2.25 \cdot 10^{-1}$
$1 \cdot 10^6$	$2.19 \cdot 10^{0}$

(b) Timing of Velocity Verlet

Figure 4: Timing of our two methods for different numbers of steps (Earth-Sun system).

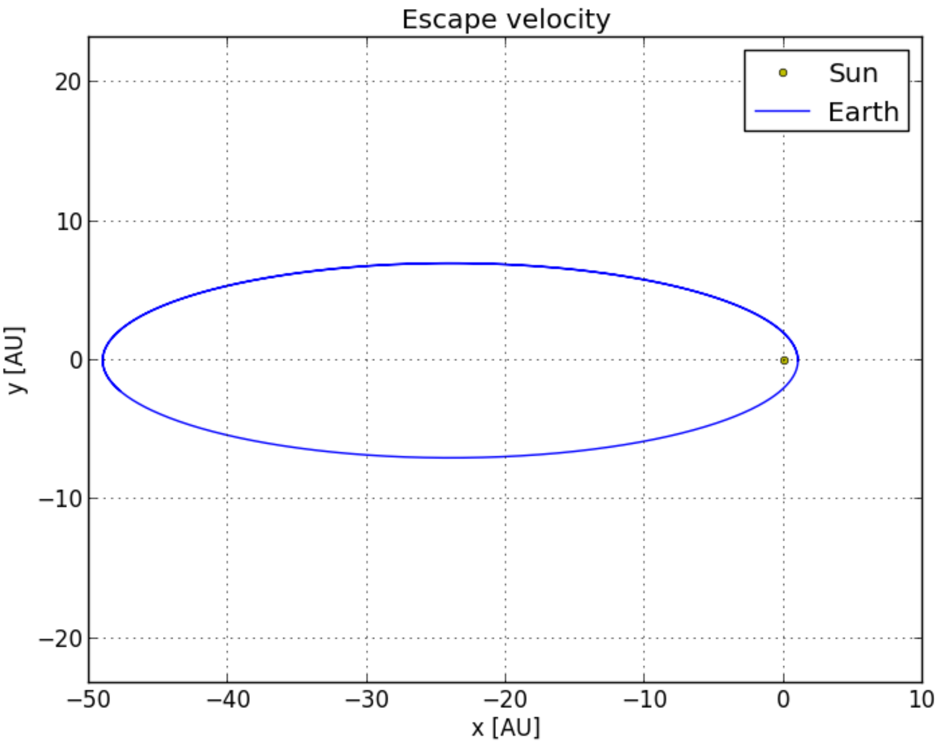

Figure 5: Earth's orbit around a fixed sun, starting at $x = 1$ AU with $v_y = 1.4 \cdot 2\pi$ AU/year (200 years, $n = 10^5$).

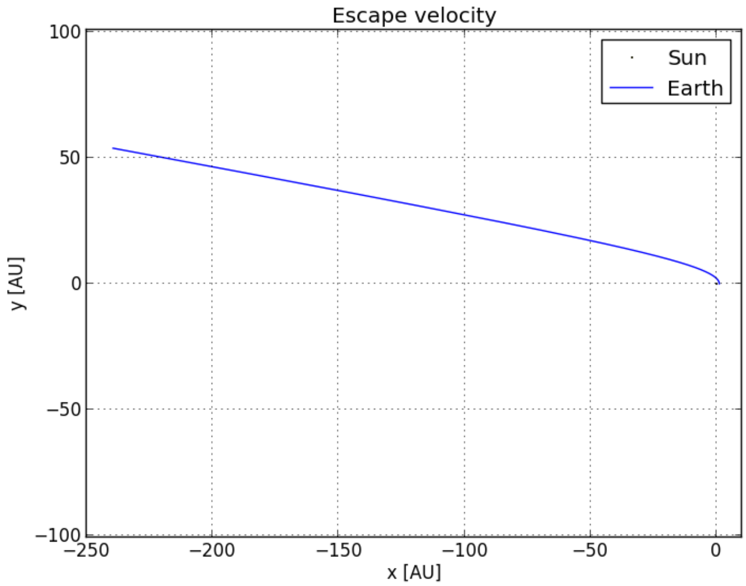

Figure 6: Earth's orbit around a fixed sun, starting at $x = 1$ AU with $v_y = 1.42 \cdot 2\pi$ AU/year (200 years, $n = 10^5$).

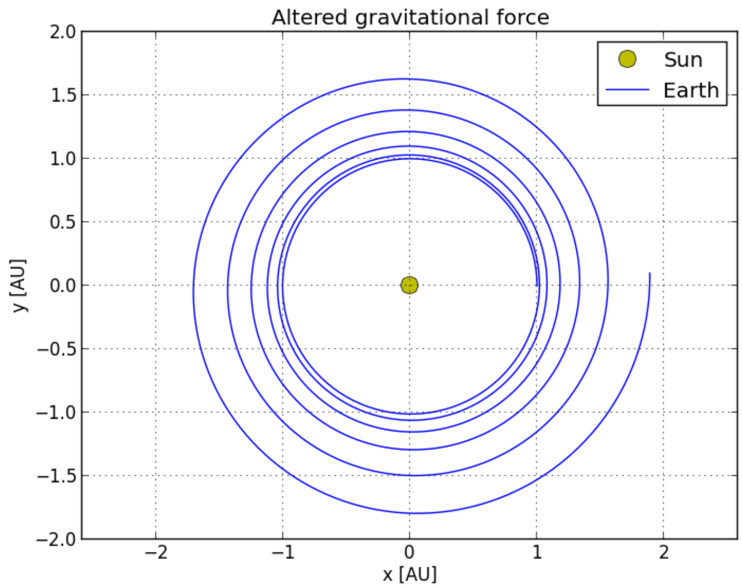

Figure 7: Earth's altered orbit with $\beta = 3$, starting at $x = 1$ AU with $v_y = 2\pi$ AU/year (10 years, $n = 10^3$).

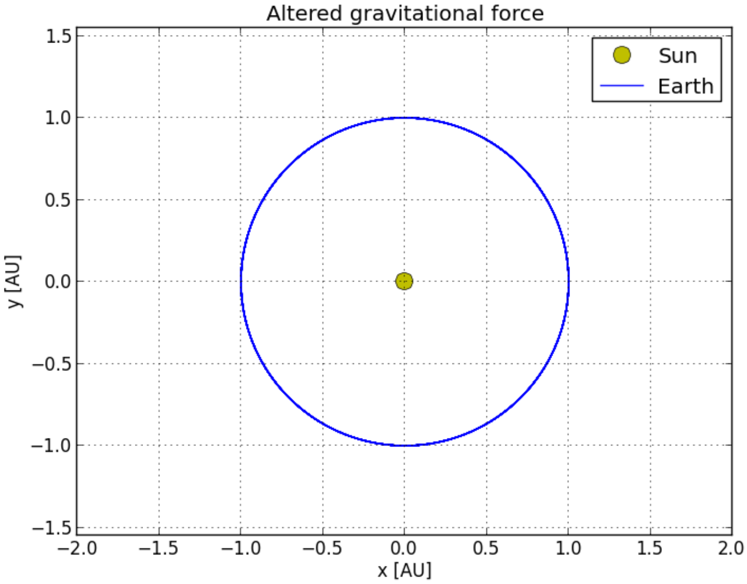

Figure 8: Earth's altered orbit with $\beta = 3$, starting at $x = 1$ AU with $v_y = 2\pi$ AU/year (10 years, $n = 10^5$).

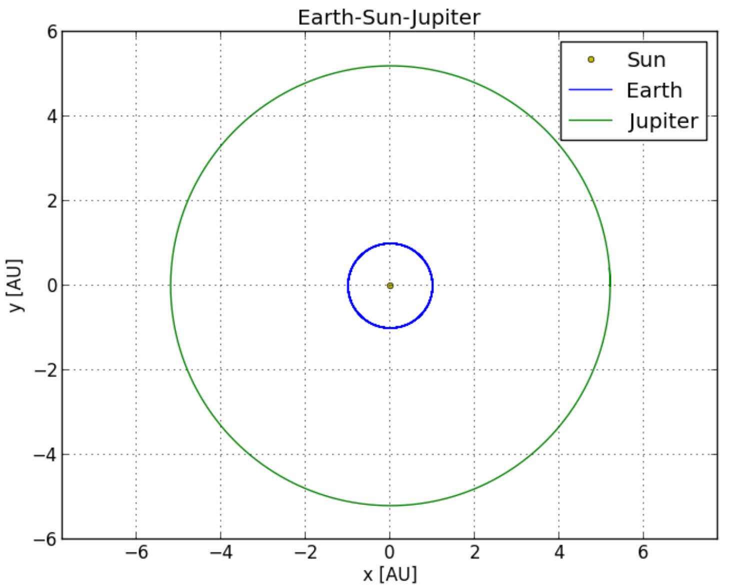

Figure 9: The Earth-Sun-Jupiter system with the sun fixed, and with Earth starting at $x = 1$ AU with $v_y = 2\pi$ AU/year and Jupiter at $x = 5.2$ AU with $v_y = 2\pi \frac{x_J}{T_J}$ AU/year (12 years, $n = 10^5$).

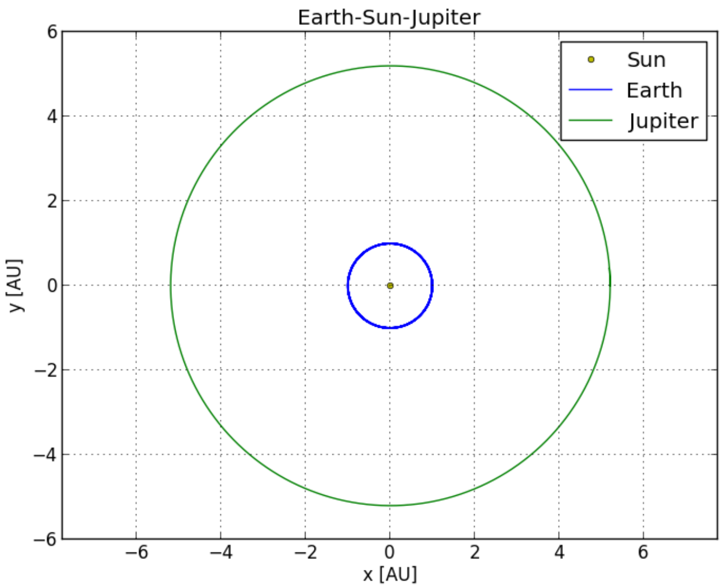

Figure 10: The Earth-Sun-Jupiter system with the sun fixed, and the mass of Jupiter increased by a factor of 10 (12 years, $n = 10^5$).

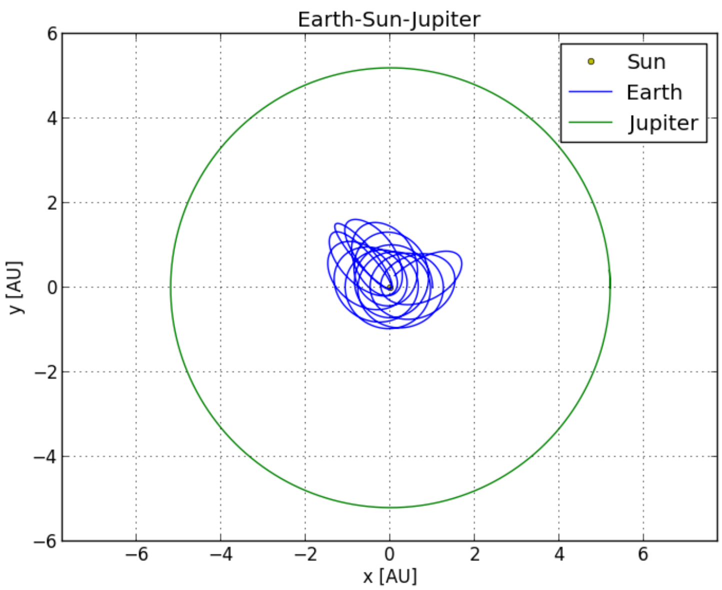

Figure 11: The Earth-Sun-Jupiter system with the sun fixed, now with the mass of Jupiter increased by a factor of 1000 (12 years, $n = 10^5$).

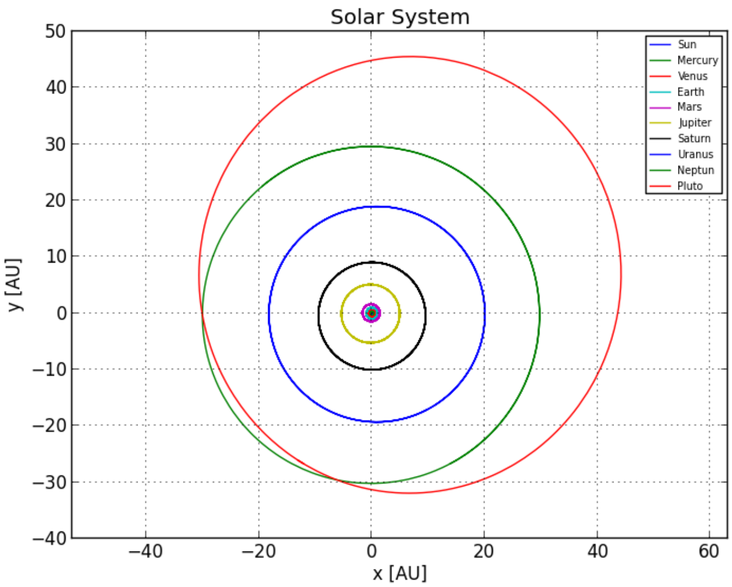

Figure 12: All planets of the solar system (including Pluto) (250 years, $n = 10^6$). From the data behind this plot, we find that the motion of the Sun is limited to around 10^{-3} AU from the system's centre of mass.

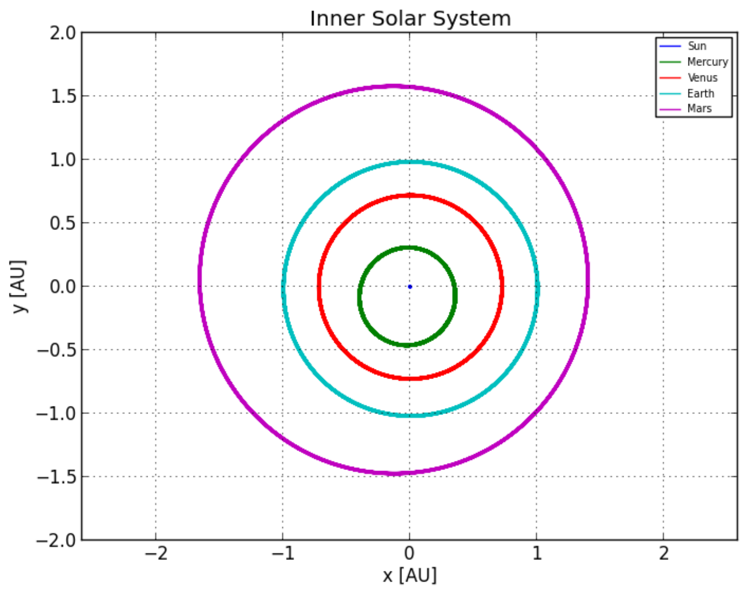

Figure 13: The planets of the inner solar system (250 years, $n = 10^6$).

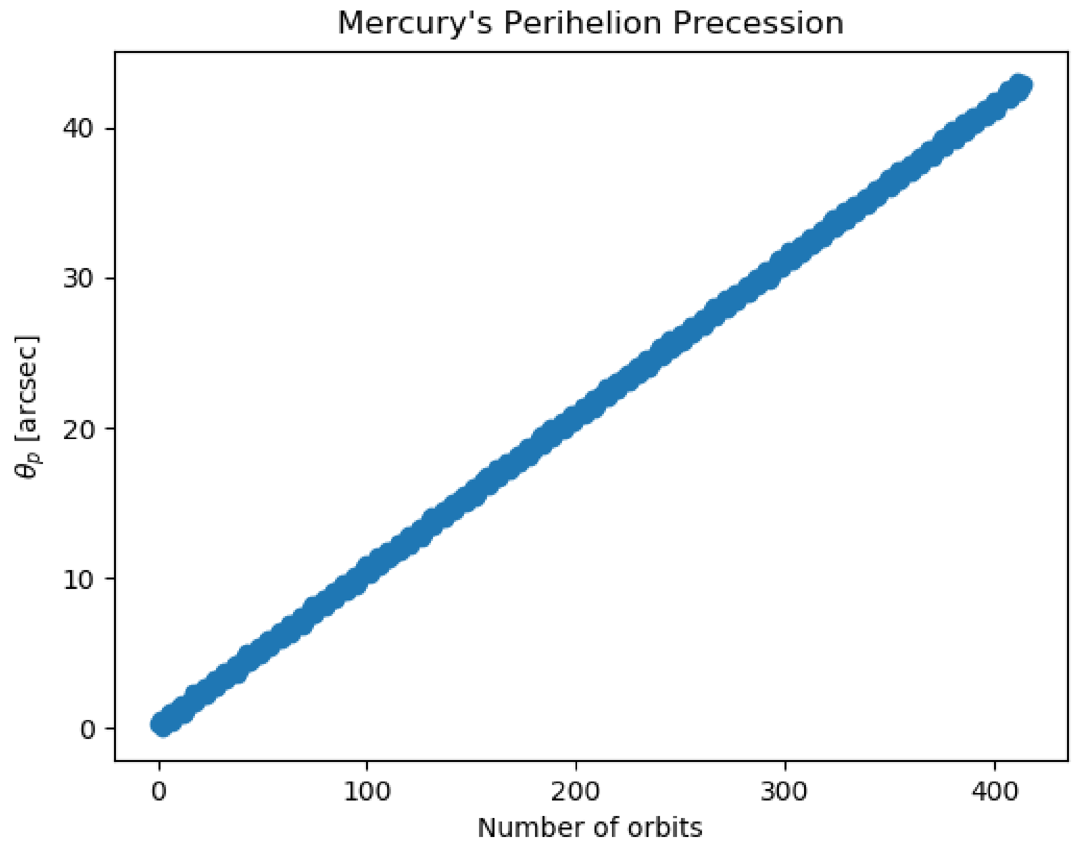

Figure 14: The perihelion precession of Mercury (100 years, $n = 10^9$). (The data behind this plot can be found in the text file mercury.txt, in the git-repo with the rest of the programs.)

5 Discussion

From Figure 1 and 2, we see that the velocity Verlet method has a much better accuracy than the forward Euler method with the same number of steps. Even with a hundred times more steps, forward Euler is still quite far away from the correct results reproduced by the velocity Verlet (see Figure 3).

In addition to having a global truncation error of the order $O(h^2)$, compared to $O(h)$ for the forward Euler method, the velocity Verlet method has the very convenient property of being symplectic (conserves energy). This keeps Earth in a stable orbit around the sun, unlike the spiralling orbit we get from the forward Euler method (Euler-Cromer, which is a simple improvement of the basic Euler method, is also symplectic).

The improved accuracy comes at quite a low price. With $8n$ FLOPs, compared to $5n$ FLOPs for forward Euler, we might expect an increase in CPU time by a factor of around 1.6. From Figure 4, we see that the CPU time actually is between twice and three times as much for the velocity Verlet method. This is probably due to the additional memory reads and writes caused by the fact that we need to store the forces in the previous points before calculating the new ones (See 2.3).

Analytically, we found that the escape velocity of a planet at a distance 1 AU from the sun is $2\pi\sqrt{2}$. From Figure 5 and 6, we see that when the velocity exceeds this value the orbit changes from an elliptic to a hyperbolic one, and the planet escapes.

We have also looked into what happens if we alter the exponent β of r in Newton's law of gravity. We found analytically that the escape velocity of a planet at a distance 1 AU from the sun approaches the circular orbit speed when $\beta \to 3$. This causes instability in the circular orbit, which we can clearly see in Figure 7. We notice that in this case the orbit becomes a spiral, not a hyperbola. When we increase the number of steps in our algorithm, the orbit appears to recover its stability (see Figure 8).

Introducing Jupiter doesn't alter Earth's stable circular motion around the Sun in any significant way (Figure 9). We don't even see any changes when we increase the mass of Jupiter by a factor of 10 (Figure 10). Eventually, by increasing the mass of Jupiter by a factor of 1000, the motion of Earth is changed dramatically. At this point, the mass of Jupiter is almost as big as the mass of the Sun. To get stable results for this last situation, we need to increase the number steps considerably (if not, we risk that the Earth comes too close to the Sun and gets repelled to infinity and beyond).

Allowing the Sun to move does not contribute to any significant changes in the orbits of Earth and Jupiter. In both the Earth-Sun-Jupiter system and the final model with all the planets, the motion of the Sun is limited to around 10^{-3} AU from the system's centre of mass. This is less than a fourth of the radius of the Sun. Hence, approximating the centre of mass of the solar system with the Sun is not very far away from the truth.

Finally, we looked at the two-body system of Mercury and the Sun with a relativistic correction to the classical gravitational force. We obtain a value for Mercury's perihelion precession angle after a century and 415 orbits of around 43", which coincides with the observed value after subtracting classical effects.

To get accurate results, we needed a pretty high time resolution. With 10^9 steps, our code took about half an hour to run, which is quite poor (at least compared to the code of a certain group teacher on Piazza). One of the reasons for this was probably the use of three-dimensional vectors, when the actual problem was limited to two dimensions.

6 Conclusion

In this project, we have demonstrated the power of object-orientated programming. In addition to enabling easy extension of programs, developing general solver classes gives us the opportunity to solve diverse problems without having to develop new code.

We have looked at the accuracy and performance of our two methods for solving differential equations. I believe it's safe to say that the improvement in accuracy of the velocity Verlet compared to the forward Euler method outweighs the possible tripling of the CPU time.

Through both analytical and numerical calculations, we have looked at escape velocities. When changing the exponent β of r in Newton's law of gravity from 2 to 3, we both alter the value of the escape velocity (from $2\pi\sqrt{2}$ to 2π for a planet 1 AU from the Sun), and the manner the planet escapes the gravitational field (hyperbolic vs. spiral orbits).

We have seen that the Sun is a good approximation of the centre of mass of the solar system.

We have confirmed that general relativity can explain the observed perihelion precession angle of Mercury, subtracting classical effects.

Through this project, we have learned a lot about handling stress and lack of sleep :-)

References

[1] JPL Solar System Dynamics – NASA. Horizons. https://ssd.jpl.nasa.gov/?horizons.

[2] Morten Hjorth-Jensen. Github Repository. https://github.com/CompPhysics/ComputationalPhysics/tree/master/doc/Projects/2018/Project2/CodeExamples.