CALCULATION OF THE WCS INVARIANT ON THE THURSTON EXAMPLE

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1. The calculation of the WCS invariant

We calculate the WCS invariant for the Thurston example in §4. Specifically, we prove

Theorem 4.1. We have

$$\int_{\overline{M_p}} \widetilde{CS}_5^W(e_1,\dots,e_5) = [2\pi p^{-1/2}\kappa]\frac{5\kappa}{128}p^2\int_0^1 (3072p^4 - 640p^2\beta^{-2} - 25\beta^{-4})d\theta_2. \quad (1.1)$$

Notation: We denote the metric $\tilde{g}$ on the Thurston example M just by g. We abbreviate $\theta_2 = \theta$, and recall that

$$\beta = 1 + \theta - \theta^2.$$

The Christoffel symbols are

$$\Gamma^a_{bc} = \frac{1}{2}g^{ae}(\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc}).$$

The curvature tensor components are

$$R_{abc}{}^d = \partial_a\Gamma^d_{bc} - \partial_b\Gamma^d_{ac} + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac}.$$

Recall that $g = \tilde{g}$ has a compatible almost complex structure J.

The matrix of g is given in (4.7), so

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \kappa^{-1}\beta^{1/2}(1+\theta) & \kappa^{-1}\beta^{1/2}\theta \\ 0 & 0 & \kappa^{-1}\beta^{1/2}\theta & \kappa^{-1}\beta^{1/2} \end{pmatrix}.$$

The long calculations below have been checked by machine calculation using Satoshi's program at `https://github.com/Egi/.....`

1.1. **The curvature tensor on $\overline{M}_p$.** Let $\overline{R}$ be the metric on $\overline{M}_p$.

Lemma 1.1. *In the notation of (4.8), we have*

$$\begin{aligned}
\overline{R}_{abc}{}^{d} &= R_{abc}{}^{d} - p^2 J_{bc} J_a{}^{d} + p^2 J_{ac} J_b{}^{d} + 2p^2 J_{ab} J_c{}^{d},\\
\overline{R}_{abc}{}^{0} &= -p\nabla_a J_{bc} + p\nabla_b J_{ac} = p\nabla_c J_{ab},\\
\overline{R}_{ab0}{}^{d} &= p\nabla_a J_b{}^{d} - p\nabla_b J_a{}^{d},\\
\overline{R}_{a0b}{}^{d} &= p\nabla_a J_b{}^{d},\\
\overline{R}_{a0b}{}^{0} &= -p^2 g_{ab}.
\end{aligned}$$

Proof. These are the local frame expressions of Lemma 3.3. □

Here we have used the identity

$$\nabla_a J_{bc} + \nabla_b J_{ca} + \nabla_c J_{ab} = 0,$$

which follows from $d\omega = 0$ and $g(JX, Y) = \omega(X, Y)$. For example, with this identity and curvature tensor symmetries, the second, third and fourth formulas in Lemma A.1 are equivalent.

1.2. **The Christoffel symbols on M.**

Lemma 1.2. *The following is the list of the Christoffel tensors*

$$\begin{aligned}
&\Gamma^2_{33} = \frac{\kappa}{4}\beta^{-3/2}(1-2\theta),\ \Gamma^2_{34} = \frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}),\ \Gamma^2_{44} = -\frac{\kappa}{4}\beta^{-3/2}(1+3\theta),\\
&\Gamma^3_{23} = -\frac{1}{4}\beta^{-1},\ \Gamma^3_{24} = -\frac{1}{2}\beta^{-1},\ \Gamma^4_{23} = -\frac{1}{2}\beta^{-1},\ \Gamma^4_{24} = \frac{1}{4}\beta^{-1}.
\end{aligned}$$

All other Christoffel symbols are zero.

Proof. We have

$$\Gamma^1_{ab} = \frac{1}{2}g^{1e}(\partial_a g_{be} + \partial_b g_{ae} - \partial_e g_{ab}) = \frac{1}{2}g^{11}(\partial_a g_{b1} + \partial_b g_{a1} - \partial_1 g_{ab}) = 0,$$

because $g_{\ell 1}$ is constant and $\partial_1 g_{ab} = 0$.

Since $\Gamma^2_{ab} = \frac{1}{2}g^{2e}(\partial_a g_{be} + \partial_b g_{ae} - \partial_e g_{ab}) = \frac{1}{2}(\partial_a g_{b2} + \partial_b g_{a2} - \partial_2 g_{ab})$ and $g_{22} = 1$, we get

$$\begin{aligned}
&\Gamma^2_{11} = \frac{1}{2}(\partial_1 g_{12} + \partial_1 g_{12} - \partial_2 g_{11}) = 0,\ \Gamma^2_{12} = \frac{1}{2}(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}) = 0,\\
&\Gamma^2_{22} = \frac{1}{2}(\partial_2 g_{22} + \partial_2 g_{22} - \partial_2 g_{22}) = 0.
\end{aligned}$$

Note that $g_{b2} = 0$, $g_{b1} = 0$ and $g_{b2} = 0$ if $b = 3, 4$. Therefore,

$$\Gamma^2_{1b} = \frac{1}{2}(\partial_1 g_{b2} + \partial_b g_{12} - \partial_2 g_{12}) = 0,\ \Gamma^2_{2b} = \frac{1}{2}(\partial_2 g_{b2} + \partial_b g_{22} - \partial_2 g_{2b}) = 0.$$

for $b = 3, 4$. We also get

$$\Gamma^2_{33} = \frac{1}{2}(\partial_3 g_{32} + \partial_3 g_{32} - \partial_2 g_{33}) = -\frac{1}{2}\partial_2(\kappa\beta^{-1/2}) = \frac{\kappa}{4}\beta^{-3/2}(1-2\theta),$$

$$\Gamma^2_{34} = \frac{1}{2}(\partial_3 g_{42} + \partial_4 g_{32} - \partial_2 g_{34}) = -\frac{1}{2}\partial_2(-\theta\kappa\beta^{-1/2}) = \frac{\kappa}{2}(\beta^{-1/2} + (-\frac{1}{2}\beta^{-3/2})\theta(1-2\theta))$$
$$= \frac{\kappa}{2}\beta^{-3/2}(1+\theta-\theta^2 - \frac{1}{2}\theta(1-2\theta)) = \frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta),$$

$$\Gamma^2_{44} = \frac{1}{2}(\partial_4 g_{42} + \partial_4 g_{42} - \partial_2 g_{44}) = -\frac{1}{2}\partial_2 g_{44} = -\frac{1}{2}\partial_2((1+\theta)\kappa\beta^{-1/2})$$
$$= -\frac{\kappa}{2}(\beta^{-1/2} + (1+\theta)(-\frac{1}{2}\beta^{-2/3}(1-2\theta)) = -\frac{\kappa}{2}\beta^{-2/3}(1+\theta-\theta^2 - \frac{1}{2}(1+\theta)(1-2\theta))$$
$$= -\frac{\kappa}{2}\beta^{-3/2}(1+\theta-\theta^2 - \frac{1}{2}(1-\theta-2\theta^2)) = -\frac{\kappa}{2}\beta^{-2/3}(\frac{1}{2}+\frac{3}{2}\theta)$$
$$= -\frac{\kappa}{4}\beta^{-3/2}(1+3\theta).$$

Since $\Gamma^3_{ab} = \frac{1}{2}g^{3e}(\partial_a g_{be} + \partial_b g_{ae} - \partial_e g_{ab})$, we have

$$\Gamma^3_{11} = \frac{1}{2}g^{3e}(\partial_1 g_{1e} + \partial_1 g_{1e} - \partial_e g_{11}) = 0,\ \Gamma^3_{12} = \frac{1}{2}g^{3e}(\partial_1 g_{2e} + \partial_2 g_{1e} - \partial_e g_{12}) = 0,$$

$$\Gamma^3_{22} = \frac{1}{2}g^{3e}(\partial_2 g_{2e} + \partial_2 g_{2e} - \partial_e g_{22}) = 0,\ \Gamma^3_{13} = \frac{1}{2}g^{3e}(\partial_1 g_{3e} + \partial_3 g_{1e} - \partial_e g_{13}) = 0,$$

$$\Gamma^3_{14} = \frac{1}{2}g^{3e}(\partial_1 g_{4e} + \partial_4 g_{1e} - \partial_e g_{14}) = 0,$$

$$\Gamma^3_{23} = \frac{1}{2}g^{3e}(\partial_2 g_{3e} + \partial_3 g_{2e} - \partial_e g_{23}) = \frac{1}{2}g^{3e}(\partial_2 g_{3e}) = \frac{1}{2}(g^{33}\partial_2 g_{33} + g^{34}\partial_2 g_{34})$$
$$= \frac{1}{2}(\kappa^{-1}(1+\theta)\beta^{-1/2}\partial_2(\kappa\beta^{-1/2}) + \theta\kappa^{-1}\beta^{-1/2}\partial_2(-\theta\kappa\beta^{-1/2}))$$
$$= \frac{1}{2}\beta^{-1/2}((1+\theta)\partial_2\beta^{-1/2} - \theta\beta^{-1/2} - \theta^2\partial_2\beta^{-1/2})$$
$$= \frac{1}{2}\beta^{-1/2}((1+\theta-\theta^2)\partial_2\beta^{-1/2} - \theta\beta^{-1/2})$$
$$= \frac{1}{2}\beta^{-1/2}(\beta(-\frac{1}{2}\beta^{-3/2}(1-2\theta)) - \theta\beta^{-1/2})$$
$$= -\frac{1}{4}\beta^{-1},$$

$$\Gamma^3_{24} = \frac{1}{2}g^{3e}(\partial_2 g_{4e} + \partial_4 g_{2e} - \partial_e g_{24}) = \frac{1}{2}g^{3e}\partial_2 g_{4e} = \frac{1}{2}(g^{33}\partial_2 g_{43} + g^{34}\partial_2 g_{44})$$
$$= \frac{1}{2}(\kappa^{-1}(1+\theta)\beta^{-1/2}\partial_2(-\theta\kappa\beta^{-1/2}) + \theta\kappa^{-1}\beta^{-1/2}\partial_2(\kappa(1+\theta)\beta^{-1/2}))$$
$$= \frac{1}{2}\beta^{-1/2}(-(1+\theta)\partial_2(\theta\beta^{-1/2}) + \theta\partial_2(1+\theta)\beta^{-1/2})$$

$$= \frac{1}{2}\beta^{-1/2}(-(1+\theta)\beta^{-1/2} - (1+\theta)\theta\partial_2\beta^{-1/2}) + \theta\beta^{-1/2} + \theta(1-\theta)\partial\beta^{-1/2}$$
$$= \frac{1}{2}\beta^{-1/2}(-\beta^{-1/2})$$
$$= -\frac{1}{2}\beta^{-1}.$$

Also, since $e = 3, 4$,

$$\Gamma^3_{33} = \frac{1}{2}g^{3e}(\partial_3 g_{3e} + \partial_3 g_{3e} - \partial_e g_{33}) = 0,\ \Gamma^3_{34} = \frac{1}{2}g^{3e}(\partial_3 g_{4e} + \partial_4 g_{3e} - \partial_e g_{34}) = 0,$$
$$\Gamma^3_{44} = \frac{1}{2}g^{3e}(\partial_4 g_{4e} + \partial_4 g_{4e} - \partial_e g_{44}) = 0.$$

Furthermore, we have $\Gamma^4_{ab} = \frac{1}{2}g^{4e}(\partial_a g_{be} + \partial_b g_{ae} - \partial_e g_{ab}) = \frac{1}{2}g^{4e}(\partial_a g_{be} + \partial_b g_{ae})$, since $e = 3, 4$. For these values of e, $g_{1e} = 0, g_{24} = 0, g_{2e} = 0$, so

$$\Gamma^4_{11} = \frac{1}{2}g^{4e}(\partial_1 g_{1e} + \partial_1 g_{1e}) = 0,\ \Gamma^4_{12} = \frac{1}{2}g^{4e}(\partial_1 g_{2e} + \partial_2 g_{1e}) = 0,$$
$$\Gamma^4_{22} = \frac{1}{2}g^{4e}(\partial_2 g_{2e} + \partial_2 g_{2e}) = 0,\ \Gamma^4_{13} = \frac{1}{2}g^{4e}(\partial_1 g_{3e} + \partial_3 g_{1e} - \partial_e g_{13}) = 0,$$
$$\Gamma^4_{14} = \frac{1}{2}g^{4e}(\partial_1 g_{4e} + \partial_4 g_{1e} - \partial_e g_{14}) = 0.$$

Also,

$$\Gamma^4_{23} = \frac{1}{2}g^{4e}(\partial_2 g_{3e} + \partial_3 g_{2e}) = \frac{1}{2}(g^{43}\partial_2 g_{33} + g^{44}\partial_2 g_{34})$$
$$= \frac{1}{2}(\theta\kappa^{-1}\beta^{-1/2}\partial_2(\kappa\beta^{-1/2}) + \kappa^{-1}\beta^{-1/2}\partial_2(-\theta\kappa\beta^{-1/2}))$$
$$= \frac{1}{2}\beta^{-1/2}(\theta\partial_2\beta^{-1/2} - \beta^{-1/2} - \theta\partial_2\beta^{-1/2})$$
$$= -\frac{1}{2}\beta^{-1},$$
$$\Gamma^4_{24} = \frac{1}{2}g^{4e}(\partial_2 g_{4e} + \partial_4 g_{2e}) = \frac{1}{2}(g^{43}\partial_2 g_{43} + g^{44}\partial_2 g_{44})$$
$$= \frac{1}{2}(\theta\kappa^{-1}\beta^{-1/2}\partial_2(-\theta\kappa\beta^{-1/2}) + \kappa^{-1}\beta^{-1/2}\partial_2((1+\theta)\kappa\beta^{-1/2}))$$
$$= \frac{1}{2}\beta^{-1/2}(-\theta(\beta^{-1/2} + \theta\partial_2\beta^{-1/2}) + \beta^{-1/2} + (1+\theta)\partial_2\beta^{-1/2})$$
$$= \frac{1}{2}(\beta^{-1/2}(1-\theta)\beta^{-1/2} + (1+\theta-\theta^2)\partial_2\beta^{-1/2})$$
$$= \frac{1}{2}\beta^{-1/2}((1-\theta)\beta^{-1/2} + \beta(-\frac{1}{2}\beta^{-3/2}(1-2\theta)))$$
$$= \frac{1}{4}\beta^{-1}.$$

Finally, we have

$$\Gamma_{33}^4 = \frac{1}{2}g^{4e}(\partial_3 g_{3e} + \partial_3 g_{3e} - \partial_e g_{33}) = 0,\ \Gamma_{34}^4 = \frac{1}{2}g^{4e}(\partial_3 g_{4e} + \partial_4 g_{3e} - \partial_e g_{34}) = 0,$$
$$\Gamma_{44}^4 = \frac{1}{2}g^{4e}(\partial_4 g_{4e} + \partial_4 g_{4e} - \partial_e g_{44}) = 0.$$

This proves the Lemma. □

1.3. **The curvature tensor on M.** We compute the curvature components ${R_{abc}}^d = \partial_a \Gamma_{bc}^d - \partial_b \Gamma_{ac}^d + \Gamma_{ae}^d \Gamma_{bc}^e - \Gamma_{be}^d \Gamma_{ac}^e$.

Lemma 1.3.

(i) ${R_{abc}}^d = 0$ *if any of* $a, b, c, d = 1$.

(ii)

$${R_{233}}^2 = -\frac{\kappa}{16}\beta^{-5/2}(9 - 16\theta + 16\theta^2),\ {R_{232}}^3 = \frac{1}{16}\beta^{-2}(9 - 8\theta),$$
$${R_{232}}^4 = \frac{1}{2}\beta^{-2}(1 - 2\theta),$$
$$R_{3232} = -{R_{233}}^a g_{a2} = -{R_{233}}^2 = \frac{\kappa}{16}\beta^{-5/2}(9 - 16\theta + 16\theta^2).$$

(iii)

$$R_{2424} = R_{4242} = {R_{424}}^a g_{a2} = {R_{424}}^2 = -{R_{244}}^2 = \frac{\kappa}{16}(1 + \theta + 24\theta^2)\beta^{-5/2},$$
$${R_{242}}^4 = \frac{1}{16}\beta^{-2}(1 + 8\theta),\ {R_{242}}^3 = \frac{1}{2}\beta^{-2}(1 - 2\theta).$$

(iv)

$${R_{234}}^2 = \frac{\kappa}{16}\beta^{-5/2}(-8 + 17\theta + 8\theta^2),$$
$$R_{2342} = {R_{234}}^a g_{a2} = {R_{234}}^2 = \frac{\kappa}{16}\beta^{-5/2}(-8 + 17\theta + 8\theta^2),$$
$$R_{3242} = -R_{2342} = -\frac{\kappa}{16}\beta^{-5/2}(-8 + 17\theta + 8\theta^2).$$

(v)

$${R_{343}}^3 = -\frac{5}{16}\kappa\theta\beta^{-5/2},\ {R_{343}}^4 = -\frac{5}{16}\kappa\beta^{-5/2},\ {R_{344}}^3 = \frac{5}{16}\kappa(1 + \theta)\beta^{-5/2},$$
$${R_{344}}^4 = \frac{5}{16}\kappa\theta\beta^{-5/2},\ R_{3434} = -R_{3443} = -\frac{5}{16}\kappa^2\beta^{-2}.$$

Proof. (i) ${R_{1bc}}^d = \partial_1\Gamma^d_{bc} - \partial_b\Gamma^d_{1c} + \Gamma^d_{1e}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{1c} = 0$, because $\Gamma^d_{1b} = 0$. Similarly,

$${R_{a1c}}^d = -{R_{1ac}}^d = 0,\ {R_{ab1}}^d = {R_{b1a}}^d + {R_{1ab}}^d = 0.$$

Using $\Gamma^1_{bc} = 0$, we get ${R_{abc}}^1 = \partial_a\Gamma^1_{bc} - \partial_b\Gamma^1_{ac} + \Gamma^1_{ae}\Gamma^e_{bc} - \Gamma^1_{be}\Gamma^e_{ac} = 0$.

(ii)

$$\begin{aligned}
{R_{233}}^2 &= \partial_2\Gamma^2_{33} - \partial_3\Gamma^2_{23} + \Gamma^2_{2e}\Gamma^e_{33} - \Gamma^2_{3e}\Gamma^e_{23} = \partial_2\Gamma^2_{33} - \Gamma^2_{33}\Gamma^3_{23} - \Gamma^2_{34}\Gamma^4_{23}\\
&= \partial_2(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta)) - (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{1}{4}\beta^{-1})\\
&\quad - (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(-\frac{1}{2}\beta^{-1})\\
&= \frac{\kappa}{4}(-2\beta^{-3/2} + (1-2\theta)\partial_2\beta^{-3/2})\\
&\qquad + \frac{\kappa}{16}\beta^{-5/2}(1-2\theta) + \frac{\kappa}{4}\beta^{-5/2}(1+\frac{\theta}{2})\\
&= \frac{\kappa}{16}\beta^{-5/2}(-8\beta + 4(1-2\theta)(-\frac{3}{2}(1-2\theta)) + (1-2\theta) + 4(1+\frac{\theta}{2}))\\
&= -\frac{\kappa}{16}\beta^{-5/2}(9 - 16\theta + 16\theta^2),
\end{aligned}$$

and

$$\begin{aligned}
{R_{232}}^3 &= \partial_2\Gamma^3_{32} - \partial_3\Gamma^3_{22} + \Gamma^3_{2e}\Gamma^e_{32} - \Gamma^3_{3e}\Gamma^e_{22}\\
&= \partial_2\Gamma^3_{32} + \Gamma^3_{23}\Gamma^3_{32} + \Gamma^3_{24}\Gamma^4_{32}\\
&= \partial_2(-\frac{1}{4}\beta^{-1}) + (-\frac{1}{4}\beta^{-1})^2 + (-\frac{1}{2}\beta^{-1})^2\\
&= -\frac{1}{4}(-\beta^{-2}(1-2\theta)) + \frac{1}{16}\beta^{-2} + \frac{1}{4}\beta^{-2}\\
&= \frac{1}{16}\beta^{-2}(9-8\theta),\\
{R_{232}}^4 &= \partial_2\Gamma^{32}_4 - \partial_3\Gamma^{22}_4 + \Gamma^{2e}_4\Gamma^{32}_e\\
&= \partial_2(-\frac{1}{2}\beta^{-1})] + (-\frac{1}{2}\beta^{-1})(-\frac{1}{4}\beta^{-1}) + (\frac{1}{4}\beta^{-1}(-\frac{1}{2}\beta^{-1})\\
&= -\frac{1}{2}(-\beta^{-2})(1-2\theta) + \frac{1}{8}\beta^{-2} - \frac{1}{8}\beta^{-2}\\
&= \frac{1}{2}\beta^{-2}(1-2\theta).
\end{aligned}$$

(iii)

$$\begin{aligned}
R_{2424} &= {R_{242}}^a g_{a4} = {R_{242}}^3 g_{34} + {R_{242}}^4 g_{44}\\
&= \frac{8}{16}\beta^{-2}(1-2\theta)(-\kappa\theta\beta^{-1/2}) + \frac{1}{16}\beta^{-2}(1+8\theta)(\kappa(1+\theta)\beta^{-1/2})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\kappa}{16}\beta^{-5/2}(1+\theta+24\theta^2),\\
R_{242}{}^{4} &= \partial_2\Gamma^4_{42} - \partial_4\Gamma^4_{22} + \Gamma^4_{2e}\Gamma^e_{42} - \Gamma^4_{4e}\Gamma^e_{22}\\
&= \partial_2\Gamma^4_{42} + \Gamma^4_{23}\Gamma^4_{24} + \Gamma^4_{24}\Gamma^4_{42}\\
&= \partial_2(\frac{1}{4}\beta^{-1}) + (-\frac{1}{2}\beta^{-1})(-\frac{1}{2}\beta^{-1}) + (\frac{1}{4}\beta^{-1})(\frac{1}{4}\beta^{-1})\\
&= \frac{1}{4}(-\beta^{-2}(1-2\theta)) + \frac{1}{4}\beta^{-2} + \frac{1}{16}\beta^{-2}\\
&= \frac{1}{16}\beta^{-2}(1+8\theta),\\
R_{242}{}^{3} &= \partial_2\Gamma^3_{42} - \partial_4\Gamma^3_{22} + \Gamma^3_{2e}\Gamma^e_{42} - \Gamma^3_{4e}\Gamma^e_{22}\\
&= \partial_2\Gamma^3_{42} + \Gamma^3_{23}\Gamma^3_{42} + \Gamma^3_{24}\Gamma^4_{42}\\
&= -\frac{1}{2}\partial_2\beta^{-1} + (-\frac{1}{4}\beta^{-1})(-\frac{1}{2}\beta^{-1}) + (-\frac{1}{2}\beta^{-1})(\frac{1}{4}\beta^{-1})\\
&= -\frac{1}{2}(-\beta^{-2}(1-2\theta)) + \frac{1}{8}\beta^{-2} - \frac{1}{8}\beta^{-2}\\
&= \frac{1}{2}\beta^{-2}(1-2\theta).
\end{aligned}
$$

(iv)

$$
\begin{aligned}
R_{234}{}^{2} &= \partial_2\Gamma^2_{34} - \partial_3\Gamma^2_{24} + \Gamma^2_{2e}\Gamma^e_{34} - \Gamma^2_{3e}\Gamma^e_{24} = \partial_2\Gamma^2_{34} - \Gamma^2_{33}\Gamma^3_{24} - \Gamma^2_{34}\Gamma^4_{24}\\
&= \frac{\kappa}{2}\partial_2(\beta^{-3/2}(1+\frac{\theta}{2})) - (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{1}{2}\beta^{-1})\\
&\quad - (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{1}{4}\beta^{-1})\\
&= \frac{\kappa}{2}(\frac{1}{2}\beta^{-3/2} + (1+\frac{\theta}{2})(-\frac{3}{2}\beta^{-5/2}(1-2\theta)))\\
&\quad + \frac{\kappa}{8}\beta^{-5/2}(1-2\theta) - \frac{\kappa}{8}\beta^{-5/2}(1+\frac{\theta}{2})\\
&= \frac{\kappa}{16}\beta^{-5/2}(-8+17\theta+8\theta^2).
\end{aligned}
$$

The other formulas follow immediately.

(v) $R_{34a}{}^{b} = \partial_3\Gamma^b_{4a} - \partial_4\Gamma^b_{3a} + \Gamma^b_{3e}\Gamma^e_{4a} - \Gamma^b_{4a}\Gamma^e_{3a} = \Gamma^b_{3e}\Gamma^e_{4a} - \Gamma^b_{4e}\Gamma^e_{3a}$, so

$$
\begin{aligned}
R_{344}{}^{2} &= \Gamma^2_{32}\Gamma^2_{44} - \Gamma^2_{43}\Gamma^3_{34} - \Gamma^2_{44}\Gamma^4_{34} = 0,\\
R_{343}{}^{4} &= \Gamma^4_{32}\Gamma^2_{43} - \Gamma^4_{42}\Gamma^2_{33} = (-\frac{1}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})) - (\frac{1}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))\\
&= -\frac{\kappa}{16}\beta^{-5/2}(4(1+\frac{\theta}{2}) + (1-2\theta)) = -\frac{5}{16}\kappa\beta^{-5/2},
\end{aligned}
$$

$$
\begin{aligned}
R_{344}{}^{3} &= \Gamma_{32}^{3}\Gamma_{44}^{2} - \Gamma_{42}^{3}\Gamma_{34}^{2} = (-\frac{1}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)) - (-\frac{1}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))\\
&= \frac{\kappa}{16}\beta^{-5/2}(1+3\theta+4(1+\frac{\theta}{2})) = \frac{5}{16}\kappa\beta^{-5/2}(1+\theta),\\
R_{344}{}^{4} &= \Gamma_{32}^{4}\Gamma_{44}^{2} - \Gamma_{42}^{4}\Gamma_{34}^{2} = (-\frac{1}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)) - (\frac{1}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))\\
&= \frac{\kappa}{8}\beta^{-5/2}(1+3\theta-(1+\frac{\theta}{2})) = \frac{5}{16}\kappa\theta\beta^{-5/2},\\
R_{343}{}^{3} &= \Gamma_{32}^{3}\Gamma_{43}^{2} - \Gamma_{42}^{3}\Gamma_{33}^{2} = (-\frac{1}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})) - (-\frac{1}{2}\beta^{-1})(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))\\
&= \frac{\kappa}{8}\beta^{-5/2}(-(1+\frac{\theta}{2}+(1-2\theta)) = -\frac{5}{16}\kappa\theta\beta^{-5/2},\\
R_{3434} &= R_{343}{}^{b}g_{b4} = R_{343}{}^{3}g_{34} + R_{343}{}^{4}g_{44}\\
&= (-\frac{5}{16}\theta\kappa\beta^{-5/2})(-\theta\kappa\beta^{-1/2}) + (-\frac{5}{16}\kappa\beta^{-5/2})(1+\theta)\kappa\beta^{-1/2}\\
&= -\frac{5}{16}\kappa^2\beta^{-3}(1+\theta-\theta^2) = -\frac{5}{16}\kappa^2\beta^{-2}.
\end{aligned}
$$

□

1.4. **The covariant derivatives of the almost complex structure on M.** We note for later use that

$$
(\omega_{ab}) = (J_a^c g_{cb}) = (J_{ab}) = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & \kappa\\ 0 & 0 & -\kappa & 0 \end{pmatrix}. \tag{1.2}
$$

We now compute the covariant derivatives of J.

Lemma 1.4. *(i)*

$$
\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0.
$$

(ii)

$$
\begin{aligned}
&\nabla_3 J_{13} = -\frac{\kappa}{4}\beta^{-3/2}(1-2\theta),\ \nabla_3 J_{14} = -\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}),\\
&\nabla_3 J_{23} = -\frac{\kappa}{2}\beta^{-1},\ \nabla_3 J_{24} = \frac{\kappa}{4}\beta^{-1}.
\end{aligned}
$$

(iii)

$$
\begin{aligned}
&\nabla_4 J_{13} = -\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}),\ \nabla_4 J_{14} = \frac{\kappa}{4}\beta^{-3/2}(1+3\theta)\\
&\nabla_4 J_{23} = \frac{\kappa}{4}\beta^{-1}.\ \nabla_4 J_{24} = \frac{\kappa}{2}\beta^{-1}.
\end{aligned}
$$

All other $\nabla_i J_{ab}$ vanish.

Proof. (i) Using $\Gamma^e_{1a} = 0$, we have $\nabla_1 J_{ab} = \partial_1 J_{ab} - \Gamma^e_{1a} J_{eb} - \Gamma^e_{1b} J_{ae} = 0$.

We also have

$$\nabla_2 J_{ab} = \partial_2 J_{ab} - \Gamma^e_{2a} J_{eb} - \Gamma^e_{2b} J_{ae} = -\Gamma^e_{2a} J_{eb} - \Gamma^e_{2b} J_{ae}.$$

Thus

$$\begin{aligned}
\nabla_2 J_{12} &= -\Gamma^e_{21} J_{e2} - \Gamma^e_{22} J_{1e} = -\Gamma^e_{22} J_{1e} = -\Gamma^2_{22} J_{12} = 0,\\
\nabla_2 J_{34} &= -\Gamma^e_{23} J_{e4} - \Gamma^e_{24} J_{3e} - \Gamma^3_{23} J_{34} - \Gamma^4_{24} J_{34}\\
&= -(-\frac{1}{4}\beta^{-1}) J_{34} - (\frac{1}{4}\beta^{-1}) J_{34} = 0,\\
\nabla_2 J_{11} &= -\Gamma^e_{21} J_{e1} - \Gamma^e_{21} J_{1e} = 0,\\
\nabla_2 J_{22} &= -\Gamma^e_{22} J_{e2} - \Gamma^e_{22} J_{2e} = 0,\\
\nabla_2 J_{aa} &= -\Gamma^e_{2a} J_{ea} - \Gamma^e_{2a} J_{ae} = 0, (\textit{Does this include } a = 1, 2 \textit{ above?})\\
\nabla_2 J_{13} &= -\Gamma^e_{21} J_{e3} - \Gamma^e_{23} J_{1e} = -\Gamma^4_{21} J_{43} - \Gamma^2_{23} J_{12} = 0,\\
\nabla_2 J_{14} &= -\Gamma^e_{21} J_{e4} - \Gamma^e_{24} J_{1e}, = -\Gamma^2_{24} J_{12} = 0,\\
\nabla_2 J_{23} &= -\Gamma^e_{23} J_{e3} - \Gamma^e_{23} J_{2e} = -\Gamma^4_{22} J_{43} - \Gamma^1_{23} J_{21} = 0,\\
\nabla_2 J_{24} &= -\Gamma^e_{22} J_{e4} - \Gamma^e_{24} J_{24} = -\Gamma^3_{22} J_{34} - \Gamma^1_{24} J_{21} = 0.
\end{aligned}$$

Therefore, $\nabla_2 J_{ab} = 0$.

(ii) In general,

$$\nabla_3 J_{ab} = -\Gamma^e_{3a} J_{eb} - \Gamma^e_{3b} J_{ae}.$$

Thus

$$\begin{aligned}
\nabla_3 J_{1b} &= -\Gamma^e_{31} J_{eb} = 0,\\
\nabla_3 J_{12} &= -\Gamma^e_{31} J_{e2} - \Gamma^e_{32} J_{1e} = -\Gamma^1_{31} J_{12} - \Gamma^2_{32} J_{12} = 0,\\
\nabla_3 J_{13} &= -\Gamma^e_{31} J_{e3} - \Gamma^e_{33} J_{1e} = -\Gamma^4_{31} J_{43} - \Gamma^2_{33} J_{12} = -\Gamma^2_{33} J_{12} = -\Gamma^2_{33}\\
&= -\frac{\kappa}{4}\beta^{-3/2}(1 - 2\theta),\\
\nabla_3 J_{14} &= -\Gamma^2_{34} J_{12} = -\frac{\kappa}{2}\beta^{-3/2}\left(1 + \frac{\theta}{2}\right),\\
\nabla_3 J_{21} &= -\Gamma^2_{32} J_{21} - \Gamma^e_{31} J_{2e} = 0,\\
\nabla_3 J_{23} &= -\Gamma^4_{32} J_{43} - \Gamma^1_{33} J_{21} = \kappa \Gamma^4_{32} = -\frac{\kappa}{2}\beta^{=1},\\
\nabla_3 J_{24} &= -\Gamma^3_{32} J_{24} - \Gamma^1_{34} J_{21} = -\kappa \Gamma^3_{32} = -\kappa(-\frac{1}{4}\beta^{-1}) = \frac{\kappa}{4}\beta^{-1},\\
\nabla_3 J_{31} &= -\Gamma^2_{33} J_{21} - \Gamma^e_{31} J_{3e} = \Gamma^2_{33} = \frac{\kappa}{4}\beta^{-3/2}(1 - 2\theta),\\
\nabla_3 J_{32} &= -\Gamma^1_{33} J_{12} - \Gamma^4_{32} J_{34} = -\Gamma^4_{32} J_{34} = -\kappa \Gamma^4_{32}, = -\kappa(-\frac{1}{2}\beta^{-1}) = \frac{\kappa}{2}\beta^{-1},
\end{aligned}$$

$$\nabla_3 J_{34} = -\Gamma^3_{33} J_{34} - \Gamma^4_{43} J_{34} = 0,$$
$$\nabla_3 J_{41} = -\Gamma^2_{34} J_{21} - \Gamma^e_{31} J_{4e} = \Gamma^2_{34} = \frac{\kappa}{2}\beta^{-3/2}\left(1+\frac{\theta}{2}\right),$$
$$\nabla_3 J_{42} = -\Gamma^1_{34} - \Gamma^3_{32} J_{43} = -\Gamma^3_{32}(-\kappa) = \kappa\Gamma^3_{32} = -\frac{\kappa}{4}\beta^{-1},$$
$$\nabla_3 J_{43} = -\Gamma^4_{34} J_{43} - \Gamma^3_{33} J_{43} = 0,$$

(iii) Since $\nabla_4 J_{ab} = -\Gamma^e_{4a} J_{eb} - \Gamma^e_{4b} J_{ae}$, we have

$$\nabla_4 J_{12} = -\Gamma^2_{43} = -\frac{\kappa}{2}\beta^{-3/2}\left(1+\frac{\theta}{2}\right),$$
$$\begin{aligned}\nabla_4 J_{14} &= -\Gamma^2_{44} J_{12} = -(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)) \\ &= \frac{\kappa}{4}\beta^{-3/2}(1+3\theta),\end{aligned}$$
$$\nabla_4 J_{21} = -\Gamma^2_{42} J_{21} - \Gamma^e_{41} J_{2e} = 0,$$
$$\nabla_4 J_{23} = -\Gamma^4_{42} J_{43} - \Gamma^1_{43} J_{21} = -\Gamma^4_{42}(-\kappa) = \kappa\Gamma^4_{42} = \frac{\kappa}{4}\beta^{-1},$$
$$\nabla_4 J_{24} = -\Gamma^3_{42} J_{34} - \Gamma^1_{44} J_{21} = -\kappa\Gamma^3_{42} = \frac{\kappa}{2}\beta^{-1},$$
$$\nabla_4 J_{31} = -\Gamma^2_{43} J_{21} - \Gamma^e_{41} J_{3e} = \Gamma^2_{43} = \frac{\kappa}{2}\beta^{-3/2}\left(1+\frac{\theta}{2}\right),$$
$$\nabla_4 J_{32} = -\Gamma^1_{43} J_{12} - \Gamma^4_{42} J_{34} = \kappa\Gamma^4_{42} = -\frac{\kappa}{4}\beta^{-1},$$
$$\nabla_4 J_{34} = -\Gamma^3_{43} J_{34} - \Gamma^4_{44} J_{34} = 0,$$
$$\nabla_4 J_{41} = -\Gamma^2_{44} J_{21} = -\frac{\kappa}{4}\beta^{-3/2}(1+3\theta),$$
$$\nabla_4 J_{42} = -\Gamma^3_{42} J_{43} = -\Gamma^3_{42}(-\kappa) = \kappa\Gamma^3_{42} = -\frac{\kappa}{2}\beta^{-1},$$
$$\nabla_4 J_{43} = -\Gamma^2_{44} J_{23} - \Gamma^3_{43} J_{43} = 0.$$

□

1.5. **The computation of (1.1). First steps.** The goal is to show

$$S := \sum_{\sigma\in\mathfrak{S}_5} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0\ell_1 0}{}^n \overline{R}_{\sigma_1\sigma_2\ell_2}{}^{\ell_1}\overline{R}_{\sigma_3\sigma_4 n}{}^{\ell_2} = \frac{\kappa}{16}p^2(3072p^4 - 640p^2\beta^{-2} - 25\beta^{-4}), \quad (1.3)$$

(see (4.8)).

For $i \in \{0,1,2,3,4\}$, we set

$$S_i = \sum_{\substack{\sigma\in\mathfrak{S}_5 \\ \sigma_i = 0}} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0\ell_1 0}{}^n \overline{R}_{\sigma_1\sigma_2\ell_2}{}^{\ell_1}\overline{R}_{\sigma_3\sigma_4 n}{}^{\ell_2},$$

so
$$S = \sum_{i=0}^{4} S_i.$$
Interchanging σ_1 and σ_2 gives $S_1 = S_2$, since both $\mathrm{sgn}(\sigma)$ and a curvature term change sign, and interchanging σ_3 and σ_4 gives $S_3 = S_4$. Thus,
$$S = S_0 + 2S_1 + 2S_3. \tag{1.4}$$
For $m = 0, 1, 3$, we set
$$S_m = S_{m,2} + S_{m,4} + S_{m,6},$$
where $S_{m,k}$ is the p^k-term of S_m. In this notation, (1.3) is equivalent to:

Proposition 1.5.
$$\begin{aligned}
\bar{S}_{[2]} &:= S_{0,2} + 2S_{1,2} + 2S_{3,2} = \frac{\kappa}{16} p^2(-25\beta^{-4}),\\
\bar{S}_{[4]} &:= S_{0,4} + 2S_{1,4} + 2S_{3,4} = \frac{\kappa}{16}(-640p^2\beta^{-2}),\\
\bar{S}_{[6]} &:= S_{0,6} + 2S_{1,6} + 2S_{3,6} = \frac{\kappa}{16} 3072p^6.
\end{aligned}$$

This will be proved in a number of steps. $\bar{S}_{[6]}$ is computed in Prop. 1.9, $\bar{S}_{[2]}$ is computed in Prop. 1.11, and $\bar{S}_{[4]}$ is computed in Prop. 1.13.

For notation, the metric $\bar{g}$ on $\bar{M}$ is given by
$$\bar{g} = (\bar{g}_{ij}) = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & \bar{g}_{ab} & & \\ 0 & & & & \end{array}\right)$$
where $i, j, k, \ldots \in \{0, 1, 2, 3, 4\}$ and $a, b, \ldots \in \{1, 2, 3, 4\}$. Thus $\bar{g}_{00} = \bar{g}(\partial_0, \partial_0), \bar{g}_{ab} = \bar{g}(\partial^L_{\theta_a}, \partial^L_{\theta_b}) = g(\partial_{\theta_a}, \partial_{\theta_b})$.

We now compute S_0, using the formulas in Lemma 1.1. In the obvious notation,
$$\begin{aligned}
S_0 &= \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \overline{R}_{0a_10}{}^{b} \overline{R}_{\sigma_1\sigma_2\ell_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 b}{}^{\ell_2}\\
&= \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \overline{R}_{0a_10}{}^{b} [\overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 b}{}^{a_2} + \overline{R}_{\sigma_1\sigma_2 0}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 b}{}^{0}].
\end{aligned}$$
Since $\overline{R}_{0a_10}{}^{b} = -p^2\delta^b_{a_1}$, we get
$$\begin{aligned}
S_0 &= -p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma)[\overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} + \overline{R}_{\sigma_1\sigma_2 0}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{0}]\\
&= -p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma)[\overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} - \overline{R}_{\sigma_1\sigma_2 a_1'}{}^{0} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{0} g^{a_1a_1'}]
\end{aligned}$$

$$
\begin{aligned}
&= -p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[\overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1}\overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} - \overline{R}_{\sigma_1\sigma_2 a_1'}{}^{0}\overline{R}_{\sigma_3\sigma_4 a_1}{}^{0} g^{a_1 a_1'}] \\
&= -p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1}\overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} + p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)\nabla_{a_1'} J_{\sigma_1\sigma_2}\nabla_{a_1} J_{\sigma_3\sigma_4} g^{a_1 a_1'} \\
&:= S_{0,*} + S_{0,4,1}.
\end{aligned}
$$

We will see below (Lemma 1.14) that

$$
S_{0,4,1} = -S_{1,4,1} = 5p^4\kappa\beta^{-2}. \tag{1.5}
$$

(For the definition of $S_{1,4,1}$, see (1.13). The first equality follows from a "change of variables" sending σ to $(01)\sigma$.) For $S_{0,*}$, we have

$$
\begin{aligned}
S_{0,*} &= -p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1}\overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} \\
&= -p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)(R_{\sigma_1\sigma_2 a_2}{}^{a_1} - p^2 J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + p^2 J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2p^2 J_{\sigma_2\sigma_2} J_{a_2}{}^{a_1}) \\
&\qquad\qquad \cdot (R_{\sigma_3\sigma_4 a_1}{}^{a_2} - p^2 J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + p^2 J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} + 2p^2 J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}) \\
&= -p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[(R_{\sigma_1\sigma_2 a_2}{}^{a_1} R_{\sigma_3\sigma_4 a_1}{}^{a_2}) \\
&\qquad\qquad + p^2(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1}) R_{\sigma_3\sigma_4 a_1}{}^{a_2} \\
&\qquad\qquad + p^2 R_{\sigma_1\sigma_2 a_2}{}^{a_1}(-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_3}{}^{a_1} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}) \\
&\qquad\qquad + p^4(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1})] \\
&\qquad\qquad \cdot (-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_3}{}^{a_1} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}).
\end{aligned}
$$

The p^4 term in $S_{0,*}$ is

$$
\begin{aligned}
S_{0,4,2} := -p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[&(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1}) R_{\sigma_3\sigma_4 a_1}{}^{a_2} \\
&+ R_{\sigma_1\sigma_2 a_2}{}^{a_1}(-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_3}{}^{a_1} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2})].
\end{aligned}
$$

We can simplify $S_{0,4,2}$ using the change of variable $\sigma \mapsto \tau = \sigma(13)(24)$ as in the calculation of (β) in Appendix B:

$$
\begin{aligned}
S_{0,4,2} &= -p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1}) R_{\sigma_3\sigma_4 a_1}{}^{a_2} \\
&\qquad\qquad + R_{\sigma_1\sigma_2 a_2}{}^{a_1}(-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2})] \qquad (1.6) \\
&= -p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[-R_{\sigma_3\sigma_4 a_1 a_2} J_{\sigma_2}{}^{a_2} J_{\sigma_1}{}^{a_1} + R_{\sigma_3\sigma_4 a_2 a_1} J_{\sigma_1}{}^{a_1} J_{\sigma_2}{}^{a_2} - 2R_{\sigma_3\sigma_4 a_1 a_2} J_{\sigma_1\sigma_2} J^{a_1 a_2} \\
&\qquad\qquad - R_{\sigma_1\sigma_2 a_2 a_1} J_{\sigma_4}{}^{a_1} J_{\sigma_3}{}^{a_2} + R_{\sigma_1\sigma_2 a_1 a_2} J_{\sigma_3}{}^{a_1} J_{\sigma_4}{}^{a_2} - 2R_{\sigma_1\sigma_2 a_2 a_1} J_{\sigma_3\sigma_4} J^{a_1 a_2}] \\
&= 4p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[R_{\sigma_1\sigma_2 a_1 a_2} J_{\sigma_3}{}^{a_1} J_{\sigma_4}{}^{a_2} + R_{\sigma_1\sigma_2 a_1 a_2} J^{a_1 a_2} J_{\sigma_3\sigma_4}]
\end{aligned}
$$

$$:= S_{0,4,2,1} + S_{0,4,2,2}.$$

where

$$S_{0,4,2,1} = 4p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) R_{\sigma_1\sigma_2 a_1 a_2} J_{\sigma_3}{}^{a_1} J_{\sigma_4}{}^{a_2}, \tag{1.7}$$

$$S_{0,4,2.2} = 4p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) R_{\sigma_1\sigma_2 a_1 a_2} J^{a_1 a_2} J_{\sigma_3\sigma_4}.$$

Here is a summary of S_0, obtained from (1.5), (1.6), (1.7).

Lemma 1.6.

$$S_{0,2} = -p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) R_{\sigma_1\sigma_2 a_2}{}^{a_1} R_{\sigma_3\sigma_4 a_1}{}^{a_2},$$

$$\begin{aligned} S_{0,4} &= p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) \nabla_{a_1'} J_{\sigma_1\sigma_2} \nabla_{a_1} J_{\sigma_3\sigma_4} g^{a_1 a_1'} \\ &\quad - 4p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) [R_{\sigma_1\sigma_2 a_1 a_2} J_{\sigma_3}{}^{a_1} J_{\sigma_4}{}^{a_2} + R_{\sigma_1\sigma_2 a_1 a_2} J^{a_1 a_2} J_{\sigma_3\sigma_4}] \end{aligned} \tag{1.8}$$

$$\begin{aligned} S_{0,6} &= -p^6 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) (-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2 J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1} \\ &\qquad \cdot (-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_3}{}^{a_1} + 2 J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}). \end{aligned}$$

We now turn to the computation of S_1.

$$\begin{aligned} S_1 &= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \overline{R}_{\sigma_0 \ell_1 0}{}^{n} \overline{R}_{0\sigma_2\ell_2}{}^{\ell_1} \overline{R}_{\sigma_3\sigma_4 n}{}^{\ell_2} = -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \bar{R}_{\sigma_0\ell_1 0 b'} \overline{R}_{0\sigma_2\ell_2}{}^{\ell_1} \overline{R}_{\sigma_3\sigma_4 b}{}^{\ell_2} g^{bb'} \\ &= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \bar{R}_{\sigma_0 0 0 b'} \overline{R}_{0\sigma_2 a}{}^{0} \overline{R}_{\sigma_3\sigma_4 b}{}^{a} g^{bb'} - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \bar{R}_{\sigma_0 a_1 0 b'} \overline{R}_{0\sigma_2\ell_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 b}{}^{\ell_2} g^{bb'} \\ &= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) (p^2 g_{\sigma_0 b'})(p^2 g_{\sigma_2 a_2}) \overline{R}_{\sigma_3\sigma_4 b}{}^{a_2} g^{bb'} - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \bar{R}_{\sigma_0 a_1 0 b'} \bar{R}_{0\sigma_2 0 a_1'} \bar{R}_{\sigma_3\sigma_4 b 0} g^{bb'} g^{a_1 a_1'} \\ &\quad - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \bar{R}_{\sigma_0 a_1 0 b'} \bar{R}_{0\sigma_2 a_2 a_1'} \overline{R}_{\sigma_2\sigma_4 b}{}^{a_2} g^{bb'} g^{a_1 a_1'} \\ &= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 R_{\sigma_3\sigma_4\sigma_0\sigma_2} - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) (-p\nabla_{b'} J_{\sigma_0 a_1})(-p^2 g_{\sigma_2 a_2'})(p\nabla_b J_{\sigma_3\sigma_4}) g^{bb'} g^{a_1 a_1'} \\ &\quad - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) (-p\nabla_{b'} J_{\sigma_0 a_1})(-p^2 \nabla_{\sigma_2} J_{a_2 a_1'}) \overline{R}_{\sigma_3\sigma_4 b}{}^{a_2} g^{bb'} g^{a_1 a_1'}. \end{aligned}$$

This can be simplified using the Bianchi identity

$$\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) R_{\sigma_3\sigma_4\sigma_0\sigma_2} = 0,$$

to give

$$\begin{aligned}
S_1 = &-\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 g^{bb'} \nabla_{b'} J_{\sigma_0\sigma_2} \nabla_b J_{\sigma_3\sigma_4} \\
&- \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^2 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} \bar{R}_{\sigma_3\sigma_4 b a_2'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
= &-\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 g^{bb'} \nabla_{b'} J_{\sigma_0\sigma_2} \nabla_b J_{\sigma_3\sigma_4} \\
&- \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^2 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3\sigma_4 b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'} \\
&- \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} \\
&\qquad \cdot (-J_{\sigma_4 b} J_{\sigma_3 a_2'} + J_{\sigma_3 b} J_{\sigma_4 a_2'} + 2 J_{\sigma_3\sigma_4} J_{b a_2'}) g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'} \\
= &-\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 g^{bb'} \nabla_b J_{\sigma_0\sigma_2} \nabla_{b'} J \sigma_3 \sigma_4 \\
&+ \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_4 b} J_{\sigma_3 a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'} \\
&- \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_3 b} J_{\sigma_4 a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'} \\
&- 2\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{sigma_2} J_{a_2 a_1'} J_{\sigma_3\sigma_4} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'} \\
&- \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^2 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3\sigma_4 b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'} \\
= &-\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 g^{bb'} \nabla_b J_{\sigma_0\sigma_2} \nabla_{b'} J_{\sigma_3\sigma_4} \\
&+ 2\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_4 b} J_{\sigma_3 a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'} \\
&- 2\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_3\sigma_4} J_{b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'} \\
&- p^2 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3\sigma_4 b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'}.
\end{aligned}$$

In summary,

Lemma 1.7.

$$S_1 = S_{1,2} + S_{1,4},$$

where

$$\begin{aligned}
S_{1,2} &= -p^2 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3\sigma_4 b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'},\\
S_{1,4} &= -p^4 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) g^{bb'} \nabla_b J_{\sigma_0\sigma_2} \nabla_{b'} J_{\sigma_3\sigma_4}\\
&\quad + 2p^4 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_4 b} J_{\sigma_3 a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'}\\
&\quad - 2p^4 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_3\sigma_4} J_{b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'}.
\end{aligned}$$

We now compute S_3. Using $\overline{R}_{\sigma_0 \ell_1 0}{}^0 = 0$, we obtain

$$\begin{aligned}
S_3 &= -\sum_{\sigma_3=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0\ell_1 0}{}^n \overline{R}_{\sigma_1\sigma_2\ell_2}{}^{\ell_1} \overline{R}_{0\sigma_4 n}{}^{\ell_2} = -\sum_{\sigma_3=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0\ell_1 0}{}^b \overline{R}_{\sigma_1\sigma_2\ell_2}{}^{\ell_1} \overline{R}_{0\sigma_4 b}{}^{\ell_2}\\
&= -\sum_{\sigma_3=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0 00}{}^b \overline{R}_{\sigma_1\sigma_2 a_2}{}^0 \overline{R}_{0\sigma_4 b}{}^{a_2} - \sum_{\sigma_3=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0 a_1 0}{}^b \overline{R}_{\sigma_1\sigma_2\ell_2}{}^{a_1} \overline{R}_{0\sigma_4 b}{}^{\ell_2}\\
&= -\sum_{\sigma_3=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0 00}{}^b \overline{R}_{\sigma_1\sigma_2 a_2}{}^0 \overline{R}_{0\sigma_4 b}{}^{a_2} - \sum_{\sigma_3=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0 a_1 0}{}^b \overline{R}_{\sigma_1\sigma_2 0}{}^{a_1} \overline{R}_{0\sigma_4 b}{}^{0}\\
&\quad - \sum_{\sigma_3=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0 a_1 0}{}^b \overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1} \overline{R}_{0\sigma_4 b}{}^{a_2}.
\end{aligned}$$

Since

$$\overline{R}_{0\sigma_4 b}{}^{a_2} = \bar{R}_b{}^{a_2}{}_0{}^{\sigma_4} = -\bar{R}_b{}^{a_2}{}_{\sigma_4 0},\ \overline{R}_{0\sigma_4 b}{}^{a_2} = -\bar{R}_b{}^{a_2}{}_{\sigma_4 0},$$

this becomes

$$\begin{aligned}
S_3 = &-\sum_{\sigma_3=0} \operatorname{sgn}(\sigma)(p^2\delta^b_{\sigma_0})(p\nabla_{a_2} J_{\sigma_1\sigma_2})(-p\nabla_{\sigma_4} J_b{}^{a_2}) - (-p^2\nabla_{\sigma_4} J_{\sigma_0 a_1})(-p\nabla^{a_1} J_{\sigma_1\sigma_2})(p^2 g_{\sigma_4 b})\\
&- \sum_{\sigma_3=0} \operatorname{sgn}(\sigma)(-p\nabla^b J_{\sigma_0 a_1})\overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1}(-p\nabla_{\sigma_4} J_b{}^{a_2}).
\end{aligned}$$

This further simplifies, using

$$\sum_{\sigma_3=0} \operatorname{sgn}(\sigma) p^4 \nabla_{a_2} J_{\sigma_1\sigma_2} \nabla_{\sigma_4} J_{\sigma_0}{}^{a_2} - \sum_{\sigma_3=0} \operatorname{sgn}(\sigma) p^4 \nabla_{\sigma_4} J_{\sigma_0 a_2} \nabla^{a_2} J_{\sigma_1\sigma_2} = 0,$$

to give

Lemma 1.8.

$$\begin{aligned}
S_3 &= S_{3,2} + S_{3,4}\\
&= -\sum_{\sigma_3=0} \operatorname{sgn}(\sigma) p^2 \nabla^b J_{\sigma_0 a_1} \nabla_{\sigma_4} J_b{}^{a_2} R_{\sigma_1\sigma_2 a_2}{}^{a_1}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\sigma_3=0} \operatorname{sgn}(\sigma) p^4 \nabla^b J_{\sigma_0 a_1} \nabla_{\sigma_4} J_b{}^{a_2} (-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J\sigma_1\sigma_2 J_{a_2}{}^{a_1}) \\
= &- \sum_{\sigma_3=0} \operatorname{sgn}(\sigma) p^2 \nabla^b J_{\sigma_0 a_1} \nabla_{\sigma_4} J_b{}^{a_2} R_{\sigma_1\sigma_2 a_2}{}^{a_1} \\
&+ 2p^4 \sum_{\sigma_3=0} \operatorname{sgn}(\sigma) \nabla^b J_{\sigma_0 a_1} \nabla_{\sigma_4} J_b{}^{a_2} [J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} - J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1}].
\end{aligned}
$$

The last equality follows from the change of variable $\sigma \mapsto \sigma(12)$.

1.6. **The p^6 term in (1.3).** We now compute the p^6 term in (1.3), or equivalently the third term in Prop. 1.5. Since $S_{1,6} = S_{3,6} = 0$ by Lemmas 1.7, 1.8, respectively, we need only compute $S_{0,6}$.

Proposition 1.9.

$$\bar{S}_{[6]} = S_{0,6} = 192p^6\kappa.$$

Proof. We have

$$
\begin{aligned}
&S_{0,6} \\
&= -p^6 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) (-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1}) \\
&\qquad \cdot (-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}) \\
&= -p^6 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) [J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} - J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} - 2J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2} \\
&\qquad - J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} + 2J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2} \\
&\qquad - 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1} J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1} J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} + 4J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1} J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}].
\end{aligned}
$$

Using (**??**), we get

$$S_{0,6} = -p^6 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)(-2\cdot 4 - 16) J_{\sigma_1\sigma_2} J_{\sigma_3\sigma_4} = p^6 \cdot 24 \cdot 2^3 \kappa = 192p^6\kappa.$$

□

1.7. **The p^2 term in (1.3).** We now compute

$$\bar{S}_{[2]} = S_{0,2} + 2S_{1,2} + 2S_{3,2}.$$

We first note that there is no contribution from $S_{0,2}$.

Lemma 1.10.

$$S_{0,2} = 0.$$

Proof. By Lemmas 1.3(i) and 1.6, we get

$$S_{0,2} = \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) R_{\sigma_1\sigma_2 a_2}{}^b R_{\sigma_3\sigma_4 b}{}^{a_2} = 0,$$

since every permutation has $\sigma_a = 1$ for some a. □

Note that the middle term in the last equation is a multiple of the first Pontryagin form, which implies that the signature of M is zero.

Proposition 1.11.
$$\bar{S}_{[2]} = 2(S_{1,2} + S_{3,2}) = -\frac{25}{16}\kappa\beta^{-4} \tag{1.9}$$

Check that the proof below is ok – compare to v2.

Proof. We have

$$\begin{aligned} S_{1,2} + S_{3,2} &= -p^2 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\nabla_{b'} J_{\sigma_0 a_1'} \nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3\sigma_4 b a_2'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\ &\quad - p^2 \sum_{\sigma_3=0} \operatorname{sgn}(\sigma)\nabla^b J_{\sigma_0 a_1} \nabla_{\sigma_4} J_b{}^{a_2} R_{\sigma_1\sigma_2 a_2}{}^{a_1} \\ &= p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)\nabla_{a_1} J_{\sigma_1 b} \nabla_{\sigma_2} J_{a_2 b'} R_{\sigma_3\sigma_4 a_2' a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\ &\quad + p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)\nabla_{b'} J_{\sigma_1 a_1} \nabla_{\sigma_2} J_{a_2' b} R_{\sigma_3\sigma_4 a_2 a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'}, \end{aligned}$$

where we use the change of variable $\sigma \mapsto (01)\sigma$ on the third line and $\sigma \mapsto (301)(24)\sigma$ on the fourth line. We know

$$(\sigma_3, \sigma_4) \in \{(2,3), (3,2), (2,4), (4,2), (3,4), (4,3)\}.$$

Case (1): $(\sigma_3, \sigma_4) = (2,3)$ or $(3,2)$

In this case, in cycle notation

$$\sigma \in \{(243), (1432), (24), (142)\},$$

with signs $+1, -1, -1, +1$, respectively. There is no contribution to $S_{1,2} + S_{3,2}$ if $\sigma = (1432)$ or $\sigma = (142)$, since $\nabla_1 J_{ab} = 0$. Thus for $(S_{1,2} + S_{3,2})(1)$ the contribution to $S_{2,2} + S_{4,2}$ from Case (1), we get

$$\begin{aligned} (S_{1,2} + S_{3,2})(1) &= 2p^2 \nabla_{a_1} J_{1b} \nabla_4 J_{a_2 b'} R_{23 a_2' a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'} \\ &\quad + 2\nabla_{b'} J_{1 a_1} \nabla_4 J_{a_2' b} R_{23 a_2 a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'} \\ &= 2p^2 \nabla_{a_1} J_{1b} \nabla_4 J_{a_2 b'} R_{23 a_2' a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'} \\ &\quad + 2p^2 \nabla_{a_1} J_{1b'} \nabla_4 J_{a_2 b} R_{23 a_2 a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'} \\ &= 4p^2 \nabla_{a_1} J_{1b} \nabla_4 J_{a_2 b} R_{23 a_2' a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'}. \end{aligned}$$

Since $\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0$, we must have $a_1 = 3$ or 4, and so

$$(S_{2,2} + S_{4,2})(1) = 4p^2 \nabla_3 J_{1b} \nabla_4 J_{a_2 b'} R_{23 a_2' b'} g^{3a_1'} g^{bb'} g^{a_2 a_2'}$$

$$
\begin{aligned}
&\quad + 4p^2\nabla_4 J_{1b}\nabla_4 J_{a_2b'} R_{23a_2'a_1'} g^{4a_1'} g^{bb'} g^{a_2a_2'} \\
&= 4p^2\nabla_3 J_{1b}\nabla_4 J_{2b'} R_{232}{}^3 g^{bb'} \\
&\quad + 4p^2\nabla_4 J_{1b}\nabla_4 J_{2b'} R_{232}{}^4 g^{bb'}.
\end{aligned}
$$

In summary,

$$
(S_{2,2}+S_{4,2})(1) = 4p^2(\nabla_3 J_{1b'}\nabla_4 J_{2b} R_{232}{}^3 g^{bb'} + \nabla_4 J_{1b'}\nabla_4 J_{2b} R_{232}{}^4 g^{bb'}). \tag{1.10}
$$

Case (2): $(\sigma_3,\sigma_4) = (2,4)$ or $(4,2)$

Now

$$
\sigma \in \{(23),(234),(132),(1342)\}
$$

with signs $-1,+1,+1,-1$, respectively. There is no contribution to $S_{2,2}+S_{4,2}$ if $\sigma = (132)$ or $\sigma = (1342)$, since $\nabla_1 J_{ab} = 0$. Thus for $(S_{2,2}+S_{4,2})(2)$ in the notation of Case (1), we get

$$
\begin{aligned}
(S_{1,2}+S_{3,2})(2) &= -2(\nabla_{a_1} J_{1b'}\nabla_3 J_{a_2'b} R_{24a_2}{}^{a_1} g^{bb'} g^{a_2a_2'} \\
&\quad + \nabla_{b'} J_{1a_1}\nabla_3 J_{a_2'b} R_{24a_2}{}^{a_1} g^{bb'} g^{a_2a_2'}).
\end{aligned}
$$

As above, we only get a nonzero contribution to this equation if $a_1 = 3$ or 4. Thus, we get

$$
\begin{aligned}
(S_{1,2}+S_{3,2})(2) &= -2(\nabla_3 J_{1b'}\nabla_3 J_{a_2'b} R_{24a_2}{}^3 g^{bb'} g^{a_2a_2'} + \nabla_4 J_{1b'}\nabla_3 J_{a_2'b} R_{24a_2}{}^4 g^{bb'} g^{a_2a_2'} \\
&\quad + \nabla_{b'} J_{13}\nabla_3 J_{a_2b} R_{24a_2}{}^3 g^{bb'} g^{a_2a_2'} + \nabla_{b'} J_{14}\nabla_3 J_{a_2b} R_{24a_2}{}^3 g^{bb'} g^{a_2a_2'}) \\
&= -4(\nabla_3 J_{1b'}\nabla_3 J_{2b} R_{242}{}^3 g^{bb'} + \nabla_4 J_{1b'}\nabla_3 J_{2b} R_{242}{}^4 g^{bb'}),
\end{aligned} \tag{1.11}
$$

because only a_2 and a_2' remain.

Case (3): $(\sigma_3,\sigma_4) = (3,4)$ or $(4,3)$

Since $\sigma_2 = 1$ or 2, we have $\nabla_{\sigma_2} J_{a_2b'} = 0$. Thus,

$$
(S_{1,2}+S_{3,2})(3) = 0.
$$

In summary, at this point we have

$$
\frac{1}{2}\bar{S}_{[2]} = S_{1,2}+S_{3,2} = (S_{1,2}+S_{3,2})(1) + (S_{1,2}+S_{3,2})(2). \tag{1.12}
$$

We continue by simplifying the explicit expressions in (1.10), (1.11).

Lemma 1.12. *(i)* $\nabla_4 J_{1b'}\nabla_4 J_{2b} g^{bb'} = 0$ *and* $\nabla_3 J_{1b'}\nabla_3 J_{2b} g^{bb'} = 0$.

(ii)

$$
\begin{aligned}
\nabla_3 J_{1b'}\nabla_4 J_{2b} g^{bb'} R_{232}{}^3 &= -\frac{5\kappa}{4^4}\beta^{-4}(9-8\theta), \\
\nabla_4 J_{1b'}\nabla_3 J_{2b} g^{bb'} R_{232}{}^4 &= \frac{5\kappa}{4^4}\beta^{-4}(1+8\theta).
\end{aligned}
$$

(iii) $\nabla_3 J_{1b'}\nabla_4 J_{2b} g^{bb'} R_{232}{}^3 - \nabla_4 J_{1b'}\nabla_3 J_{2b} g^{bb'} R_{242}{}^4 = -\frac{50}{4^4}\kappa\beta^{-4}$.

Proof. (i) By Lem. 1.4,

$$
\begin{aligned}
&\nabla_4 J_{1b'}\nabla_4 J_{2b} g^{bb'}\\
&= \nabla_4 J_{13} + \nabla_4 J_{23} g^{33} + \nabla_4 J_{13}\nabla_4 J_{24} g^{34} + \nabla_4 J_{14}\nabla_4 J_{23} g^{43} + \nabla_4 J_{14} + \nabla_4 J_{24} g^{44}\\
&= (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2}))\\
&= \kappa\beta^{-3}(-\frac{1}{8}(1+\frac{\theta}{2})(1+\theta) - \frac{1}{4}(1+\frac{\theta}{2})\theta + \frac{1}{16}(1+3\theta)\theta + \frac{1}{8}(1+3\theta))\\
&= \frac{\kappa}{16}\beta^{-3}(-2(1+\frac{3\theta}{2}+\frac{1}{2}\theta^2) - 4\theta - 2\theta^2 + \theta + 3\theta^2 + 2 + 6\theta)\\
&= \frac{\kappa}{16}\beta^{-3}(-2 - 3\theta - \theta^2 - 4\theta - 2\theta^2 + \theta + 3\theta^2 + 2 + 6\theta)\\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
&\nabla_3 J_{1b'}\nabla_3 J_{2b} g^{bb'}\\
&= \nabla_3 J_{13} + \nabla_3 J_{23} g^{33} + \nabla_3 J_{13} + \nabla_3 J_{24} g^{34} + \nabla_3 J_{14}\nabla_3 J_{23} g^{43} + \nabla_3 J_{14}\nabla_3 J_{24} g^{44}\\
&= (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(-\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
&= \kappa\beta^{-3}(\frac{1}{8}(1-2\theta)(1+\theta) - \frac{1}{16}(1-2\theta)\theta + \frac{\theta}{4}(1+\frac{\theta}{2}) - \frac{1}{8}(1+\frac{\theta}{2}))\\
&= \frac{\kappa}{16}\beta^{-3}(2(1-2\theta)(1+\theta) - \theta(1-2\theta) + 4\theta(1+\frac{\theta}{2}) - 2(1+\frac{\theta}{2}))\\
&= \frac{\kappa}{16}\beta^{-3}(2(1-\theta-2\theta^2) - \theta + 2\theta^2 + 4\theta + 2\theta^2 - 2 - \theta)\\
&= \frac{\kappa}{4}(2 - 2\theta - 4\theta^2 - \theta + 2\theta^2 + 4\theta + 2\theta^2 - 2 - \theta)\\
&= 0.
\end{aligned}
$$

(ii) From

$$\nabla_3 J_{1b'}\nabla_4 J_{2b} g^{bb'}$$

$$
\begin{aligned}
&= \nabla_3 J_{13}\nabla_4 J_{23} g^{33} + \nabla_3 J_{14}\nabla_4 J_{24} g^{43} + \nabla_3 J_{13}\nabla_4 J_{24} g^{34} + \nabla_3 J_{14}\nabla_4 J_{24} g^{44}\\
&= (-\frac{\kappa}{4}(1-2\theta))(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1})\\
&\quad + (-\frac{\kappa}{2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{2}(1+\frac{\theta}{2}))(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
&= \kappa\beta^{-3}(1\frac{1}{16}(1-2\theta)(1+\theta) - \frac{1}{8}(1+\theta)\theta - \frac{1}{8}\theta(1-2\theta) + \frac{1}{4}(1+\frac{\theta}{2}))\\
&= -\frac{\kappa}{16}\beta^{-1}((1-2\theta)(1+\theta) + 2(1+\frac{\theta}{2})\theta + 2\theta(1-2\theta) + 4(1+\frac{\theta}{2}))\\
&= -\frac{5}{16}\kappa\beta^{-3}(1+\theta-\theta^2)\\
&= -\frac{5}{16}\kappa\beta^{-2},
\end{aligned}
$$

we get

$$
\nabla_3 J_{1b'}\nabla_4 J_{2b} g^{bb'} R_{232}{}^3 = -\frac{5}{16}\kappa\beta^{-2}\frac{1}{16}\beta^{-2}(9-8\theta) = -\frac{5}{4^4}\kappa\beta^{-4}(9-8\theta).
$$

Similarly, from

$$
\begin{aligned}
&\nabla_4 J_{1b'}\nabla_3 J_{2b} g^{bb'}\\
&= \nabla_4 J_{13}\nabla_3 J_{23} g^{33} + \nabla_4 J_{14}\nabla_3 J_{23} g^{43} + \nabla_4 J_{13}\nabla_3 J_{24} g^{43} + \nabla_4 J_{14}\nabla_3 J_{24} g^{44}\\
&= (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(-\frac{\kappa}{2})(\kappa^{-1}(1+\theta)\beta^{-1/2}\\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
&= \kappa\beta{-}3(\frac{1}{4}(1+\frac{\theta}{2})(1+\theta) - \frac{1}{8}\theta(1+3\theta) - \frac{1}{8}(1+\frac{\theta}{2})\theta + \frac{1}{16}(1+3\theta))\\
&= \frac{\kappa}{16}\beta^{-3}(4(1+\frac{3}{2}\theta+\frac{1}{2}\theta^2) - 2\theta - 6\theta^2 - 2\theta - \theta^2 1 + 3\theta)\\
&= \frac{\kappa}{16}\beta^{-3}(5+5\theta-5\theta^2)\\
&= \frac{5}{16}\kappa\beta^{-2},
\end{aligned}
$$

we get

$$\nabla_4 J_{1b'}\nabla_3 J_{2b} g^{bb'} R_{242}{}^4 = \frac{5}{16}\kappa\beta^{-2}(\frac{1}{16}\beta^{-2}(1+8\theta)) = \frac{5}{4^4}\kappa\beta^{-4}(1+8\theta).$$

(iii)

$$\begin{aligned}&\nabla_3 J_{1b'}\nabla_4 J_{2b} g^{bb'} R_{232}{}^3 - \nabla_4 J_{1b'}\nabla_3 J_{2b} g^{bb'} R_{242}{}^4 \\ &\qquad = -\frac{5}{4^4}\kappa\beta^{-4}(9-8\theta) - \frac{5}{4^4}\beta^{-4}(1+8\theta) = \frac{5}{4^4}\kappa\beta(-10\theta) = -\frac{50}{4^4}\kappa\beta^{-4}.\end{aligned}$$

□

Combining (1.10), (1.11), (1.12), we obtain

$$\begin{aligned}S_{2,2}+S_{4,2} &= (S_{2,2}+S_{4,2})(1) + (S_{2,2}+S_{4,2})(2) = 4p^2(\frac{5}{4^4}\kappa\beta^{-4}(-10)) \\ &= -\frac{25}{16}\kappa\beta^{-4},\end{aligned}$$

which finishes the proof of Prop. A.11. □

1.8. **The p^4 term in (1.3).** We compute the p^4-term of S by the following Proposition, which finishes the proof of (1.3) and hence Theorem 4.1.

Proposition 1.13. $\bar{S}_{[4]} = -\frac{640}{16}\kappa\beta p^4$.

The proof is a long calculation. We know

$$\bar{S}_{[4]} = S_{0,4} + 2S_{1,4} + 2S_{3,4},$$

where by Lemmas 1.6, 1.7, 1.8, we have

(1) $S_{0,4} = S_{0,4,1} + S_{0,4,2}$, with

$$S_{0,4,1} = p^4 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma)\nabla_{a_1'} J_{\sigma_1\sigma_2}\nabla_{a_1} J_{\sigma_3\sigma_4} g^{a_1a_1'},$$

$$\begin{aligned}S_{0,4,2} &= 4p^4 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) R_{\sigma_1\sigma_2a_2b} J_{\sigma_4}{}^b J_{\sigma_3}{}^{a_2} - 4p^4 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) R_{\sigma_1\sigma_2a_2}{}^b J_b{}^{a_2} J_{\sigma_3\sigma_4} \\ &:= S_{0,4,2,1} + S_{0,4,2,2}.\end{aligned}$$

(2)

$$\begin{aligned}S_{1,4} = &-p^4 \sum_{\sigma_1=0} \mathrm{sgn}(\sigma) g^{bb'}\nabla_b J_{\sigma_1\sigma_2}\nabla_{b'} J_{\sigma_3\sigma_4} \\ &+ 2p^4 \sum_{\sigma_1=0} \mathrm{sgn}(\sigma)\nabla_{b'} J_{\sigma_1a_1}\nabla_{\sigma_2} J_{a_2a_1'} J_{\sigma_4b} J_{\sigma_3a_2'} g^{bb'} g^{a_2a_2'} g^{a_1a_1'} \\ &- 2p^4 \sum_{\sigma_1=0} \mathrm{sgn}(\sigma)\nabla_{b'} J_{\sigma_1a_1}\nabla_{\sigma_2} J_{a_2a_1'} J_{\sigma_3\sigma_4} J_{ba_2'} g^{bb'} g^{a_2a_2'} g^{a_1a_1'} \\ := &\, S_{1,4,1} + S_{1,4,2} + S_{1,4,3}. \qquad (1.13)\end{aligned}$$

Check the last term.

(3)

$$\begin{aligned} S_{3,4} &= 2p^4 \sum_{\sigma_3=0} \operatorname{sgn}(\sigma) \nabla^b J_{\sigma_1 a_1} \nabla_{\sigma_4} J_b{}^{a_2} J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} \\ &\quad - 2p^4 \sum_{\sigma_3=0} \operatorname{sgn}(\sigma) \nabla^b J_{\sigma_3 a_1} \nabla_{\sigma_4} J_b{}^{a_2} J_{\sigma_1 \sigma_2} J_{a_2}{}^{a_1} \\ &:= S_{3,4,1} + S_{3,4,2}. \end{aligned}$$

These terms are computed as follows.

Lemma 1.14.
(i) $S_{0,4,1} = 5\kappa\beta^{-2}p^4,\ S_{0,4,2,1} = -10\kappa\beta^{-2}p^4,\ S_{0,4,2,2} = -10\kappa\beta^{-2}p^4$;
(ii)(a) $S_{1,4,1} = -5\kappa\beta^{-2}p^4$, *(b)* $S_{1,4,2} = -\frac{5}{2}\kappa\beta^{-2}p^4$, *(c)* $S_{1,4,3} = 5\kappa\beta^{-2}p^4$;
(iii)(a) $S_{3,4,1} = -5\kappa\beta^{-2}p^4$, *(b)* $S_{3,4,2} = -5\kappa\beta^{-2}p^4$.

Can some of the results in this Lemma be derived from other results in the Lemma by a change of variables?

Assuming the Lemma, we get

$$\begin{aligned} \bar{S}_{[4]} &= (S_{0,4,1} + S_{0,4,2}) + 2(S_{1,4,1} + S_{1,4,2} + S_{1,4,3}) + 2(S_{3,4,1} + S_{3,4,2}) \\ &= -40p^4\kappa\beta^{-2} = -\frac{640}{16}p^4\kappa\beta^{-2}, \end{aligned}$$

finishing the proof of Proposition 1.13.

Proof of Lemma 1.14. (i) It suffices to prove the last two equalities. For $S_{0,4,2,1} = p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) R_{\sigma_1\sigma_2 a_2 b} J_{\sigma_3}{}^{a_2} J_{\sigma_4}{}^b$, the only possible σ have $(\sigma_1, \sigma_2) \in \{(2,3), (2,4), (3,4)\}$. This gives

$$\sigma \in \{(123), (1234), (13)(24)\},$$

with signs $+1, -1, +1$, respectively. (Associated to e.g., (123) is another permissible permutation $(14)(123) = (1234)$ switching the assignment of 1 and 4, but these permutations give the same contribution to $S_{0,4,2,1}$.) Thus, we have

$$\begin{aligned} S_{0,4,2,1} &= 4p^4[4R_{23a_2b}J_1{}^{a_2}J_4{}^b - 4R_{24a_2b}J_1{}^{a_2}J_3{}^b + 4R_{34a_2b}J_1{}^{a_2}J_2{}^b] \\ &= 16p^4[R_{232b}J_4{}^b - R_{242b}J_3{}^b + R_{342}{}^bJ_2{}^b] \\ &= 16p^4[R_{2323}J_4{}^3 + R_{232}{}^4J_4{}^4 - R_{2423}J_3{}^3 - R_{2424}J_3{}^4 + R_{3423}J_2{}^3 + R_{3434}J_2{}^4] \\ &= 16p^4[R_{232b}J_4{}^b - R_{242b}J_3{}^b + R_{342}{}^bJ_2{}^b] \\ &= 16p^4[R_{2323}J_4{}^3 + R_{232}{}^4J_4{}^4 - R_{2423}J_3{}^3 - R_{2424}J_3{}^4], \end{aligned}$$

since $J_2{}^3 = J_2{}^4 = 0$. Therefore,

$$S_{0,4,2,1} = 16p^4\left[\frac{\kappa}{16}\beta^{-5/2}(9 - 16\theta + 16\theta^2)(-(1+\theta)\beta^{-1/2})\right.$$

$$
\begin{aligned}
&+ (-\frac{\kappa}{16}\beta^{-5/2}(-8+17\theta+8\theta^2))(-\theta\beta^{-1/2}) \\
&- (-\frac{\kappa}{16}\beta^{-5/2}(-8+17\theta+8\theta^2))(\theta\beta^{-1/2}) \\
&- (\frac{\kappa}{16}(1+\theta+24\theta^2)\beta^{-5/2})(\beta^{-1/2})\Big] \\
= {}& -10\kappa\beta^{-2}p^4.
\end{aligned}
$$

For $S_{0,4,2,2} = -4p^4\sum_{\sigma_0=0}\operatorname{sgn}(\sigma)R_{\sigma_1\sigma_2 a_2}{}^b J_b{}^{a_2}J_{\sigma_3\sigma_4}$, the only possible σ have $(\sigma_1,\sigma_2)\in\{(2,3),(2,4),(3,4)\}$. This gives

$$\sigma\in\{(123),(1243),(13(24)\},$$

with signs +1, -1, +1, respectively. (As above, each permutation has a partner switching the assignment of 3 and 4.) This gives

$$
\begin{aligned}
S_{0,4,2,2} &= -4p^4[4R_{23a_2}{}^b J_b{}^{a_2}J_{14} - 4R_{24a_2}{}^b J_b{}^{a_2}J_{13} + 4R_{34a_2}{}^b J_b{}^{a_2}J_{12}] \\
&= -4p^4\cdot 4R_{34a_2}{}^b J_b{}^{a_2} \\
&= -16p^4[R_{343}{}^4 J_3{}^3 + R_{343}{}^4 J_4{}^3 + R_{344}{}^3 J_3{}^4 + R_{344}{}^4 J_4{}^4] \\
&= -16p^4\Big[(-\frac{5}{16}\theta\kappa\beta^{-5/2})(\theta\beta^{-1/2}) + (-\frac{5}{16}\kappa\beta^{-5/2})(-(1+\theta)\beta^{-1/2}) \\
&\qquad + (\frac{5}{16}\kappa(1+\theta)\beta^{-5/2})(\beta^{-1/2}) + (\frac{5}{16}\theta\kappa\beta^{-5/2})(-\theta\beta^{-1/2})]\Big] \\
&= -16p^4(\frac{5}{16}\kappa\beta^{-3})[-\theta^2+(1+\theta)+(1+\theta)-\theta^2] \\
&= -10\kappa\beta^{-2}p^4.
\end{aligned}
$$

(ii)(a) We have

$$
\begin{aligned}
S_{1,4,1} = {}& -4p^4\big(g^{bb'}\nabla_b J_{12}\nabla_{b'}J_{34} - g^{bb'}\nabla_b J_{13}\nabla_{b'}J_{24} + g^{bb'}\nabla_b J_{14}\nabla_{b'}J_{23} - g^{bb'}\nabla_b J_{23}\nabla_{b'}J_{41} \\
&- g^{bb'}\nabla_b J_{24}\nabla_{b'}J_{13} + g^{bb'}\nabla_b J_{34}\nabla_{b'}J_{12}\big) \\
= {}& -8p^4\big(-g^{bb'}\nabla_b J_{13}\nabla b'24 + g^{bb'}\nabla_b J_{14}\nabla_{b'}J_{23}\big) \\
= {}& -8p^4\big(-g^{33}\nabla_3 J_{13}\nabla_3 J_{24} - g^{34}\nabla_3 J_{13}\nabla_4 J_{24} - g^{43}\nabla_4 J_{13}\nabla_3 J_{24} - g^{44}\nabla_4 J_{13}\nabla_4 J_{24} \\
&+ g^{33}\nabla_3 J_{14}\nabla_3 J_{23} + g^{34}\nabla_3 J_{14}\nabla_4 J_{23} + g^{43}\nabla_4 J_{14}\nabla_3 J_{23} + g^{44}\nabla_4 J_{14}\nabla_4 J_{23}\big).
\end{aligned}
$$

Recall that

$$
g^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(1+\theta)\kappa}{\beta^{1/2}\kappa^2} & \frac{\theta\kappa}{\beta^{1/2}\kappa^2} \\ 0 & 0 & \frac{\theta\kappa}{\beta^{1/2}\kappa^2} & \frac{\kappa}{\beta^{1/2}\kappa^2} \end{pmatrix} \tag{1.14}
$$

After plugging in from (1.14) and Lemma 1.4, we find

$$
\begin{aligned}
S_{1,4,1} &= -8p^4\Big[-(\kappa^{-1}(1+\theta)\beta^{-1/2})(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta)(\frac{\kappa}{4}\beta^{-1})\\
&\quad -(\theta\kappa^{-1}\beta^{-1/2})(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{2}\beta^{-1})\\
&\quad -(\theta\kappa^{-1}\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-1})\\
&\quad -(\kappa^{-1}\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(\frac{\kappa}{2}\beta^{-1})\\
&\quad +(\kappa^{-1}(1+\theta)\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(-\frac{\kappa}{2}\beta^{-1})\\
&\quad +(\theta\kappa^{-1}\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-1})\\
&\quad +(\theta\kappa^{-1}\beta^{-1/2})(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{2}\beta^{-1})\\
&\quad +(\kappa^{-1}\beta^{-1/2})(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-1})\Big]\\
&= -8p^4\kappa\beta^{-3}\Big[(1+\theta))(\frac{1}{4^2}(1-2\theta))+\frac{1}{2}\theta(\frac{1}{4}(1-2\theta))+\theta(\frac{1}{2}(1+\frac{\theta}{2}))(\frac{1}{4})\\
&\quad +(\frac{1}{2}(1+\frac{\theta}{2}))(\frac{1}{2})+(1+\theta)(\frac{1}{2}(1+\frac{\theta}{2}))(\frac{1}{2})-\theta(\frac{1}{2}(1+\frac{\theta}{2}))(\frac{1}{4})\\
&\quad -\theta(\frac{1}{4}(1+3\theta))(\frac{1}{2})+\frac{1}{4}(1+3\theta)(\frac{1}{4})\Big]\\
&= -8p^4\kappa\beta^{-3}\Big[\frac{1}{16}(1+\theta)(1-2\theta)+\frac{1}{8}\theta(1-2\theta)+\frac{1}{8}\theta(1+\frac{\theta}{2})+\frac{1}{4}(1+\frac{\theta}{2})\\
&\quad +\frac{1}{4}(1+\theta)(1+\frac{\theta}{2})-\frac{1}{8}\theta(1+\frac{\theta}{2})-\frac{1}{8}\theta(1+3\theta)+\frac{1}{16}(1+3\theta)\Big]\\
&= -\frac{8}{16}p^4\kappa\beta^{-3}\Big[(1+\theta)(1-2\theta)+2\theta(1-2\theta)+2\theta(1+\frac{\theta}{2})+4(1+\frac{\theta}{2})\\
&\quad +4(1+\theta)(1+\frac{\theta}{2})-2\theta(1+\frac{\theta}{2})-2\theta(1+3\theta)+(1+3\theta)\Big]\\
&= -\frac{8}{16}p^4\kappa\beta^{-3}\Big[1-\theta-2\theta^2+2\theta-4\theta^2+2\theta+\theta^2+4+2\theta+4+2\cdot 3\theta\\
&\quad +2\theta^2-2\theta-\theta^2-2\theta-6\theta^2+1+3\theta\Big]\\
&= -\frac{8}{16}p^4\kappa\beta^{-3}\big[10+10\theta-10\theta^2\big]
\end{aligned}
$$

$$= -5\kappa\beta^{-2}p^4.$$

(ii)(b) For $S_{1,4,2} = 2p^4 \sum_{\sigma_1=0} \mathrm{sgn}(\sigma)\nabla_{b'} J_{\sigma_1 a_1} \nabla_{\sigma_2} J_{a_2 a'_1} J_{\sigma_4 b} J_{\sigma_3 a'_2} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1}$, we use $\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0$ to obtain

$$\begin{aligned} S_{1,4,2} &= 2p^4 \sum_{\sigma_1=0,\sigma_2=3} \mathrm{sgn}(\sigma)\nabla_{b'} J_{\sigma_1 a_1} \nabla_3 J_{a_2 a'_1} J_{\sigma_4 b} J_{\sigma_3 a'_2} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} \\ &\quad + 2p^4 \sum_{\sigma_1=0,\sigma_2=4} \mathrm{sgn}(\sigma)\nabla_{b'} J_{\sigma_1 a_1} \nabla_4 J_{a_2 a'_1} J_{\sigma_4 b} J_{\sigma_3 a'_2} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} \\ &:= S_{1,4,2}(\sigma_2 = 3) + S_{1,4,2}(\sigma_2 = 4). \end{aligned} \tag{1.15}$$

For $S_{1,4,2}(\sigma_2 = 3)$, the possible σ are $\sigma \in \{(243), (24), (1423), (14)(23), (123), (124)\}$, with signs $-1, +1, -1, +1, +1, -1$, respectively. (Here we include all possible σ, not just half of them.) Thus

$$\begin{aligned} S_{1,4,2}(\sigma_2 = 3) &= 2p^4 \left(-\nabla_{b'} J_{1a_1} \nabla_3 J_{1a'_1} J_{43} J_{21} g^{3b'} g^{a_1 a'_1} + \nabla_{b'} J_{2a_1} \nabla_3 J_{2a'_1} J_{43} J_{12} g^{3b'} g^{a_1 a'_1} \right) \\ &= -2p^4 J_{34} J_{12} \left(\nabla_{b'} J_{1a_1} \nabla_3 J_{1a'_1} g^{3b'} g^{a_1 a'_1} + \nabla_{b'} J_{2a_1} \nabla_3 J_{2a'_1} g^{3b'} g^{a_1 a'_1} \right). \end{aligned} \tag{1.16}$$

Similarly, we have

$$\begin{aligned} S_{1,4,2}(\sigma_2 = 4) = 2p^4 \cdot \Big(&\nabla_{b'} J_{1a_1} \nabla_4 J_{a_2 a'_1} J_{3b} J_{21} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} \\ &- \nabla_{b'} J_{1a_1} \nabla_4 J_{a_2 a'_1} J_{2b} J_{3a'_2} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} \\ &(\pm - -which?) \nabla_{b'} J_{3a_1} \nabla_4 J_{a_2 a'_1} J_{2b} J_{2a'_1} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} \\ &- \nabla_{b'} J_{3a_1} \nabla_4 J_{a_2 a'_1} J_{1b} J_{2a'_2} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} \\ &- \nabla_{b'} J_{2a_1} \nabla_4 J_{a_2 a'_1} J_{3b} J_{1a'_2} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} \\ &+ \nabla_{b'} J_{2a_1} \nabla_4 J_{a_2 a'_1} J_{1b} J_{3a'_2} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} \Big) \\ = -2p^4 J_{34} J_{12} (&\nabla_{b'} J_{1a_1} \nabla_4 J_{1a'_1} g^{4b'} g^{a_1 a'_1} + \nabla_{b'} J_{2a_1} \nabla_4 J_{2a'_1} g^{4b'} g^{a_1 a'_1}). \end{aligned} \tag{1.17}$$

Combining (1.15), (1.16), (1.17), we get

$$\begin{aligned} S_{1,4,2} &= -2p^4 J_{34} J_{12} \Big(\nabla_{b'} J_{1a_1} \nabla_3 J_{1a'_1} g^{3b'} g^{a_1 a'_1} + \nabla_{b'} J_{2a_1} \nabla_3 J_{2a'_1} g^{3b'} g^{a_1 a'_1} \\ &\quad + \nabla_{b'} J_{1a_1} \nabla_4 J_{1a'_1} g^{4b'} g^{a_1 a'_1} + \nabla_{b'} J_{2a_1} \nabla_4 J_{2a'_1} g^{4b'} g^{a_1 a'_1} \Big) \\ &= -2p^4 J_{34} J_{12} \Big(g^{33} \nabla_3 J_{1a_1} \nabla_3 J_{1a'_1} g^{a_1 a'_1} + g^{34} \nabla_4 J_{1a_1} \nabla_3 J_{1a'_1} g^{a_1 a'_1} \\ &\quad + g^{33} \nabla_3 J_{2a_1} \nabla_3 J_{2a'_1} g^{a_1 a'_1} + g^{34} \nabla_4 J_{2a_1} \nabla_3 J_{2a'_1} g^{a_1 a'_1} \end{aligned} \tag{1.18}$$

$$
\begin{aligned}
&+ g^{43}\nabla_3 J_{1a_1}\nabla_4 J_{1a_1'} g^{a_1a_1'} + g^{44}\nabla_4 J_{1a_1}\nabla_4 J_{1a_1'} g^{a_1a_1'} \\
&+ g^{43}\nabla_3 J_{2a_1}\nabla_4 J_{1a_1'} g^{a_1a_1'} + g^{44}\nabla_4 J_{2a_1}\nabla_4 J_{2a_1'} g^{a_1a_1'}\Big) \\
:= &-2p^4 J_{34}J_{12}(\langle 1\rangle + 2\langle 2\rangle + 2\langle 3\rangle + \langle 4\rangle + \langle 5\rangle + \langle 6\rangle)
\end{aligned}
$$

where

$$
\begin{aligned}
\langle 1\rangle &= g^{33}\nabla_3 J_{1a_1}\nabla_3 J_{1a_1'} g^{a_1a_1'},\ \langle 2\rangle = g^{34}\nabla_4 J_{1a_1}\nabla_3 J_{1a_1'} g^{a_1a_1'},\\
\langle 3\rangle &= g^{34}\nabla_4 J_{2a_1}\nabla_3 J_{2a_1'} g^{a_1a_1'},\ \langle 4\rangle = g^{33}\nabla_3 J_{2a_1}\nabla_3 J_{2a_1'} g^{a_1a_1'},\\
\langle 5\rangle &= g^{44}\nabla_4 J_{1a_1}\nabla_4 J_{1a_1'} g^{a_1a_1'},\ \langle 6\rangle = g^{44}\nabla_4 J_{2a_1}\nabla_4 J_{2a_1'} g^{a_1a_1'}.
\end{aligned}
$$

We claim that

Claim 1.15.

$$
\begin{aligned}
&\langle 1\rangle = \frac{5}{16}(1+\theta)\beta^{-3},\ \langle 2\rangle = -\frac{5}{16}\theta^2\beta^{-3},\ \langle 3\rangle = -\frac{5}{16}\theta^2\beta^{-3},\\
&\langle 4\rangle = \frac{5}{16}(1+\theta)\beta^{-3},\ \langle 5\rangle = \frac{5}{16}(1+\theta)\beta^{-3},\ \langle 6\rangle = \frac{5}{16}(1+\theta)\beta^{-3}.
\end{aligned}
\tag{1.19}
$$

Assume the claim. Plugging (1.19) into (1.18) and using (1.2) for J_{34}, J_{12}, we obtain

$$
\begin{aligned}
S_{1,4,2} &= -2p^4 J_{34}J_{12}\bigg(\frac{5}{16}(1+\theta)\beta^{-3} + 2(-\frac{5}{16}\theta^2\beta^{-3}) + 2(-\frac{5}{16}\theta^2\beta^{-3}) + \frac{5}{16}(1+\theta)\beta^{-3}\\
&\qquad + \frac{5}{16}(1+\theta)\beta^{-3} + \frac{5}{16}(1+\theta)\beta^{-3}\bigg)\\
&= -2p^4 J_{34}J_{12}\frac{5}{16}\beta^{-3}\Big((1+\theta) - 2\theta^2 - 2\theta^2 + (1+\theta) + (1+\theta) + (1+\theta)\Big)\\
&= -2p^4 J_{34}J_{12}\Big((\frac{5}{16}\cdot 4\cdot\beta^{-3}(1+\theta-\theta^2)\Big)\\
&= -\frac{5}{16}\cdot 8\cdot\beta^{-2}J_{34}J_{12}\\
&= -\frac{5}{2}p^4\beta^{-2}\kappa.
\end{aligned}
$$

We now prove Claim 1.15, which will finish (ii)(b).

Proof of Claim 1.15. The proofs are all direct calculations.

$$
\begin{aligned}
\langle 1\rangle &= g^{33}\nabla_3 J_{1a_1}\nabla_3 J_{1a_1'} g^{a_1a_1'}\\
&= g^{33}(\nabla_3 J_{13}\nabla_3 J_{13} g^{33} + \nabla_3 J_{13}\nabla_3 J_{14} g^{34} + \nabla_3 J_{14}\nabla_3 J_{13} g^{43} + \nabla_3 J_{14}\nabla_3 J_{14} g^{44})\\
&= g^{33}(\nabla_3 J_{13}\nabla_3 J_{13} g^{33} + 2\nabla_3 J_{13}\nabla_3 J_{14} g^{34} + \nabla_3 J_{14}\nabla_3 J_{14} g^{44})\\
&= \kappa(1+\theta)\beta^{-1/2}\Big(\kappa^{-1}(1+\theta)\beta^{-1/2}\big(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta)\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\theta\kappa^{-1}\beta^{-1/2}\big(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big)\big(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big) \\
&+ \kappa^{-1}\beta^{-1/2}\big(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big)\big(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big)\Big) \\
&= \beta^{-4}(1+\theta)\bigg(\frac{1}{16}(1+\theta)(1-2\theta)^2 + \frac{1}{4}\theta(1-2\theta)(1+\frac{\theta}{2}) + \frac{1}{4}(1+\frac{\theta}{2})^2\bigg) \\
&= \frac{1}{16}\beta^{-4}(1+\theta)\bigg((1+\theta)(1-2\theta)^2 + 4\theta(1-2\theta)(1+\frac{\theta}{2}) + 4(1+\frac{\theta}{2})^2\bigg) \\
&= \frac{5}{16}(1+\theta)(1+\theta-\theta^2) \\
&= \frac{5}{16}\beta^{-3}(1+\theta).
\end{aligned}
$$

$$
\begin{aligned}
\langle 2\rangle &= g^{34}\nabla_4 J_{1a_1}\nabla_3 J_{1a_1'}g^{a_1a_1'} \\
&= g^{34}\big(\nabla_4 J_{13}\nabla_3 J_{13}g^{33} + \nabla_4 J_{13}\nabla_3 J_{14}g^{34} + \nabla_4 J_{14}\nabla_3 J_{13}g^{43} + \nabla_4 J_{14}\nabla_3 J_{14}g^{44}\big) \\
&= g^{34}\bigg((-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2}) \\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\kappa^{-1}\theta\beta^{-1/2}) \\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}\theta\beta^{-1/2}) \\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\kappa^{-1}\beta^{-1/2})\bigg) \\
&= g^{34}\kappa\beta^{-7/2}\bigg(\frac{1}{8}(1+\frac{\theta}{2})(1-2\theta)(1+\theta) + \frac{1}{4}(1+\frac{\theta}{2})(1+\frac{\theta}{2})\theta - 1\frac{1}{16}(1+3\theta)(1-2\theta)\theta \\
&\quad - \frac{1}{8}(1+3\theta)(1+\frac{\theta}{2})\bigg) \\
&= g^{34}\kappa\beta^{-7/2}\frac{1}{16}\bigg(2(1+\frac{\theta}{2})(1-2\theta)(1+\theta) \\
&\quad + 4(1+\frac{\theta}{2})(1+\frac{\theta}{2})\theta - (1+3\theta)(1-2\theta)\theta - 2(1+3\theta)(1+\frac{\theta}{2})\bigg) \\
&= -\frac{5}{16}\theta^2\beta^{-3},
\end{aligned}
$$

where the last line follows from (1.14) and a direct calculation.

$$
\langle 3\rangle = g^{34}\nabla_4 J_{2a_1}\nabla_3 J_{2a_1'}g^{a_1a_1'}
$$

$$
\begin{aligned}
&= g^{34}\Bigg(\nabla_4 J_{23}\nabla_3 J_{23}g^{33} + \nabla_4 J_{23}\nabla_3 J_{24}g^{34} + \nabla_4 J_{24}\nabla_3 J_{23}g^{34} + \nabla_4 J_{24}\nabla_3 J_{24}g^{44}\Bigg)\\
&= g^{34}\Bigg((\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2}) + (\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1/2}) + (\frac{\kappa}{2})(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\Bigg)\\
&= g^{34}\kappa\beta^{-5/2}(-\frac{1}{8}(1+\theta) + \frac{1}{16}\theta - \frac{1}{4}\theta + \frac{1}{8}))\\
&= -\frac{5}{16}\theta^2\beta^{-3}.
\end{aligned}
$$

$$
\begin{aligned}
\langle 4\rangle &= g^{33}\nabla_3 J_{2a_1}\nabla_3 J_{2a_1'}g^{a_1a_1'}\\
&= g^{33}(\nabla_3 J_{23}\nabla_3 J_{23}g^{33} + \nabla_3 J_{23}\nabla_3 J_{24}g^{34} + \nabla_3 J_{24}\nabla_3 J_{23}g^{43} + \nabla_3 J_{24}\nabla_3 J_{24}g^{44})\\
&= g^{33}\Bigg((-\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2}) + (-\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad + (\frac{\kappa}{4})(-\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2}) + (\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1})\Bigg)\\
&= g^{33}\kappa\beta^{-5/2}\left(\frac{1}{4}(1+\theta) - \frac{1}{8}\theta - \frac{1}{8} + \frac{1}{16}\right)\\
&= \frac{5}{16}(1+\theta)\beta^{-3}.
\end{aligned}
$$

$$
\begin{aligned}
\langle 5\rangle &= g^{44}\nabla_4 J_{1a_1}\nabla_4 J_{1a_1'}g^{a_1a_1'}\\
&= g^{44}(\nabla_4 J_{13}\nabla_4 J_{13}g^{33} + \nabla_4 J_{13}\nabla_4 J_{14}g^{34} + \nabla_4 J_{14}\nabla_4 J_{13}g^{43} + \nabla_4 J_{14}\nabla_4 J_{14}g^{44})\\
&= g^{44}\Bigg((-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})\Bigg)\\
&= g^{44}\kappa\beta^{-7/2}\Bigg(\frac{1}{4}(1+\frac{\theta}{2})(1+\frac{\theta}{2})(1+\theta) - \frac{1}{8}(1+\frac{\theta}{2})(1+3\theta)\theta\\
&\qquad - \frac{1}{8}(1+3\theta)(1+\frac{\theta}{2})\theta + \frac{1}{16}(1+3\theta)(1+3\theta)\Bigg)
\end{aligned}
$$

$$= \frac{1}{16} g^{44} \kappa \beta^{-7/2} \Big(4(1+\frac{\theta}{2}(1+\frac{\theta}{2})(1+\theta) - 2(1+\frac{\theta}{2})(1+3\theta)\theta$$
$$- 2(1+3\theta)(1+\frac{\theta}{2})\theta + (1+3\theta)(1+3\theta) \Big)$$
$$= \frac{5}{16}(1+\theta)\beta^{-3},$$

after some calculation.

$$\begin{aligned}
\langle 6 \rangle &= g^{44} \nabla_4 J_{2a_1} \nabla_4 J_{2a_1'} g^{a_1 a_1'} \\
&= g^{44} (\nabla_4 J_{23} \nabla_4 J_{23} g^{33} + \nabla_4 J_{23} \nabla_4 J_{24} g^{34} + \nabla_4 J_{24} \nabla_4 J_{23} g^{43} + \nabla_4 J_{24} \nabla_4 J_{24} g^{44}) \\
&= g^{44} (\nabla_4 J_{23} \nabla_4 J_{23} g^{33} + 2 \nabla_4 J_{23} \nabla_4 J_{24} g^{34} + \nabla_4 J_{24} \nabla_4 J_{24} g^{44}) \\
&= g^{44} \Big((\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2}) + 2(\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2}) \\
&\quad + (\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2}) \Big) \\
&= g^{44} \kappa \beta^{-5/2} (\frac{1}{16}(1+\theta) + 2 \cdot \frac{1}{8}\theta + \frac{1}{4}) \\
&= \frac{5}{16}(1+\theta)\beta^{-3}.
\end{aligned}$$

□

(ii)(c) We have

$$S_{1,4,3} = -2p^4 \sum_{\sigma_1 = 0} \operatorname{sgn}(\sigma) \nabla_{b'} J_{\sigma_1 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_3 \sigma_4} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}. \tag{1.20}$$

If (σ_1, σ_2) equals $(1,2)$ or $(2,1$, then $\nabla_1 J = \nabla_2 J = 0$ implies that the summand in (1.20) vanishes. Thus, $(\sigma_3, \sigma_4) \in \{(1,2),(2,1)\}$, so

$$\sigma \in \{(13)(24), (1324), (1423), (14)(23)\},$$

with signs $+1, -1, -1, +1$, respectively. Thus by an easy symmetry argument,

$$\begin{aligned}
&S_{1,4,3} \\
&= -4p^4 (\nabla_{b'} J_{3a_1} \nabla_4 J_{a_2 a_1'} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'} - \nabla_{b'} J_{4a_1} \nabla_3 J_{a_2 a_1'} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}) \\
&- 4p^4 \big(\nabla_{b'} J_{31} \nabla_4 J_{a_2 a_1'} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1' 1} + \nabla_{b'} J_{32} \nabla_4 J_{a_2 a_1'} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1' 1} \\
&\quad - \nabla_{b'} J_{41} \nabla_3 J_{a_2 a_1'} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1' 1} - \nabla_{b'} J_{42} \nabla_3 J_{a_2 a_1'} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1' 1} \big) \\
&= -4p^2 \big(\nabla_{b'} J_{31} \nabla_4 J_{31} J_{b a_2'} g^{bb'} g^{3 a_2'} + \nabla_{b'} J_{31} \nabla_4 J_{41} J_{b a_2'} g^{bb'} g^{4 a_2'} + \nabla_{b'} J_{32} \nabla_4 J_{32} J_{b a_2'} g^{bb'} g^{3 a_2'} \\
&\quad + \nabla_{b'} J_{32} \nabla_4 J_{42} J_{b a_2'} g^{bb'} g^{4 a_2'} - \nabla_{b'} J_{41} \nabla_3 J_{31} J_{b a_2'} g^{bb'} g^{a_2' 3} - \nabla_{b'} J_{41} \nabla_3 J_{41} J_{b a_2'} g^{bb'} g^{a_2' 4} \\
&\quad - \nabla_{b'} J_{42} \nabla_3 J_{32} J_{b a_2'} g^{bb'} g^{a_2' 3} - \nabla_{b'} J_{42} \nabla_3 J_{42} J_{b a_2'} g^{bb'} g^{a_2' 4} \big)
\end{aligned}$$

$$
\begin{aligned}
&= -4p^4\big(\nabla_3 J_{31}\nabla_4 J_{31} J_{ba_2'} g^{b3} g^{3a_2'} + \nabla_4 J_{31}\nabla_4 J_{31} J_{ba_2'} g^{b4} g^{3a_2'} + \nabla_3 J_{31}\nabla_4 J_{41} J_{ba_2'} g^{b3} g^{4a_2'}\\
&\quad + \nabla_4 J_{31}\nabla_4 J_{41} J_{ba_2'} g^{b4} g^{4a_2'} + \nabla_3 J_{32}\nabla_4 J_{32} J_{ba_2'} g^{b3} g^{3a_2'} + \nabla_4 J_{32}\nabla_4 J_{32} J_{ba_2'} g^{b4} g^{3a_2'}\\
&\quad + \nabla_3 J_{32}\nabla_4 J_{42} J_{ba_2'} g^{b3} g^{4a_2'} + \nabla_4 J_{32}\nabla_4 J_{42} J_{ba_2'} g^{b4} g^{4a_2'} - \nabla_3 J_{41}\nabla_3 J_{31} J_{ba_2'} g^{b3} g^{a_2'3}\\
&\quad - \nabla_4 J_{41}\nabla_3 J_{31} J_{ba_2'} g^{b4} g^{a_2'3} - \nabla_3 J_{41}\nabla_3 J_{41} J_{ba_2'} g^{b3} g^{a_2'4} - \nabla_4 J_{41}\nabla_3 J_{41} J_{ba_2'} g^{b4} g^{a_2'4}\\
&\quad - \nabla_3 J_{42}\nabla_3 J_{32} J_{ba_2'} g^{b3} g^{a_2'3} - \nabla_4 J_{42}\nabla_3 J_{32} J_{ba_2'} g^{b4} g^{a_2'3} - \nabla_3 J_{42}\nabla_3 J_{42} J_{ba_2'} g^{b3} g^{a_2'4}\\
&\quad - \nabla_4 J_{42}\nabla_3 J_{42} J_{ba_2'} g^{b4} g^{a_2'4}\Big).
\end{aligned}
$$

Note that

$$
\begin{aligned}
J_{ba_2'} g^{b3} g^{3a_2'} &= J_{34} g^{33} g^{34} + J_{43} g^{43} g^{33} = 0,\ \ J_{ba_2'} g^{b4} g^{4a_2'} = J_{34} g^{34} g^{44} + J_{43} g^{44} g^{43} = 0,\\
J_{ba_2'} g^{b4} g^{3a_2'} &= J_{34} g^{34} g^{34} - J_{43} g^{44} g^{33} = J_{34}((g^{34})^2 - g^{44} g^{33})\\
&= -\kappa^{-2} J_{34} = -\kappa^{-1},\\
J_{ba_2'} g^{b3} g^{4a_2'} &= \kappa^{-2} J_{34} = \kappa^{-1}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
S_{1,4,3} &= -4p^2\big(-\kappa^{-2}\nabla_4 J_{31}\nabla_4 J_{31} + \kappa^{-2}\nabla_3 J_{31}\nabla_4 J_{41} - \kappa^{-2}\nabla_4 J_{32}\nabla_4 J_{32}\\
&\qquad + \kappa^{-2}\nabla_3 J_{32}\nabla_4 J_{42} + \kappa^{-2}\nabla_4 J_{41}\nabla_3 J_{31} - \kappa^{-2}\nabla_3 J_{41}\nabla_3 J_{41}\\
&\qquad + \kappa^{-2}\nabla_4 J_{42}\nabla_3 J_{32} - \kappa^{-2}\nabla_3 J_{42}\nabla_3 J_{42}\big)\\
&= -4p^2\kappa^{-1}\Bigg(-(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))\\
&\qquad + (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)) - (-\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})\\
&\qquad + (\frac{\kappa}{2})(-\frac{\kappa}{2}\beta^{-1}) + (-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))\\
&\qquad - (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})) + (-\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-1})\\
&\qquad - (-\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})\Bigg)\\
&= -4p^4\cdot 2\beta^{-3}\kappa\left(-\frac{1}{4}(1+\frac{\theta}{2})^2 - \frac{1}{16}(1-2\theta)(1+3\theta) - \frac{1}{16}\beta - \frac{1}{4}\beta\right)\\
&= -\frac{\kappa}{16}\cdot 8p^4\beta^{-3}(-10\beta)\\
&= 5p^4\beta^{-2}\kappa.
\end{aligned}
$$

(iii)(a) We compute $S_{3,4,1}$ as follows:

$$
S_{3,4,1} = 2p^4\sum_{\sigma_3=0}\mathrm{sgn}(\sigma)\nabla_b J_{\sigma_3 a_1}\nabla_{\sigma_4} J_{b'a_2} J_{\sigma_2 a_2} J_{\sigma_1 a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'}
$$

$$
\begin{aligned}
&= 2p^4 \sum_{\sigma_3=0,\sigma_4=3} \operatorname{sgn}(\sigma) \nabla_b J_{\sigma_3 a_1} \nabla_3 J_{b' a_2'} J_{\sigma_2 a_2} J_{\sigma_1 a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
&\qquad + 2p^4 \sum_{\sigma_3=0,\sigma_4=4} \operatorname{sgn}(\sigma) \nabla_b J_{\sigma_3 a_1} \nabla_4 J_{b' a_2'} J_{\sigma_2 a_2} J_{\sigma_1 a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
&:= S_{3,4,1}(\sigma_4 = 3) + S_{3.4.1}(\sigma_4 = 4).
\end{aligned}
$$

For $S_{3,4,1}(\sigma_4 = 3)$, the possible σ are $\sigma \in \{(34), (12)(34), (1432), (243), (143), (1243)\}$, with signs $-1, +1, -1, +1, +1, -1$, respectively. Thus

$$
\begin{aligned}
&S_{3,4,1}(\sigma_4 = 3) \\
&= 2p^4 \Big(-\nabla_b J_{4a_1'} \nabla_3 J_{b' a_2'} J_{1a_2} J_{1a_1} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} + \nabla_b J_{4a_1'} \nabla_3 J_{b' a_2'} J_{1a_2} J_{2a_1} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
&\quad - \nabla_b J_{2a_1'} \nabla_3 J_{b' a_2'} J_{1a_2} J_{4a_1} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} + \nabla_b J_{2a_1'} \nabla_3 J_{b' a_2'} J_{4a_2} J_{1a_1} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
&\quad + \nabla_b J_{1a_1'} \nabla_3 J_{b' a_2'} J_{2a_2} J_{4a_1} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} - \nabla_b J_{1a_1'} \nabla_3 J_{b' a_2'} J_{4a_2} J_{2a_1} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \Big).
\end{aligned}
$$

From $\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0$, we conclude

$$
\nabla_b J_{2a_1'} \nabla_3 J_{b' a_2'} J_{4a_2} J_{1a_1} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} = 0, \ \nabla_b J_{1a_1'} \nabla_3 J_{b' a_2'} J_{4a_2} J_{2a_1} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} = 0.
$$

Therefore,

$$
\begin{aligned}
&S_{3,4,1}(\sigma_4 = 3) \\
&\qquad = 2p^4 \Big(-\nabla_b J_{42} \nabla_3 J_{b'1} J_{21} J_{12} g^{bb'} g^{22} g^{11} - \nabla_b J_{42} \nabla_3 J_{b'1} J_{21} J_{12} g^{bb'} g^{22} g^{11} \\
&\qquad\quad - \nabla_b J_{42} \nabla_3 J_{b'1} J_{21} J_{12} g^{bb'} g^{22} g^{11} - \nabla_b J_{42} \nabla_3 J_{b'1} J_{21} J_{12} g^{bb'} g^{22} g^{11} \Big) \\
&\qquad := 2p^4 \Big(\langle\langle 1 \rangle\rangle + \langle\langle 2 \rangle\rangle + \langle\langle 3 \rangle\rangle + \langle\langle 4 \rangle\rangle \Big).
\end{aligned}
$$

Similarly, for $S_{3,4,1}(\sigma_4 = 4)$, we have $\sigma \in \{\mathrm{id}, (12), (123), (13), (23), (132)\}$, with signs $+1, -1, +1, -1, -1, +1$, respectively. Thus

$$
\begin{aligned}
&S_{3,4,1}(\sigma_4 = 4) \\
&= 2p^4 \Big(\nabla_b J_{3a_1} \nabla_4 J_{b' a_2'} J_{2a_1} J_{1a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} - \nabla_b J_{3a_1} \nabla_4 J_{b' a_2'} J_{1a_2} J_{2a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
&\quad + \nabla_b J_{1a_1} \nabla_4 J_{b' a_2'} J_{3a_2} J_{2a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} - \nabla_b J_{1a_1} \nabla_4 J_{b' a_2'} J_{2a_2} J_{3a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
&\quad - \nabla_b J_{2a_1} \nabla_4 J_{b' a_2'} J_{3a_2} J_{1a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} + \nabla_b J_{2a_1} \nabla_4 J_{b' a_2'} J_{1a_2} J_{3a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \Big)
\end{aligned}
$$

Using $\nabla_b J_{11} = \nabla_b J_{22} = 0$, we obtain

$$
\nabla_b J_{1a_1} \nabla_4 J_{b' a_2'} J_{3a_2} J_{2a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} = 0, \ \nabla_b J_{2a_1} \nabla_4 J_{b' a_2'} J_{3a_2} J_{1a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} = 0.
$$

Therefore,

$$
\begin{aligned}
&S_{3,4,1}(\sigma_4 = 4)\\
&= 2p^4\Big(\nabla_b J_{32}\nabla 4b'1 J_{21}J_{12}g^{bb'}g^{22}g^{11} - \nabla_b J_{31}\nabla 4b'2 J_{12}J_{21}g^{bb'}g^{11}g^{22}\\
&\qquad - \nabla_b J_{1a_1}\nabla 4b'1 J_{21}J_{34}g^{bb'}g^{a_14}g^{11} + \nabla_b J_{2a_1}\nabla 4b'2 J_{12}J_{34}g^{bb'}g^{a_14}g^{22}\Big)\\
&:= 2p^4\Big(\langle\langle 5\rangle\rangle + \langle\langle 6\rangle\rangle + \langle\langle 7\rangle\rangle + \langle\langle 8\rangle\rangle\Big).
\end{aligned}
$$

We now compute $\langle\langle 1\rangle\rangle - \langle\langle 8\rangle\rangle$.

$$
\begin{aligned}
&\langle\langle 1\rangle\rangle\\
&= (J_{12})^2\nabla_b J_{42}\nabla_3 J_{b'}g^{bb'}\\
&= (J_{12})^2\big(\nabla_3 J_{42}\nabla_3 J_{31}g^{33} + \nabla_3 J_{42}\nabla_3 J_{41}g^{34}\nabla_4 J_{42}\nabla_3 J_{31}g^{43} + \nabla_4 J_{42}\nabla_3 J_{41}g^{44}\big)\\
&= (-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad + (-\frac{1}{2}\kappa\beta^{-1})(\frac{1}{4}\beta^{-3/2}(1-2\theta))(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad + (-\frac{1}{2}\kappa\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\kappa^{-1}\beta^{-1/2})\\
&= \kappa\beta^{-3}\Big(-\frac{1}{16}(1-2\theta)(1+\theta) - \frac{1}{8}(1+\frac{\theta}{2})\theta - \frac{1}{8}(1-2\theta)\theta - \frac{1}{4}(1+\frac{\theta}{2})\Big)\\
&= \kappa\beta^{-3}\frac{1}{16}\Big(-(1-2\theta)(1+\theta) - 2\theta(1+\frac{\theta}{2}) - 2\theta(1-2\theta) - 4(1+\frac{\theta}{2})\Big)\\
&= \frac{\kappa}{16}(-5-5\theta+5\theta^2)\\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

$$
\begin{aligned}
&\langle\langle 2\rangle\rangle\\
&= \nabla_b J_{41}\nabla_3 J_{b'2}(-J_{12})^2 g^{bb\prime}\\
&= -(J_{12})^2\Big(\nabla_3 J_{41}\nabla_3 J_{32}g^{33} + \nabla_4 J_{41}\nabla_3 J_{42}g^{34} + \nabla_4 J_{41}\nabla_3 J_{32}g^{43} + \nabla_4 J_{41}\nabla_3 J_{42}g^{44}\Big)\\
&= (-1)\big(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big)(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})
\end{aligned}
$$

$$
\begin{aligned}
&+\big(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big)(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&+\big(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)\big)(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&+\big(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)\big)(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
=&(-1)\kappa\beta^{-3}\left(\frac{1}{4}(1+\frac{\theta}{2})(1+\theta)-\frac{1}{8}(1+\frac{\theta}{2})\theta-\frac{1}{8}(1+3\theta)\theta+\frac{1}{16}(1+3\theta)\right)\\
=&-\frac{1}{16}\kappa\beta^{-3}(4-6\theta+2\theta^2-2\theta-\theta^2-2\theta-6\theta^2+1+3\theta)\\
=&-\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

$$
\begin{aligned}
&\langle\langle 3\rangle\rangle\\
&=-\nabla_b J_{2a_1'}\nabla_3 J_{b'2}J_{12}J_{43}g^{a_1'3}g^{bb'}\\
&=(-\kappa)(-1)(g^{33}\nabla_b J_{23}\nabla_3 J_{b'2}g^{bb'}+g^{34}\nabla_b J_{24}\nabla_3 J_{b'2}g^{bb'})\\
&=\kappa\Big(g^{33}(\nabla_3 J_{23}\nabla_3 J_{32}g^{33}+\nabla_3 J_{23}\nabla_3 J_{42}g^{34}+\nabla_4 J_{23}\nabla_3 J_{32}g^{43}+\nabla_4 J_{23}\nabla_3 J_{42}g^{44})\\
&\qquad+g^{34}(\nabla_3 J_{24}\nabla_3 J_{32}g^{33}\nabla_3 J_{24}\nabla_3 J_{42}g^{34}+\nabla_4 J_{24}\nabla_3 J_{32}g^{43}+\nabla_4 J_{24}\nabla_3 J_{42}g^{44})\Big)\\
&=\kappa\Big[\kappa^{-1}(1+\theta)\beta^{-1/2}\Big((-\frac{1}{2}\kappa\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad\qquad+(-\frac{\kappa}{2}\beta^{-1})(-\frac{1}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1})+(\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1})\\
&\qquad\qquad+(\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1})\Big)\Big]\\
&\qquad+\kappa\theta\beta^{-1/2}\Big[(\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})+(\frac{\kappa}{4}\beta^{-1})(-\frac{1}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad\qquad+(\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})+(\frac{\kappa}{2}\beta^{-1})(-\frac{1}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\Big]\\
&=\kappa\beta^{-3}\Big((1+\theta)(-\frac{1}{4}(1+\theta)+\frac{1}{8}\theta+\frac{1}{8}\theta-\frac{1}{16})\\
&\qquad+\theta(\frac{1}{8}(1+\theta)-\frac{1}{16}\theta+\frac{1}{4}\theta-\frac{1}{8})\Big)\\
&=\frac{1}{16}\kappa\beta^{-3}\Big((1+\theta)(-4(1+\theta)-4\theta-1)+(2(1+\theta)-\theta+4\theta-2)\theta\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{16}\kappa\beta^{-3}(-5)(1+\theta-\theta^2)\\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

$\langle\langle 4\rangle\rangle$

$$
\begin{aligned}
&= -\nabla_b J_{42}\nabla_3 J_{b'1}J_{21}J_{12}g^{bb'}g^{22}g^{11}\\
&= \kappa\Bigg[\kappa^{-1}(1+\theta)\beta^{-1/2}\Bigg((-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (\kappa^{-1}\theta\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}(1-2\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\beta^{-1/2})\Bigg)\Bigg]\\
&= \kappa\beta^{-4}\Bigg((1+\theta)\left(-\frac{1}{16}\right)(1-2\theta)^2(1+\theta) - \frac{1}{8}(1-2\theta)(1+\frac{1}{2}\theta)\theta - \frac{1}{8}(1+\frac{1}{2}\theta)(1-2\theta)\theta\\
&\qquad - \frac{1}{4}(1+\frac{1}{2}\theta)^2 + \theta\left(-\frac{1}{8}\right)(1+\frac{1}{2}\theta)(1-2\theta)(1+\theta) - \frac{1}{4}(1+\frac{1}{2}\theta)^2\theta\\
&\qquad + \frac{1}{16}(1+3\theta)(1-2\theta)\theta + \frac{1}{8}(1+3\theta)(1+\frac{1}{2}\theta)\Bigg)\\
&= -\frac{1}{16}\kappa\beta^{-4}\Bigg((1+\theta)(1-2\theta)^2(1+\theta) + (1-2\theta)(2+\theta)\theta + (2+\theta)(1-2\theta)\theta\\
&\qquad\quad + (2+\theta)^2 + \theta((2+\theta)(1-2\theta)(1+\theta) + (2+\theta)^2\theta)\\
&\qquad\quad - (1+3\theta)(1-2\theta)\theta - (1+3\theta)(2+\theta)\Bigg)\\
&= -\frac{1}{16}\kappa\beta^{-4}\Bigg((1+\theta)\big[(1+\theta)(1-2\theta)^2 + 2\theta(2+\theta)(1-2\theta) + (2+\theta)^2\big]\\
&\qquad\quad + \theta\big[(2+\theta)(1-2\theta)(1+\theta) + (2+\theta)^2\theta - (1+3\theta)(1-2\theta)\theta\\
&\qquad\quad - (1+3\theta)(2+\theta)\big]\Bigg)
\end{aligned}
$$

$$= -\frac{1}{16}\kappa\beta^{-4}\cdot 5\cdot(1+\theta-\theta^2)^2)$$
$$= -\frac{5}{16}\kappa\beta^{-2}.$$

$$\begin{aligned}
&\langle\langle 5\rangle\rangle\\
&= \nabla_b J_{32}\nabla_4 J_{b'1}(-(J_{12})^2)g^{bb'} = -1\nabla_b J_{32}\nabla_4 J_{b'1}g^{bb'}\\
&= -(\nabla_3 J_{32}\nabla_4 J_{31}g^{33} + \nabla_3 J_{32}\nabla_4 J_{41}g^{34} + \nabla_4 J_{32}\nabla_4 J_{31}g^{43} + \nabla_4 J_{32}\nabla_4 J_{41}g^{44})\\
&= -(\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad -(\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad -(-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad -(-\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})\\
&= -\kappa\beta^{-3}\left(\frac{1}{4}(1+\frac{1}{2}\theta)(1+\theta) - \frac{1}{8}\theta(1+3\theta) - \frac{1}{8}(1+\frac{1}{2}\theta)\theta + \frac{1}{16}(1+3\theta)\right)\\
&= -\frac{\kappa}{16}(4+6\theta;2\theta^2-2\theta-6\theta^2-2\theta-\theta^2+1+3\theta)\\
&= -\frac{1}{16}\kappa\beta^{-3}(5+5\theta-5\theta^2)\\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}$$

$$\begin{aligned}
&\langle\langle 6\rangle\rangle\\
&= -\nabla_b J_{31}\nabla_4 J_{b'2}(-(J_{12})^2)g^{bb'} = \nabla_b J_{31}\nabla_4 J_{b'2}g^{bb'}\\
&= \nabla_3 J_{31}\nabla_4 J_{32}g^{33} + \nabla_3 J_{31}\nabla_4 J_{42}g^{34} + \nabla_4 J_{31}\nabla_4 J_{32}g^{43} + \nabla_4 J_{31}\nabla_4 J_{42}g^{44}\\
&= (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad +(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad +(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta)(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad +(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
&= \kappa\beta^{-3}\left(-\frac{1}{16}(1-2\theta)(1+\theta) - \frac{1}{8}(1-2\theta)\theta - \frac{1}{8}(1+\frac{1}{2}\theta)\theta - \frac{1}{4}(1+\frac{1}{2}\theta)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16}\kappa\beta^{-3}(-5-5\theta+5\theta^2)\\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

$\langle\langle 7\rangle\rangle$

$$
\begin{aligned}
&= -\nabla_b J_{1a_1}\nabla_4 J_{b'1}J_{21}J_{34}g^{bb'}g^{a_14} = \kappa\nabla_b J_{1a_1}\nabla_4 J_{b'1}g^{bb'}g^{a_14}\\
&= \kappa(g^{34}\nabla_b J_{13}\nabla_4 J_{b'1}g^{bb'} + g^{44}\nabla_b J_{14}\nabla_4 J_{b'1}g^{bb'})\\
&= \kappa\Big(g^{34}\big(\nabla_3 J_{13}\nabla_4 J_{31}g^{33} + \nabla_3 J_{13}\nabla_4 J_{41}g^{34} + \nabla_4 J_{13}\nabla_4 J_{31}g^{43} + \nabla_4 J_{13}\nabla_4 J_{41}g^{44}\big)\\
&\qquad + g^{44}\big(\nabla_3 J_{14}\nabla_4 J_{31}g^{33} + \nabla_3 J_{14}\nabla_4 J_{41}g^{34} + \nabla_4 J_{14}\nabla_4 J_{31}g^{43} + \nabla_4 J_{14}\nabla_4 J_{41}g^{44}\big)\Big)\\
&= \kappa\Big[\kappa^{-1}\theta\beta^{-1/2}\Big((-\frac{\kappa}{4}\beta^{-3/2})(1-2\theta)(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})\Big)\\
&\qquad + \kappa^{-1}\beta^{-1/2}\Big((-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})\Big)\Big]\\
&= \kappa\beta^{-4}\Big[\Big(\theta(-\frac{1}{8}(1-2\theta)(1+\frac{1}{2}\theta)(1+\theta) + \frac{1}{16}(1-2\theta)(1+3\theta)\theta - \frac{1}{4}(1+\frac{1}{2}\theta)(1+\frac{1}{2}\theta)\theta\\
&\qquad\qquad + \frac{1}{8}(1+\frac{1}{2}\theta)(1+3\theta)\Big)\\
&\qquad + \Big(-\frac{1}{4}(1+\frac{1}{2}\theta)(1+\frac{1}{2}\theta)(1+\theta) + \frac{1}{8}(1+\frac{1}{2}\theta)(1+3\theta)\theta + \frac{1}{8}(1+3\theta)(1+\frac{1}{2}\theta)\\
&\qquad\qquad - \frac{1}{16}(1+3\theta)(1+3\theta)\Big)\Big]
\end{aligned}
$$

$$= -\frac{5}{16}\kappa\beta^{-4}(1+2\theta-\theta^2-2\theta^3+\theta^4) = -\frac{5}{16}\kappa\beta^{-4}\beta^2 = -\frac{5}{16}\kappa\beta^{-2}.$$

$\langle\langle 8\rangle\rangle$

$$\begin{aligned}
&= \nabla_b J_{2a_1}\nabla_4 J_{b'2}J_{12}J_{34}g^{bb'}g^{a_14} = J_{12}J_{34}(\nabla_b J_{23}\nabla_4 J_{b'2}g^{bb'}g^{34} + \nabla_b J_{24}\nabla_4 J_{b'2}g^{bb'}g^{44})\\
&= \kappa(\nabla_b J_{23}\nabla_4 J_{b'2}g^{bb'}g^{34} + \nabla_b J_{24} + \nabla_4 J_{b'2}g^{bb'}g^{44})\\
&= \kappa\bigg((\nabla_3 J_{23}\nabla_4 J_{32}g^{33}\nabla_3 J_{23}\nabla_4 J_{42}g^{34} + \nabla_4 J_{23}\nabla_4 J_{32}g^{43} + \nabla_4 J_{23}\nabla_4 J_{42}g^{44})g^{34}\\
&\qquad + (\nabla_3 J_{24}\nabla_4 J_{32}g^{33} + \nabla_3 J_{24}\nabla_4 J_{42}g^{34} + \nabla_4 J_{24}\nabla_4 J_{32}g^{43} + \nabla_4 J_{24}\nabla_4 J_{42}g^{44})g^{44}\bigg)\\
&= \kappa\bigg[\kappa^{-1}\theta\beta^{-1/2}\bigg((-\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2}) + (-\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad\qquad + (\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2}) + (\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\bigg)\\
&\qquad + \kappa^{-1}\beta^{-1/2}\bigg((\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1}(\kappa^{-1}(1+\theta)\beta^{-1/2}) + (\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad\qquad + (\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2}) + (\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\bigg)\bigg]\\
&= \kappa\beta^{-3}\bigg[\theta\bigg(\frac{1}{8}(1+\theta) + \frac{1}{4}\theta - \frac{1}{16}\theta - \frac{1}{8}\bigg) + \bigg(-\frac{1}{16}(1+\theta) - \frac{1}{8}\theta - \frac{1}{8}\theta - \frac{1}{4}\bigg)\bigg]\\
&= -\frac{5}{16}\kappa\beta^{-3}\beta = -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}$$

Thus

$$S_{3,4,1} = 2p^4(\langle\langle 1\rangle\rangle + \ldots + \langle\langle 8\rangle\rangle) = 2p^4\cdot 8\cdot\left(-\frac{5}{16}\kappa\beta^{-2}\right) = -5p^4\beta^{-2}.$$

(iii)(b) We compute $S_{3,4,2}$ as follows.

$$\begin{aligned}
S_{3,4,2} &= -2p^4\sum_{\sigma_3=0}\operatorname{sgn}(\sigma)\nabla_b J_{\sigma_3 a_1}\nabla_{\sigma_4}J_{b'}{}^{a_2}J_{\sigma_1\sigma_2}J_{a_2}{}^{a_1}g^{bb'}\\
&= -2p^4\sum_{\sigma_3=0}\operatorname{sgn}(\sigma)\nabla_b J_{\sigma_3 a_1}\nabla_{\sigma_4}J_{b'a_2'}J_{a_2a_1'}J_{\sigma_1\sigma_2}g^{bb'}g^{a_1a_1'}g^{a_2a_2'}.
\end{aligned}$$

We must have $(\sigma_1,\sigma_2)\in\{(1,2),(2,1),(3,4),(4,3)\}$ From the form of the symplectic structure, it follows that $\sigma\in S=\{\mathrm{id},(34),(13)(24),(1324)\}$. (There are also permutations of the form $(12)\tau,(34)\tau$ for $\tau\in S$, but by the skew-symmetry of J_{ab}, only the terms in S contribute to $S_{3,4,2}$.) The signs of the permutations in S are $+1,-1,+1,-1$, respectively.

Keeping track of the extra permutations and their signs, we have

$$
\begin{aligned}
&S_{3,4,2}\\
&= -2p^4\cdot 2\Big(\nabla_b J_{3a_1}\nabla_4 J_{b'a_2'}J_{a_2a_1'}J_{12}g^{bb'}g^{a_2a_2'}g^{a_1a_1'} - \nabla_b J_{4a_1}\nabla_3 J_{b'a_2'}J_{a_2a_1'}J_{12}g^{bb'}g^{a_2a_2'}g^{a_1a_1'}\\
&\qquad + \nabla_b J_{1a_1}\nabla_2 J_{b'a_2'}J_{a_2a_1'}J_{34}g^{bb'}g^{a_2a_2'}g^{a_1a_1'} - \nabla_b J_{2a_1}\nabla_1 J_{b'a_2'}J_{a_2a_1'}J_{34}g^{bb'}g^{a_2a_2'}g^{a_1a_1'}\Big).
\end{aligned}
$$

Using $\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0$, and $J_{12} = 1$, we get

$$
\begin{aligned}
&S_{3,4,2}\\
&= -4p^4\Big(\nabla_b J_{3a_1}\nabla_4 J_{b'a_2'}J_{a_2a_1'}g^{bb'}g^{a_2a_2'}g^{a_1a_1'} - \nabla_b J_{4a_1}\nabla_3 J_{b'a_2'}J_{a_2a_1'}g^{bb'}g^{a_2a_2'}g^{a_1a_1'}\Big)\\
&= 4p^4\Big(\nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{a_1a_2}g^{bb'}g^{a_2a_2'}g^{a_1a_1'} - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{a_1a_2}g^{bb'}g^{a_2a_2'}g^{a_1a_1'}\Big).
\end{aligned}
$$

The only nonzero terms come from $(a_1, a_2) \in \{(1,2),(2,1),(3,4),(4,3)\}$. Therefore,

$$
\begin{aligned}
&S_{3,4,2}\\
&= -4p^4\Big(\nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{12}g^{bb'}g^{2a_2'}g^{1a_1'} + \nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{21}g^{bb'}g^{1a_2'}g^{2a_1'}\\
&\qquad - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{12}g^{bb'}g^{2a_2'}g^{1a_1'} - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{21}g^{bb'}g^{1a_2'}g^{2a_1'}\\
&\qquad + \nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{34}g^{bb'}g^{4a_2'}g^{3a_1'} + \nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{43}g^{bb'}g^{3a_2'}g^{4a_1'}\\
&\qquad - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{34}g^{bb'}g^{4a_2'}g^{3a_1'} - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{43}g^{bb'}g^{3a_2'}g^{4a_1'}\Big)\\
&= 4p^4\Big(\nabla_b J_{31}\nabla_4 J_{b'2}g^{bb'} - \nabla_b J_{32}\nabla_4 J_{b'1}g^{bb'} - \nabla_b J_{41}\nabla_3 J_{b'2}g^{bb'} + \nabla_b J_{42}\nabla_3 J_{b'1}g^{bb'}\\
&\qquad + \nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}g^{bb'}\kappa g^{4a_2'}g^{3a_1'} - \nabla_b J_{3a_1}\nabla_4 J_{b'a_2'}g^{bb'}\kappa g^{3a_2'}g^{4a_1'}\\
&\qquad - \nabla_b J_{4a_1'}\nabla_3 J_{ba_2'}g^{bb'}\kappa g^{4a_2'}g^{3a_1'} + \nabla_b J_{4a_1'}\nabla_3 J_{ba_2'}g^{bb'}\kappa g^{3a_2'}g^{4a_1'}\Big)\\
&:= 4p^4\Big([[1]]+[[2]]+[[3]]+[[4]]+[[5]]+[[6]]+[[7]]+[[8]]\Big).
\end{aligned}
$$

We now compute [[1]]–[[8]].

$$
\begin{aligned}
&[[1]]\\
&= \nabla_b J_{31}\nabla_4 J_{b'2}g^{bb'}\\
&= \nabla_3 J_{31}\nabla_4 J_{32}g^{33} + \nabla_3 J_{31}\nabla_4 J_{42}g^{34} + \nabla_4 J_{31}\nabla_4 J_{32}g^{43} + \nabla_4 J_{31}\nabla_4 J_{42}g^{44}\\
&= \Big(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta)\Big)\Big(-\frac{\kappa}{4}\beta^{-1}\Big)\Big(\kappa^{-1}(1+\theta)\beta^{-1/2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&+(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-3/2})\\
&+(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
=&\ \kappa\beta^{-3}\left(-\frac{1}{16}(1-2\theta)(1+\theta)-\frac{1}{8}(1-2\theta)\theta-\frac{1}{8}\theta(1+\frac{1}{2}\theta)-\frac{1}{4}(1+\frac{1}{2}\theta)\right)\\
=&\ \frac{1}{16}\kappa\beta^{-3}(-5-5\theta+5\theta^2)\\
=&\ -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

[[2]]

$$
\begin{aligned}
&= -\nabla_b J_{32}\nabla_4 J_{b'1}g^{bb'}\\
&= -\left(\nabla_3 J_{32}\nabla_4 J_{31}g^{33}+\nabla_3 J_{32}\nabla_4 J_{41}g^{34}+\nabla_4 J_{32}\nabla_4 J_{31}g^{43}+\nabla_4 J_{32}\nabla_4 J_{41}g^{44}\right)\\
&= -\left((\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\right.\\
&\quad +(\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\quad +(-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\quad \left.+(-\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})\right)\\
&= -\kappa\beta^{-3}\left(\frac{1}{4}(1+\frac{1}{2}\theta)(1+\theta)-\frac{1}{8}(1+3\theta)\theta-\frac{1}{8}(1+\frac{1}{2}\theta)\theta+\frac{1}{16}(1+3\theta)\right)\\
&= \frac{1}{16}\kappa\beta^{-3}(5+5\theta-5\theta^2)\\
&- \frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

[[3]]

$$
\begin{aligned}
&= -\nabla_b J_{41}\nabla_3 J_{b'2}g^{bb'}\\
&= -\left(\nabla_3 J_{41}\nabla_3 J_{32}g^{33}+\nabla_3 J_{41}\nabla_3 J_{42}g^{34}+\nabla_4 J_{41}\nabla_3 J_{32}g^{43}+\nabla_4 J_{41}\nabla_3 J_{42}g^{44}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\Big((\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2}) \\
&\quad + (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2}) \\
&\quad + (-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2}) \\
&\quad + (-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\Big) \\
&= -\frac{1}{16}\kappa\beta^{-3}\Big(4(1+\frac{1}{2}\theta)(1+\theta) - 2(1+\frac{1}{2}\theta)\theta - 2(1+3\theta)\theta + (1+3\theta)\Big) \\
&= -\frac{1}{16}\kappa\beta^{-3}(5+5\theta-5\theta^2) \\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

$$
\begin{aligned}
&[[4]] \\
&= \nabla_b J_{42}\nabla_3 J_{b'1}g^{bb'} \\
&= \nabla_3 J_{42}\nabla_3 J_{31}g^{33} + \nabla_3 J_{42}\nabla_3 J_{41}g^{43} + \nabla_4 J_{42}\nabla_3 J_{31}g^{43} + \nabla_4 J_{42}\nabla_3 J_{41}g^{4} \\
&= (-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2}) \\
&\quad + (-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2}) \\
&\quad + (-\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}\theta\beta^{-1/2}) \\
&\quad + (-\frac{\kappa}{2}\beta\ -1)(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\beta^{-1/2}) \\
&= \frac{1}{16}\kappa\beta^{-3}\Big(-(1-2\theta)(1+\theta) - 2(1+\frac{1}{2}\theta)\theta - 2(1-2\theta)\theta - 4(1+\frac{1}{2}\theta)\Big) \\
&= \frac{1}{16}\kappa\beta^{-3}(-5-5\theta+5\theta^2) \\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

We have

$$[[5]] = \kappa\nabla_b J_{3a_1'}\nabla_4 J_{b'a_2}g^{bb'}g^{4a_2'}g^{3a_1'} = 0,$$

since we must have $a_1' = 3$ or 4, and $\nabla_b J_{33} = \nabla_b J_{34} = 0$. For the same reason,

$$[[6]] = -\kappa\nabla_b J_{3a_1'}\nabla_4 J_{b'a_2}g^{bb'}g^{3a_2'}g^{4a_1'} = 0.$$

Similarly,

$$[[7]] = -\kappa\nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}g^{bb'}g^{4a_2'}g^{3a_1'} = 0,$$

since $a_1' = 3$ or 4, and $\nabla_b J_{43} = 0$, $\nabla_b J_{44} = 0$. For the same reason,

$$[[8]] = \kappa \nabla_b J_{4a_1'} \nabla_3 J_{b'a_2} g^{bb'} g^{3a_2'} g^{4a_1'} = 0.$$

Thus, we have

$$S_{3,4,2} = 4p^4 \left(-\frac{5}{16} \kappa \beta^{-2} \right) \cdot 4 = -5\kappa \beta^{-2} p^4.$$

This finishes the proofs of Lemma 1.14 and Proposition 1.13.

As explained below Proposition 1.5, this finishes the demonstration of (1.3).

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