# CALCULATION OF THE WCS INVARIANT ON THE THURSTON EXAMPLE

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## 1. The calculation of the WCS invariant

We calculate the WCS invariant for the Thurston example in §4. Specifically, we prove

**Theorem 4.1.** We have

$$\int_{\overline{M_p}} \widetilde{CS}_5^W(e_1,\dots,e_5) = [2\pi p^{-1/2}\kappa]\frac{5\kappa}{128}p^2\int_0^1 (3072p^4 - 640p^2\beta^{-2} - 25\beta^{-4})d\theta_2. \quad (1.1)$$

**Notation:** We denote the metric $\tilde{g}$ on the Thurston example $M$ just by $g$. We abbreviate $\theta_2 = \theta$, and recall that

$$\beta = 1 + \theta - \theta^2.$$

The Christoffel symbols are

$$\Gamma^a_{bc} = \frac{1}{2}g^{ae}(\partial_b g_{ce} + \partial_c g_{be} - \partial_e g_{bc}).$$

The curvature tensor components are

$$R_{abc}{}^d = \partial_a\Gamma^d_{bc} - \partial_b\Gamma^d_{ac} + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac}.$$

Recall that $g = \tilde{g}$ has a compatible almost complex structure $J$.

The matrix of $g$ is given in (4.7), so

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \kappa^{-1}\beta^{-1/2}(1+\theta) & \kappa^{-1}\beta^{-1/2}\theta \\ 0 & 0 & \kappa^{-1}\beta^{-1/2}\theta & \kappa^{-1}\beta^{-1/2} \end{pmatrix}.$$

The long calculations below have been checked by machine calculation at [4] in the references.

1.1. **The curvature tensor on $\overline{M}_p$.** Let $\overline{R}$ be the metric on $\overline{M}_p$.

**Lemma 1.1.** *In the notation of (4.8), we have*

$$\begin{aligned}
\overline{R}_{abc}{}^d &= R_{abc}{}^d - p^2 J_{bc} J_a{}^d + p^2 J_{ac} J_b{}^d + 2p^2 J_{ab} J_c{}^d,\\
\overline{R}_{abc}{}^0 &= -p\nabla_a J_{bc} + p\nabla_b J_{ac} = p\nabla_c J_{ab},\\
\overline{R}_{ab0}{}^d &= p\nabla_a J_b{}^d - p\nabla_b J_a{}^d,\\
\overline{R}_{a0b}{}^d &= p\nabla_a J_b{}^d,\\
\overline{R}_{a0b}{}^0 &= -p^2 g_{ab}.
\end{aligned}$$

*Proof.* These are the local frame expressions of Lemma 3.3. □

Here we have used the identity

$$\nabla_a J_{bc} + \nabla_b J_{ca} + \nabla_c J_{ab} = 0,$$

which follows from $d\omega = 0$ and $g(JX, Y) = \omega(X, Y)$. For example, with this identity and curvature tensor symmetries, the second, third and fourth formulas in Lemma A.1 are equivalent.

1.2. **The Christoffel symbols on $M$.**

**Lemma 1.2.** *The following is the list of the Christoffel tensors*

$$\begin{aligned}
&\Gamma^2_{33} = \frac{\kappa}{4}\beta^{-3/2}(1-2\theta),\ \Gamma^2_{34} = \frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}),\ \Gamma^2_{44} = -\frac{\kappa}{4}\beta^{-3/2}(1+3\theta),\\
&\Gamma^3_{23} = -\frac{1}{4}\beta^{-1},\ \Gamma^3_{24} = -\frac{1}{2}\beta^{-1},\ \Gamma^4_{23} = -\frac{1}{2}\beta^{-1},\ \Gamma^4_{24} = \frac{1}{4}\beta^{-1}.
\end{aligned}$$

*All other Christoffel symbols are zero.*

*Proof.* We have

$$\Gamma^1_{ab} = \frac{1}{2}g^{1e}(\partial_a g_{be} + \partial_b g_{ae} - \partial_e g_{ab}) = \frac{1}{2}g^{11}(\partial_a g_{b1} + \partial_b g_{a1} - \partial_1 g_{ab}) = 0,$$

because $g_{\ell 1}$ is constant and $\partial_1 g_{ab} = 0$.

Since $\Gamma^2_{ab} = \frac{1}{2}g^{2e}(\partial_a g_{be} + \partial_b g_{ae} - \partial_e g_{ab}) = \frac{1}{2}(\partial_a g_{b2} + \partial_b g_{a2} - \partial_2 g_{ab})$ and $g_{22} = 1$, we get

$$\Gamma^2_{11} = \Gamma^2_{12} = \Gamma^2_{12} = 0$$

Note that $g_{b2} = 0$, $g_{b1} = 0$ and $g_{b2} = 0$ if $b = 3, 4$.

Therefore,

$$\Gamma^2_{1b} = \Gamma^2_{2b} = 0$$

for $b = 3, 4$. We also get

$$\Gamma^2_{33} = \frac{1}{2}(\partial_3 g_{32} + \partial_3 g_{32} - \partial_2 g_{33}) = -\frac{1}{2}\partial_2(\kappa\beta^{-1/2}) = \frac{\kappa}{4}\beta^{-3/2}(1-2\theta),$$

$$\begin{aligned}\Gamma^2_{34} &= \frac{1}{2}(\partial_3 g_{42} + \partial_4 g_{32} - \partial_2 g_{34}) = -\frac{1}{2}\partial_2(-\theta\kappa\beta^{-1/2}) \\ &= \frac{\kappa}{2}\beta^{-3/2}(1 + \frac{1}{2}\theta), \\ \Gamma^2_{44} &= \frac{1}{2}(\partial_4 g_{42} + \partial_4 g_{42} - \partial_2 g_{44}) = -\frac{1}{2}\partial_2 g_{44} = -\frac{1}{2}\partial_2((1+\theta)\kappa\beta^{-1/2}) \\ &= -\frac{\kappa}{4}\beta^{-3/2}(1 + 3\theta).\end{aligned}$$

Since $\Gamma^3_{ab} = \frac{1}{2}g^{3e}(\partial_a g_{be} + \partial_b g_{ae} - \partial_e g_{ab})$, we have

$$\Gamma^3_{11} = \Gamma^3_{12} = \Gamma^3_{22} == \Gamma^3_{13} = \Gamma^3_{14} = 0$$

$$\begin{aligned}\Gamma^3_{23} &= \frac{1}{2}g^{3e}(\partial_2 g_{3e} + \partial_3 g_{2e} - \partial_e g_{23}) = \frac{1}{2}g^{3e}(\partial_2 g_{3e}) = \frac{1}{2}(g^{33}\partial_2 g_{33} + g^{34}\partial_2 g_{34}) \\ &= -\frac{1}{4}\beta^{-1}, \\ \Gamma^3_{24} &= \frac{1}{2}g^{3e}(\partial_2 g_{4e} + \partial_4 g_{2e} - \partial_e g_{24}) = \frac{1}{2}g^{3e}\partial_2 g_{4e} = \frac{1}{2}(g^{33}\partial_2 g_{43} + g^{34}\partial_2 g_{44}) \\ &= -\frac{1}{2}\beta^{-1}.\end{aligned}$$

Also, since $e = 3, 4$,

$$\Gamma^3_{33} = \Gamma^3_{34} = \Gamma^3_{44} = 0$$

Furthermore, we have $\Gamma^4_{ab} == \frac{1}{2}g^{4e}(\partial_a g_{be} + \partial_b g_{ae} - \partial_e g_{ab}) = \frac{1}{2}g^{4e}(\partial_a g_{be} + \partial_b g_{ae})$, since $e = 3, 4$. For these values of $e$, $g_{1e} = 0, g_{24} = 0, g_{2e} = 0$, so

$$\Gamma^4_{11} = \Gamma^4_{12} = \Gamma^4_{22} = \Gamma^4_{13} = \Gamma^4_{14} = 0.$$

Also,

$$\begin{aligned}\Gamma^4_{23} &= \frac{1}{2}g^{4e}(\partial_2 g_{3e} + \partial_3 g_{2e}) = \frac{1}{2}(g^{43}\partial_2 g_{33} + g^{44}\partial_2 g_{34}) \\ &= \frac{1}{2}(\theta\kappa^{-1}\beta^{-1/2}\partial_2(\kappa\beta^{-1/2}) + \kappa^{-1}\beta^{-1/2}\partial_2(-\theta\kappa\beta^{-1/2})) \\ &= -\frac{1}{2}\beta^{-1}, \\ \Gamma^4_{24} &= \frac{1}{2}g^{4e}(\partial_2 g_{4e} + \partial_4 g_{2e}) = \frac{1}{2}(g^{43}\partial_2 g_{43} + g^{44}\partial_2 g_{44}) \\ &= \frac{1}{2}(\theta\kappa^{-1}\beta^{-1/2}\partial_2(-\theta\kappa\beta^{-1/2}) + \kappa^{-1}\beta^{-1/2}\partial_2((1+\theta)\kappa\beta^{-1/2})) \\ &= \frac{1}{4}\beta^{-1}.\end{aligned}$$

Finally, we have

$$\Gamma^4_{33} = \Gamma^4_{34} = \Gamma^4_{44} = 0.$$

This proves the Lemma. □

1.3. **The curvature tensor on $M$.** We compute the curvature components ${R_{abc}}^d = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^d_{ae}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{ac}$.

**Lemma 1.3.**

*(i)* ${R_{abc}}^d = 0$ *if any of* $a, b, c, d = 1$.

*(ii)*

$$\begin{aligned}
&{R_{233}}^2 = -\frac{\kappa}{16}\beta^{-5/2}(9 - 16\theta + 16\theta^2),\ {R_{232}}^3 = \frac{1}{16}\beta^{-2}(9 - 8\theta),\\
&{R_{232}}^4 = \frac{1}{2}\beta^{-2}(1 - 2\theta),\\
&R_{3232} = -{R_{233}}^a g_{a2} = -{R_{233}}^2 = \frac{\kappa}{16}\beta^{-5/2}(9 - 16\theta + 16\theta^2).
\end{aligned}$$

*(iii)*

$$\begin{aligned}
&R_{2424} = R_{4242} = {R_{424}}^a g_{a2} = {R_{424}}^2 = -{R_{244}}^2 = \frac{\kappa}{16}(1 + \theta + 24\theta^2)\beta^{-5/2},\\
&{R_{242}}^4 = \frac{1}{16}\beta^{-2}(1 + 8\theta),\ {R_{242}}^3 = \frac{1}{2}\beta^{-2}(1 - 2\theta).
\end{aligned}$$

*(iv)*

$$\begin{aligned}
&{R_{234}}^2 = \frac{\kappa}{16}\beta^{-5/2}(-8 + 17\theta + 8\theta^2),\\
&R_{2342} = {R_{234}}^a g_{a2} = {R_{234}}^2 = \frac{\kappa}{16}\beta^{-5/2}(-8 + 17\theta + 8\theta^2),\\
&R_{3242} = -R_{2342} = -\frac{\kappa}{16}\beta^{-5/2}(-8 + 17\theta + 8\theta^2).
\end{aligned}$$

*(v)*

$$\begin{aligned}
&{R_{343}}^3 = -\frac{5}{16}\kappa\theta\beta^{-5/2},\ {R_{343}}^4 = -\frac{5}{16}\kappa\beta^{-5/2},\ {R_{344}}^3 = \frac{5}{16}\kappa(1 + \theta)\beta^{-5/2},\\
&{R_{344}}^4 = \frac{5}{16}\kappa\theta\beta^{-5/2},\ R_{3434} = -R_{3443} = -\frac{5}{16}\kappa^2\beta^{-2}.
\end{aligned}$$

*Proof.* (i) ${R_{1bc}}^d = \partial_1\Gamma^d_{bc} - \partial_b\Gamma^d_{1c} + \Gamma^d_{1e}\Gamma^e_{bc} - \Gamma^d_{be}\Gamma^e_{1c} = 0$, because $\Gamma^d_{1b} = 0$. Similarly,

$${R_{a1c}}^d = -{R_{1ac}}^d = 0,\ {R_{ab1}}^d = {R_{b1a}}^d + {R_{1ab}}^d = 0.$$

Using $\Gamma^1_{bc} = 0$, we get ${R_{abc}}^1 = \partial_a\Gamma^1_{bc} - \partial_b\Gamma^1_{ac} + \Gamma^1_{ae}\Gamma^e_{bc} - \Gamma^1_{be}\Gamma^e_{ac} = 0$.

(ii)

$$\begin{aligned}
{R_{233}}^2 &= \partial_2\Gamma^2_{33} - \partial_3\Gamma^2_{23} + \Gamma^2_{2e}\Gamma^e_{33} - \Gamma^2_{3e}\Gamma^e_{23} = \partial_2\Gamma^2_{33} - \Gamma^2_{33}\Gamma^3_{23} - \Gamma^2_{34}\Gamma^4_{23} \\
&= \partial_2(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta)) - (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{1}{4}\beta^{-1}) \\
&\quad - (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(-\frac{1}{2}\beta^{-1}) \\
&= -\frac{\kappa}{16}\beta^{-5/2}(9-16\theta+16\theta^2),
\end{aligned}$$

and

$$\begin{aligned}
{R_{232}}^3 &= \partial_2\Gamma^3_{32} - \partial_3\Gamma^3_{22} + \Gamma^3_{2e}\Gamma^e_{32} - \Gamma^3_{3e}\Gamma^e_{22} \\
&= \partial_2\Gamma^3_{32} + \Gamma^3_{23}\Gamma^3_{32} + \Gamma^3_{24}\Gamma^4_{32} \\
&= \partial_2(-\frac{1}{4}\beta^{-1}) + (-\frac{1}{4}\beta^{-1})^2 + (-\frac{1}{2}\beta^{-1})^2 \\
&= \frac{1}{16}\beta^{-2}(9-8\theta), \\
{R_{232}}^4 &= \partial_2\Gamma^{32}_4 - \partial_3\Gamma^{22}_4 + \Gamma^{2e}_4\Gamma^{32}_e \\
&= \frac{1}{2}\beta^{-2}(1-2\theta).
\end{aligned}$$

(iii)

$$\begin{aligned}
R_{2424} &= {R_{242}}^a g_{a4} = {R_{242}}^3 g_{34} + {R_{242}}^4 g_{44} \\
&= \frac{8}{16}\beta^{-2}(1-2\theta)(-\kappa\theta\beta^{-1/2}) + \frac{1}{16}\beta^{-2}(1+8\theta)(\kappa(1+\theta)\beta^{-1/2}) \\
&= \frac{\kappa}{16}\beta^{-5/2}(1+\theta+24\theta^2), \\
{R_{242}}^4 &= \partial_2\Gamma^4_{42} - \partial_4\Gamma^4_{22} + \Gamma^4_{2e}\Gamma^e_{42} - \Gamma^4_{4e}\Gamma^e_{22} \\
&= \partial_2\Gamma^4_{42} + \Gamma^4_{23}\Gamma^4_{24} + \Gamma^4_{24}\Gamma^4_{42} \\
&= \partial_2(\frac{1}{4}\beta^{-1}) + (-\frac{1}{2}\beta^{-1})(-\frac{1}{2}\beta^{-1}) + (\frac{1}{4}\beta^{-1})(\frac{1}{4}\beta^{-1}) \\
&= \frac{1}{16}\beta^{-2}(1+8\theta), \\
{R_{242}}^3 &= \partial_2\Gamma^3_{42} - \partial_4\Gamma^3_{22} + \Gamma^3_{2e}\Gamma^e_{42} - \Gamma^3_{4e}\Gamma^e_{22} \\
&= \partial_2\Gamma^3_{42} + \Gamma^3_{23}\Gamma^3_{42} + \Gamma^3_{24}\Gamma^4_{42} \\
&= -\frac{1}{2}\partial_2\beta^{-1} + (-\frac{1}{4}\beta^{-1})(-\frac{1}{2}\beta^{-1}) + (-\frac{1}{2}\beta^{-1})(\frac{1}{4}\beta^{-1}) \\
&= \frac{1}{2}\beta^{-2}(1-2\theta).
\end{aligned}$$

(iv)

$$\begin{aligned} R_{234}{}^2 &= \partial_2\Gamma^2_{34} - \partial_3\Gamma^2_{24} + \Gamma^2_{2e}\Gamma^e_{34} - \Gamma^2_{3e}\Gamma^e_{24} = \partial_2\Gamma^2_{34} - \Gamma^2_{33}\Gamma^3_{24} - \Gamma^2_{34}\Gamma^4_{24} \\ &= \frac{\kappa}{2}\partial_2(\beta^{-3/2}(1+\frac{\theta}{2})) - (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{1}{2}\beta^{-1}) \\ &\quad - (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{1}{4}\beta^{-1}) \\ &= \frac{\kappa}{16}\beta^{-5/2}(-8+17\theta+8\theta^2). \end{aligned}$$

The other formulas follow immediately.

(v) $R_{34a}{}^b = \partial_3\Gamma^b_{4a} - \partial_4\Gamma^b_{3a} + \Gamma^b_{3e}\Gamma^e_{4a} - \Gamma^b_{4a}\Gamma^e_{3a} = \Gamma^b_{3e}\Gamma^e_{4a} - \Gamma^b_{4e}\Gamma^e_{3a}$, so

$$\begin{aligned} R_{344}{}^2 &= \Gamma^2_{32}\Gamma^2_{44} - \Gamma^2_{43}\Gamma^3_{34} - \Gamma^2_{44}\Gamma^4_{34} = 0, \\ R_{343}{}^4 &= \Gamma^4_{32}\Gamma^2_{43} - \Gamma^4_{42}\Gamma^2_{33} = (-\frac{1}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})) - (\frac{1}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta)) \\ &= -\frac{\kappa}{16}\beta^{-5/2}(4(1+\frac{\theta}{2}) + (1-2\theta)) = -\frac{5}{16}\kappa\beta^{-5/2}, \\ R_{344}{}^3 &= \Gamma^3_{32}\Gamma^2_{44} - \Gamma^3_{42}\Gamma^2_{34} = (-\frac{1}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)) - (-\frac{1}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})) \\ &= \frac{\kappa}{16}\beta^{-5/2}(1+3\theta+4(1+\frac{\theta}{2})) = \frac{5}{16}\kappa\beta^{-5/2}(1+\theta), \\ R_{344}{}^4 &= \Gamma^4_{32}\Gamma^2_{44} - \Gamma^4_{42}\Gamma^2_{34} = (-\frac{1}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)) - (\frac{1}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})) \\ &= \frac{\kappa}{8}\beta^{-5/2}(1+3\theta-(1+\frac{\theta}{2})) = \frac{5}{16}\kappa\theta\beta^{-5/2}, \\ R_{343}{}^3 &= \Gamma^3_{32}\Gamma^2_{43} - \Gamma^3_{42}\Gamma^2_{33} = (-\frac{1}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})) - (-\frac{1}{2}\beta^{-1})(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta)) \\ &= \frac{\kappa}{8}\beta^{-5/2}(-(1+\frac{\theta}{2}+(1-2\theta)) = -\frac{5}{16}\kappa\theta\beta^{-5/2}, \\ R_{3434} &= R_{343}{}^b g_{b4} = R_{343}{}^3 g_{34} + R_{343}{}^4 g_{44} \\ &= (-\frac{5}{16}\theta\kappa\beta^{-5/2})(-\theta\kappa\beta^{-1/2}) + (-\frac{5}{16}\kappa\beta^{-5/2})(1+\theta)\kappa\beta^{-1/2} \\ &= -\frac{5}{16}\kappa^2\beta^{-3}(1+\theta-\theta^2) = -\frac{5}{16}\kappa^2\beta^{-2}. \end{aligned}$$

$\square$

1.4. **The covariant derivatives of the almost complex structure on $M$.** We note for later use that

$$(\omega_{ab}) = (J_a^c g_{cb}) = (J_{ab}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa \\ 0 & 0 & -\kappa & 0 \end{pmatrix}. \tag{1.2}$$

We now compute the covariant derivatives of $J$.

**Lemma 1.4.** *(i)*

$$\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0.$$

*(ii)*

$$\begin{aligned} \nabla_3 J_{13} &= -\frac{\kappa}{4}\beta^{-3/2}(1-2\theta),\ \nabla_3 J_{14} = -\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}), \\ \nabla_3 J_{23} &= -\frac{\kappa}{2}\beta^{-1},\ \nabla_3 J_{24} = \frac{\kappa}{4}\beta^{-1}. \end{aligned}$$

*(iii)*

$$\begin{aligned} \nabla_4 J_{13} &= -\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}),\ \nabla_4 J_{14} = \frac{\kappa}{4}\beta^{-3/2}(1+3\theta) \\ \nabla_4 J_{23} &= \frac{\kappa}{4}\beta^{-1}.\ \nabla_4 J_{24} = \frac{\kappa}{2}\beta^{-1}. \end{aligned}$$

*All other $\nabla_i J_{ab}$ vanish.*

*Proof.* (i) Using $\Gamma_{1a}^e = 0$, we have $\nabla_1 J_{ab} = \partial_1 J_{ab} - \Gamma_{1a}^e J_{eb} - \Gamma_{1b}^e J_{ae} = 0$.
We also have

$$\nabla_2 J_{ab} = \partial_2 J_{ab} - \Gamma_{2a}^e J_{eb} - \Gamma_{2b}^e J_{ae} = -\Gamma_{2a}^e J_{eb} - \Gamma_{2b}^e J_{ae}.$$

Thus

$$\begin{aligned} \nabla_2 J_{12} &= -\Gamma_{21}^e J_{e2} - \Gamma_{22}^e J_{1e} = -\Gamma_{22}^e J_{1e} = -\Gamma_{22}^2 J_{12} = 0, \\ \nabla_2 J_{34} &= -\Gamma_{23}^e J_{e4} - \Gamma_{24}^e J_{3e} - \Gamma_{23}^3 J_{34} - \Gamma_{24}^4 J_{34} \\ &= -(-\frac{1}{4}\beta^{-1})J_{34} - (\frac{1}{4}\beta^{-1})J_{34} = 0, \\ \nabla_2 J_{11} &= -\Gamma_{21}^e J_{e1} - \Gamma_{21}^e J_{1e} = 0, \\ \nabla_2 J_{22} &= -\Gamma_{22}^e J_{e2} - \Gamma_{22}^e J_{2e} = 0, \\ \nabla_2 J_{aa} &= -\Gamma_{2a}^e J_{ea} - \Gamma_{2a}^e J_{ae} = 0, \\ \nabla_2 J_{13} &= -\Gamma_{21}^e J_{e3} - \Gamma_{23}^e J_{1e} = -\Gamma_{21}^4 J_{43} - \Gamma_{23}^2 J_{12} = 0, \\ \nabla_2 J_{14} &= -\Gamma_{21}^e J_{e4} - \Gamma_{24}^e J_{1e}, = -\Gamma_{24}^2 J_{12} = 0, \\ \nabla_2 J_{23} &= -\Gamma_{23}^e J_{e3} - \Gamma_{23}^e J_{2e} = -\Gamma_{22}^4 J_{43} - \Gamma_{23}^1 J_{21} = 0, \\ \nabla_2 J_{24} &= -\Gamma_{22}^e J_{e4} - \Gamma_{24}^e J_{24} = -\Gamma_{22}^3 J_{34} - \Gamma_{24}^1 J_{21} = 0. \end{aligned}$$

Therefore, $\nabla_2 J_{ab} = 0$.

(ii) In general,

$$\nabla_3 J_{ab} = -\Gamma^e_{3a} J_{eb} - \Gamma^e_{3b} J_{ae}.$$

Thus

$$\begin{aligned}
\nabla_3 J_{1b} &= -\Gamma^e_{31} J_{eb} = 0,\\
\nabla_3 J_{12} &= -\Gamma^e_{31} J_{e2} - \Gamma^e_{32} J_{1e} = -\Gamma^1_{31} J_{12} - \Gamma^2_{32} J_{12} = 0,\\
\nabla_3 J_{13} &= -\Gamma^e_{31} J_{e3} - \Gamma^e_{33} J_{1e} = -\Gamma^4_{31} J_{43} - \Gamma^2_{33} J_{12} = -\Gamma^2_{33} J_{12} = -\Gamma^2_{33}\\
&= -\frac{\kappa}{4}\beta^{-3/2}(1-2\theta),\\
\nabla_3 J_{14} &= -\Gamma^2_{34} J_{12} = -\frac{\kappa}{2}\beta^{-3/2}\left(1+\frac{\theta}{2}\right),\\
\nabla_3 J_{21} &= -\Gamma^2_{32} J_{21} - \Gamma^e_{31} J_{2e} = 0,\\
\nabla_3 J_{23} &= -\Gamma^4_{32} J_{43} - \Gamma^1_{33} J_{21} = \kappa\Gamma^4_{32} = -\frac{\kappa}{2}\beta^{=1},\\
\nabla_3 J_{24} &= -\Gamma^3_{32} J_{24} - \Gamma^1_{34} J_{21} = -\kappa\Gamma^3_{32} = -\kappa(-\frac{1}{4}\beta^{-1}) = \frac{\kappa}{4}\beta^{-1},\\
\nabla_3 J_{31} &= -\Gamma^2_{33} J_{21} - \Gamma^e_{31} J_{3e} = \Gamma^2_{33} = \frac{\kappa}{4}\beta^{-3/2}(1-2\theta),\\
\nabla_3 J_{32} &= -\Gamma^1_{33} J_{12} - \Gamma^4_{32} J_{34} = -\Gamma^4_{32} J_{34} = -\kappa\Gamma^4_{32}, = -\kappa(-\frac{1}{2}\beta^{-1}) = \frac{\kappa}{2}\beta^{-1},\\
\nabla_3 J_{34} &= -\Gamma^3_{33} J_{34} - \Gamma^4_{43} J_{34} = 0,\\
\nabla_3 J_{41} &= -\Gamma^2_{34} J_{21} - \Gamma^e_{31} J_{4e} = \Gamma^2_{34} = \frac{\kappa}{2}\beta^{-3/2}\left(1+\frac{\theta}{2}\right),\\
\nabla_3 J_{42} &= -\Gamma^1_{34} - \Gamma^3_{32} J_{43} = -\Gamma^3_{32}(-\kappa) = \kappa\Gamma^3_{32} = -\frac{\kappa}{4}\beta^{-1},\\
\nabla_3 J_{43} &= -\Gamma^4_{34} J_{43} - \Gamma^3_{33} J_{43} = 0,
\end{aligned}$$

(iii) Since $\nabla_4 J_{ab} = -\Gamma^e_{4a} J_{eb} - \Gamma^e_{4b} J_{ae}$, we have

$$\begin{aligned}
\nabla_4 J_{12} &= -\Gamma^2_{43} = -\frac{\kappa}{2}\beta^{-3/2}\left(1+\frac{\theta}{2}\right),\\
\nabla_4 J_{14} &= -\Gamma^2_{44} J_{12} = -(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))\\
&= \frac{\kappa}{4}\beta^{-3/2}(1+3\theta),\\
\nabla_4 J_{21} &= -\Gamma^2_{42} J_{21} - \Gamma^e_{41} J_{2e} = 0,\\
\nabla_4 J_{23} &= -\Gamma^4_{42} J_{43} - \Gamma^1_{43} J_{21} = -\Gamma^4_{42}(-\kappa) = \kappa\Gamma^4_{42} = \frac{\kappa}{4}\beta^{-1},\\
\nabla_4 J_{24} &= -\Gamma^3_{42} J_{34} - \Gamma^1_{44} J_{21} = -\kappa\Gamma^3_{42} = \frac{\kappa}{2}\beta^{-1},
\end{aligned}$$

$$\begin{aligned}
\nabla_4 J_{31} &= -\Gamma_{43}^2 J_{21} - \Gamma_{41}^e J_{3e} = \Gamma_{43}^2 = \frac{\kappa}{2}\beta^{-3/2}\left(1+\frac{\theta}{2}\right),\\
\nabla_4 J_{32} &= -\Gamma_{43}^1 J_{12} - \Gamma_{42}^4 J_{34} = \kappa\Gamma_{42}^4 = -\frac{\kappa}{4}\beta^{-1},\\
\nabla_4 J_{34} &= -\Gamma_{43}^3 J_{34} - \Gamma_{44}^4 J_{34} = 0,\\
\nabla_4 J_{41} &= -\Gamma_{44}^2 J_{21} = -\frac{\kappa}{4}\beta^{-3/2}(1+3\theta),\\
\nabla_4 J_{42} &= -\Gamma_{42}^3 J_{43} = -\Gamma_{42}^3(-\kappa) = \kappa\Gamma_{42}^3 = -\frac{\kappa}{2}\beta^{-1},\\
\nabla_4 J_{43} &= -\Gamma_{44}^2 J_{23} - \Gamma_{43}^3 J_{43} = 0.
\end{aligned}$$

□

1.5. **The computation of (1.1). First steps.** The goal is to show

$$S := \sum_{\sigma\in\mathfrak{S}_5} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0\ell_1 0}{}^{n}\overline{R}_{\sigma_1\sigma_2\ell_2}{}^{\ell_1}\overline{R}_{\sigma_3\sigma_4 n}{}^{\ell_2} = \frac{\kappa}{16}p^2(3072p^4 - 640p^2\beta^{-2} - 25\beta^{-4}), \quad (1.3)$$

(see (4.8)).

For $i \in \{0,1,2,3,4\}$, we set

$$S_i = \sum_{\substack{\sigma\in\mathfrak{S}_5\\ \sigma_i=0}} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0\ell_1 0}{}^{n}\overline{R}_{\sigma_1\sigma_2\ell_2}{}^{\ell_1}\overline{R}_{\sigma_3\sigma_4 n}{}^{\ell_2},$$

so

$$S = \sum_{i=0}^{4} S_i.$$

Interchanging $\sigma_1$ and $\sigma_2$ gives $S_1 = S_2$, since both $\operatorname{sgn}(\sigma)$ and a curvature term change sign, and interchanging $\sigma_3$ and $\sigma_4$ gives $S_3 = S_4$. Thus,

$$S = S_0 + 2S_1 + 2S_3. \quad (1.4)$$

For $m = 0,1,3$, we set

$$S_m = S_{m,2} + S_{m,4} + S_{m,6},$$

where $S_{m,k}$ is the $p^k$-term of $S_m$. In this notation, (1.3) is equivalent to:

**Proposition 1.5.**

$$\begin{aligned}
\bar{S}_{[2]} &:= S_{0,2} + 2S_{1,2} + 2S_{3,2} = \frac{\kappa}{16}p^2(-25\beta^{-4}),\\
\bar{S}_{[4]} &:= S_{0,4} + 2S_{1,4} + 2S_{3,4} = \frac{\kappa}{16}(-640p^2\beta^{-2}),\\
\bar{S}_{[6]} &:= S_{0,6} + 2S_{1,6} + 2S_{3,6} = \frac{\kappa}{16}3072p^6.
\end{aligned}$$

This will be proved in a number of steps. $\bar{S}_{[6]}$ is computed in Prop. 1.9, $\bar{S}_{[2]}$ is computed in Prop. 1.11, and $\bar{S}_{[4]}$ is computed in Prop. 1.13.

For notation, the metric $\bar{g}$ on $\bar{M}$ is given by

$$\bar{g} = (\bar{g}_{ij}) = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & \bar{g}_{ab} & & \\ 0 & & & & \end{array}\right)$$

where $i, j, k, \ldots \in \{0, 1, 2, 3, 4\}$ and $a, b, \ldots \in \{1, 2, 3, 4\}$. Thus $\bar{g}_{00} = \bar{g}(\partial_0, \partial_0), \bar{g}_{ab} = \bar{g}(\partial^L_{\theta_a}, \partial^L_{\theta_b}) = g(\partial_{\theta_a}, \partial_{\theta_b})$.

We now compute $S_0$, using the formulas in Lemma 1.1. In the obvious notation,

$$\begin{aligned} S_0 &= \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \overline{R}_{0a_10}{}^{b} \overline{R}_{\sigma_1\sigma_2\ell_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 b}{}^{\ell_2} \\ &= \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \overline{R}_{0a_10}{}^{b} [\overline{R}_{\sigma_1\sigma_2a_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 b}{}^{a_2} + \overline{R}_{\sigma_1\sigma_2 0}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 b}{}^{0}]. \end{aligned}$$

Since $\overline{R}_{0a_10}{}^{b} = -p^2 \delta^b_{a_1}$, we get

$$\begin{aligned} S_0 &= -p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) [\overline{R}_{\sigma_1\sigma_2a_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} + \overline{R}_{\sigma_1\sigma_2 0}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{0}] \\ &= -p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) [\overline{R}_{\sigma_1\sigma_2a_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} - \overline{R}_{\sigma_1\sigma_2 a_1'}{}^{0} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{0} g^{a_1a_1'}] \\ &= -p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) [\overline{R}_{\sigma_1\sigma_2a_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} - \overline{R}_{\sigma_1\sigma_2 a_1'}{}^{0} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{0} g^{a_1a_1'}] \\ &= -p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \overline{R}_{\sigma_1\sigma_2a_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} + p^4 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \nabla_{a_1'} J_{\sigma_1\sigma_2} \nabla_{a_1} J_{\sigma_3\sigma_4} g^{a_1a_1'} \\ &:= S_{0,*} + S_{0,4,1}. \end{aligned}$$

We will see below (Lemma 1.14) that

$$S_{0,4,1} = -S_{1,4,1} = 5p^4\kappa\beta^{-2}. \tag{1.5}$$

(For the definition of $S_{1,4,1}$, see (1.13). The first equality follows from a "change of variables" sending $\sigma$ to $(01)\sigma$.) For $S_{0,*}$, we have

$$\begin{aligned} S_{0,*} &= -p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \overline{R}_{\sigma_1\sigma_2a_2}{}^{a_1} \overline{R}_{\sigma_3\sigma_4 a_1}{}^{a_2} \\ &= -p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) (R_{\sigma_1\sigma_2a_2}{}^{a_1} - p^2 J_{\sigma_2a_2} J_{\sigma_1}{}^{a_1} + p^2 J_{\sigma_1a_2} J_{\sigma_2}{}^{a_1} + 2p^2 J_{\sigma_2\sigma_2} J_{a_2}{}^{a_1}) \\ &\qquad \cdot (R_{\sigma_3\sigma_4a_1}{}^{a_2} - p^2 J_{\sigma_4a_1} J_{\sigma_3}{}^{a_2} + p^2 J_{\sigma_3a_1} J_{\sigma_4}{}^{a_2} + 2p^2 J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}) \end{aligned}$$

$$
\begin{aligned}
= -p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[&(R_{\sigma_1\sigma_2 a_2}{}^{a_1} R_{\sigma_3\sigma_4 a_1}{}^{a_2}) \\
&+ p^2(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1}) R_{\sigma_3\sigma_4 a_1}{}^{a_2} \\
&+ p^2 R_{\sigma_1\sigma_2 a_2}{}^{a_1}(-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_3}{}^{a_1} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}) \\
&+ p^4(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1})] \\
&\cdot (-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_3}{}^{a_1} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}).
\end{aligned}
$$

The $p^4$ term in $S_{0,*}$ is

$$
\begin{aligned}
S_{0,4,2} := -p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[&(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1}) R_{\sigma_3\sigma_4 a_1}{}^{a_2} \\
&+ R_{\sigma_1\sigma_2 a_2}{}^{a_1}(-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_3}{}^{a_1} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2})].
\end{aligned}
$$

We can simplify $S_{0,4,2}$ using the change of variable $\sigma \mapsto \tau = \sigma(13)(24)$ as in the calculation of $(\beta)$ in Appendix B:

$$
\begin{aligned}
S_{0,4,2} &= -p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1}) R_{\sigma_3\sigma_4 a_1}{}^{a_2} \\
&\qquad + R_{\sigma_1\sigma_2 a_2}{}^{a_1}(-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2})] \qquad (1.6) \\
&= -p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[-R_{\sigma_3\sigma_4 a_1 a_2} J_{\sigma_2}{}^{a_2} J_{\sigma_1}{}^{a_1} + R_{\sigma_3\sigma_4 a_2 a_1} J_{\sigma_1}{}^{a_1} J_{\sigma_2}{}^{a_2} - 2R_{\sigma_3\sigma_4 a_1 a_2} J_{\sigma_1\sigma_2} J^{a_1 a_2} \\
&\qquad - R_{\sigma_1\sigma_2 a_2 a_1} J_{\sigma_4}{}^{a_1} J_{\sigma_3}{}^{a_2} + R_{\sigma_1\sigma_2 a_1 a_2} J_{\sigma_3}{}^{a_1} J_{\sigma_4}{}^{a_2} - 2R_{\sigma_1\sigma_2 a_2 a_1} J_{\sigma_3\sigma_4} J^{a_1 a_2}] \\
&= 4p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[R_{\sigma_1\sigma_2 a_1 a_2} J_{\sigma_3}{}^{a_1} J_{\sigma_4}{}^{a_2} + R_{\sigma_1\sigma_2 a_1 a_2} J^{a_1 a_2} J_{\sigma_3\sigma_4}] \\
&:= S_{0,4,2,1} + S_{0,4,2,2}.
\end{aligned}
$$

where

$$
\begin{aligned}
S_{0,4,2,1} &= 4p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) R_{\sigma_1\sigma_2 a_1 a_2} J_{\sigma_3}{}^{a_1} J_{\sigma_4}{}^{a_2}, \qquad (1.7) \\
S_{0,4,2.2} &= 4p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) R_{\sigma_1\sigma_2 a_1 a_2} J^{a_1 a_2} J_{\sigma_3\sigma_4}.
\end{aligned}
$$

Here is a summary of $S_0$, obtained from (1.5), (1.6), (1.7).

**Lemma 1.6.**

$$
\begin{aligned}
S_{0,2} &= -p^2 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) R_{\sigma_1\sigma_2 a_2}{}^{a_1} R_{\sigma_3\sigma_4 a_1}{}^{a_2}, \\
S_{0,4} &= p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma) \nabla_{a_1'} J_{\sigma_1\sigma_2} \nabla_{a_1} J_{\sigma_3\sigma_4} g^{a_1 a_1'} \\
&\qquad - 4p^4 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)[R_{\sigma_1\sigma_2 a_1 a_2} J_{\sigma_3}{}^{a_1} J_{\sigma_4}{}^{a_2} + R_{\sigma_1\sigma_2 a_1 a_2} J^{a_1 a_2} J_{\sigma_3\sigma_4}] \qquad (1.8)
\end{aligned}
$$

$$S_{0,6} = -p^6 \sum_{\sigma_0=0} \operatorname{sgn}(\sigma)(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1}$$
$$\cdot (-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_3}{}^{a_1} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}).$$

We now turn to the computation of $S_1$.

$$\begin{aligned}
S_1 &= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\overline{R}_{\sigma_0\ell_1 0}{}^{n}\overline{R}_{0\sigma_2\ell_2}{}^{\ell_1}\overline{R}_{\sigma_3\sigma_4 n}{}^{\ell_2} = -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\bar{R}_{\sigma_0\ell_1 0b'}\overline{R}_{0\sigma_2\ell_2}{}^{\ell_1}\overline{R}_{\sigma_3\sigma_4 b}{}^{\ell_2} g^{bb'}\\
&= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\bar{R}_{\sigma_0 00b'}\overline{R}_{0\sigma_2 a}{}^{0}\overline{R}_{\sigma_3\sigma_4 b}{}^{a} g^{bb'} - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\bar{R}_{\sigma_0 a_1 0b'}\overline{R}_{0\sigma_2\ell_2}{}^{a_1}\overline{R}_{\sigma_3\sigma_4 b}{}^{\ell_2} g^{bb'}\\
&= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma)(p^2 g_{\sigma_0 b'})(p^2 g_{\sigma_2 a_2})\overline{R}_{\sigma_3\sigma_4 b}{}^{a_2} g^{bb'} - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\bar{R}_{\sigma_0 a_1 0b'}\bar{R}_{0\sigma_2 0a_1'}\bar{R}_{\sigma_3\sigma_4 b0} g^{bb'} g^{a_1 a_1'}\\
&\quad - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)\bar{R}_{\sigma_0 a_1 0b'}\bar{R}_{0\sigma_2 a_2 a_1'}\overline{R}_{\sigma_2\sigma_4 b}{}^{a_2} g^{bb'} g^{a_1 a_1'}\\
&= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma)p^4 R_{\sigma_3\sigma_4\sigma_0\sigma_2} - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)(-p\nabla_{b'} J_{\sigma_0 a_1})(-p^2 g_{\sigma_2 a_2'})(p\nabla_b J_{\sigma_3\sigma_4}) g^{bb'} g^{a_1 a_1'}\\
&\quad - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)(-p\nabla_{b'} J_{\sigma_0 a_1})(-p^2\nabla_{\sigma_2} J_{a_2 a_1'})\overline{R}_{\sigma_3\sigma_4 b}{}^{a_2} g^{bb'} g^{a_1 a_1'}.
\end{aligned}$$

This can be simplified using the Bianchi identity

$$\sum_{\sigma_1=0} \operatorname{sgn}(\sigma) R_{\sigma_3\sigma_4\sigma_0\sigma_2} = 0,$$

to give

$$\begin{aligned}
S_1 &= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma)p^4 g^{bb'}\nabla_{b'} J_{\sigma_0\sigma_2}\nabla_b J_{\sigma_3\sigma_4}\\
&\quad - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)p^2\nabla_{b'} J_{\sigma_0 a_1}\nabla_{\sigma_2} J_{a_2 a_1'}\bar{R}_{\sigma_3\sigma_4 b a_2'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'}\\
&= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma)p^4 g^{bb'}\nabla_{b'} J_{\sigma_0\sigma_2}\nabla_b J_{\sigma_3\sigma_4}\\
&\quad - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)p^2\nabla_{b'} J_{\sigma_0 a_1}\nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3\sigma_4 b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'}\\
&\quad - \sum_{\sigma_1=0} \operatorname{sgn}(\sigma)p^4\nabla_{b'} J_{\sigma_0 a_1}\nabla_{\sigma_2} J_{a_2 a_1'}\\
&\qquad \cdot (-J_{\sigma_4 b} J_{\sigma_3 a_2'} + J_{\sigma_3 b} J_{\sigma_4 a_2'} + 2J_{\sigma_3\sigma_4} J_{b a_2'}) g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}\\
&= -\sum_{\sigma_1=0} \operatorname{sgn}(\sigma)p^4 g^{bb'}\nabla_b J_{\sigma_0\sigma_2}\nabla_{b'} J\sigma_3\sigma_4
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_4 b} J_{\sigma_3 a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'} \\
&- \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_3 b} J_{\sigma_4 a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'} \\
&- 2 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{sigma_2} J_{a_2 a_1'} J_{\sigma_3 \sigma_4} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'} \\
&- \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^2 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3 \sigma_4 b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'} \\
= &- \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 g^{bb'} \nabla_b J_{\sigma_0 \sigma_2} \nabla_{b'} J_{\sigma_3 \sigma_4} \\
&+ 2 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_4 b} J_{\sigma_3 a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'} \\
&- 2 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) p^4 \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_3 \sigma_4} J_{b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'} \\
&- p^2 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3 \sigma_4 b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'}.
\end{aligned}
$$

In summary,

**Lemma 1.7.**

$$S_1 = S_{1,2} + S_{1,4},$$

*where*

$$
\begin{aligned}
S_{1,2} &= -p^2 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3 \sigma_4 b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'}, \\
S_{1,4} &= -p^4 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) g^{bb'} \nabla_b J_{\sigma_0 \sigma_2} \nabla_{b'} J_{\sigma_3 \sigma_4} \\
&\quad + 2p^4 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_4 b} J_{\sigma_3 a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'} \\
&\quad - 2p^4 \sum_{\sigma_1=0} \operatorname{sgn}(\sigma) \nabla_{b'} J_{\sigma_0 a_1} \nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_3 \sigma_4} J_{b a_2'} g^{a_2 a_2'} g^{bb'} g^{a_1 a_1'}.
\end{aligned}
$$

We now compute $S_3$. Using $\overline{R}_{\sigma_0 \ell_1 0}{}^0 = 0$, we obtain

$$
\begin{aligned}
S_3 &= -\sum_{\sigma_3=0} \operatorname{sgn}(\sigma) \overline{R}_{\sigma_0 \ell_1 0}{}^n \overline{R}_{\sigma_1 \sigma_2 \ell_2}{}^{\ell_1} \overline{R}_{0 \sigma_4 n}{}^{\ell_2} = -\sum_{\sigma_3=0} \operatorname{sgn}(\sigma) \overline{R}_{\sigma_0 \ell_1 0}{}^b \overline{R}_{\sigma_1 \sigma_2 \ell_2}{}^{\ell_1} \overline{R}_{0 \sigma_4 b}{}^{\ell_2} \\
&= -\sum_{\sigma_3=0} \operatorname{sgn}(\sigma) \overline{R}_{\sigma_0 0 0}{}^b \overline{R}_{\sigma_1 \sigma_2 a_2}{}^0 \overline{R}_{0 \sigma_4 b}{}^{a_2} - \sum_{\sigma_3=0} \operatorname{sgn}(\sigma) \overline{R}_{\sigma_0 a_1 0}{}^b \overline{R}_{\sigma_1 \sigma_2 \ell_2}{}^{a_1} \overline{R}_{0 \sigma_4 b}{}^{\ell_2}
\end{aligned}
$$

$$= -\sum_{\sigma_3=0} \mathrm{sgn}(\sigma)\overline{R}_{\sigma_0 00}{}^{b}\overline{R}_{\sigma_1\sigma_2 a_2}{}^{0}\overline{R}_{0\sigma_4 b}{}^{a_2} - \sum_{\sigma_3=0} \mathrm{sgn}(\sigma)\overline{R}_{\sigma_0 a_1 0}{}^{b}\overline{R}_{\sigma_1\sigma_2 0}{}^{a_1}\overline{R}_{0\sigma_4 b}{}^{0}$$
$$-\sum_{\sigma_3=0} \mathrm{sgn}(\sigma)\overline{R}_{\sigma_0 a_1 0}{}^{b}\overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1}\overline{R}_{0\sigma_4 b}{}^{a_2}.$$

Since

$$\overline{R}_{0\sigma_4 b}{}^{a_2} = \bar{R}_b{}^{a_2}{}_0{}^{\sigma_4} = -\bar{R}_b{}^{a_2}{}_{\sigma_4 0},\ \overline{R}_{0\sigma_4 b}{}^{a_2} = -\bar{R}_b{}^{a_2}{}_{\sigma_4 0},$$

this becomes

$$S_3 = -\sum_{\sigma_3=0} \mathrm{sgn}(\sigma)(p^2\delta^b_{\sigma_0})(p\nabla_{a_2}J_{\sigma_1\sigma_2})(-p\nabla_{\sigma_4}J_b{}^{a_2}) - (-p^2\nabla_{\sigma_4}J_{\sigma_0 a_1})(-p\nabla^{a_1}J_{\sigma_1\sigma_2})(p^2 g_{\sigma_4 b})$$
$$-\sum_{\sigma_3=0} \mathrm{sgn}(\sigma)(-p\nabla^b J_{\sigma_0 a_1})\overline{R}_{\sigma_1\sigma_2 a_2}{}^{a_1}(-p\nabla_{\sigma_4}J_b{}^{a_2}).$$

This further simplifies, using

$$\sum_{\sigma_3=0} \mathrm{sgn}(\sigma)p^4\nabla_{a_2}J_{\sigma_1\sigma_2}\nabla_{\sigma_4}J_{\sigma_0}{}^{a_2} - \sum_{\sigma_3=0} \mathrm{sgn}(\sigma)p^4\nabla_{\sigma_4}J_{\sigma_0 a_2}\nabla^{a_2}J_{\sigma_1\sigma_2} = 0,$$

to give

**Lemma 1.8.**

$$\begin{aligned} S_3 &= S_{3,2} + S_{3,4} \\ &= -\sum_{\sigma_3=0} \mathrm{sgn}(\sigma)p^2\nabla^b J_{\sigma_0 a_1}\nabla_{\sigma_4}J_b{}^{a_2}R_{\sigma_1\sigma_2 a_2}{}^{a_1} \\ &\quad -\sum_{\sigma_3=0} \mathrm{sgn}(\sigma)p^4\nabla^b J_{\sigma_0 a_1}\nabla_{\sigma_4}J_b{}^{a_2}(-J_{\sigma_2 a_2}J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2}J_{\sigma_2}{}^{a_1} + 2J\sigma_1\sigma_2 J_{a_2}{}^{a_1}) \\ &= -\sum_{\sigma_3=0} \mathrm{sgn}(\sigma)p^2\nabla^b J_{\sigma_0 a_1}\nabla_{\sigma_4}J_b{}^{a_2}R_{\sigma_1\sigma_2 a_2}{}^{a_1} \\ &\quad + 2p^4\sum_{\sigma_3=0} \mathrm{sgn}(\sigma)\nabla^b J_{\sigma_0 a_1}\nabla_{\sigma_4}J_b{}^{a_2}[J_{\sigma_2 a_2}J_{\sigma_1}{}^{a_1} - J_{\sigma_1\sigma_2}J_{a_2}{}^{a_1}]. \end{aligned}$$

The last equality follows from the change of variable $\sigma \mapsto \sigma(12)$.

1.6. **The $p^6$ term in (1.3).** We now compute the $p^6$ term in (1.3), or equivalently the third term in Prop. 1.5. Since $S_{1,6} = S_{3,6} = 0$ by Lemmas 1.7, 1.8, respectively, we need only compute $S_{0,6}$.

**Proposition 1.9.**

$$\bar{S}_{[6]} = S_{0,6} = 192p^6\kappa.$$

*Proof.* We have

$S_{0,6}$

$$= -p^6 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma)(-J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1})$$
$$\cdot (-J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} + 2J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2})$$
$$= -p^6 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma)[J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} - J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} - 2J_{\sigma_2 a_2} J_{\sigma_1}{}^{a_1} J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}$$
$$- J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} + 2J_{\sigma_1 a_2} J_{\sigma_2}{}^{a_1} J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}$$
$$- 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1} J_{\sigma_4 a_1} J_{\sigma_3}{}^{a_2} + 2J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1} J_{\sigma_3 a_1} J_{\sigma_4}{}^{a_2} + 4J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1} J_{\sigma_3\sigma_4} J_{a_1}{}^{a_2}].$$

Using (**??**), we get

$$S_{0,6} = -p^6 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma)(-2\cdot 4 - 16) J_{\sigma_1\sigma_2} J_{\sigma_3\sigma_4} = p^6 \cdot 24 \cdot 2^3 \kappa = 192 p^6 \kappa.$$

□

1.7. **The $p^2$ term in (1.3).** We now compute

$$\bar{S}_{[2]} = S_{0,2} + 2S_{1,2} + 2S_{3,2}.$$

We first note that there is no contribution from $S_{0,2}$.

**Lemma 1.10.**

$$S_{0,2} = 0.$$

*Proof.* By Lemmas 1.3(i) and 1.6, we get

$$S_{0,2} = \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) R_{\sigma_1\sigma_2 a_2}{}^{b} R_{\sigma_3\sigma_4 b}{}^{a_2} = 0,$$

since every permutation has $\sigma_a = 1$ for some $a$. □

Note that the middle term in the last equation is a multiple of the first Pontryagin form, which implies that the signature of $M$ is zero.

**Proposition 1.11.**

$$\bar{S}_{[2]} = 2(S_{1,2} + S_{3,2}) = -\frac{25}{16}\kappa\beta^{-4} \tag{1.9}$$

*Proof.* We have

$$S_{1,2} + S_{3,2} = -p^2 \sum_{\sigma_1=0} \mathrm{sgn}(\sigma) \nabla_{b'} J_{\sigma_0 a_1'} \nabla_{\sigma_2} J_{a_2 a_1'} R_{\sigma_3\sigma_4 b a_2'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'}$$
$$- p^2 \sum_{\sigma_3=0} \mathrm{sgn}(\sigma) \nabla^b J_{\sigma_0 a_1} \nabla_{\sigma_4} J_b{}^{a_2} R_{\sigma_1\sigma_2 a_2}{}^{a_1}$$
$$= p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \nabla_{a_1} J_{\sigma_1 b} \nabla_{\sigma_2} J_{a_2 b'} R_{\sigma_3\sigma_4 a_2' a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'}$$
$$+ p^2 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \nabla_{b'} J_{\sigma_1 a_1} \nabla_{\sigma_2} J_{a_2' b} R_{\sigma_3\sigma_4 a_2 a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'},$$

where we use the change of variable $\sigma \mapsto (01)\sigma$ on the third line and $\sigma \mapsto (301)(24)\sigma$ on the fourth line. We know

$$(\sigma_3, \sigma_4) \in \{(2,3), (3,2), (2,4), (4,2), (3,4), (4,3)\}.$$

**Case (1)**: $(\sigma_3, \sigma_4) = (2,3)$ or $(3,2)$

In this case, in cycle notation

$$\sigma \in \{(243), (1432), (24), (142)\},$$

with signs $+1, -1, -1, +1$, respectively. There is no contribution to $S_{1,2} + S_{3,2}$ if $\sigma = (1432)$ or $\sigma = (142)$, since $\nabla_1 J_{ab} = 0$. Thus for $(S_{1,2} + S_{3,2})(1)$ the contribution to $S_{2,2} + S_{4,2}$ from Case (1), we get

$$\begin{aligned}(S_{1,2} + S_{3,2})(1) &= 2p^2 \nabla_{a_1} J_{1b} \nabla_4 J_{a_2 b'} R_{23a_2' a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'} \\ &\quad + 2\nabla_{b'} J_{1a_1} \nabla_4 J_{a_2' b} R_{23a_2 a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'} \\ &= 2p^2 \nabla_{a_1} J_{1b} \nabla_4 J_{a_2 b'} R_{23a_2' a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'} \\ &\quad + 2p^2 \nabla_{a_1} J_{1b'} \nabla_4 J_{a_2 b} R_{23a_2 a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'} \\ &= 4p^2 \nabla_{a_1} J_{1b} \nabla_4 J_{a_2 b} R_{23a_2' a_1'} g^{a_1 a_1'} g^{bb'} g^{a_2 a_2'}.\end{aligned}$$

Since $\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0$, we must have $a_1 = 3$ or $4$, and so

$$\begin{aligned}(S_{2,2} + S_{4,2})(1) &= 4p^2 \nabla_3 J_{1b} \nabla_4 J_{a_2 b'} R_{23a_2' b'} g^{3a_1'} g^{bb'} g^{a_2 a_2'} \\ &\quad + 4p^2 \nabla_4 J_{1b} \nabla_4 J_{a_2 b'} R_{23a_2' a_1'} g^{4a_1'} g^{bb'} g^{a_2 a_2'} \\ &= 4p^2 \nabla_3 J_{1b} \nabla_4 J_{2b'} R_{232}{}^3 g^{bb'} \\ &\quad + 4p^2 \nabla_4 J_{1b} \nabla_4 J_{2b'} R_{232}{}^4 g^{bb'}.\end{aligned}$$

In summary,

$$(S_{2,2} + S_{4,2})(1) = 4p^2 (\nabla_3 J_{1b'} \nabla_4 J_{2b} R_{232}{}^3 g^{bb'} + \nabla_4 J_{1b'} \nabla_4 J_{2b} R_{232}{}^4 g^{bb'}). \tag{1.10}$$

**Case (2)**: $(\sigma_3, \sigma_4) = (2,4)$ or $(4,2)$

Now

$$\sigma \in \{(23), (234), (132), (1342)\}$$

with signs $-1, +1, +1, -1$, respectively. There is no contribution to $S_{2,2} + S_{4,2}$ if $\sigma = (132)$ or $\sigma = (1342)$, since $\nabla_1 J_{ab} = 0$. Thus for $(S_{2,2} + S_{4,2})(2)$ in the notation of Case (1), we get

$$\begin{aligned}(S_{1,2} + S_{3,2})(2) &= -2(\nabla_{a_1} J_{1b'} \nabla_3 J_{a_2' b} R_{24a_2}{}^{a_1} g^{bb'} g^{a_2 a_2'} \\ &\quad + \nabla_{b'} J_{1a_1} \nabla_3 J_{a_2' b} R_{24a_2}{}^{a_1} g^{bb'} g^{a_2 a_2'}).\end{aligned}$$

As above, we only get a nonzero contribution to this equation if $a_1 = 3$ or $4$. Thus, we get

$$\begin{aligned}(S_{1,2}+S_{3,2})(2) &= -2(\nabla_3 J_{1b'}\nabla_3 J_{a_2'b}R_{24a_2}{}^3 g^{bb'}g^{a_2a_2'} + \nabla_4 J_{1b'}\nabla_3 J_{a_2'b}R_{24a_2}{}^4 g^{bb'}g^{a_2a_2'} \\ &\qquad + \nabla_{b'}J_{13}\nabla_3 J_{a_2b}R_{24a_2}{}^3 g^{bb'}g^{a_2a_2'} + \nabla_{b'}J_{14}\nabla_3 J_{a_2b}R_{24a_2}{}^3 g^{bb'}g^{a_2a_2'}) \\ &= -4(\nabla_3 J_{1b'}\nabla_3 J_{2b}R_{242}{}^3 g^{bb'} + \nabla_4 J_{1b'}\nabla_3 J_{2b}R_{242}{}^4 g^{bb'}), \qquad (1.11)\end{aligned}$$

because only $a_2$ and $a_2'$ remain.

**Case (3)**: $(\sigma_3,\sigma_4) = (3,4)$ or $(4,3)$

Since $\sigma_2 = 1$ or $2$, we have $\nabla_{\sigma_2}J_{a_2b'} = 0$. Thus,

$$(S_{1,2}+S_{3,2})(3) = 0.$$

In summary, at this point we have

$$\frac{1}{2}\bar{S}_{[2]} = S_{1,2}+S_{3,2} = (S_{1,2}+S_{3,2})(1) + (S_{1,2}+S_{3,2})(2). \qquad (1.12)$$

We continue by simplifying the explicit expressions in (1.10), (1.11).

**Lemma 1.12.** *(i)* $\nabla_4 J_{1b'}\nabla_4 J_{2b}g^{bb'} = 0$ *and* $\nabla_3 J_{1b'}\nabla_3 J_{2b}g^{bb'} = 0$.

*(ii)*

$$\begin{aligned}\nabla_3 J_{1b'}\nabla_4 J_{2b}g^{bb'}R_{232}{}^3 &= -\frac{5\kappa}{4^4}\beta^{-4}(9-8\theta), \\ \nabla_4 J_{1b'}\nabla_3 J_{2b}g^{bb'}R_{232}{}^4 &= \frac{5\kappa}{4^4}\beta^{-4}(1+8\theta).\end{aligned}$$

*(iii)* $\nabla_3 J_{1b'}\nabla_4 J_{2b}g^{bb'}R_{232}{}^3 - \nabla_4 J_{1b'}\nabla_3 J_{2b}g^{bb'}R_{242}{}^4 = -\frac{50}{4^4}\kappa\beta^{-4}$.

*Proof.* (i) By Lem. 1.4,

$$\begin{aligned}&\nabla_4 J_{1b'}\nabla_4 J_{2b}g^{bb'} \\ &\quad = \nabla_4 J_{13} + \nabla_4 J_{23}g^{33} + \nabla_4 J_{13}\nabla_4 J_{24}g^{34} + \nabla_4 J_{14}\nabla_4 J_{23}g^{43} + \nabla_4 J_{14} + \nabla_4 J_{24}g^{44} \\ &\quad = (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2}) \\ &\qquad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2}) \\ &\qquad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2}) \\ &\qquad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2})) \\ &\quad = \kappa\beta^{-3}(-\frac{1}{8}(1+\frac{\theta}{2})(1+\theta) - \frac{1}{4}(1+\frac{\theta}{2})\theta + \frac{1}{16}(1+3\theta)\theta + \frac{1}{8}(1+3\theta)) \\ &\quad = \frac{\kappa}{16}\beta^{-3}(-2(1+\frac{3\theta}{2}+\frac{1}{2}\theta^2) - 4\theta - 2\theta^2 + \theta + 3\theta^2 + 2 + 6\theta)\end{aligned}$$

$$\begin{aligned}
&= \frac{\kappa}{16}\beta^{-3}(-2-3\theta-\theta^2-4\theta-2\theta^2+\theta+3\theta^2+2+6\theta)\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
&\nabla_3 J_{1b'}\nabla_3 J_{2b} g^{bb'}\\
&\quad = \nabla_3 J_{13} + \nabla_3 J_{23} g^{33} + \nabla_3 J_{13} + \nabla_3 J_{24} g^{34} + \nabla_3 J_{14}\nabla_3 J_{23} g^{43} + \nabla_3 J_{14}\nabla_3 J_{24} g^{44}\\
&\quad = (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(-\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
&\quad = \kappa\beta^{-3}(\frac{1}{8}(1-2\theta)(1+\theta) - \frac{1}{16}(1-2\theta)\theta + \frac{\theta}{4}(1+\frac{\theta}{2}) - \frac{1}{8}(1+\frac{\theta}{2}))\\
&\quad = \frac{\kappa}{16}\beta^{-3}(2(1-2\theta)(1+\theta) - \theta(1-2\theta) + 4\theta(1+\frac{\theta}{2}) - 2(1+\frac{\theta}{2}))\\
&\quad = \frac{\kappa}{16}\beta^{-3}(2(1-\theta-2\theta^2) - \theta + 2\theta^2 + 4\theta + 2\theta^2 - 2 - \theta)\\
&\quad = \frac{\kappa}{4}(2-2\theta-4\theta^2-\theta+2\theta^2+4\theta+2\theta^2-2-\theta)\\
&\quad = 0.
\end{aligned}$$

(ii) From

$$\begin{aligned}
&\nabla_3 J_{1b'}\nabla_4 J_{2b} g^{bb'}\\
&\quad = \nabla_3 J_{13}\nabla_4 J_{23} g^{33} + \nabla_3 J_{14}\nabla_4 J_{24} g^{43} + \nabla_3 J_{13}\nabla_4 J_{24} g^{34} + \nabla_3 J_{14}\nabla_4 J_{24} g^{44}\\
&\quad = (-\frac{\kappa}{4}(1-2\theta))(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1})\\
&\qquad + (-\frac{\kappa}{2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad + (-\frac{\kappa}{2}(1+\frac{\theta}{2}))(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
&\quad = \kappa\beta^{-3}(1\frac{1}{16}(1-2\theta)(1+\theta) - \frac{1}{8}(1+\theta)\theta - \frac{1}{8}\theta(1-2\theta) + \frac{1}{4}(1+\frac{\theta}{2}))\\
&\quad = -\frac{\kappa}{16}\beta^{-1}((1-2\theta)(1+\theta) + 2(1+\frac{\theta}{2})\theta + 2\theta(1-2\theta) + 4(1+\frac{\theta}{2}))\\
&\quad = -\frac{5}{16}\kappa\beta^{-3}(1+\theta-\theta^2)
\end{aligned}$$

$$= -\frac{5}{16}\kappa\beta^{-2},$$

we get

$$\nabla_3 J_{1b'}\nabla_4 J_{2b} g^{bb'} R_{232}{}^3 = -\frac{5}{16}\kappa\beta^{-2}\frac{1}{16}\beta^{-2}(9-8\theta) = -\frac{5}{4^4}\kappa\beta^{-4}(9-8\theta).$$

Similarly, from

$$\begin{aligned}
&\nabla_4 J_{1b'}\nabla_3 J_{2b} g^{bb'} \\
&= \nabla_4 J_{13}\nabla_3 J_{23} g^{33} + \nabla_4 J_{14}\nabla_3 J_{23} g^{43} + \nabla_4 J_{13}\nabla_3 J_{24} g^{43} + \nabla_4 J_{14}\nabla_3 J_{24} g^{44} \\
&= (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(-\frac{\kappa}{2})(\kappa^{-1}(1+\theta)\beta^{-1/2} \\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2}) \\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2}) \\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2}) \\
&= \kappa\beta{-}3(\frac{1}{4}(1+\frac{\theta}{2})(1+\theta) - \frac{1}{8}\theta(1+3\theta) - \frac{1}{8}(1+\frac{\theta}{2})\theta + \frac{1}{16}(1+3\theta)) \\
&= \frac{\kappa}{16}\beta^{-3}(4(1+\frac{3}{2}\theta+\frac{1}{2}\theta^2) - 2\theta - 6\theta^2 - 2\theta - \theta^2 1 + 3\theta) \\
&= \frac{\kappa}{16}\beta^{-3}(5+5\theta-5\theta^2) \\
&= \frac{5}{16}\kappa\beta^{-2},
\end{aligned}$$

we get

$$\nabla_4 J_{1b'}\nabla_3 J_{2b} g^{bb'} R_{242}{}^4 = \frac{5}{16}\kappa\beta^{-2}(\frac{1}{16}\beta^{-2}(1+8\theta)) = \frac{5}{4^4}\kappa\beta^{-4}(1+8\theta).$$

(iii)

$$\begin{aligned}
&\nabla_3 J_{1b'}\nabla_4 J_{2b} g^{bb'} R_{232}{}^3 - \nabla_4 J_{1b'}\nabla_3 J_{2b} g^{bb'} R_{242}{}^4 \\
&\qquad = -\frac{5}{4^4}\kappa\beta^{-4}(9-8\theta) - \frac{5}{4^4}\beta^{-4}(1+8\theta) = \frac{5}{4^4}\kappa\beta(-10\theta) = -\frac{50}{4^4}\kappa\beta^{-4}.
\end{aligned}$$

□

Combining (1.10), (1.11), (1.12), we obtain

$$\begin{aligned}
S_{2,2} + S_{4,2} &= (S_{2,2}+S_{4,2})(1) + (S_{2,2}+S_{4,2})(2) = 4p^2(\frac{5}{4^4}\kappa\beta^{-4}(-10)) \\
&= -\frac{25}{16}\kappa\beta^{-4},
\end{aligned}$$

which finishes the proof of Prop. A.11. □

1.8. **The $p^4$ term in (1.3).** We compute the $p^4$-term of $S$ by the following Proposition, which finishes the proof of (1.3) and hence Theorem 4.1.

**Proposition 1.13.** $\bar{S}_{[4]} = -\frac{640}{16}\kappa\beta p^4$.

The proof is a long calculation. We know

$$\bar{S}_{[4]} = S_{0,4} + 2S_{1,4} + 2S_{3,4},$$

where by Lemmas 1.6, 1.7, 1.8, we have

(1) $S_{0,4} = S_{0,4,1} + S_{0,4,2}$, with

$$\begin{aligned} S_{0,4,1} &= p^4 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) \nabla_{a_1'} J_{\sigma_1\sigma_2} \nabla_{a_1} J_{\sigma_3\sigma_4} g^{a_1a_1'}, \\ S_{0,4,2} &= 4p^4 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) R_{\sigma_1\sigma_2a_2b} J_{\sigma_4}{}^b J_{\sigma_3}{}^{a_2} - 4p^4 \sum_{\sigma_0=0} \mathrm{sgn}(\sigma) R_{\sigma_1\sigma_2a_2}{}^b J_b{}^{a_2} J_{\sigma_3\sigma_4} \\ &:= S_{0,4,2,1} + S_{0,4,2,2}. \end{aligned}$$

(2)

$$\begin{aligned} S_{1,4} = &-p^4 \sum_{\sigma_1=0} \mathrm{sgn}(\sigma) g^{bb'} \nabla_b J_{\sigma_1\sigma_2} \nabla_{b'} J_{\sigma_3\sigma_4} \\ &+ 2p^4 \sum_{\sigma_1=0} \mathrm{sgn}(\sigma) \nabla_{b'} J_{\sigma_1a_1} \nabla_{\sigma_2} J_{a_2a_1'} J_{\sigma_4b} J_{\sigma_3a_2'} g^{bb'} g^{a_2a_2'} g^{a_1a_1'} \\ &- 2p^4 \sum_{\sigma_1=0} \mathrm{sgn}(\sigma) \nabla_{b'} J_{\sigma_1a_1} \nabla_{\sigma_2} J_{a_2a_1'} J_{\sigma_3\sigma_4} J_{ba_2'} g^{bb'} g^{a_2a_2'} g^{a_1a_1'} \\ := &\ S_{1,4,1} + S_{1,4,2} + S_{1,4,3}. \end{aligned} \tag{1.13}$$

(3)

$$\begin{aligned} S_{3,4} = &\ 2p^4 \sum_{\sigma_3=0} \mathrm{sgn}(\sigma) \nabla^b J_{\sigma_1a_1} \nabla_{\sigma_4} J_b{}^{a_2} J_{\sigma_2a_2} J_{\sigma_1}{}^{a_1} \\ &- 2p^4 \sum_{\sigma_3=0} \mathrm{sgn}(\sigma) \nabla^b J_{\sigma_3a_1} \nabla_{\sigma_4} J_b{}^{a_2} J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1} \\ := &\ S_{3,4,1} + S_{3,4,2}. \end{aligned}$$

These terms are computed as follows.

**Lemma 1.14.**
*(i)* $S_{0,4,1} = 5\kappa\beta^{-2}p^4$, $S_{0,4,2,1} = -10\kappa\beta^{-2}p^4$, $S_{0,4,2,2} = -10\kappa\beta^{-2}p^4$;
*(ii)(a)* $S_{1,4,1} = -5\kappa\beta^{-2}p^4$, *(b)* $S_{1,4,2} = -\frac{5}{2}\kappa\beta^{-2}p^4$, *(c)* $S_{1,4,3} = 5\kappa\beta^{-2}p^4$;
*(iii)(a)* $S_{3,4,1} = -5\kappa\beta^{-2}p^4$, *(b)* $S_{3,4,2} = -5\kappa\beta^{-2}p^4$.

Assuming the Lemma, we get

$$\bar{S}_{[4]} = (S_{0,4,1} + S_{0,4,2}) + 2(S_{1,4,1} + S_{1,4,2} + S_{1,4,3}) + 2(S_{3,4,1} + S_{3,4,2})$$

$$= -40p^4\kappa\beta^{-2} = -\frac{640}{16}p^4\kappa\beta^{-2},$$

finishing the proof of Proposition 1.13.

*Proof of Lemma 1.14.* (i) It suffices to prove the last two equalities. For $S_{0,4,2,1} = p^4\sum_{\sigma_0=0}\mathrm{sgn}(\sigma)R_{\sigma_1\sigma_2a_2b}J_{\sigma_3}{}^{a_2}J_{\sigma_4}{}^{b}$, the only possible $\sigma$ have $(\sigma_1,\sigma_2)\in\{(2,3),(2,4),(3,4)\}$. This gives

$$\sigma\in\{(123),(1234),(13)(24)\},$$

with signs $+1,-1,+1$, respectively. (Associated to e.g., (123) is another permissible permutation $(14)(123)=(1234)$ switching the assignment of 1 and 4, but these permutations give the same contribution to $S_{0,4,2,1}$.) Thus, we have

$$\begin{aligned}
S_{0,4,2,1} &= 4p^4[4R_{23a_2b}J_1{}^{a_2}J_4{}^b - 4R_{24a_2b}J_1{}^{a_2}J_3{}^b + 4R_{34a_2b}J_1{}^{a_2}J_2{}^b]\\
&= 16p^4[R_{232b}J_4{}^b - R_{242b}J_3{}^b + R_{342}{}^bJ_2{}^b]\\
&= 16p^4[R_{2323}J_4{}^3 + R_{232}{}^4J_4{}^4 - R_{2423}J_3{}^3 - R_{2424}J_3{}^4 + R_{3423}J_2{}^3 + R_{3434}J_2{}^4]\\
&= 16p^4[R_{232b}J_4{}^b - R_{242b}J_3{}^b + R_{342}{}^bJ_2{}^b]\\
&= 16p^4[R_{2323}J_4{}^3 + R_{232}{}^4J_4{}^4 - R_{2423}J_3{}^3 - R_{2424}J_3{}^4],
\end{aligned}$$

since $J_2{}^3 = J_2{}^4 = 0$. Therefore,

$$\begin{aligned}
S_{0,4,2,1} = 16p^4\Big[&\frac{\kappa}{16}\beta^{-5/2}(9-16\theta+16\theta^2)(-(1+\theta)\beta^{-1/2})\\
&+(-\frac{\kappa}{16}\beta^{-5/2}(-8+17\theta+8\theta^2))(-\theta\beta^{-1/2})\\
&-(-\frac{\kappa}{16}\beta^{-5/2}(-8+17\theta+8\theta^2))(\theta\beta^{-1/2})\\
&-(\frac{\kappa}{16}(1+\theta+24\theta^2)\beta^{-5/2})(\beta^{-1/2})\Big]\\
= -10\kappa\beta^{-2}p^4.&
\end{aligned}$$

For $S_{0,4,2,2} = -4p^4\sum_{\sigma_0=0}\mathrm{sgn}(\sigma)R_{\sigma_1\sigma_2a_2}{}^bJ_b{}^{a_2}J_{\sigma_3\sigma_4}$, the only possible $\sigma$ have $(\sigma_1,\sigma_2)\in\{(2,3),(2,4),(3,4)\}$. This gives

$$\sigma\in\{(123),(1243),(13(24)\},$$

with signs +1, -1, +1, respectively. (As above, each permutation has a partner switching the assignment of 3 and 4.) This gives

$$\begin{aligned}
S_{0,4,2,2} &= -4p^4[4R_{23a_2}{}^bJ_b{}^{a_2}J_{14} - 4R_{24a_2}{}^bJ_b{}^{a_2}J_{13} + 4R_{34a_2}{}^bJ_b{}^{a_2}J_{12}]\\
&= -4p^4\cdot 4R_{34a_2}{}^bJ_b{}^{a_2}\\
&= -16p^4[R_{343}{}^4J_3{}^3 + R_{343}{}^4J_4{}^3 + R_{344}{}^3J_3{}^4 + R_{344}{}^4J_4{}^4]\\
&= -16p^4\Big[(-\frac{5}{16}\theta\kappa\beta^{-5/2})(\theta\beta^{-1/2}) + (-\frac{5}{16}\kappa\beta^{-5/2})(-(1+\theta)\beta^{-1/2})
\end{aligned}$$

$$+ (\frac{5}{16}\kappa(1+\theta)\beta^{-5/2})(\beta^{-1/2}) + (\frac{5}{16}\theta\kappa\beta^{-5/2})(-\theta\beta^{-1/2})]\Big]$$
$$= -16p^4(\frac{5}{16}\kappa\beta^{-3})[-\theta^2 + (1+\theta) + (1+\theta) - \theta^2]$$
$$= -10\kappa\beta^{-2}p^4.$$

(ii)(a) We have

$$\begin{aligned} S_{1,4,1} &= -4p^4\big(g^{bb'}\nabla_b J_{12}\nabla_{b'}J_{34} - g^{bb'}\nabla_b J_{13}\nabla_{b'}J_{24} + g^{bb'}\nabla_b J_{14}\nabla_{b'}J_{23} - g^{bb'}\nabla_b J_{23}\nabla_{b'}J_{41} \\ &\qquad - g^{bb'}\nabla_b J_{24}\nabla_{b'}J_{13} + g^{bb'}\nabla_b J_{34}\nabla_{b'}J_{12}\big) \\ &= -8p^4\big(-g^{bb'}\nabla_b J_{13}\nabla b'24 + g^{bb'}\nabla_b J_{14}\nabla_{b'}J_{23}\big) \\ &= -8p^4\big(-g^{33}\nabla_3 J_{13}\nabla_3 J_{24} - g^{34}\nabla_3 J_{13}\nabla_4 J_{24} - g^{43}\nabla_4 J_{13}\nabla_3 J_{24} - g^{44}\nabla_4 J_{13}\nabla_4 J_{24} \\ &\quad + g^{33}\nabla_3 J_{14}\nabla_3 J_{23} + g^{34}\nabla_3 J_{14}\nabla_4 J_{23} + g^{43}\nabla_4 J_{14}\nabla_3 J_{23} + g^{44}\nabla_4 J_{14}\nabla_4 J_{23}\big). \end{aligned}$$

Recall that

$$g^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{(1+\theta)\kappa}{\beta^{1/2}\kappa^2} & \frac{\theta\kappa}{\beta^{1/2}\kappa^2} \\ 0 & 0 & \frac{\theta\kappa}{\beta^{1/2}\kappa^2} & \frac{\kappa}{\beta^{1/2}\kappa^2} \end{pmatrix} \tag{1.14}$$

After plugging in from (1.14) and Lemma 1.4, we find

$$\begin{aligned} S_{1,4,1} = -8p^4\Big[&-(\kappa^{-1}(1+\theta)\beta^{-1/2})(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta)(\frac{\kappa}{4}\beta^{-1}) \\ &- (\theta\kappa^{-1}\beta^{-1/2})(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{2}\beta^{-1}) \\ &- (\theta\kappa^{-1}\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-1}) \\ &- (\kappa^{-1}\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(\frac{\kappa}{2}\beta^{-1}) \\ &+ (\kappa^{-1}(1+\theta)\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(-\frac{\kappa}{2}\beta^{-1}) \\ &+ (\theta\kappa^{-1}\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-1}) \\ &+ (\theta\kappa^{-1}\beta^{-1/2})(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{2}\beta^{-1}) \\ &+ (\kappa^{-1}\beta^{-1/2})(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-1})\Big] \end{aligned}$$
$$= -\frac{8}{16}p^4\kappa\beta^{-3}\big[10 + 10\theta - 10\theta^2\big]$$
$$= -5\kappa\beta^{-2}p^4.$$

(ii)(b) For $S_{1,4,2} = 2p^4 \sum_{\sigma_1=0} \mathrm{sgn}(\sigma)\nabla_{b'} J_{\sigma_1 a_1}\nabla_{\sigma_2} J_{a_2 a_1'} J_{\sigma_4 b} J_{\sigma_3 a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}$, we use $\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0$ to obtain

$$\begin{aligned}
S_{1,4,2} &= 2p^4 \sum_{\sigma_1=0,\sigma_2=3} \mathrm{sgn}(\sigma)\nabla_{b'} J_{\sigma_1 a_1}\nabla_3 J_{a_2 a_1'} J_{\sigma_4 b} J_{\sigma_3 a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}\\
&\quad + 2p^4 \sum_{\sigma_1=0,\sigma_2=4} \mathrm{sgn}(\sigma)\nabla_{b'} J_{\sigma_1 a_1}\nabla_4 J_{a_2 a_1'} J_{\sigma_4 b} J_{\sigma_3 a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}\\
&:= S_{1,4,2}(\sigma_2 = 3) + S_{1,4,2}(\sigma_2 = 4).
\end{aligned} \tag{1.15}$$

For $S_{1,4,2}(\sigma_2 = 3)$, the possible $\sigma$ are $\sigma \in \{(243), (24), (1423), (14)(23), (123), (124)\}$, with signs $-1, +1, -1, +1, +1, -1$, respectively. (Here we include all possible $\sigma$, not just half of them.) Thus

$$\begin{aligned}
S_{1,4,2}(\sigma_2 = 3) &= 2p^4 \left(-\nabla_{b'} J_{1a_1}\nabla_3 J_{1a_1'} J_{43} J_{21} g^{3b'} g^{a_1 a_1'} + \nabla_{b'} J_{2a_1}\nabla_3 J_{2a_1'} J_{43} J_{12} g^{3b'} g^{a_1 a_1'}\right)\\
&= -2p^4 J_{34} J_{12}\left(\nabla_{b'} J_{1a_1}\nabla_3 J_{1a_1'} g^{3b'} g^{a_1 a_1'} + \nabla_{b'} J_{2a_1}\nabla_3 J_{2a_1'} g^{3b'} g^{a_1 a_1'}\right).
\end{aligned} \tag{1.16}$$

Similarly, we have

$$\begin{aligned}
S_{1,4,2}(\sigma_2 = 4) = 2p^4 \cdot \Big(&\nabla_{b'} J_{1a_1}\nabla_4 J_{a_2 a_1'} J_{3b} J_{21} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}\\
&- \nabla_{b'} J_{1a_1}\nabla_4 J_{a_2 a_1'} J_{2b} J_{3a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}\\
&(\pm - -which?)\nabla_{b'} J_{3a_1}\nabla_4 J_{a_2 a_1'} J_{2b} J_{2a_1'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}\\
&- \nabla_{b'} J_{3a_1}\nabla_4 J_{a_2 a_1'} J_{1b} J_{2a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}\\
&- \nabla_{b'} J_{2a_1}\nabla_4 J_{a_2 a_1'} J_{3b} J_{1a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}\\
&+ \nabla_{b'} J_{2a_1}\nabla_4 J_{a_2 a_1'} J_{1b} J_{3a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1 a_1'}\Big)\\
= -2p^4 J_{34} J_{12} (&\nabla_{b'} J_{1a_1}\nabla_4 J_{1a_1'} g^{4b'} g^{a_1 a_1'} + \nabla_{b'} J_{2a_1}\nabla_4 J_{2a_1'} g^{4b'} g^{a_1 a_1'}).
\end{aligned} \tag{1.17}$$

Combining (1.15), (1.16), (1.17), we get

$$\begin{aligned}
S_{1,4,2} &= -2p^4 J_{34} J_{12}\Big(\nabla_{b'} J_{1a_1}\nabla_3 J_{1a_1'} g^{3b'} g^{a_1 a_1'} + \nabla_{b'} J_{2a_1}\nabla_3 J_{2a_1'} g^{3b'} g^{a_1 a_1'}\\
&\quad + \nabla_{b'} J_{1a_1}\nabla_4 J_{1a_1'} g^{4b'} g^{a_1 a_1'} + \nabla_{b'} J_{2a_1}\nabla_4 J_{2a_1'} g^{4b'} g^{a_1 a_1'}\Big)\\
&= -2p^4 J_{34} J_{12}\Big(g^{33}\nabla_3 J_{1a_1}\nabla_3 J_{1a_1'} g^{a_1 a_1'} + g^{34}\nabla_4 J_{1a_1}\nabla_3 J_{1a_1'} g^{a_1 a_1'}\\
&\quad + g^{33}\nabla_3 J_{2a_1}\nabla_3 J_{2a_1'} g^{a_1 a_1'} + g^{34}\nabla_4 J_{2a_1}\nabla_3 J_{2a_1'} g^{a_1 a_1'}\\
&\quad + g^{43}\nabla_3 J_{1a_1}\nabla_4 J_{1a_1'} g^{a_1 a_1'} + g^{44}\nabla_4 J_{1a_1}\nabla_4 J_{1a_1'} g^{a_1 a_1'}
\end{aligned} \tag{1.18}$$

$$+ g^{43}\nabla_3 J_{2a_1}\nabla_4 J_{1a_1'}g^{a_1a_1'} + g^{44}\nabla_4 J_{2a_1}\nabla_4 J_{2a_1'}g^{a_1a_1'}\Big)$$
$$:= -2p^4 J_{34}J_{12}(\langle 1\rangle + 2\langle 2\rangle + 2\langle 3\rangle + \langle 4\rangle + \langle 5\rangle + \langle 6\rangle)$$

where

$$\langle 1\rangle = g^{33}\nabla_3 J_{1a_1}\nabla_3 J_{1a_1'}g^{a_1a_1'},\ \langle 2\rangle = g^{34}\nabla_4 J_{1a_1}\nabla_3 J_{1a_1'}g^{a_1a_1'},$$
$$\langle 3\rangle = g^{34}\nabla_4 J_{2a_1}\nabla_3 J_{2a_1'}g^{a_1a_1'},\ \langle 4\rangle = g^{33}\nabla_3 J_{2a_1}\nabla_3 J_{2a_1'}g^{a_1a_1'},$$
$$\langle 5\rangle = g^{44}\nabla_4 J_{1a_1}\nabla_4 J_{1a_1'}g^{a_1a_1'},\ \langle 6\rangle = g^{44}\nabla_4 J_{2a_1}\nabla_4 J_{2a_1'}g^{a_1a_1'}.$$

We claim that

**Claim 1.15.**

$$\langle 1\rangle = \frac{5}{16}(1+\theta)\beta^{-3},\ \langle 2\rangle = -\frac{5}{16}\theta^2\beta^{-3},\ \langle 3\rangle = -\frac{5}{16}\theta^2\beta^{-3},$$
$$\langle 4\rangle = \frac{5}{16}(1+\theta)\beta^{-3},\ \langle 5\rangle = \frac{5}{16}(1+\theta)\beta^{-3},\ \langle 6\rangle = \frac{5}{16}(1+\theta)\beta^{-3}. \tag{1.19}$$

Assume the claim. Plugging (1.19) into (1.18) and using (1.2) for $J_{34}$, $J_{12}$, we obtain

$$\begin{aligned} S_{1,4,2} &= -2p^4 J_{34}J_{12}\bigg(\frac{5}{16}(1+\theta)\beta^{-3} + 2(-\frac{5}{16}\theta^2\beta^{-3}) + 2(-\frac{5}{16}\theta^2\beta^{-3}) + \frac{5}{16}(1+\theta)\beta^{-3} \\ &\qquad + \frac{5}{16}(1+\theta)\beta^{-3} + \frac{5}{16}(1+\theta)\beta^{-3}\bigg) \\ &= -\frac{5}{2}p^4\beta^{-2}\kappa. \end{aligned}$$

We now prove Claim 1.15, which will finish (ii)(b).

*Proof of Claim 1.15.* The proofs are all direct calculations.

$$\begin{aligned} \langle 1\rangle &= g^{33}\nabla_3 J_{1a_1}\nabla_3 J_{1a_1'}g^{a_1a_1'} \\ &= g^{33}(\nabla_3 J_{13}\nabla_3 J_{13}g^{33} + \nabla_3 J_{13}\nabla_3 J_{14}g^{34} + \nabla_3 J_{14}\nabla_3 J_{13}g^{43} + \nabla_3 J_{14}\nabla_3 J_{14}g^{44}) \\ &= g^{33}(\nabla_3 J_{13}\nabla_3 J_{13}g^{33} + 2\nabla_3 J_{13}\nabla_3 J_{14}g^{34} + \nabla_3 J_{14}\nabla_3 J_{14}g^{44}) \\ &= \kappa(1+\theta)\beta^{-1/2}\bigg(\kappa^{-1}(1+\theta)\beta^{-1/2}\big(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta)\big) \\ &\qquad + 2\theta\kappa^{-1}\beta^{-1/2}\big(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big)\big(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big) \\ &\qquad + \kappa^{-1}\beta^{-1/2}\big(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big)\big(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})\big)\bigg) \\ &= \frac{5}{16}\beta^{-3}(1+\theta). \end{aligned}$$

$$
\begin{aligned}
\langle 2\rangle &= g^{34}\nabla_4 J_{1a_1}\nabla_3 J_{1a_1'}g^{a_1a_1'}\\
&= g^{34}\big(\nabla_4 J_{13}\nabla_3 J_{13}g^{33}+\nabla_4 J_{13}\nabla_3 J_{14}g^{34}+\nabla_4 J_{14}\nabla_3 J_{13}g^{43}+\nabla_4 J_{14}\nabla_3 J_{14}g^{44}\big)\\
&= g^{34}\Bigg((-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad +(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad +(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad +(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\kappa^{-1}\beta^{-1/2})\Bigg)\\
&= -\frac{5}{16}\theta^2\beta^{-3},
\end{aligned}
$$

where the last line follows from (1.14) and a direct calculation.

$$
\begin{aligned}
\langle 3\rangle &= g^{34}\nabla_4 J_{2a_1}\nabla_3 J_{2a_1'}g^{a_1a_1'}\\
&= g^{34}\Bigg(\nabla_4 J_{23}\nabla_3 J_{23}g^{33}+\nabla_4 J_{23}\nabla_3 J_{24}g^{34}+\nabla_4 J_{24}\nabla_3 J_{23}g^{34}+\nabla_4 J_{24}\nabla_3 J_{24}g^{44}\Bigg)\\
&= g^{34}\Bigg((\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})+(\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad +(\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1/2})+(\frac{\kappa}{2})(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\Bigg)\\
&= -\frac{5}{16}\theta^2\beta^{-3}.
\end{aligned}
$$

$$
\begin{aligned}
\langle 4\rangle &= g^{33}\nabla_3 J_{2a_1}\nabla_3 J_{2a_1'}g^{a_1a_1'}\\
&= g^{33}(\nabla_3 J_{23}\nabla_3 J_{23}g^{33}+\nabla_3 J_{23}\nabla_3 J_{24}g^{34}+\nabla_3 J_{24}\nabla_3 J_{23}g^{43}+\nabla_3 J_{24}\nabla_3 J_{24}g^{44})\\
&= g^{33}\Bigg((-\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})+(-\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\qquad +(\frac{\kappa}{4})(-\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})+(\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1})\Bigg)\\
&= \frac{5}{16}(1+\theta)\beta^{-3}.
\end{aligned}
$$

$$
\begin{aligned}
\langle 5\rangle &= g^{44}\nabla_4 J_{1a_1}\nabla_4 J_{1a_1'}g^{a_1a_1'}\\
&= g^{44}(\nabla_4 J_{13}\nabla_4 J_{13}g^{33}+\nabla_4 J_{13}\nabla_4 J_{14}g^{34}+\nabla_4 J_{14}\nabla_4 J_{13}g^{43}+\nabla_4 J_{14}\nabla_4 J_{14}g^{44})
\end{aligned}
$$

$$
\begin{aligned}
&= g^{44}\Big( (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(\kappa^{-1}(1+\theta)\beta^{-1/2}) \\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\theta\beta^{-1/2}) \\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2})(\kappa^{-1}\theta\beta^{-1/2}) \\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})\Big) \\
&= g^{44}\kappa\beta^{-7/2}\Big(\frac{1}{4}(1+\frac{\theta}{2})(1+\frac{\theta}{2})(1+\theta) - \frac{1}{8}(1+\frac{\theta}{2})(1+3\theta)\theta \\
&\quad - \frac{1}{8}(1+3\theta)(1+\frac{\theta}{2})\theta + \frac{1}{16}(1+3\theta)(1+3\theta)\Big) \\
&= \frac{1}{16}g^{44}\kappa\beta^{-7/2}\Big(4(1+\frac{\theta}{2}(1+\frac{\theta}{2})(1+\theta) - 2(1+\frac{\theta}{2})(1+3\theta)\theta \\
&\quad - 2(1+3\theta)(1+\frac{\theta}{2})\theta + (1+3\theta)(1+3\theta)\Big) \\
&= \frac{5}{16}(1+\theta)\beta^{-3},
\end{aligned}
$$

after some calculation.

$$
\begin{aligned}
\langle 6\rangle &= g^{44}\nabla_4 J_{2a_1}\nabla_4 J_{2a_1'}g^{a_1a_1'} \\
&= g^{44}(\nabla_4 J_{23}\nabla_4 J_{23}g^{33} + \nabla_4 J_{23}\nabla_4 J_{24}g^{34} + \nabla_4 J_{24}\nabla_4 J_{23}g^{43} + \nabla_4 J_{24}\nabla_4 J_{24}g^{44}) \\
&= \frac{5}{16}(1+\theta)\beta^{-3}.
\end{aligned}
$$

□

(ii)(c) We have

$$
S_{1,4,3} = -2p^4\sum_{\sigma_1=0}\mathrm{sgn}(\sigma)\nabla_{b'}J_{\sigma_1 a_1}\nabla_{\sigma_2}J_{a_2a_1'}J_{\sigma_3\sigma_4}J_{ba_2'}g^{bb'}g^{a_2a_2'}g^{a_1a_1'}. \tag{1.20}
$$

If $(\sigma_1,\sigma_2)$ equals $(1,2)$ or $(2,1$, then $\nabla_1 J = \nabla_2 J = 0$ implies that the summand in (1.20) vanishes. Thus, $(\sigma_3,\sigma_4)\in\{(1,2),(2,1)\}$, so

$$
\sigma\in\{(13)(24),(1324),(1423),(14)(23)\},
$$

with signs $+1,-1,-1,+1$, respectively. Thus by an easy symmetry argument,

$$
\begin{aligned}
&S_{1,4,3} \\
&= -4p^4(\nabla_{b'}J_{3a_1}\nabla_4 J_{a_2a_1'}J_{ba_2'}g^{bb'}g^{a_2a_2'}g^{a_1a_1'} - \nabla_{b'}J_{4a_1}\nabla_3 J_{a_2a_1'}J_{ba_2'}g^{bb'}g^{a_2a_2'}g^{a_1a_1'}) \\
&- 4p^4\big(\nabla_{b'}J_{31}\nabla_4 J_{a_2a_1'}J_{ba_2'}g^{bb'}g^{a_2a_2'}g^{a_1'1} + \nabla_{b'}J_{32}\nabla_4 J_{a_2a_1'}J_{ba_2'}g^{bb'}g^{a_2a_2'}g^{a_1'1}
\end{aligned}
$$

$$
\begin{aligned}
&- \nabla_{b'} J_{41} \nabla_3 J_{a_2 a_1'} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1' 1} - \nabla_{b'} J_{42} \nabla_3 J_{a_2 a_1'} J_{b a_2'} g^{bb'} g^{a_2 a_2'} g^{a_1' 1}\big) \\
= {} & -4p^2 \big( \nabla_{b'} J_{31} \nabla_4 J_{31} J_{b a_2'} g^{bb'} g^{3 a_2'} + \nabla_{b'} J_{31} \nabla_4 J_{41} J_{b a_2'} g^{bb'} g^{4 a_2'} + \nabla_{b'} J_{32} \nabla_4 J_{32} J_{b a_2'} g^{bb'} g^{3 a_2'} \\
&+ \nabla_{b'} J_{32} \nabla_4 J_{42} J_{b a_2'} g^{bb'} g^{4 a_2'} - \nabla_{b'} J_{41} \nabla_3 J_{31} J_{b a_2'} g^{bb'} g^{a_2' 3} - \nabla_{b'} J_{41} \nabla_3 J_{41} J_{b a_2'} g^{bb'} g^{a_2' 4} \\
&- \nabla_{b'} J_{42} \nabla_3 J_{32} J_{b a_2'} g^{bb'} g^{a_2' 3} - \nabla_{b'} J_{42} \nabla_3 J_{42} J_{b a_2'} g^{bb'} g^{a_2' 4}\big) \\
= {} & -4p^4 \big( \nabla_3 J_{31} \nabla_4 J_{31} J_{b a_2'} g^{b3} g^{3 a_2'} + \nabla_4 J_{31} \nabla_4 J_{31} J_{b a_2'} g^{b4} g^{3 a_2'} + \nabla_3 J_{31} \nabla_4 J_{41} J_{b a_2'} g^{b3} g^{4 a_2'} \\
&+ \nabla_4 J_{31} \nabla_4 J_{41} J_{b a_2'} g^{b4} g^{4 a_2'} + \nabla_3 J_{32} \nabla_4 J_{32} J_{b a_2'} g^{b3} g^{3 a_2'} + \nabla_4 J_{32} \nabla_4 J_{32} J_{b a_2'} g^{b4} g^{3 a_2'} \\
&+ \nabla_3 J_{32} \nabla_4 J_{42} J_{b a_2'} g^{b3} g^{4 a_2'} + \nabla_4 J_{32} \nabla_4 J_{42} J_{b a_2'} g^{b4} g^{4 a_2'} - \nabla_3 J_{41} \nabla_3 J_{31} J_{b a_2'} g^{b3} g^{a_2' 3} \\
&- \nabla_4 J_{41} \nabla_3 J_{31} J_{b a_2'} g^{b4} g^{a_2' 3} - \nabla_3 J_{41} \nabla_3 J_{41} J_{b a_2'} g^{b3} g^{a_2' 4} - \nabla_4 J_{41} \nabla_3 J_{41} J_{b a_2'} g^{b4} g^{a_2' 4} \\
&- \nabla_3 J_{42} \nabla_3 J_{32} J_{b a_2'} g^{b3} g^{a_2' 3} - \nabla_4 J_{42} \nabla_3 J_{32} J_{b a_2'} g^{b4} g^{a_2' 3} - \nabla_3 J_{42} \nabla_3 J_{42} J_{b a_2'} g^{b3} g^{a_2' 4} \\
&- \nabla_4 J_{42} \nabla_3 J_{42} J_{b a_2'} g^{b4} g^{a_2' 4} \Big).
\end{aligned}
$$

Note that

$$
\begin{aligned}
J_{b a_2'} g^{b3} g^{3 a_2'} &= J_{34} g^{33} g^{34} + J_{43} g^{43} g^{33} = 0, \;\; J_{b a_2'} g^{b4} g^{4 a_2'} = J_{34} g^{34} g^{44} + J_{43} g^{44} g^{43} = 0, \\
J_{b a_2'} g^{b4} g^{3 a_2'} &= J_{34} g^{34} g^{34} - J_{43} g^{44} g^{33} = J_{34}((g^{34})^2 - g^{44} g^{33}) \\
&= -\kappa^{-2} J_{34} = -\kappa^{-1}, \\
J_{b a_2'} g^{b3} g^{4 a_2'} &= \kappa^{-2} J_{34} = \kappa^{-1}.
\end{aligned}
$$

This gives

$$
\begin{aligned}
S_{1,4,3} = {} & -4p^2 \big( -\kappa^{-2} \nabla_4 J_{31} \nabla_4 J_{31} + \kappa^{-2} \nabla_3 J_{31} \nabla_4 J_{41} - \kappa^{-2} \nabla_4 J_{32} \nabla_4 J_{32} \\
&+ \kappa^{-2} \nabla_3 J_{32} \nabla_4 J_{42} + \kappa^{-2} \nabla_4 J_{41} \nabla_3 J_{31} - \kappa^{-2} \nabla_3 J_{41} \nabla_3 J_{41} \\
&+ \kappa^{-2} \nabla_4 J_{42} \nabla_3 J_{32} - \kappa^{-2} \nabla_3 J_{42} \nabla_3 J_{42} \big) \\
= {} & -4p^2 \kappa^{-1} \Bigg( -(\frac{\kappa}{2} \beta^{-3/2}(1 + \frac{\theta}{2}))(\frac{\kappa}{2} \beta^{-3/2}(1 + \frac{\theta}{2})) \\
&+ (\frac{\kappa}{4} \beta^{-3/2}(1 - 2\theta))(-\frac{\kappa}{4} \beta^{-3/2}(1 + 3\theta)) - (-\frac{\kappa}{4} \beta^{-1})(-\frac{\kappa}{4} \beta^{-1}) \\
&+ (\frac{\kappa}{2})(-\frac{\kappa}{2} \beta^{-1}) + (-\frac{\kappa}{4} \beta^{-3/2}(1 + 3\theta))(\frac{\kappa}{4} \beta^{-3/2}(1 - 2\theta)) \\
&- (\frac{\kappa}{2} \beta^{-3/2}(1 + \frac{\theta}{2}))(\frac{\kappa}{2} \beta^{-3/2}(1 + \frac{\theta}{2})) + (-\frac{\kappa}{2} \beta^{-1})(\frac{\kappa}{2} \beta^{-1}) \\
&- (-\frac{\kappa}{4} \beta^{-1})(-\frac{\kappa}{4} \beta^{-1}) \Bigg) \\
= {} & 5p^4 \beta^{-2} \kappa.
\end{aligned}
$$

(iii)(a) We compute $S_{3,4,1}$ as follows:

$$\begin{aligned}
S_{3,4,1} &= 2p^4 \sum_{\sigma_3=0} \operatorname{sgn}(\sigma) \nabla_b J_{\sigma_3 a_1} \nabla_{\sigma_4} J_{b' a_2} J_{\sigma_2 a_2} J_{\sigma_1 a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
&= 2p^4 \sum_{\sigma_3=0,\sigma_4=3} \operatorname{sgn}(\sigma) \nabla_b J_{\sigma_3 a_1} \nabla_3 J_{b' a_2'} J_{\sigma_2 a_2} J_{\sigma_1 a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
&\quad + 2p^4 \sum_{\sigma_3=0,\sigma_4=4} \operatorname{sgn}(\sigma) \nabla_b J_{\sigma_3 a_1} \nabla_4 J_{b' a_2'} J_{\sigma_2 a_2} J_{\sigma_1 a_1'} g^{bb'} g^{a_1 a_1'} g^{a_2 a_2'} \\
&:= S_{3,4,1}(\sigma_4 = 3) + S_{3.4.1}(\sigma_4 = 4).
\end{aligned}$$

For $S_{3,4,1}(\sigma_4 = 3)$, the possible $\sigma$ are $\sigma \in \{(34), (12)(34), (1432), (243), (143), (1243)\}$, with signs $-1, +1, -1, +1, +1, -1$, respectively. Thus

$$\begin{aligned}
&S_{3,4,1}(\sigma_4 = 3) \\
&= 2p^4 \Big( -\nabla_b J_{4a_1'} \nabla_3 J_{b'a_2'} J_{1a_2} J_{1a_1} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} + \nabla_b J_{4a_1'} \nabla_3 J_{b'a_2'} J_{1a_2} J_{2a_1} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} \\
&\quad - \nabla_b J_{2a_1'} \nabla_3 J_{b'a_2'} J_{1a_2} J_{4a_1} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} + \nabla_b J_{2a_1'} \nabla_3 J_{b'a_2'} J_{4a_2} J_{1a_1} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} \\
&\quad + \nabla_b J_{1a_1'} \nabla_3 J_{b'a_2'} J_{2a_2} J_{4a_1} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} - \nabla_b J_{1a_1'} \nabla_3 J_{b'a_2'} J_{4a_2} J_{2a_1} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} \Big).
\end{aligned}$$

From $\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0$, we conclude

$$\nabla_b J_{2a_1'} \nabla_3 J_{b'a_2'} J_{4a_2} J_{1a_1} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} = 0, \ \nabla_b J_{1a_1'} \nabla_3 J_{b'a_2'} J_{4a_2} J_{2a_1} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} = 0.$$

Therefore,

$$\begin{aligned}
&S_{3,4,1}(\sigma_4 = 3) \\
&\qquad = 2p^4 \Big( -\nabla_b J_{42} \nabla_3 J_{b'1} J_{21} J_{12} g^{bb'} g^{22} g^{11} - \nabla_b J_{42} \nabla_3 J_{b'1} J_{21} J_{12} g^{bb'} g^{22} g^{11} \\
&\qquad\quad - \nabla_b J_{42} \nabla_3 J_{b'1} J_{21} J_{12} g^{bb'} g^{22} g^{11} - \nabla_b J_{42} \nabla_3 J_{b'1} J_{21} J_{12} g^{bb'} g^{22} g^{11} \Big) \\
&\qquad := 2p^4 \Big( \langle\langle 1 \rangle\rangle + \langle\langle 2 \rangle\rangle + \langle\langle 3 \rangle\rangle + \langle\langle 4 \rangle\rangle \Big).
\end{aligned}$$

Similarly, for $S_{3,4,1}(\sigma_4 = 4)$, we have $\sigma \in \{\mathrm{id}, (12), (123), (13), (23), (132)\}$, with signs $+1, -1, +1, -1, -1, +1$, respectively. Thus

$$\begin{aligned}
&S_{3,4,1}(\sigma_4 = 4) \\
&= 2p^4 \Big( \nabla_b J_{3a_1} \nabla_4 J_{b'a_2'} J_{2a_1} J_{1a_1'} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} - \nabla_b J_{3a_1} \nabla_4 J_{b'a_2'} J_{1a_2} J_{2a_1'} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} \\
&\quad + \nabla_b J_{1a_1} \nabla_4 J_{b'a_2'} J_{3a_2} J_{2a_1'} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} - \nabla_b J_{1a_1} \nabla_4 J_{b'a_2'} J_{2a_2} J_{3a_1'} g^{bb'} g^{a_1a_1'} g^{a_2a_2'}
\end{aligned}$$

$$- \nabla_b J_{2a_1} \nabla_4 J_{b'a_2'} J_{3a_2} J_{1a_1'} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} + \nabla_b J_{2a_1} \nabla_4 J_{b'a_2'} J_{1a_2} J_{3a_1'} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} \Big)$$

Using $\nabla_b J_{11} = \nabla_b J_{22} = 0$, we obtain

$$\nabla_b J_{1a_1} \nabla_4 J_{b'a_2'} J_{3a_2} J_{2a_1'} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} = 0, \ \nabla_b J_{2a_1} \nabla_4 J_{b'a_2'} J_{3a_2} J_{1a_1'} g^{bb'} g^{a_1a_1'} g^{a_2a_2'} = 0.$$

Therefore,

$$\begin{aligned}
& S_{3,4,1}(\sigma_4 = 4) \\
&= 2p^4 \Big( \nabla_b J_{32} \nabla 4b'1 J_{21} J_{12} g^{bb'} g^{22} g^{11} - \nabla_b J_{31} \nabla 4b'2 J_{12} J_{21} g^{bb'} g^{11} g^{22} \\
&\qquad - \nabla_b J_{1a_1} \nabla 4b'1 J_{21} J_{34} g^{bb'} g^{a_14} g^{11} + \nabla_b J_{2a_1} \nabla 4b'2 J_{12} J_{34} g^{bb'} g^{a_14} g^{22} \Big) \\
&:= 2p^4 \Big( \langle\langle 5 \rangle\rangle + \langle\langle 6 \rangle\rangle + \langle\langle 7 \rangle\rangle + \langle\langle 8 \rangle\rangle \Big).
\end{aligned}$$

We now compute $\langle\langle 1 \rangle\rangle$ – $\langle\langle 8 \rangle\rangle$.

$$\begin{aligned}
& \langle\langle 1 \rangle\rangle \\
&= (J_{12})^2 \nabla_b J_{42} \nabla_3 J_{b'} g^{bb'} \\
&= (J_{12})^2 \big( \nabla_3 J_{42} \nabla_3 J_{31} g^{33} + \nabla_3 J_{42} \nabla_3 J_{41} g^{34} \nabla_4 J_{42} \nabla_3 J_{31} g^{43} + \nabla_4 J_{42} \nabla_3 J_{41} g^{44} \big) \\
&= (-\frac{\kappa}{4} \beta^{-1})(\frac{\kappa}{4} \beta^{-3/2}(1 - 2\theta))(\kappa^{-1}(1 + \theta)\beta^{-1/2}) \\
&\qquad + (-\frac{\kappa}{4} \beta^{-1})(\frac{\kappa}{2} \beta^{-3/2}(1 + \frac{\theta}{2}))(\theta \kappa^{-1} \beta^{-1/2}) \\
&\qquad + (-\frac{1}{2} \kappa \beta^{-1})(\frac{1}{4} \beta^{-3/2}(1 - 2\theta))(\theta \kappa^{-1} \beta^{-1/2}) \\
&\qquad + (-\frac{1}{2} \kappa \beta^{-1})(\frac{\kappa}{2} \beta^{-3/2}(1 + \frac{\theta}{2}))(\kappa^{-1} \beta^{-1/2}) \\
&= -\frac{5}{16} \kappa \beta^{-2}.
\end{aligned}$$

$$\begin{aligned}
& \langle\langle 2 \rangle\rangle \\
&= \nabla_b J_{41} \nabla_3 J_{b'2} (-J_{12})^2 g^{bb\prime} \\
&= -(J_{12})^2 \Big( \nabla_3 J_{41} \nabla_3 J_{32} g^{33} + \nabla_4 J_{41} \nabla_3 J_{42} g^{34} + \nabla_4 J_{41} \nabla_3 J_{32} g^{43} + \nabla_4 J_{41} \nabla_3 J_{42} g^{44} \Big) \\
&= (-1) \big( \frac{\kappa}{2} \beta^{-3/2} (1 + \frac{\theta}{2}) \big) (\frac{\kappa}{2} \beta^{-1})(\kappa^{-1}(1 + \theta)\beta^{-1/2}) \\
&\qquad + \big( \frac{\kappa}{2} \beta^{-3/2} (1 + \frac{\theta}{2}) \big) (-\frac{\kappa}{4} \beta^{-1})(\kappa^{-1} \theta \beta^{-1/2})
\end{aligned}$$

$$
\begin{aligned}
&\quad + \big(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)\big)(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&\quad + \big(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta)\big)(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

$$
\begin{aligned}
&\langle\langle 3\rangle\rangle\\
&= -\nabla_b J_{2a_1'}\nabla_3 J_{b'2}J_{12}J_{43}g^{a_1'3}g^{bb'}\\
&= (-\kappa)(-1)(g^{33}\nabla_b J_{23}\nabla_3 J_{b'2}g^{bb'} + g^{34}\nabla_b J_{24}\nabla_3 J_{b'2}g^{bb'})\\
&= \kappa\Big(g^{33}(\nabla_3 J_{23}\nabla_3 J_{32}g^{33} + \nabla_3 J_{23}\nabla_3 J_{42}g^{34} + \nabla_4 J_{23}\nabla_3 J_{32}g^{43} + \nabla_4 J_{23}\nabla_3 J_{42}g^{44})\\
&\qquad + g^{34}(\nabla_3 J_{24}\nabla_3 J_{32}g^{33}\nabla_3 J_{24}\nabla_3 J_{42}g^{34} + \nabla_4 J_{24}\nabla_3 J_{32}g^{43} + \nabla_4 J_{24}\nabla_3 J_{42}g^{44})\Big)\\
&= \kappa\Big[\kappa^{-1}(1+\theta)\beta^{-1/2}\Big((-\frac{1}{2}\kappa\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad\quad + (-\frac{\kappa}{2}\beta^{-1})(-\frac{1}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1}) + (\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1})\\
&\qquad\quad + (\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1})\Big)\Big]\\
&\qquad + \kappa\theta\beta^{-1/2}\Big[(\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2}) + (\frac{\kappa}{4}\beta^{-1})(-\frac{1}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad\quad + (\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2}) + (\frac{\kappa}{2}\beta^{-1})(-\frac{1}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\Big]\\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}
$$

$$
\begin{aligned}
&\langle\langle 4\rangle\rangle\\
&= -\nabla_b J_{42}\nabla_3 J_{b'1}J_{21}J_{12}g^{bb'}g^{22}g^{11}\\
&= \kappa\Big[\kappa^{-1}(1+\theta)\beta^{-1/2}\Big((-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\qquad\quad + (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\qquad\quad + (\kappa^{-1}\theta\beta^{-1/2})(-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{\theta}{2}))(\frac{\kappa}{4}(1-2\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})
\end{aligned}
$$

$$+ (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2})$$
$$+ (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}\theta\beta^{-1/2})$$
$$+ (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\beta^{-1/2})\Big)\Big]$$
$$= -\frac{5}{16}\kappa\beta^{-2}.$$

$\langle\langle 5\rangle\rangle$

$$= \nabla_b J_{32}\nabla_4 J_{b'1}(-(J_{12})^2)g^{bb'} = -1\nabla_b J_{32}\nabla_4 J_{b'1}g^{bb'}$$
$$= -(\nabla_3 J_{32}\nabla_4 J_{31}g^{33} + \nabla_3 J_{32}\nabla_4 J_{41}g^{34} + \nabla_4 J_{32}\nabla_4 J_{31}g^{43} + \nabla_4 J_{32}\nabla_4 J_{41}g^{44})$$
$$= -(\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})$$
$$- (\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\theta\kappa^{-1}\beta^{-1/2})$$
$$- (-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\theta\kappa^{-1}\beta^{-1/2})$$
$$- (-\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})$$
$$= -\kappa\beta^{-3}\left(\frac{1}{4}(1+\frac{1}{2}\theta)(1+\theta) - \frac{1}{8}\theta(1+3\theta) - \frac{1}{8}(1+\frac{1}{2}\theta)\theta + \frac{1}{16}(1+3\theta)\right)$$
$$= -\frac{\kappa}{16}(4+6\theta; 2\theta^2 - 2\theta - 6\theta^2 - 2\theta - \theta^2 + 1 + 3\theta)$$
$$= -\frac{1}{16}\kappa\beta^{-3}(5+5\theta-5\theta^2)$$
$$= -\frac{5}{16}\kappa\beta^{-2}.$$

$\langle\langle 6\rangle\rangle$

$$= -\nabla_b J_{31}\nabla_4 J_{b'2}(-(J_{12})^2)g^{bb'} = \nabla_b J_{31}\nabla_4 J_{b'2}g^{bb'}$$
$$= \nabla_3 J_{31}\nabla_4 J_{32}g^{33} + \nabla_3 J_{31}\nabla_4 J_{42}g^{34} + \nabla_4 J_{31}\nabla_4 J_{32}g^{43} + \nabla_4 J_{31}\nabla_4 J_{42}g^{44}$$
$$= (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})$$
$$+ (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})$$
$$+ (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta)(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})$$

$$+ (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2})$$
$$= -\frac{5}{16}\kappa\beta^{-2}.$$

$\langle\langle 7\rangle\rangle$

$$\begin{aligned}
&= -\nabla_b J_{1a_1}\nabla_4 J_{b'1}J_{21}J_{34}g^{bb'}g^{a_14} = \kappa\nabla_b J_{1a_1}\nabla_4 J_{b'1}g^{bb'}g^{a_14}\\
&= \kappa(g^{34}\nabla_b J_{13}\nabla_4 J_{b'1}g^{bb'} + g^{44}\nabla_b J_{14}\nabla_4 J_{b'1}g^{bb'})\\
&= \kappa\Big(g^{34}\big(\nabla_3 J_{13}\nabla_4 J_{31}g^{33} + \nabla_3 J_{13}\nabla_4 J_{41}g^{34} + \nabla_4 J_{13}\nabla_4 J_{31}g^{43} + \nabla_4 J_{13}\nabla_4 J_{41}g^{44}\big)\\
&\quad + g^{44}\big(\nabla_3 J_{14}\nabla_4 J_{31}g^{33} + \nabla_3 J_{14}\nabla_4 J_{41}g^{34} + \nabla_4 J_{14}\nabla_4 J_{31}g^{43} + \nabla_4 J_{14}\nabla_4 J_{41}g^{44}\big)\Big)\\
&= \kappa\Big[\kappa^{-1}\theta\beta^{-1/2}\Big((-\frac{\kappa}{4}\beta^{-3/2})(1-2\theta)(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})\Big)\\
&\quad + \kappa^{-1}\beta^{-1/2}\Big((-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\quad + (-\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2})\\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})\Big)\Big]\\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}$$

$\langle\langle 8\rangle\rangle$

$$\begin{aligned}
&= \nabla_b J_{2a_1}\nabla_4 J_{b'2}J_{12}J_{34}g^{bb'}g^{a_14} = J_{12}J_{34}(\nabla_b J_{23}\nabla_4 J_{b'2}g^{bb'}g^{34} + \nabla_b J_{24}\nabla_4 J_{b'2}g^{bb'}g^{44})\\
&= \kappa(\nabla_b J_{23}\nabla_4 J_{b'2}g^{bb'}g^{34} + \nabla_b J_{24} + \nabla_4 J_{b'2}g^{bb'}g^{44})\\
&= \kappa\Big((\nabla_3 J_{23}\nabla_4 J_{32}g^{33}\nabla_3 J_{23}\nabla_4 J_{42}g^{34} + \nabla_4 J_{23}\nabla_4 J_{32}g^{43} + \nabla_4 J_{23}\nabla_4 J_{42}g^{44})g^{34}
\end{aligned}$$

$$+ (\nabla_3 J_{24}\nabla_4 J_{32}g^{33} + \nabla_3 J_{24}\nabla_4 J_{42}g^{34} + \nabla_4 J_{24}\nabla_4 J_{32}g^{43} + \nabla_4 J_{24}\nabla_4 J_{42}g^{44})g^{44}\Big)$$

$$= \kappa\Big[\kappa^{-1}\theta\beta^{-1/2}\Big((-\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2}) + (-\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})$$

$$+ (\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2}) + (\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\Big)$$

$$+ \kappa^{-1}\beta^{-1/2}\Big((\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1}(\kappa^{-1}(1+\theta)\beta^{-1/2}) + (\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})$$

$$+ (\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2}) + (\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})\Big)\Big]$$

$$= -\frac{5}{16}\kappa\beta^{-2}.$$

Thus

$$S_{3,4,1} = 2p^4(\langle\langle 1\rangle\rangle + \ldots + \langle\langle 8\rangle\rangle) = 2p^4 \cdot 8 \cdot \left(-\frac{5}{16}\kappa\beta^{-2}\right) = -5p^4\beta^{-2}.$$

(iii)(b) We compute $S_{3,4,2}$ as follows.

$$S_{3,4,2} = -2p^4 \sum_{\sigma_3=0} \mathrm{sgn}(\sigma)\nabla_b J_{\sigma_3 a_1}\nabla_{\sigma_4} J_{b'}{}^{a_2} J_{\sigma_1\sigma_2} J_{a_2}{}^{a_1} g^{bb'}$$
$$= -2p^4 \sum_{\sigma_3=0} \mathrm{sgn}(\sigma)\nabla_b J_{\sigma_3 a_1}\nabla_{\sigma_4} J_{b'a'_2} J_{a_2 a'_1} J_{\sigma_1\sigma_2} g^{bb'} g^{a_1 a'_1} g^{a_2 a'_2}.$$

We must have $(\sigma_1,\sigma_2) \in \{(1,2),(2,1),(3,4),(4,3)\}$ From the form of the symplectic structure, it follows that $\sigma \in S = \{\mathrm{id}, (34), (13)(24), (1324)\}$. (There are also permutations of the form $(12)\tau, (34)\tau$ for $\tau \in S$, but by the skew-symmetry of $J_{ab}$, only the terms in $S$ contribute to $S_{3,4,2}$.) The signs of the permutations in $S$ are $+1, -1, +1, -1$, respectively.

Keeping track of the extra permutations and their signs, we have

$$S_{3,4,2}$$
$$= -2p^4 \cdot 2\Big(\nabla_b J_{3a_1}\nabla_4 J_{b'a'_2} J_{a_2 a'_1} J_{12} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} - \nabla_b J_{4a_1}\nabla_3 J_{b'a'_2} J_{a_2 a'_1} J_{12} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1}$$
$$+ \nabla_b J_{1a_1}\nabla_2 J_{b'a'_2} J_{a_2 a'_1} J_{34} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} - \nabla_b J_{2a_1}\nabla_1 J_{b'a'_2} J_{a_2 a'_1} J_{34} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1}\Big).$$

Using $\nabla_1 J_{ab} = \nabla_2 J_{ab} = 0$, and $J_{12} = 1$, we get

$$S_{3,4,2}$$
$$= -4p^4\Big(\nabla_b J_{3a_1}\nabla_4 J_{b'a'_2} J_{a_2 a'_1} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1} - \nabla_b J_{4a_1}\nabla_3 J_{b'a'_2} J_{a_2 a'_1} g^{bb'} g^{a_2 a'_2} g^{a_1 a'_1}\Big)$$

$$= 4p^4\left(\nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{a_1a_2}g^{bb'}g^{a_2a_2'}g^{a_1a_1'} - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{a_1a_2}g^{bb'}g^{a_2a_2'}g^{a_1a_1'}\right).$$

The only nonzero terms come from $(a_1, a_2) \in \{(1,2),(2,1),(3,4),(4,3)\}$. Therefore,

$$\begin{aligned}
&S_{3,4,2}\\
&= -4p^4\Big(\nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{12}g^{bb'}g^{2a_2'}g^{1a_1'} + \nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{21}g^{bb'}g^{1a_2'}g^{2a_1'}\\
&\qquad - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{12}g^{bb'}g^{2a_2'}g^{1a_1'} - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{21}g^{bb'}g^{1a_2'}g^{2a_1'}\\
&\qquad + \nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{34}g^{bb'}g^{4a_2'}g^{3a_1'} + \nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}J_{43}g^{bb'}g^{3a_2'}g^{4a_1'}\\
&\qquad - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{34}g^{bb'}g^{4a_2'}g^{3a_1'} - \nabla_b J_{4a_1'}\nabla_3 J_{b'a_2'}J_{43}g^{bb'}g^{3a_2'}g^{4a_1'}\Big)\\
&= 4p^4\Big(\nabla_b J_{31}\nabla_4 J_{b'2}g^{bb'} - \nabla_b J_{32}\nabla_4 J_{b'1}g^{bb'} - \nabla_b J_{41}\nabla_3 J_{b'2}g^{bb'} + \nabla_b J_{42}\nabla_3 J_{b'1}g^{bb'}\\
&\qquad + \nabla_b J_{3a_1'}\nabla_4 J_{b'a_2'}g^{bb'}\kappa g^{4a_2'}g^{3a_1'} - \nabla_b J_{3a_1}\nabla_4 J_{b'a_2'}g^{bb'}\kappa g^{3a_2'}g^{4a_1'}\\
&\qquad - \nabla_b J_{4a_1'}\nabla_3 J_{ba_2'}g^{bb'}\kappa g^{4a_2'}g^{3a_1'} + \nabla_b J_{4a_1'}\nabla_3 J_{ba_2'}g^{bb'}\kappa g^{3a_2'}g^{4a_1'}\Big)\\
&:= 4p^4\Big([[1]]+[[2]]+[[3]]+[[4]]+[[5]]+[[6]]+[[7]]+[[8]]\Big).
\end{aligned}$$

We now compute [[1]]–[[8]].

$$\begin{aligned}
&[[1]]\\
&= \nabla_b J_{31}\nabla_4 J_{b'2}g^{bb'}\\
&= \nabla_3 J_{31}\nabla_4 J_{32}g^{33} + \nabla_3 J_{31}\nabla_4 J_{42}g^{34} + \nabla_4 J_{31}\nabla_4 J_{32}g^{43} + \nabla_4 J_{31}\nabla_4 J_{42}g^{44}\\
&= (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})\\
&\quad + (\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(-\frac{\kappa}{2}\beta^{-1})(\theta\kappa^{-1}\beta^{-1/2})\\
&\quad + (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{4}\beta^{-1})(\theta\kappa^{-1}\beta^{-3/2})\\
&\quad + (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\\
&= -\frac{5}{16}\kappa\beta^{-2}.
\end{aligned}$$

$$\begin{aligned}
&[[2]]\\
&= -\nabla_b J_{32}\nabla_4 J_{b'1}g^{bb'}
\end{aligned}$$

$$= -\bigg(\nabla_3 J_{32}\nabla_4 J_{31} g^{33} + \nabla_3 J_{32}\nabla_4 J_{41} g^{34} + \nabla_4 J_{32}\nabla_4 J_{31} g^{43} + \nabla_4 J_{32}\nabla_4 J_{41} g^{44}\bigg)$$

$$= -\bigg( (\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})$$
$$+ (\frac{\kappa}{2}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\theta\beta^{-1/2})$$
$$+ (-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2})$$
$$+ (-\frac{\kappa}{4}\beta^{-1})(-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\kappa^{-1}\beta^{-1/2})\bigg)$$
$$= \frac{5}{16}\kappa\beta^{-2}.$$

[[3]]

$$= -\nabla_b J_{41}\nabla_3 J_{b'2} g^{bb'}$$
$$= -\bigg(\nabla_3 J_{41}\nabla_3 J_{32} g^{33} + \nabla_3 J_{41}\nabla_3 J_{42} g^{34} + \nabla_4 J_{41}\nabla_3 J_{32} g^{43} + \nabla_4 J_{41}\nabla_3 J_{42} g^{44}\bigg)$$
$$= -\bigg( (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}(1+\theta)\beta^{-1/2})$$
$$+ (\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})$$
$$+ (-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(\frac{\kappa}{2}\beta^{-1})(\kappa^{-1}\theta\beta^{-1/2})$$
$$+ (-\frac{\kappa}{4}\beta^{-3/2}(1+3\theta))(-\frac{\kappa}{4}\beta^{-1})(\kappa^{-1}\beta^{-1/2})\bigg)$$
$$= -\frac{5}{16}\kappa\beta^{-2}.$$

[[4]]

$$= \nabla_b J_{42}\nabla_3 J_{b'1} g^{bb'}$$
$$= \nabla_3 J_{42}\nabla_3 J_{31} g^{33} + \nabla_3 J_{42}\nabla_3 J_{41} g^{43} + \nabla_4 J_{42}\nabla_3 J_{31} g^{43} + \nabla_4 J_{42}\nabla_3 J_{41} g^{4}$$
$$= (-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}(1+\theta)\beta^{-1/2})$$
$$+ (-\frac{\kappa}{4}\beta^{-1})(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\theta\beta^{-1/2})$$
$$+ (-\frac{\kappa}{2}\beta^{-1})(\frac{\kappa}{4}\beta^{-3/2}(1-2\theta))(\kappa^{-1}\theta\beta^{-1/2})$$

$$+(-\frac{\kappa}{2}\beta\ -1)(\frac{\kappa}{2}\beta^{-3/2}(1+\frac{1}{2}\theta))(\kappa^{-1}\beta^{-1/2})$$
$$= -\frac{5}{16}\kappa\beta^{-2}.$$

We have

$$[[5]] = \kappa\nabla_b J_{3a'_1}\nabla_4 J_{b'a_2} g^{bb'} g^{4a'_2} g^{3a'_1} = 0,$$

since we must have $a'_1 = 3$ or 4, and $\nabla_b J_{33} = \nabla_b J_{34} = 0$. For the same reason,

$$[[6]] = -\kappa\nabla_b J_{3a'_1}\nabla_4 J_{b'a_2} g^{bb'} g^{3a'_2} g^{4a'_1} = 0.$$

Similarly,

$$[[7]] = -\kappa\nabla_b J_{4a'_1}\nabla_3 J_{b'a'_2} g^{bb'} g^{4a'_2} g^{3a'_1} = 0,$$

since $a'_1 = 3$ or 4, and $\nabla_b J_{43} = 0$, $\nabla_b J_{44} = 0$. For the same reason,

$$[[8]] = \kappa\nabla_b J_{4a'_1}\nabla_3 J_{b'a_2} g^{bb'} g^{3a'_2} g^{4a'_1} = 0.$$

Thus, we have

$$S_{3,4,2} = 4p^4\left(-\frac{5}{16}\kappa\beta^{-2}\right)\cdot 4 = -5\kappa\beta^{-2}p^4.$$

This finishes the proofs of Lemma 1.14 and Proposition 1.13.

As explained below Proposition 1.5, this finishes the demonstration of (1.3).

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