Inhomogeneous Probabilistic Zero Forcing on Graphs

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Background and Research Plan

Graphs, also called networks, can be seen as a set of vertices and a set of edges. They are studied both in pure mathematics (as part of combinatorics) and mathematical statistics (as part of network science). Zero forcing is an iterative vertex-coloring process on a graph G. In this process, we begin with a subset of vertices of G colored blue iteratively apply a color change rule, called a “force,” until no further forces can occur. In standard zero forcing, the color change rule is as follows: if a blue vertex has a unique non-blue neighbor, then the non-blue vertex can be “forced” blue by the blue vertex (see Fig. 1). If an initial set of blue vertices is capable of forcing *all* vertices in G blue, then that initial subset of vertices is called a zero forcing set [1]. Two associated numbers are of interest: the size of the smallest zero forcing set, $Z(G)$, and the length of time it takes for a graph to become entirely forced, $\mathrm{pt}(G)$ This process can be used to model the spread of information or resources through a network.

Phase 1 Phase 2

Fig. 1: An example of standard zero forcing. The leftmost vertex, which is blue in phase 1, forces its unique non-blue neighbor blue, represented in phase 2. The process will continue since more forces can occur.

Thus far, standard zero forcing has been used to model situations in the physical and computer sciences, while a variation, leaky zero forcing, has been studied to simulate failures in information networks [2]. However, its use as a social network model is diminished by its rigidity. Social networks are dynamic, and at any point vertices and edges may join or disappear from a graph representing one. Thus, the instinct arises to incorporate this dynamic nature into the zero forcing model. Two attempts have been proposed to accomplish this. Firstly, probabilistic zero forcing has been studied, which assigns a probabilistic color-change rule:

$$\mathbb{P}(u \text{ forces } v) = \frac{|N[u] \cap Z|}{\deg(u)}$$

where $N[u]$ denotes the set of vertices adjacent to u. This color change rule has been characterized as a Markov process [3]. Furthermore, $\mathbb{E}\mathrm{pt}(G)$ has been found for several families of graphs, and this has been extended to characterize confidence intervals for $\mathrm{pt}(G)$ [4]. Secondly, fuzzy zero forcing was introduced, which applies zero forcing rules to fuzzy graphs [5]. A fuzzy graph is a graph $G = (V, \sigma, \rho)$ with an underlying vertex set V, a probabilistic function $\sigma : V \to [0, 1]$ defining the probability that vertex $v \in V$ appears in G, and a similar function $\rho : V \times V \to [0, 1]$ defining the probability that edge $e \in V \times V$ appears in G. The fuzzy graph is closely related to the Erdős-Rényi random graph, but adds the existence of a probabilistic function for the vertices.

Both of the above methods are important steps in reaching an approximation to real-world systems, but assume homogeneity of probabilities. Thus, the first aim of my project would be **to characterize inhomogeneous probabilistic zero forcing** by defining an inhomogeneous probabilistic zero forcing rule based on edge weights. Instead of treating each adjacency as equally influential within a graph, a weighted-edge color change rule would allow for dynamic changes to occur in the spread of colorings throughout a graph. The second aim would be **to test this inhomogeneous model against simulated data**. By defining a rule dependent on weight and building a dynamic graph system with appearing and disappearing vertices and edges, a model can be derived tracking the spread of information across a dynamic network. This research would be computationally and theoretically driven, drawing on algorithms introduced for finding zero forcing sets [5, 6] as well as models common in statistical network science, including the inhomogeneous Erdős-Rényi model, the Random Dot Product Graph Model (RDPGM), and the Latent Space Model (LSM) [7].

Motivation and Intellectual Merit

Zero forcing in graph theory arose from several disparate applied fields of research: power domination of electrical networks, quantum control, and fast-mixed graph searching in theoretical computer science [1]. It has grown into its own active research area due to its connection with the Inverse Eigenvalue Problem of a Graph (IEP-G), which seeks to characterize the inverse eigenvalue problem using spectral graph theory. In short, **zero forcing is an active research area with applications across mathematics and the sciences**. However, the intuitive social process analogous to zero-forcing—the spread of information, ideas, or rumors through a network of people—has been limited due to its deterministic nature, as opposed to the chaotic nature of social data [8]. Research into chaotic zero forcing, via probabilistic and fuzzy zero forcing, could generate new areas of spectral graph theory, especially in relation to statistical network science.

Furthermore, the **social applications of statistical network science** are vast. From the spread of respiratory disease to sharing posts on social media, humans consistently produce dynamic, time-dependent networks, with probabilistically approximated relationships. This extension of zero forcing will allow for greater diversity in the current models available for social information spread. Models built to address these problems may be used to predict and address spread of misinformation, one of the most rampant social problems in today's world.

Broader Impacts and Results

My ultimate goal is to become a professor of mathematics with a focus in probability and statistics. As mentioned in my personal statement, I am passionate about equitable education and futures in STEM and believe that a career as a statistician working with social data is a fulfilling extension of these passions. Furthermore, the most meaningful experiences I have had in mathematics have been due to my position as a mentor, and because of this, I wish to continue teaching throughout my career. The combination of these goals leads naturally to a career as a professor of statistics.

Another deeply meaningful aspect of my experience in mathematics thus far has been attending conferences and interacting with mathematicians from around the world. I have been fortunate to present work at two conferences, SACNAS National Diversity in STEM conference (NDiSTEM) 2023 in Portland, Oregon, and the Joint Mathematics Meetings (JMM) 2024 in San Francisco, California. At both conferences, I was overcome with a sense of community, curiosity, and possibility. These experiences were only possible due to my participation in an REU during the summer of 2023. Thus, in pursuing this project, I would seek ways to include undergraduate researchers, so that they might discover the powerful methods of graph theory and gain access to the invaluable resource that is the mathematical research community. Of particular interest would be collaboration with the currently active Inverse Eigenvalue Problem of a Graph - Zero Forcing AIM Research Community, and their associated events at JMM.

References

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