Positive Semidefinite Leaky Zero Forcing

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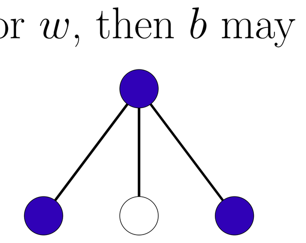
PSD and Leaky Forcing

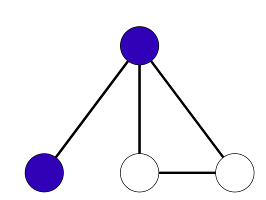
Zero forcing is a graph-coloring process in which an initial set of blue vertices may force other vertices to become colored as well. With its roots in the long-standing inverse eigenvalue problem, zero forcing has constituted its own area of research in recent years [4]. In particular, positive semidefinite (PSD) forcing and leaky forcing have been studied independently in depth. [1, 2, 4]

The Positive Semidefinite (PSD) forcing process.

1. Color any number of vertices on a graph G blue. Let B_0 denote the initial blue vertices.
2. A white component is a set of connected vertices with no blue vertices obstructing them. Let $W_1, W_2, \ldots, W_k$ denote the white components.
3. For a single white component W_i, observe its blue neighbors.
4. If, within W_i, a blue vertex $b \in B_0$ has only one white neighbor w, then b may force w blue.



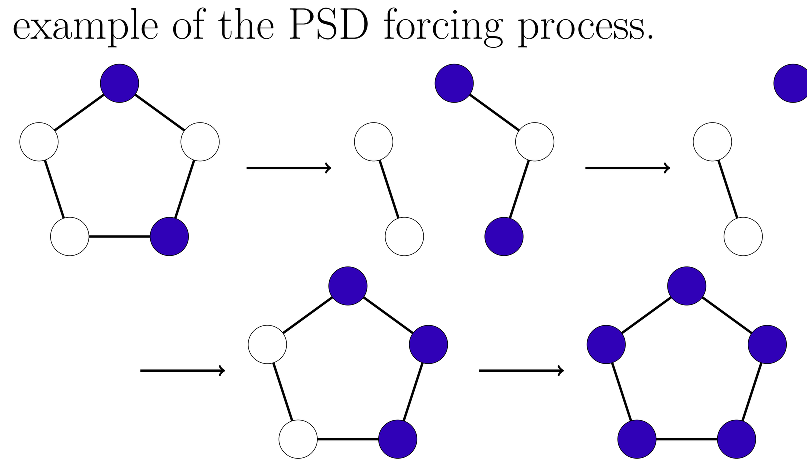
Forces can occur.



No forces can occur.

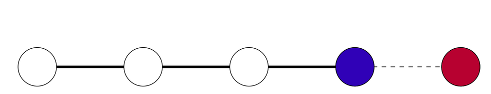
5. Repeat this process for all vertices within W_i, then again in all other white components.
6. Once all forces with the initial set B have been completed, begin another round of forces with the vertex set B_1, which includes the initial blue vertices and those forced in the first round.
7. Repeat steps 2-6 until no more forces are possible.

If B_0 is able to force all vertices in G after any number of rounds, then it is called a **PSD forcing set**. The following is an example of the PSD forcing process.



The **PSD forcing number** of a graph G is the size of a smallest possible forcing set.

Leaks are additions to a graph that make certain vertices unable to force. They are imaginary vertices connected to existing vertices in the graph by one new edge. Leaks will be drawn with dotted edges and in red.



Notice, no PSD forces can occur.

PSD-Leaky Forcing

- The **PSD-Leaky forcing process** follows PSD forcing rules with the added obstacle of leaks. Leaks may occur on any vertices after the initial coloring.
- A **PSD ℓ-leaky forcing set** on a graph G is a set of initially blue vertices that will be able to force all of G blue in the presence of any ℓ leaks.
- The **PSD ℓ-leaky forcing number**, denoted $Z^+_{(\ell)}(G)$, is the size of the smallest possible forcing set on a graph G with ℓ leaks.

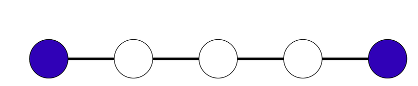
Research Questions

- Which families of graphs have different PSD-leaky forcing numbers compared to standard-leaky?
- What could be causing these differences?

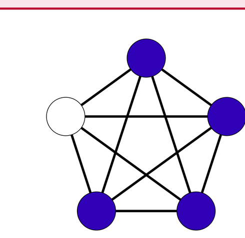
Families with Equal Standard- and PSD-Leaky Forcing Numbers

The following theorem, first proved for standard-leaky forcing in [2] and adapted and proved similarly by us for PSD-leaky forcing, can be used to determine $Z^+_{(\ell)}(G)$ for graphs below.

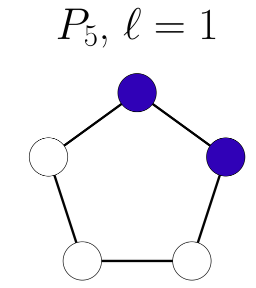
Theorem 1 *For any graph G with ℓ leaks, any PSD forcing set has at least the vertices of degree ℓ or less.*



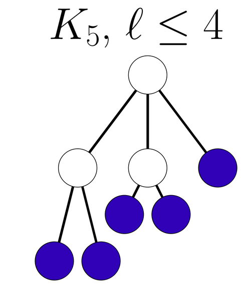
P_5, $\ell = 1$



K_5, $\ell \leq 4$



C_5, $\ell \leq 1$



Tree, $\ell = 1$

We showed the following graph families have the same PSD-leaky forcing numbers as standard-leaky forcing numbers [2]. However, specifically for trees, $Z^+_{(0)}(T) = 1$, unlike the standard forcing [1].

- Paths (P_n)
- Cycles (C_n)
- Complete graphs (K_n)
- Trees

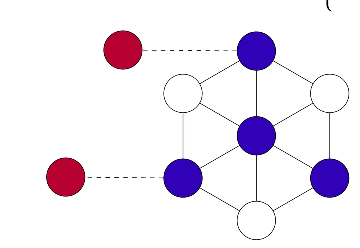
This motivates our pursuit of graphs where the PSD-leaky number is different than that of standard-leaky forcing.

Families Different from Standard

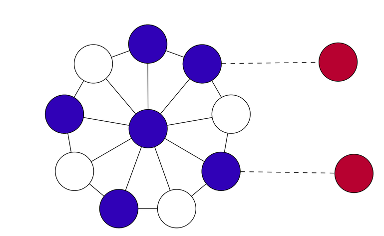
These graph families were studied under standard-leaky forcing in [2, 3] and we provide results for PSD-leaky forcing.

Wheels. Let W_n denote the wheel graph on $n+1$ vertices.

$$Z^+_{(\ell)}(W_n) = \begin{cases} 3 & \ell = 0, 1 \\ \lceil \frac{n}{2} \rceil + 1 & \ell = 2 \\ n & 2 < \ell < n \\ n+1 & \ell = n, n+1 \end{cases}$$



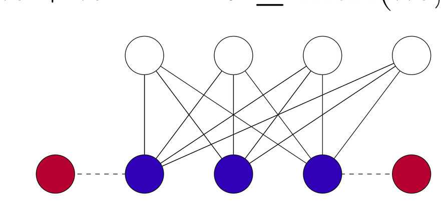
$Z^+_{(2)}(W_6) = 4$



$Z^+_{(2)}(W_9) = 6$

Complete Bipartites. Let $K_{m,n}$ denote the complete bipartite graph with parts of size m and n.

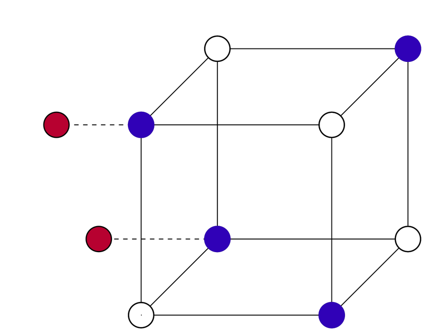
$$Z^+_{(\ell)}(K_{m,n}) = \begin{cases} \min(m,n) & \ell < \min(m,n) \\ \max(m,n) & \min(m,n) \leq \ell < \max(m,n) \\ m+n & \ell \geq \max(m,n) \end{cases}$$



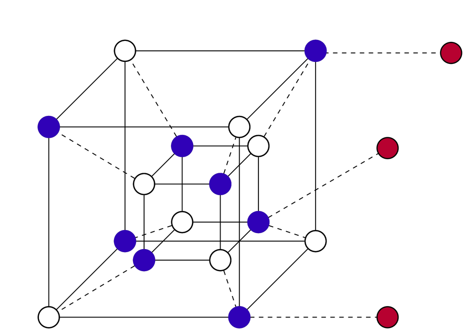
$Z^+_{(2)}(K_{3,4}) = 3$

Hypercubes. Let Q_d denote the hypercube of dimension d, on 2^d vertices.

$$Z^+_{(\ell)}(Q_d) = \begin{cases} 2^{d-1} & \ell \leq d-1 \\ 2^d & d \leq \ell \leq 2^d \end{cases}$$



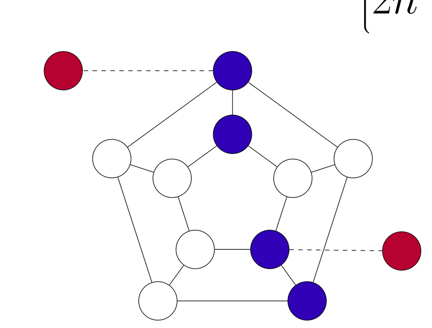
$Z^+_{(2)}(Q_3) = 4$



$Z^+_{(3)}(Q_4) = 8$

Prisms. Let $GP(n,1)$ denote the prism on $2n$ vertices.

$$Z^+_{(\ell)}(GP(n,1)) = \begin{cases} 3 & \ell = 0, 1, \ n = 3 \\ 4 & \ell = 2, \ n = 3 \\ 4 & \ell = 0, 1, \ n \geq 4 \\ 2(\lfloor \frac{n-1}{3} \rfloor) + 2 & \ell = 2, \ n \geq 4 \\ 2n & \ell \geq 3 \end{cases}$$

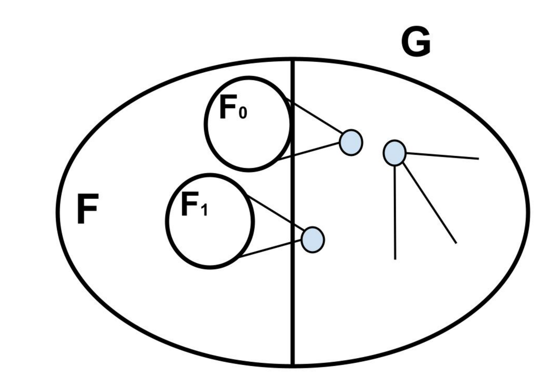


$Z^+_{(2)}(GP(5,1)) = 4$

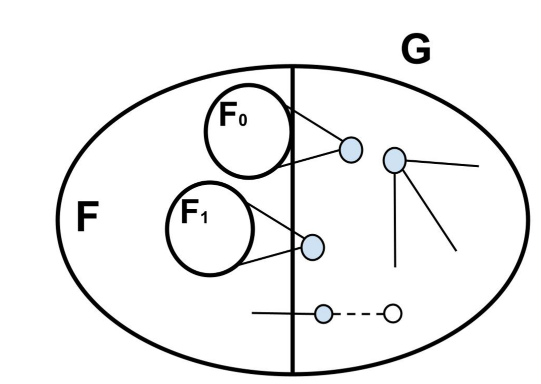
PSD-Leaky Forts

Forts are vertex subsets that prevent forcing. Our thought is there may be fort-related properties within families of graphs that create differences between the leaky forcing numbers.

Definition 2 *A **PSD fort** in a graph G is a subset of vertices, $F_+ \subset V(G)$ such that, for every vertex set F_i in $G[F_+]$ (the induced subgraph of F_+), and $\forall v \notin F_+$, $d_{F_i}(v) = 0$ or $d_{F_i}(v) \geq 2$.*



Definition 3 *A **PSD ℓ-leaky fort** in a graph G is a subset of vertices, $F^+_{(\ell)} \subset V(G)$, such that, for all components F_i in $G[F^+_{(\ell)}]$, and $\forall v \notin F^+_{(\ell)}$, $d_{F_i}(v) = 0$, $d_{F_i}(v) \geq 2$, or $|\{v \in V(G) \backslash F^+_{(\ell)} : |N_G(v) \cap F_i| = 1\}| \leq \ell$.*



Future Research Questions

- What structural properties of forts could point to differences in leaky forcing numbers?
- Are there other graph structures similar to forts that could explain differences in leaky forcing numbers?

References & Acknowledgements

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[2] Shannon Dillman and Franklin Kenter. *Leaky Forcing: A New Variation of Zero Forcing.* 2019. arXiv: 1910.00168 [math.CO].

[3] Rebekah Herrman. "The (d-2)-leaky forcing number of Q_d and ℓ-leaky forcing number of $G(n, 1)$". In: *Discrete Optimization* 46 (2022).

[4] Leslie Hogben, Jephian C.-H. Lin, and Bryan L. Shader. *Inverse Problems and Zero Forcing For Graphs.* Vol. 270. Mathematical Surveys and Monographs, 2022.

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