Natural Sciences Tripos, Part IA
Mathematical Methods II, Course B

Example Sheet 2
Functions of More than One Variable

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This example sheet is for course B. Starred questions are intended as extras; do them if you have time, but try to complete the others first. Please communicate any errors to `N.G.Berloff@damtp.cam.ac.uk`.

Skills section

Questions in this section are intended to give practice in routine calculations.

S1. For each of the following functions, evaluate the partial derivatives $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}$, and verify that $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)$.

(a) $f = x^3 - 2x^2y + 3xy^2 - 4y^3$ (b) $f = \exp(-x^2y^2)$

(c) $f = \exp(-x/y)$ (d) $f = \sin(x+y)$

(e) $f = 1/(x^2 + xy + 2y^2)$

S2. For each of the following functions, write out the differential expression $\mathrm{d}f = P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$ showing the explicit form of the functions $P(x,y)$ and $Q(x,y)$.

(a) $f = \exp[-1/(x+y)]$ (b) $f = \sinh x / \sinh y$

(c) $f = (x^2+y^2)^{1/2}$ (d) $f = \arctan(y/x)$

(e) $f = x^y$

S3. For each of the functions in question S1, find the locations of any stationary points. (You need not determine the type of the stationary points.)

Standard questions

Partial differentiation

4. For the function
$$f(x,y) = x(y^2 + 2y - 1)\,,$$
find the components of the vector $(\partial f/\partial x, \partial f/\partial y)$, known as the gradient vector, at the points $(-1,0)$, $(1,0)$, $(-1,1)$ and $(1,1)$. Make a sketch showing the directions of the gradient vector at these points.

5. The period T of a simple pendulum of length ℓ swinging in a gravitational field g is given by
$$T = 2\pi(\ell/g)^{1/2}\,.$$
Estimate the percentage error in a measurement of g resulting from

   (a) a 0.1 % error in measuring ℓ,

   (b) a 0.1 % error in measuring T.

   [*Hint: take logs before differentiating!*]

6. For $f(x,y) = \exp(-xy)$, find $(\partial f/\partial x)_y$ and $(\partial f/\partial y)_x$. Then find $(\partial f/\partial r)_\phi$ and $(\partial f/\partial \phi)_r$

   (i) by using the chain rule,

   (ii) by first expressing f in terms of polar coordinates (r,ϕ),

   and check that the two methods give the same results.

7. If $xyz + x^3 + y^4 + z^5 = 0$ (an implicit equation for any of the variables x, y, z in terms of the other two), find
$$\left(\frac{\partial x}{\partial y}\right)_z\,, \quad \left(\frac{\partial y}{\partial z}\right)_x\,, \quad \left(\frac{\partial z}{\partial x}\right)_y\,,$$
and show that their product is -1.

8. Van der Waals's equation
$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$
is an early (and in many ways a remarkably successful) attempt to represent the relation between the pressure p, volume V and temperature T of a real gas. (R, a and b are constants for a given mass of the gas.) Find expressions for $(\partial p/\partial V)_T$, $(\partial V/\partial T)_p$ and $(\partial T/\partial p)_V$, and verify that their product is -1.

9. $f(x,y)$ is a scalar function of position on the (x,y) plane. Position may also be specified by Cartesian coordinates u, v which are referred to axes rotated by an angle θ from the x and y axes. Show that
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\,,$$
i.e. the two-dimensional ∇^2 operator is invariant under a rotation of axes.

Exact differentials

10. Determine whether the following differentials $P\,\mathrm{d}x + Q\,\mathrm{d}y$ are exact. For those that are exact, find a function f such that $\mathrm{d}f = P\,\mathrm{d}x + Q\,\mathrm{d}y$. For those that are not, try to find an integrating factor $\mu(x)$ or $\mu(y)$. In each case, find the general solution of the differential equation $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$.

    (a) $y\,\mathrm{d}x + x\,\mathrm{d}y$

    (b) $y\,\mathrm{d}x + x^2\,\mathrm{d}y$

    (c) $(x+y)\,\mathrm{d}x + (x-y)\,\mathrm{d}y$

    (d) $(\cosh x \cos y + \cosh y \cos x)\,\mathrm{d}x - (\sinh x \sin y - \sinh y \sin x)\,\mathrm{d}y$

    (e) $(\cos x - \sin x)\,\mathrm{d}x + (\sin x + \cos x)\,\mathrm{d}y$

    (f) $(x\,\mathrm{d}y - y\,\mathrm{d}x)/(x^2+y^2)$

    (You may wish to discuss with your supervisor why case (f) presents difficulties.)

11. The enthalpy of a gas is defined by $H = U + pV$, where U satisfies $\mathrm{d}U = T\,\mathrm{d}S - p\,\mathrm{d}V$. Determine a relationship between the differentials of H, S and p. Hence show that

$$\left(\frac{\partial V}{\partial S}\right)_p = \left(\frac{\partial T}{\partial p}\right)_S .$$

    By regarding U as a function of p and V and considering two expressions for $\partial^2 U/\partial p\,\partial V$, show that

$$\left(\frac{\partial S}{\partial V}\right)_p \left(\frac{\partial T}{\partial p}\right)_V - \left(\frac{\partial S}{\partial p}\right)_V \left(\frac{\partial T}{\partial V}\right)_p = 1 .$$

12. Given that

$$\mathrm{d}U = T\,\mathrm{d}S - p\,\mathrm{d}V ,$$

    find a function G such that

$$\mathrm{d}G = V\,\mathrm{d}p - S\,\mathrm{d}T .$$

    Hence show that

$$\left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p .$$

13. The pressure p can be considered as a function of the variables V and T or as a function of the variables V and S.

    (i) Find an expression for

$$\left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S \quad \text{in terms of} \quad \left(\frac{\partial S}{\partial V}\right)_T \quad \text{and} \quad \left(\frac{\partial S}{\partial p}\right)_V .$$

    (ii) Hence, using $T\,\mathrm{d}S = \mathrm{d}U + p\,\mathrm{d}V$, show that

$$\left(\frac{\partial \ln p}{\partial \ln V}\right)_T - \left(\frac{\partial \ln p}{\partial \ln V}\right)_S = \left(\frac{\partial (pV)}{\partial T}\right)_V \left[\frac{p^{-1}(\partial U/\partial V)_T + 1}{(\partial U/\partial T)_V}\right] . \qquad (\star)$$

    $\left[\textit{Hint: } \left(\frac{\partial \ln p}{\partial \ln V}\right)_T = \frac{V}{p}\left(\frac{\partial p}{\partial V}\right)_T .\right]$

    (iii) For one mole of an ideal gas, you may assume that $U = C_{\mathrm{v}}T$, $pV = RT$, and pV^γ depends only on S, where C_{v}, R and γ are constants.
    Use equation $(\star)$ to find an expression for γ in terms of C_{v} and R.

    (iv) What is the value of γ for a monatomic gas for which $C_{\mathrm{v}} = \frac{3}{2}R$?

Stationary values

14. The height h of each point (x, y) of an area of land is given by

$$h(x, y) = \frac{a(x + y)}{x^2 + y^2 + a^2} ,$$

where a is a positive constant. Find the locations and heights of the highest and lowest points of the terrain, and also those along the x and y axes. Sketch a map of the region by showing contours of constant h in the (x, y) plane.

15. (i) Find the stationary points of the function

$$z = (x^2 - y^2)\,\mathrm{e}^{-x^2-y^2} .$$

    (ii) Find the contours on which $z = 0$ and examine the behaviour of z on the x and y axes. Hence, or otherwise, determine the character of the stationary points. Sketch the contours.

16. For each of the functions

    (a) $(x^2 + y^2 + 1)^{-1}$

    (b) $\sin x \sin y \quad (0 < x < \pi,\ 0 < y < \pi)$

    (c) $(xy - y)\exp(2x - x^2 - y^2) \qquad$ [*more difficult*]

find the stationary points and determine their character.

Conditional stationary values

17. Find, using Lagrange multipliers, the stationary values subject to the constraint $x^2 + y^2 = 1$ of

    (a)
    $$xy^2$$

    (b)
    $$\mathrm{e}^{-xy}$$

    and the points at which they occur. Check your answers by applying the constraint using the substitution $x = \cos\theta$, $y = \sin\theta$.

18. Show that the largest volume V of any cuboid inscribed inside the ellipsoid
    $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
    is $8abc/\sqrt{27}$.

19. The area A of a triangle with sides a, b and c is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. Show, using Lagrange multipliers, that of all triangles of given perimeter $2s$, the triangle of largest area is equilateral.

    Find (in terms of the perimeter) the largest possible area of a right-angled triangle of given perimeter.

20*. Show that the maximum value, r, of $(x^2+y^2+z^2)^{1/2}$, subject to the conditions
    $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,, \qquad \ell x + my + nz = 0\,,$$
    satisfies the equation
    $$\frac{\ell^2 a^2}{a^2 - r^2} + \frac{m^2 b^2}{b^2 - r^2} + \frac{n^2 c^2}{c^2 - r^2} = 0\,.$$
    Interpret the problem geometrically and state the geometrical significance of one of the Lagrange multipliers.

Boltzmann distribution

21. Let

$$W = \prod_{s=1}^{N} \frac{(g_s - 1 + n_s)!}{(g_s - 1)!\, n_s!} \qquad (\dagger)$$

and let the numbers n_s be subject to the conditions

$$\sum_{s=1}^{N} n_s = n\,, \qquad \sum_{s=1}^{N} n_s E_s = E\,.$$

If *both* g_s and n_s are assumed to be large, so that Stirling's approximation $\ln n! \approx n \ln n - n$ may be applied, show that $\ln W$, considered as a function of the numbers n_s, is greatest when

$$n_s = g_s \left[\mathrm{e}^{\beta(E_s - \mu)} - 1 \right]^{-1}\,, \qquad (\ddagger)$$

where β and $-\beta\mu$ are Lagrange multipliers.

Equation ($\ddagger$) is in fact the Bose–Einstein distribution and follows from the formula ($\dagger$) which takes into account the effects of quantum indistinguishability for bosons. However you do not need to know any quantum mechanics to do this question!

For fermions the formula for W turns out to be

$$W = \prod_{s=1}^{N} \frac{g_s!}{n_s!\,(g_s - n_s)!}\,.$$

Find the value of n_s for which this expression for W is maximized subject to the same conditions. This is the Fermi–Dirac distribution.

22*. A finite container is filled with a monatomic ideal gas of constant internal energy (total kinetic energy). It can be shown that the degeneracy of the kinetic energy states is proportional to $E^{1/2}$.

The number of particles n of kinetic energy E is therefore distributed as

$$n \propto E^{1/2}\, \mathrm{e}^{-\beta E - \alpha}\,,$$

where α and β are constants. Determine the most probable value and the expected value of the kinetic energy of a molecule.