ECE313 Cheat Sheet

The set of events $\mathcal{F}$ satisfy. $A \in \mathcal{F} \rightarrow A^c \in \mathcal{F}$; $A, B \in \mathcal{F} \rightarrow A \cup B \in \mathcal{F}$; $\Omega, \phi \in \mathcal{F}$. $A, B \in \mathcal{F} \rightarrow AB \in \mathcal{F}$

$E[X+b]=E[X]+b$; $Var(X+b)=Var(X)$; $E[aX]=aE[X]$; $Var(aX)=a^2 Var(X)$ $P(B|A)=\frac{P(AB)}{P(A)}$

A. B. C are independent if they are pairwise independent and $P(ABC)=P(A)P(B)P(C)$

Bernoulli distribution. $E[X]=p$, $Var(X)=p(1-p)$

Binomial distribution: $P_X(k)=\binom{n}{k}p^k(1-p)^{n-k}$ $(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k$ $E[X]=np$ $Var(X)=np(1-p)$

$k^*=\lfloor (n+1)p \rfloor$ k^* is the value that maximize $p(k)$

Geometric distribution: $P_L(k)=(1-p)^{k-1}p$ for $k \geq 1$ median: $p\{L \geq k\}=(1-p)^{k-1}$ $((1-p)^{k-1} \geq 1/2,\ (1-p)^k \leq 1/2)$

$p\{L>k\}=(1-p)^k$, $E[L]=1/p$. $Var(L)=\frac{1-p}{p^2}$

Bernoulli process. $X: 0\,1\,0\,0\,0\,1$; $C: \underset{\underset{C_0}{\uparrow}}{0}\,0\,1\,1\,1\,1\,2$; $L: 2, 4$; $S: \underset{\underset{S_0}{\uparrow}}{0}.\,2.\,6$ L_i has geometric distribution with p

Negative binomial distribution: $S_r: p(n)=\binom{n-1}{r-1}p^r(1-p)^{n-r}$ for $n \geq r$, $E[S_r]=\frac{r}{p}$, $Var(S_r)=\frac{r(1-p)}{p^2}$

Poisson distribution: $p(k)=\frac{e^{-\lambda}\lambda^k}{k!}$ $E[Y]=Var(Y)=\lambda=np$

Markov inequalities: $P\{Y \geq c\} \leq \frac{E[Y]}{c}$ Chebychev inequalities: $P\{|X-\mu| \geq d\} \leq \frac{\sigma^2}{d^2}$, $P\{|X-\mu| \geq a\sigma\} \leq \frac{1}{a^2}$

Confidence Interval: $P\{p \in (\hat{p}-\frac{a}{2\sqrt{n}}, \hat{p}+\frac{a}{2\sqrt{n}})\} \geq 1-\frac{1}{a^2}$ where $\hat{p}=\frac{X}{n}$ (mean)

Law of total probability. $P(A)=P(A|E_1)P(E_1)+\cdots+P(A|E_k)P(E_k)$

Bayes' formula: $P(E_i|A)=\frac{P(A|E_i)P(E_i)}{P(A)}$

Binary hypothesis testing: $P_{false\ alarm}=P(\text{declare } H_1 \text{ true}|H_0)$. $P_{miss}=P(\text{declare } H_0 \text{ true}|H_1)$. $P_e=P_{false\ alarm}+P_{mis}$

Maximum likelihood (ML) decision rule: $\Lambda(k)=\frac{P_1(k)}{P_0(k)}$ $\Lambda(X)\begin{cases} >1 & \text{declare } H_1 \text{ true} \\ <1 & \text{declare } H_0 \text{ true} \end{cases}$ or substitute 1 with $\tau=\frac{\pi_0}{\pi_1}$,

that is Maximum a posteriori probability (MAP) decision rule.

Union Bound: $P(A \cup B)=P(A)+P(B)-P(AB) \leq P(A)+P(B)$ trick: $P(A^cB)=P(B)-P(AB)$

$P(A_1 \cup A_2 \cup \cdots \cup A_m) \leq P(A_1)+P(A_2)+\cdots+P(A_m)$

a function is the CDF if ① F is nondecreasing ② $\lim_{c\to\infty}F(c)=1$, $\lim_{c\to-\infty}F(c)=0$ ③ $F_X(c)=F_X(c^+)$

$P\{|X-5|<0.1\}=P\{4.9 \leq X \leq 5.1\}=P\{X \leq 5.1\}-P\{X<4.9\}=F_X(5.1)-F_X(4.9^-)$ $\mu_X=\int_{-\infty}^{\infty}uf_X(u)du$ $E[g(X)]=\int_{-\infty}^{\infty}g(u)f_X(u)du$

eg. $E[aX^2+bX+c]=aE[X^2]+bE[X]+c$, $Var(X)=E[X^2]-\mu_X^2$, $Var(X^2)=E[X^4]-E[X^2]^2$, $E[X^2]=\int_{-\infty}^{\infty}u^2f_X(u)du$

uniform distribution: $f_X(u)=\begin{cases}\frac{1}{b-a} & a \leq u \leq b \\ 0 & \text{else}\end{cases}$ $E[X]=\frac{a+b}{2}$. $E[X^2]=\frac{a^2+ab+b^2}{3}$, $\mu_Y=\frac{(a-b)^2}{12}$

when $a=0, b=1$. $E[X^k]=\int_0^1 u^k du=\frac{1}{k+1}$

Exponential distribution:

$f_T(t)=\begin{cases}\lambda e^{-\lambda t} & t \geq 0 \\ 0 & \text{else}\end{cases}$ CDF: $F_T(t)=\begin{cases}1-e^{-\lambda t} & t \geq 0 \\ 0 & t<0\end{cases}$ $F_T^c(t)=P\{T>t\}=\begin{cases}e^{-\lambda t} & t \geq 0 \\ 1 & t<0\end{cases}$

$E[T^n]=\frac{n}{\lambda}E[T^{n-1}]$. $E[T]=\frac{1}{\lambda}$. $E[T^n]=\frac{n!}{\lambda^n}$. $Var(T)=\frac{1}{\lambda^2}$. $P(T>s+t|T>s)=e^{-\lambda t}=P(T>t)$

Poisson process: $P\{X=n\}=\frac{e^{-\lambda T}(\lambda T)^n}{n!}$, $P\{X_1=i|X=n\}=\binom{n}{i}p^i(1-p)^{n-i}$, $P\{X=n|X_1=i\}=\frac{e^{-\lambda(T-\tau)}(\lambda(T-\tau))^{n-i}}{(n-i)!}$ $\begin{pmatrix} 0 \to \tau . X_1 \\ \tau \to T : X_2 \\ 0 \to T : X \end{pmatrix}$

Erlang distribution: T_r denotes the time of r^{th} count of a Poisson process. $f_{T_r}(t)=\frac{\exp(-\lambda t)\lambda^r t^{r-1}}{(r-1)!}$, $E[T_r]=\frac{r}{\lambda}$, $Var(T_r)=\frac{r}{\lambda^2}$

Scaling rule for pdfs: $Y = aX + b \Rightarrow f_Y(v) = f_X(\frac{v-b}{a})\frac{1}{a}$ $F_Y(v) = F_X(\frac{v-b}{a})$

Gaussian (normal) distribution: $f(u) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(u-\mu)^2}{2\sigma^2})$ standard normal distribution: $N(0,1)$ denote by Φ

$Q(u) = 1 - \Phi(u) = \Phi(-u)$ $Y = \sigma X + \mu$. $P(X \geq c) = P\{\frac{X-\mu}{\sigma} \geq \frac{c-\mu}{\sigma}\} = Q(\frac{c-\mu}{\sigma})$ $\geq$ use Q, $\leq$ use Φ

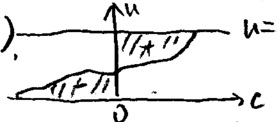
$Q(u) = \int_u^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{v^2}{2}) dv$. $f_X(u) = \Phi'(u) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})$

DeMoivre-Laplace limit theorem: $\lim_{n\to\infty} P\{\frac{S_{n,p} - np}{\sqrt{np(1-p)}} \leq c\} = \Phi(c)$. $S_{n,p}$ is a binomial random variable
(one version of central limit theorem)

Gaussian approximation with continuity correction: $\begin{cases} P\{X \leq k\} \approx P\{\hat{X} \leq k + 0.5\} \\ P\{X \geq k\} \approx P\{\hat{X} \geq k - 0.5\} \end{cases}$

Distribution of a function of a random variable: 1. Scope the problem $\rightarrow$ 2. Find the CDF of Y 3. Differentiate F_Y to find its derivative, f_Y
If Y is discrete type $\rightarrow$ 2. Find the pmf of Y

Suppose X is a continuous-type random variable with CDF F_X, $X \rightarrow F_X(X) \rightarrow U$ (uniform distributed with 0, 1).
$U \rightarrow F_X^{-1}(U) = X$. often ask to find $F_X^{-1}(U)$.



The area rule for expectation base on the CDF: $E[X] = \text{area}(+) - \text{area}(-) = \int_0^1 F_X^{-1}(u) du$.

Failure rate function: $h(t)$ (the possibility that the system fail in the next time unit) $= \frac{f_T(t)}{1 - F_T(t)}$ $\underset{\substack{\uparrow \\ \text{distribution}}}{F}(t) = 1 - e^{-\int_0^t \underset{\substack{\uparrow \\ \text{failure rate}}}{h(s)} ds}$

Binary hypothesis testing: $\Lambda(u) = \frac{f_1(u)}{f_0(u)}$, $\Lambda(X) \begin{cases} > \tau & \text{declare } H_1 \text{ is true} \\ < \tau & \text{declare } H_0 \text{ is true.} \end{cases}$

Joint CDF: $F_X(u) = F_{X,Y}(u, \infty)$, $F_Y(v) = F_{X,Y}(\infty, u)$ $\lim_{u\to-\infty} F(u,v) = 0$ for each v, $\lim_{v\to-\infty} F(u,v) = 0$ for each u, $\lim_{u\to\infty}\lim_{v\to\infty} F(u,v) =$

$E[aX + bY + c] = aE[X] + bE[Y] +$

Joint pdf: $F_{X,Y}(u_0, v_0) = \int_{-\infty}^{u_0} \int_{-\infty}^{v_0} f_{X,Y}(u,v) dv du$ $E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v) f_{X,Y}(u,v) du dv$. eg. $E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u f_{X,Y}(u,v) du dv$

Uniform joint pdfs: $f_{X,Y}(u,v) = \begin{cases} \frac{1}{\text{area of } S} & \text{if } (u,v) \in S \\ 0 & \text{else} \end{cases}$ $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v) dv$ $F_X(u_0) = P\{X \leq u_0\} = \int_{-\infty}^{u_0}[\int_{-\infty}^{\infty} f_{X,Y}(u,v) dv] d$
$\mu_X = (a+b)/2$, $Var(X) = (b-a)^2/12$ $f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v) du$

$P_{Y|X}(v|u_0) = \frac{P_{X,Y}(u_0,v)}{P_X(u_0)}$ $f_{Y|X}(v|u_0) = \frac{f_{X,Y}(u_0,v)}{f_X(u_0)}$ $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v) dv$, $f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v) du$ $E[Y|X=u] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u) dv$.

Independence: $F_{X,Y}(u_0, v_0) = F_X(u_0) F_Y(v_0)$, $f_{X,Y}(u,v) = f_X(u) f_Y(v)$ requirement: $f_X(u) = 0$ or $f_{Y|X}(v|u) = f_Y(v)$ for all $v \in R$

Product set: A product set has swap property: if $(a,b) \in S$, $(c,d) \in S$, then $(a,d), (b,c) \in S$ (always like a rectangle)
(X,Y) is uniformly distributed over set S. X. Y are independent iff S is a product set.

Distribution of sums: discrete: $p_S(k) = \sum_j p_X(j) p_Y(k-j)$ continuous: $f_S(c) = \int_{-\infty}^{\infty} f_X(u) f_Y(c-u) du$ ($f_X * f_Y(c)$).

if X, Y are both Gaussian Distribution: $f_{X+Y}(c) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-c^2/2\sigma^2)$ $\sigma^2 = \sigma_1^2 + \sigma_2^2$ $F_S(c) = P\{S \leq c\} = \int_{-\infty}^{\infty}(\int_{-\infty}^{c-u} f_{X,Y}(u,v) dv) du$.

Transformation of pdfs: $\begin{pmatrix} W \\ Z \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$ where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\begin{cases} W = aX + bY \\ Z = cX + dY \end{cases}$ suppose $\binom{X}{Y}$ is in u-v plane, $\binom{W}{Z}$ is in α-β plane. $f_{W,Z}(\alpha,\beta) = \frac{1}{|\det A|} f_{X,Y}(u,v)$

$f_{W,Z}(\alpha,\beta) = \frac{1}{|\det A|} f_{X,Y}(A^{-1}\binom{\alpha}{\beta})$ if $\binom{W}{Z} = g(\binom{X}{Y})$, $f_{W,Z}(\alpha,\beta) = \frac{1}{|\det J|} f_{X,Y}(g^{-1}(\binom{\alpha}{\beta}))$. $|\det(J)| = \frac{\text{area}(S) \leftarrow \text{image of } R \text{ under mapping}}{\text{area}(R) \leftarrow \text{set } R \text{ near } (u,v)}$
(g is a one to one mapping)

$J = J(u,v) = \begin{pmatrix} \frac{\partial g_1(u,v)}{\partial u} & \frac{\partial g_1(u,v)}{\partial v} \\ \frac{\partial g_2(u,v)}{\partial u} & \frac{\partial g_2(u,v)}{\partial v} \end{pmatrix}$

$Var(X + 2Y) = Var(X) + 4Cov(X,Y) + 4Var(Y)$

Correlation: the correlation $E[XY]$, the covariance $Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$, correlation coefficient $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$Cov(X,Y) = E[XY] - E[X]E[Y]$, $Cov(X+Y, U+V) = Cov(X,U) + Cov(X,V) + Cov(Y,U) + Cov(Y,V)$

$Cov(aX+b, cY+d) = ac\,Cov(X,Y)$, $Var(X \pm Y) = Var(X) + Var(Y) \pm 2Cov(X,Y)$ $Var(X) = Cov(X,X)$

$S_n = X_1 + X_2 + \cdots + X_n$ (X_i are uncorrelated, $Cov(X_i, X_j) = 0$ if $i \neq j$). $E[X_i] = \mu$. $Var(X_i) = \sigma^2$. $E[S_n] = n\mu$. $Var(S_n) = n\sigma^2$

Standardized version of S_n is $\frac{S_n - n\mu}{\sqrt{n\sigma^2}}$

$Cov(\frac{X - E[X]}{\sigma_X}, \frac{Y - E[Y]}{\sigma_Y}) = \rho_{X,Y}$, $\rho_{aX+b, cY+d} = \rho_{X,Y}$ $\rho_{X,Y} > 0$ means X, Y have the same trend, $\rho_{X,Y} < 0$ means they are opposite

$\rho_{X,Y} = \pm 1 \rightarrow Y = aX + b$

Schwarz's inequality $|E[XY]| \le \sqrt{E[X^2]E[Y^2]}$ if $E[X^2] \neq 0$ ($P\{Y=cX\}=1$ for some const c),

$|E[XY]| = \sqrt{E[X^2]E[Y^2]}$

if $Var(X), Var(Y) \neq 0$, $\begin{cases} |\rho_{X,Y}| \le 1 \\ \rho_{X,Y}=1, Y=aX+b, a>0 \\ \rho_{X,Y}=-1, Y=aX+b, a<0 \end{cases}$

For two random X, Y. $|Cov(X,Y)| \le \sqrt{Var(X)Var(Y)}$ if $Var(X) \neq 0$, $Y=aX+b$, equality holds

Minimum mean square error estimation ($MSE = E[(Y-g(X))^2]$, make MSE small)

① Constant estimator $\delta = E[Y]$. minimum $MSE = Var(Y)$.

② unconstrained estimator $g^*(u) = E[Y|X=u] = \int_{-\infty}^{\infty} v f_{Y|X}(v|u)dv$ minimum $MSE = E[(Y-E[Y|X])^2] = E[Y^2] - E[(E[Y|X])^2]$

③ linear estimator $L^*(X) = \hat{E}[Y|X] = \mu_Y + \left(\frac{Cov(Y,X)}{Var(X)}\right)(X-\mu_X) = \mu_Y + \sigma_Y \rho_{XY}\left(\frac{X-\mu_X}{\sigma_X}\right)$

Minimum $MSE = \sigma_Y^2 - \frac{(Cov(X,Y))^2}{Var(X)} = \sigma_Y^2(1-\rho_{X,Y}^2) = \sigma_Y^2 - Var(\hat{E}[Y|X]) = E[Y^2] - E[\hat{E}[Y|X]^2]$

$\left(Var(\hat{E}[Y|X]) = \left(\frac{\sigma_Y \rho_{XY}}{\sigma_X}\right) \right.$

$\left. Var(X) = \sigma_Y^2 \rho_{XY}^2 \right.$

② ≤ ③ ≤ ①, ①②③ all holds: $E[aY+bZ+c] = aE[Y] + bE[Z] + c$, $E[aY+bZ+c|X] = aE[Y|X] + bE[Z|X] + c$

$\hat{E}[aY+bZ+c|X] = a\hat{E}[Y|X] + b\hat{E}[Z|X] + c$

Law of large numbers. $X_1, X_2 \cdots$ is a sequence of uncorrelated random varialle with $E[X_k]=\mu$, $Var(X_k) \le C$, for any $\delta > 0$

$P\left\{\left|\frac{S_n}{n} - \mu\right| \ge \delta\right\} \le \frac{C}{n\delta^2} \xrightarrow{n\to\infty} 0$. $Var(S_n) = Cov(X_1+\cdots+X_n, X_1+\cdots+X_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} Cov(X_i, X_j) = \sum_{i=1}^{n-1} Cov(X_i, X_{i+1}) + \sum_{i=1}^{n} Cov(X_i, X_i)$

Central limit theorem. $X_1, X_2 \cdots$ are independent, identically distributed random variables, with mean μ and σ^2, $S_n = X_1+\cdots+X_n$ $+ \sum_{i=2}^{n} Cov(X_i, X_{i-1})$

$\lim_{n\to\infty} P\left\{\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \le c\right\} = \Phi(c)$

Joint Gaussian distribution: recognize whether a pdf is a bivariate normal: $f_{X,Y}(u,v) = C \cdot \exp(-P(u,v))$, $P(u,v) = au^2+buv+cv^2+du+e$

$P(u,v) \to +\infty$ as $|u|+|v| \to \infty$, $b^2-4ac<0$.

$f_{XY}(u,v) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{\left(\frac{u-\mu_X}{\sigma_X}\right)^2 + \left(\frac{v-\mu_Y}{\sigma_Y}\right)^2 - 2\rho\left(\frac{u-\mu_X}{\sigma_X}\right)\left(\frac{v-\mu_Y}{\sigma_Y}\right)}{2(1-\rho^2)}\right)$

standard normal pdf: $f_{W,Z}(\alpha,\beta) = \left(\frac{e^{-\frac{\alpha^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}}\right) = \frac{e^{-\frac{\alpha^2+\beta^2}{2}}}{2\pi}$

Suppose X, Y have bivariate normal pdf with parameter $\mu_X, \mu_Y, \sigma_X, \sigma_Y$ and ρ.

a) X has the $N(\mu_X, \sigma_X^2)$ distribution, Y has the $N(\mu_Y, \sigma_Y^2)$ distribution

b) $aX+bY$ is a Gaussian random variable

c) X and Y are independent iff $\rho=0$

d) for estimation of Y from X, $L^*(X) = g^*(X)$, $E[Y|X] = \hat{E}[Y|X]$, i.e. ② = ③

e) the conditional distribution of Y given $X=u$ is $N(\hat{E}[Y|X=u], \sigma_e^2)$, where σ_e^2 is the MSE for $\hat{E}[Y|X]$

if $X_1, X_2, \cdots X_n$ are mutually indepedent, $Y = \sum_{i=1}^{n} c_i X_i$ follows the normal distribution $N\left(\sum_{i=1}^{n} c_i\mu_i, \sum_{i=1}^{n} c_i^2\sigma_i^2\right)$

for $\mu=0, \sigma=1$, $f_{X,Y}(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{u^2+v^2-2\rho uv}{2(1-\rho^2)}\right) = \underbrace{\left[\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)\right]}_{f_X(u)\ N(0,1)} \underbrace{\left[\frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{(v-\rho u)^2}{2(1-\rho^2)}\right)\right]}_{f_{Y|X}(v|u)}$ (Y has mean ρu, variance $1-\rho^2$)

$\boxed{\text{use } Var(X) = Cov(X,X)\,!}$

Trick: $E[Z^2] = E[Z]^2 + Var(Z)$

$\sum_{k=1}^{n} r^k = \frac{r(1-r^n)}{1-r}$, $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$