GEODESY
and
GEODESIC
TRANSFORMATIONS

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Introduction

In MM, structuring elements can be defined in different ways:

- By their geometry
- In an explicit way (list of points)
- With a distance:

$$B_\lambda (z) = \{y, d(z,y) \le \lambda\}$$

→ Trivial when the euclidean distance is used

→ Much more interesting when a non euclidean distance is used (geodesic distance)

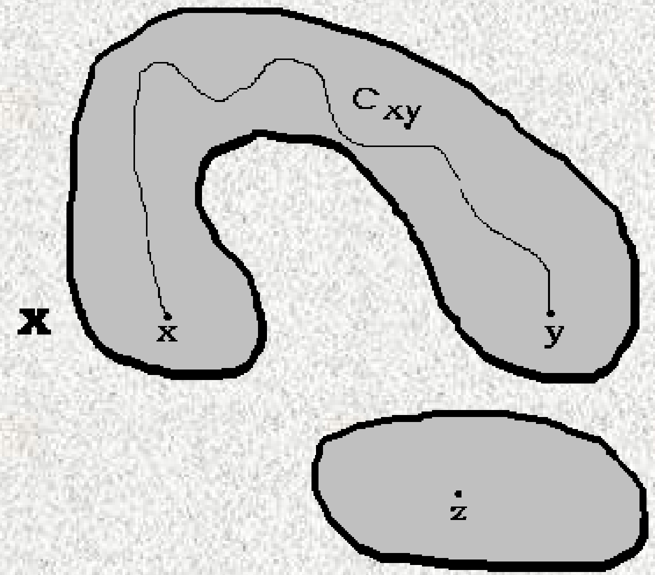
Lecture contents

- The geodesic distance
- Set geodesic erosions and dilations
- Set reconstruction, applications
- Extension to numerical functions
- Reconstruction of functions
- Generalised geodesic distance

Notion of path

Definition

If X is a topological space and if x and y are two points of X, a **path from x to y** is defined as any continuous application $\gamma\ [0,1] \rightarrow E$ such that $\gamma(0)=x$ and $\gamma(1)=y$.
Points x and y are linked if and only if there exists a path starting in x and ending in y.

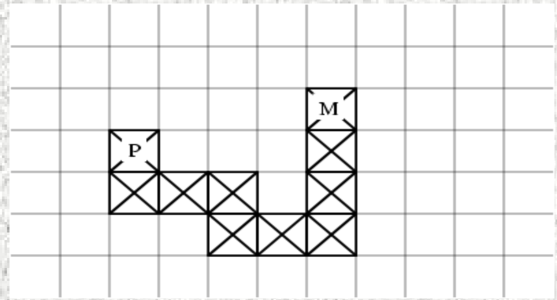


Digital path

X, binary image, subset of Z^2, equipped with a neighbourhood relationship ν (reflexive and symmetrical relation).
A path of length k - 1 is a sequence of k points such that:

$$\forall i, \quad 1 \leq i \leq k-1 \quad p_{i+1} \nu\ p_i$$

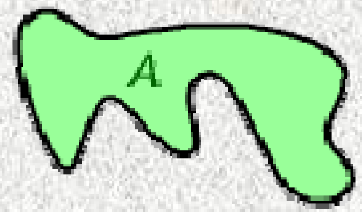
Example : path between P and M in 4-connectivity.



Connectivity, connected component

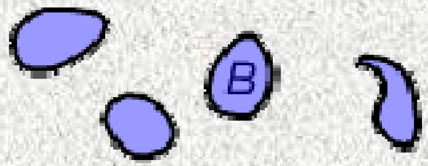
Connected set

Let X be a topological space. X is connected if it is not the union of two disjoint and non empty open sets (or of two disjoint and non empty closed sets).
X is « all in one piece »



Arc-connected sets

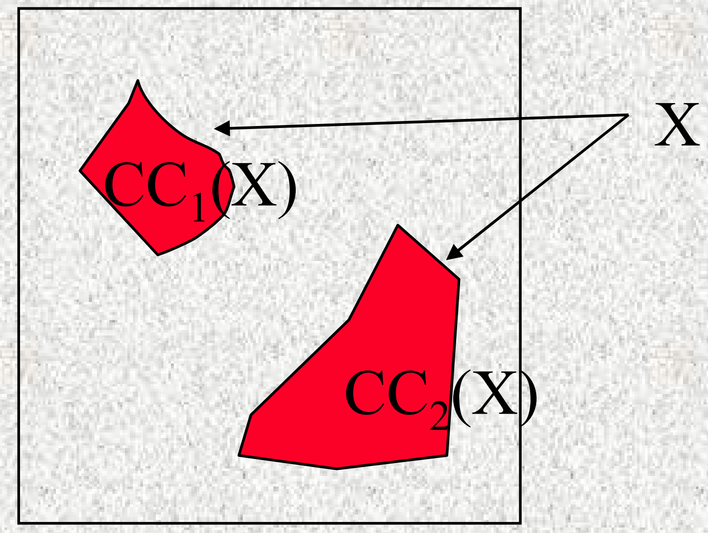
A set X is arc-connected if and only if every couple of points of X is linked by a path.



A subset Y of X is arc-connected if and only if any couple of Y points is linked by a path included in Y.

Connected component

Given a point x in a set X, the largest connected part containing x is called connected component C_x of x in X.
« to be connected» defines an equivalence relationship.
The equivalence classes are called connected components of X.



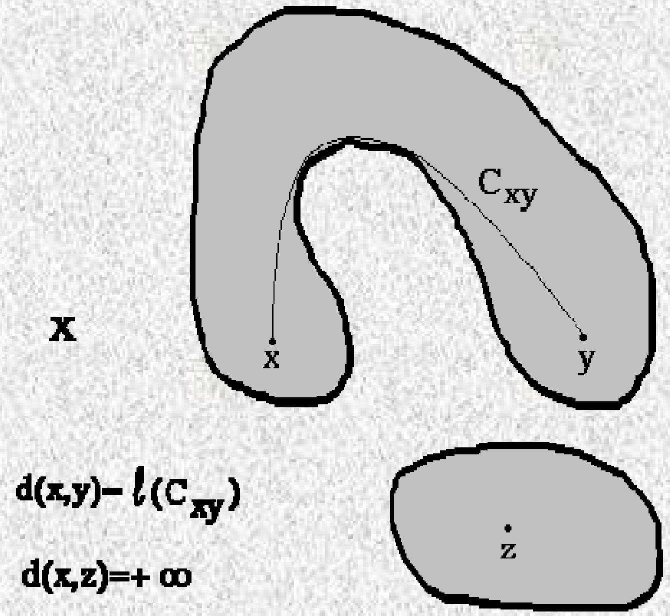
Geodesic distance

Definition

The geodesic distance d_X: E x E $\rightarrow$ R+, is defined in the geodesic space X by:

$d_X(x,y)$ = Inf. of lengths of paths with extremities x and y included in X

$d_X(x,y) = +\infty$, if no path exists



Properties

1) d_X is a distance:

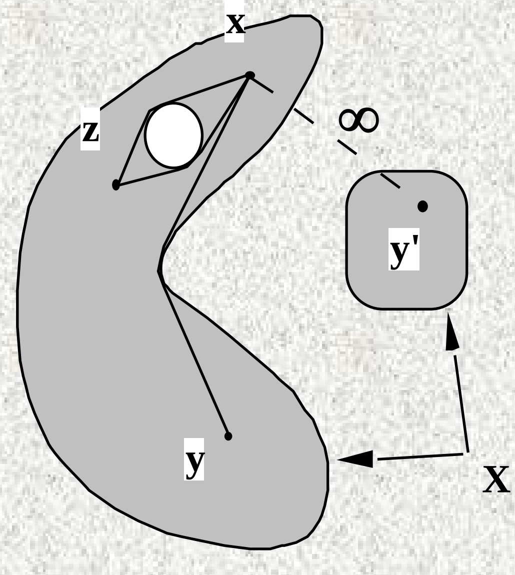
$d_X(x,y) = d_X(y,x)$

$d_X(x,y) = 0 \Leftrightarrow x = y$

$d_X(x,z) \leq d_X(x,y) + d_X(y,z)$

2) The geodesic distance is always greater than (or equal to) the euclidean distance.

3) A minimal geodesic path is not necessarily unique.

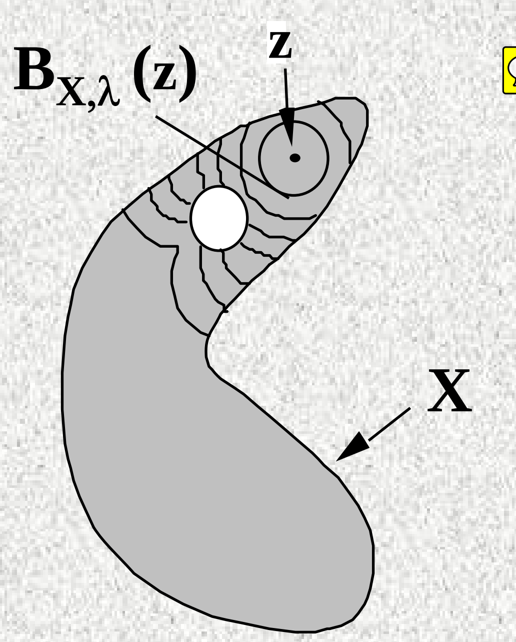


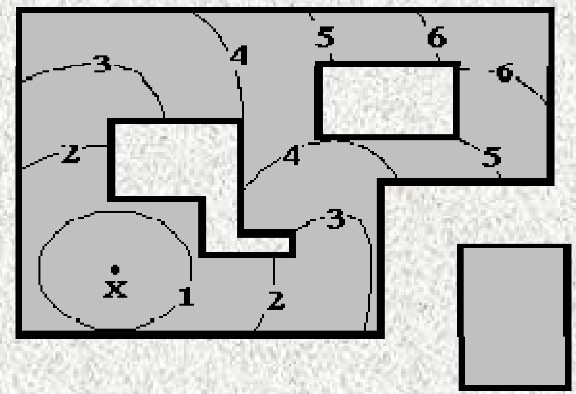
Geodesic balls

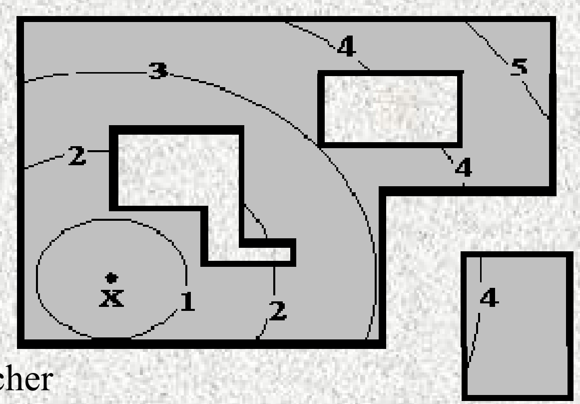
- The introduction of the geodesic distance allows to define the notion of geodesic ball:

$$B_{X,\lambda}(z) = \{y,\ d_X(z,y) \leq \lambda\}$$

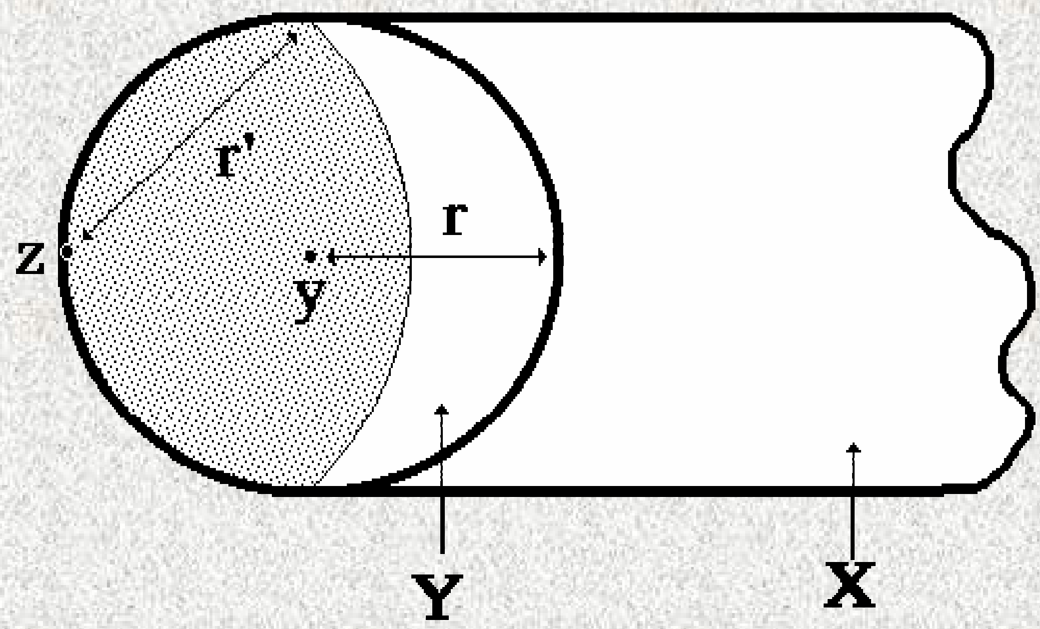
- When the radius r increases, the boundaries of the successive balls draws a propagation front in medium X.
- For a given radius λ, the balls $B_{X,\lambda}$ can be seen as structuring elements whose shape varies from place to place.







Characteristics of geodesic balls



X, geodesic space

Attention!

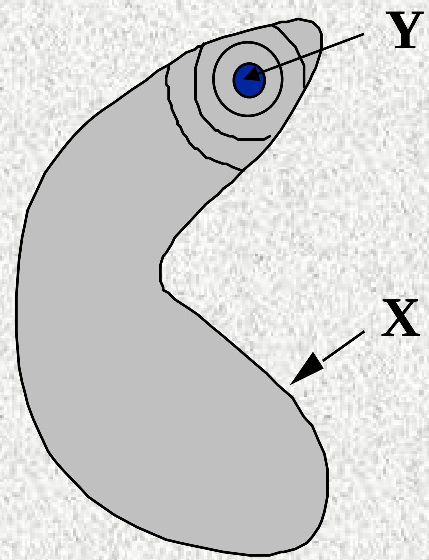
A geodesic ball $B_X(y,r)$ of radius r and placed at point y may contain another ball $B_X(z,r')$ of radius r' and placed at point z **<u>even if</u>** r'>r.

Set geodesic dilation

The size λ geodesic dilation of Y in the geodesic space X is defined by:

$$\delta_{X,\lambda}(Y) = \cup \{B_{X,\lambda}(y),\ y \in Y\}$$

$$\delta_{X,\lambda}(Y) = \{y \in Y : B_{X,\lambda}(y) \cap Y \neq \emptyset\}$$

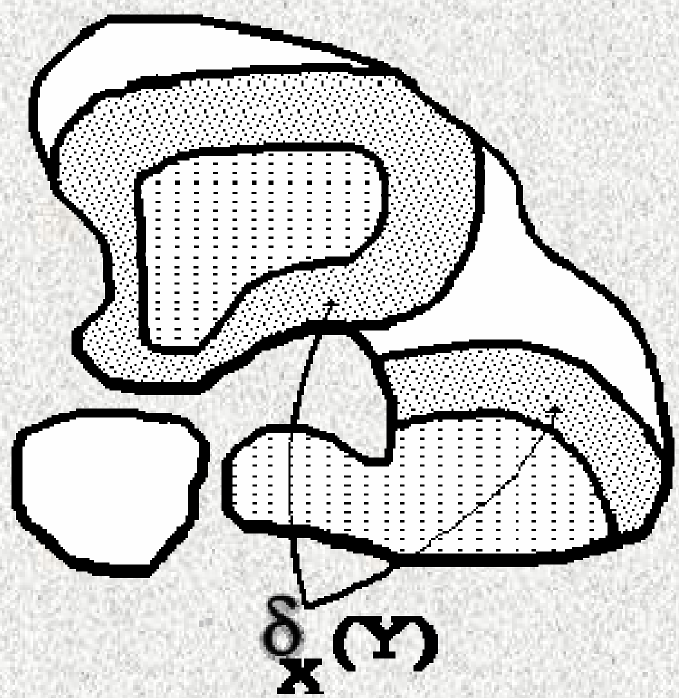


- this transformation fulfils the following property:

$$\delta_{X,\lambda+\mu} = \delta_{X,\lambda}[\,\delta_{X,\mu}]$$

- δ is increasing and extensive
- δ is also increasing when it is considered as a transformation applied to the geodesic space X (Y fixed)

$$X \subset X' : \delta_X(Y) \subset \delta_{X'}(Y)$$



Digital geodesic dilation

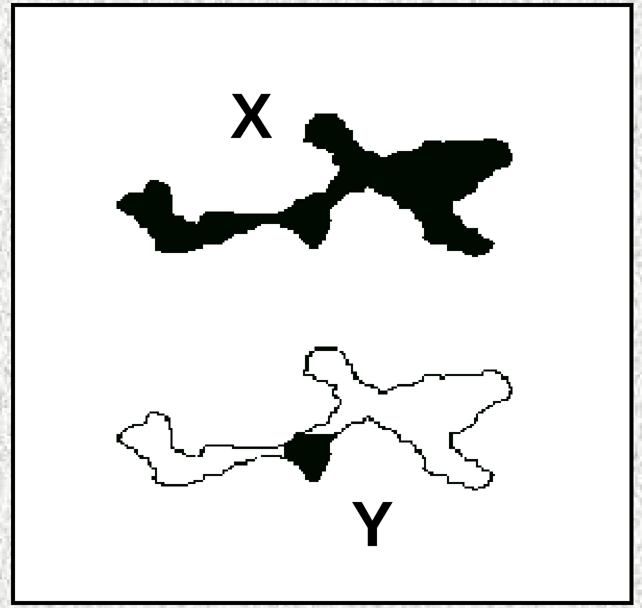
- When E is a digital metric space and when d(x) is the dilation by a unit ball centered at point x, then the unit geodesic dilation is defined by the formula:

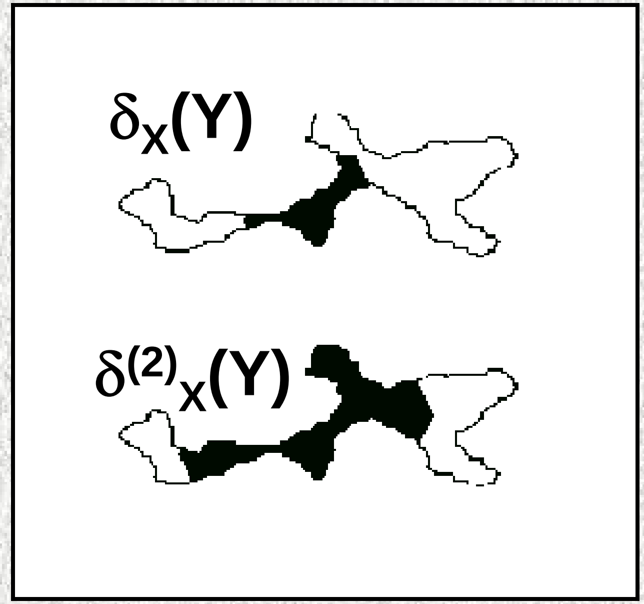
$$\delta_X(Y) = \delta(Y) \cap X$$

- The size n dilation is built by iteration:

$$\delta_{X,n}(Y) = \delta(\ldots\delta(\delta(Y) \cap X) \cap X)\ldots \cap X$$

- Note that geodesic dilations are not translation invariant.





Geodesic erosion

The geodesic erosion is defined by:

$$\varepsilon_{X,\lambda}(Y) = \{y \in Y : B_{X,\lambda}(y) \subset Y\}$$

It can be defined by duality (by adjunction or by complementation).
The complement is defined with respect to the geodesic space X ($Y \rightarrow X \setminus Y = X \cap Y^C$):

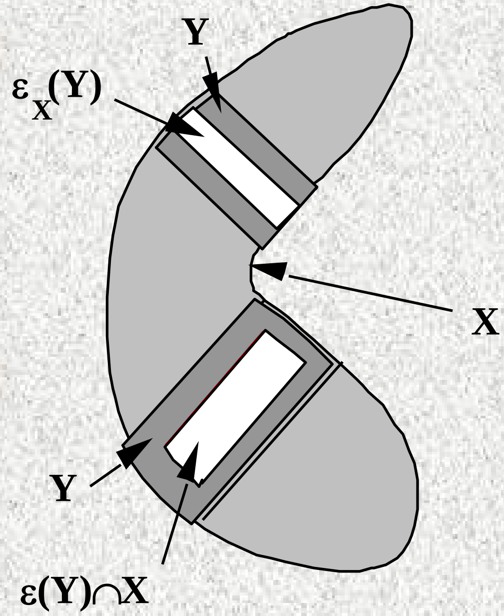
$$\varepsilon_X(Y) = X \setminus \delta_X (X \setminus Y)$$

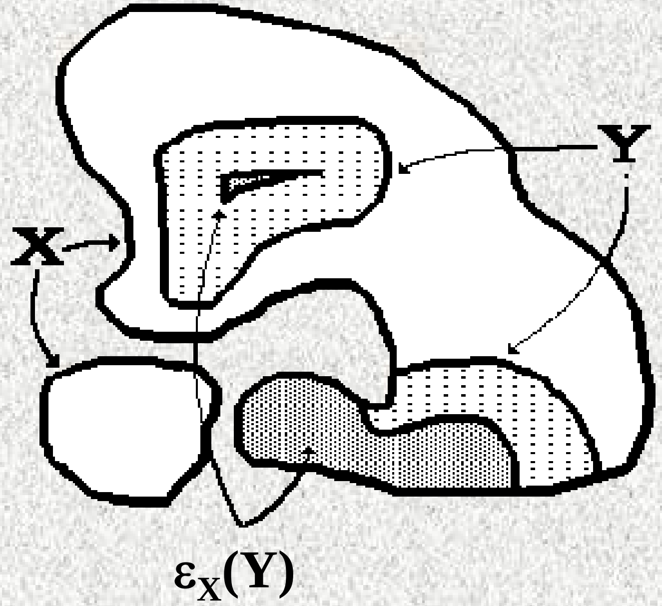
The elementary digital geodesic erosion is defined by:

$$\varepsilon_X(Y) = \varepsilon (X^c \cup Y) \cap X$$

ε is the elementary euclidean erosion.

Note the difference between $\varepsilon_X(Y)$ and $\varepsilon(Y) \cap X$.



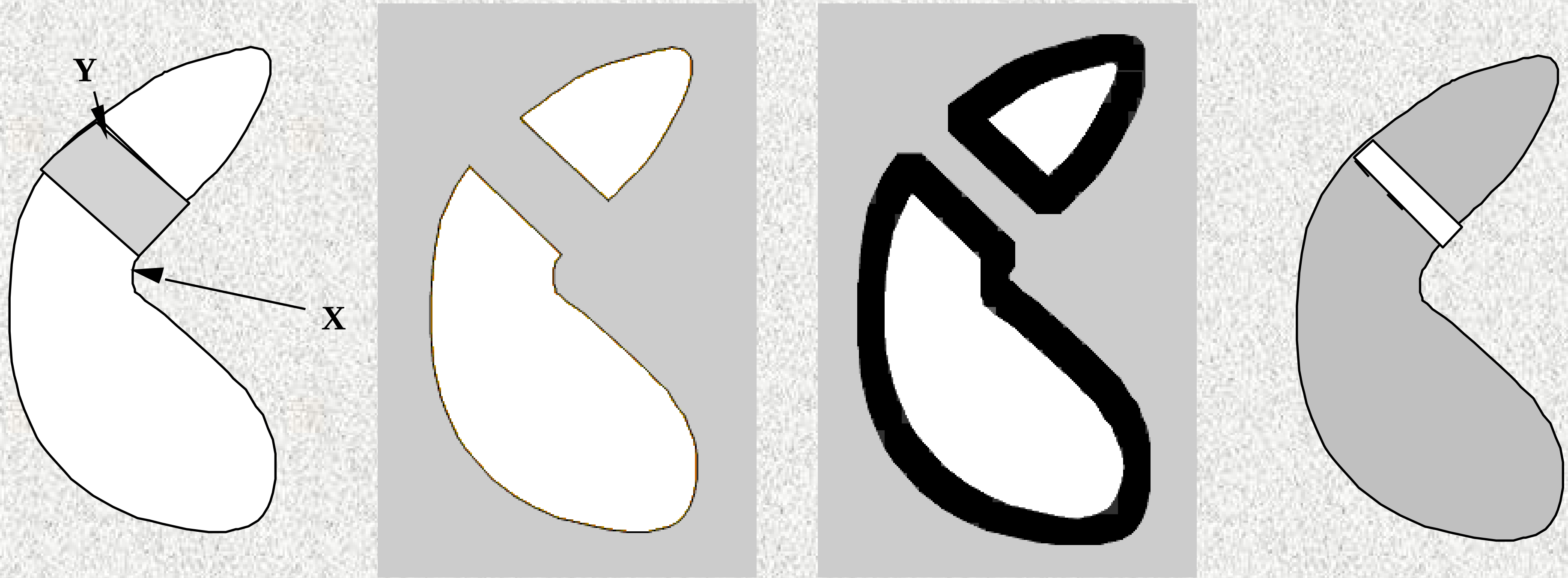


Geodesic erosion, construction

$$\varepsilon_X(Y) = X \setminus \delta_X (X \setminus Y) = X \cap \{\delta (X \cap Y^C) \cap X\}^C$$

$$\varepsilon_X(Y) = X \cap \{[\delta (X \cap Y^C)]^C \cup X^C\} = \{X \cap X^C\} \cup \{X \cap [\delta (X \cap Y^C)]^C\}$$

$$\varepsilon_X(Y) = X \cap \varepsilon (X^c \cup Y)$$



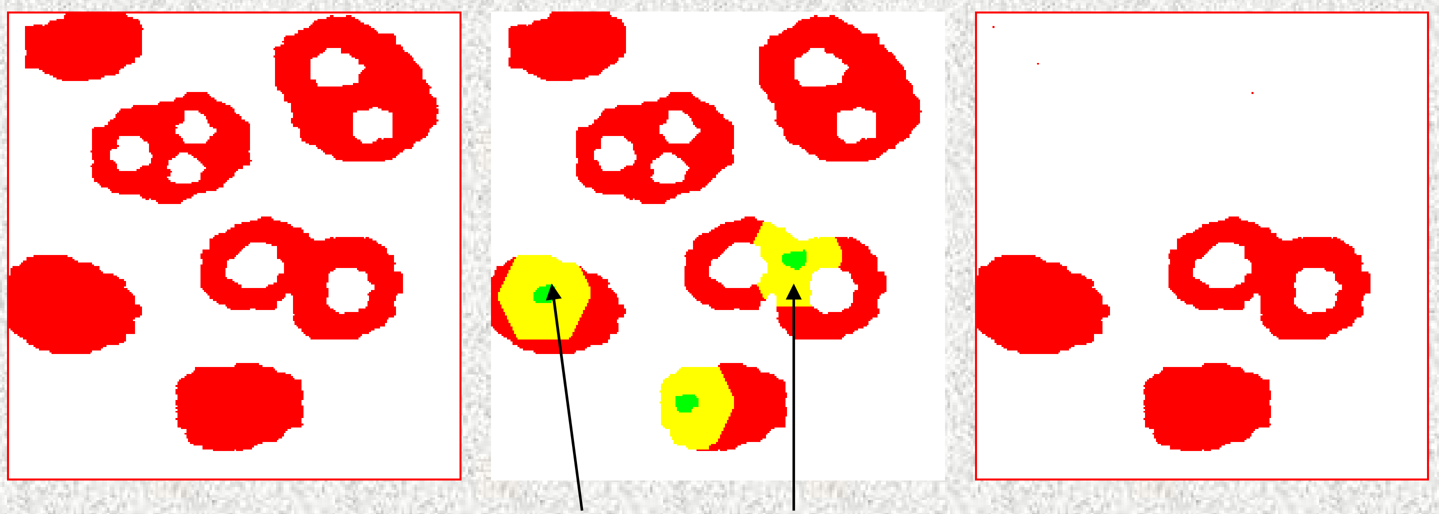
$X^c \cup Y$ | **$\varepsilon (X^c \cup Y)$** | **$X \cap \varepsilon (X^c \cup Y)$**

In practice, this formula is not used because it requires the complementary set X^c of the geodesic space which is not always available (in particular when X is identical to the analysis field).

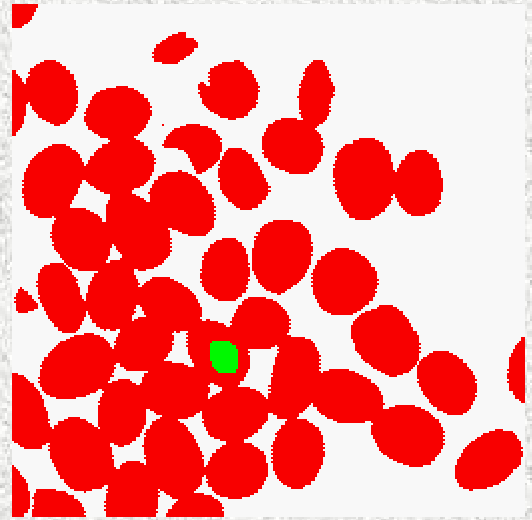
Geodesic reconstruction

Iteration of geodesic dilations until idempotence

$$R_X(Y) = \delta_X^{+\infty}(Y) = \lim_{\lambda \to +\infty} \delta_{X,\lambda}(Y)$$



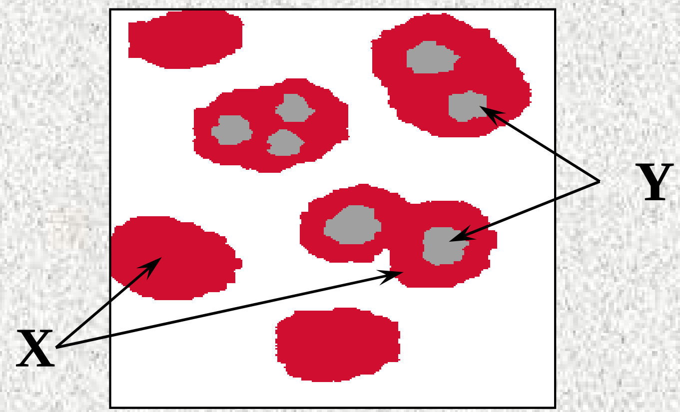
X $\quad Y \quad \delta_{X,\lambda}(Y) \quad R_X(Y)$

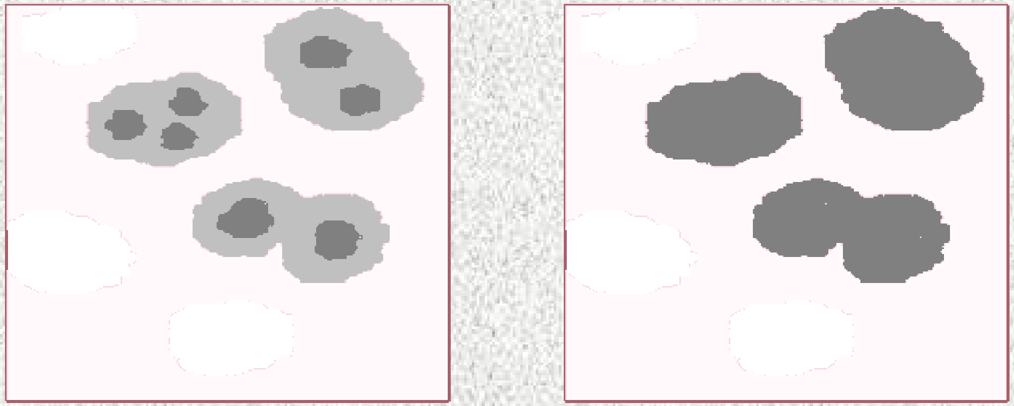


This operator allows the reconstruction of all the connected components of X marked by Y (reconstruction of X by Y).

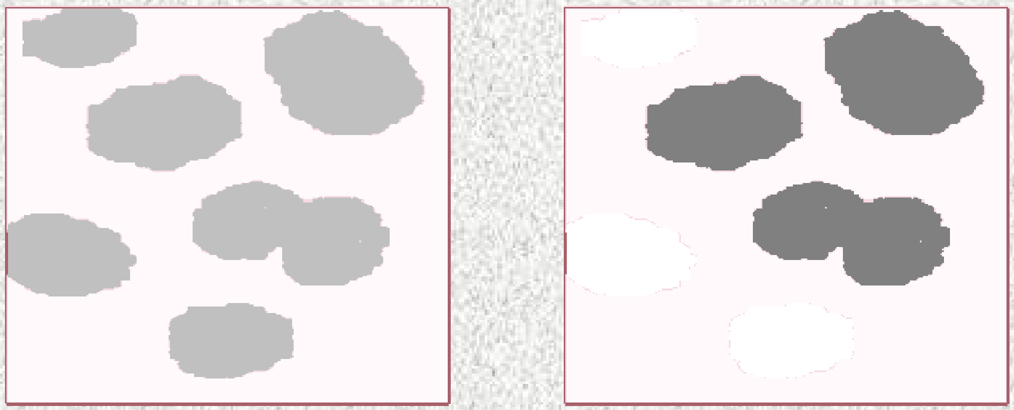
Reconstruction and opening

- Given X, the geodesic reconstruction of X by Y is a closing with respect to Y.





- But if we consider this reconstruction as an operation on the (varying) set X, with Y fixed, then this transformation is an opening



Using reconstruction

- If Y is an isolated point → point opening, individual particle analysis
- If Y is an erosion (or an opening) → Opening by reconstruction
- If Y is the intersection of the image edge with X → Selection of the connected components of X touching the edge of the image
- If Y is the intersection of the image edge with the complementary of X → Background reconstruction

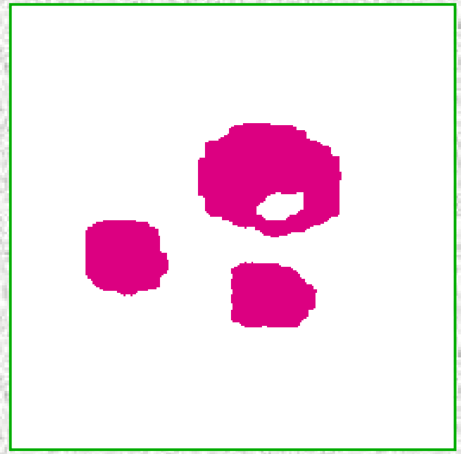
Particle individual analysis

Algorithm

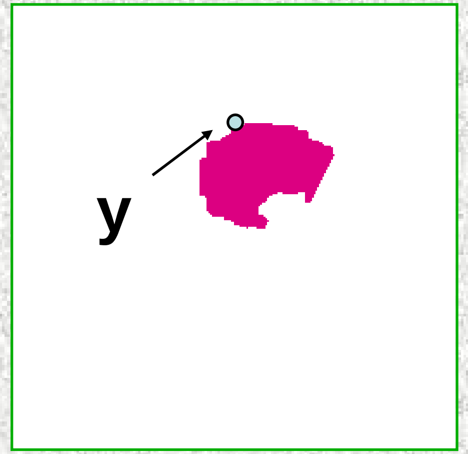
While the set is not empty
{ - extract y, first point of X (video scan order or any other else);

- Z = $R_X(y)$ reconstruction of X by y;
- Analysis of Z;

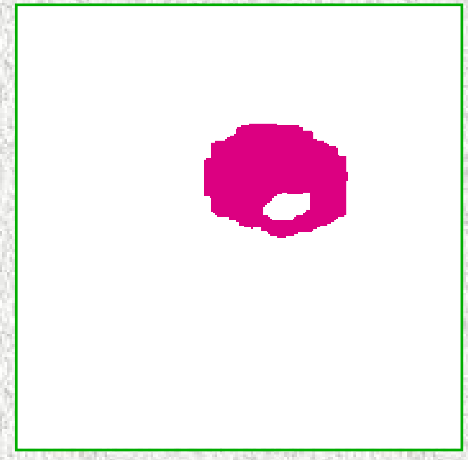
-X := X\Y (difference)
}



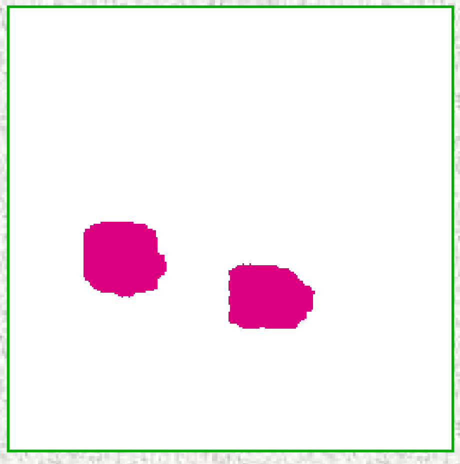
X



$\delta_{X,\lambda}(y)$



$R_X(y)$

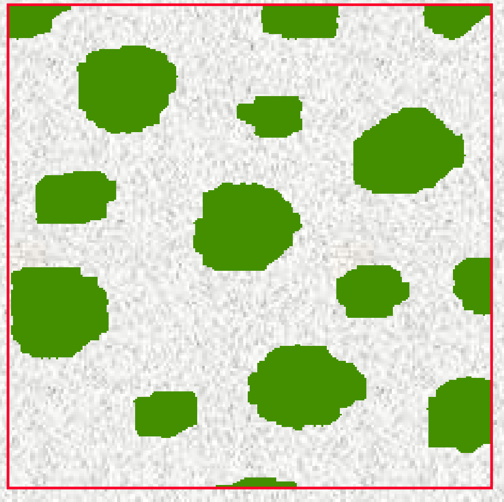


X := X\Y

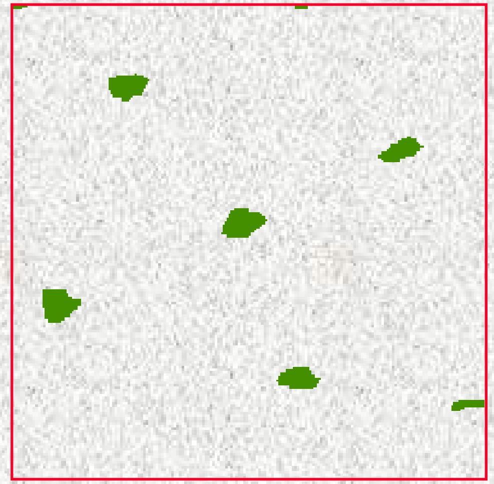
Filter by erosion-reconstruction

- The erosion $X \Theta B_\lambda$ removes first the connected components of X of size less than λ (they cannot contain a disk of size λ)

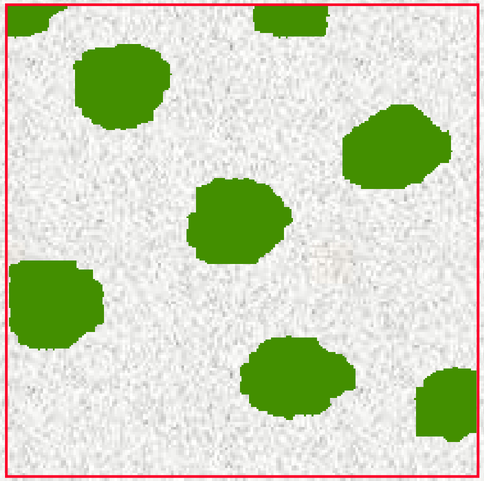
- Then the opening $\gamma^{Rec}{}_X(Y) = R_X(Y)$ by the marker set $Y = X \Theta B_\lambda$ rebuilds the remaining connected components



Initial image



Erosion of X by a disk of size λ

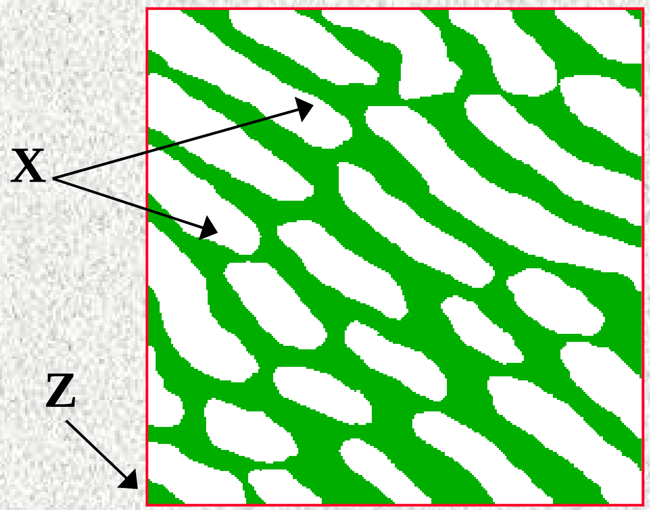


Reconstruction

The operation is the same if $Y = \gamma_\lambda(X) = (X \Theta B_\lambda) \oplus B_\lambda$

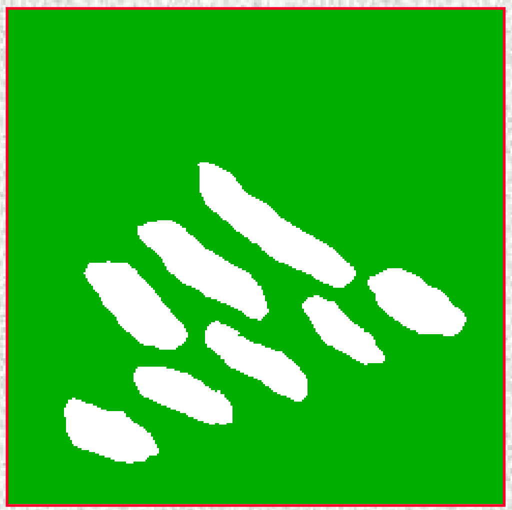
Particles at the edge of the field

- Let Z be the edges of the image and X the particles under study
- The set Y is the reconstruction of X by $Z \cap X$
- The difference between X and Y extracts the interior particles.



Initial image

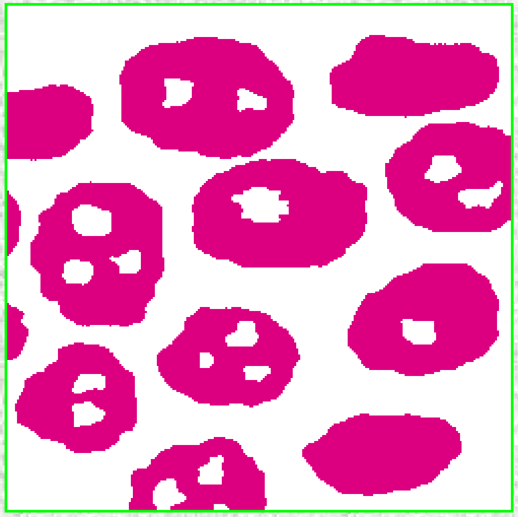
Particles touching the edges



Difference

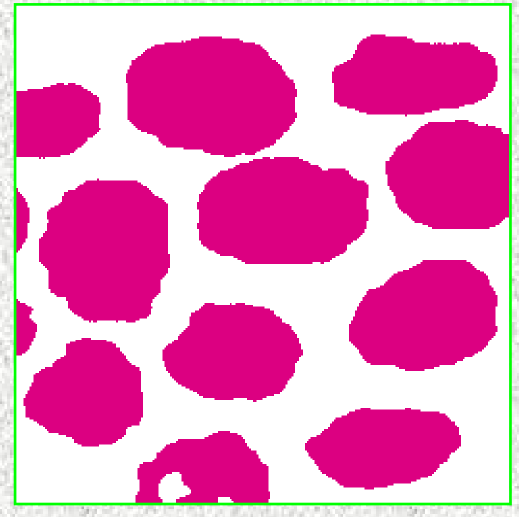
Filling holes

- Let Z be the edges of the image and X the grains under study
- The set Y is the reconstruction of X by $Z \cap X^c$
- The complementation of Y fills the holes.



Initial image X

Edges included in X^c



Inverted Y

Some particles cutting the edges are not correctly filled….
Enhancement suggestions?

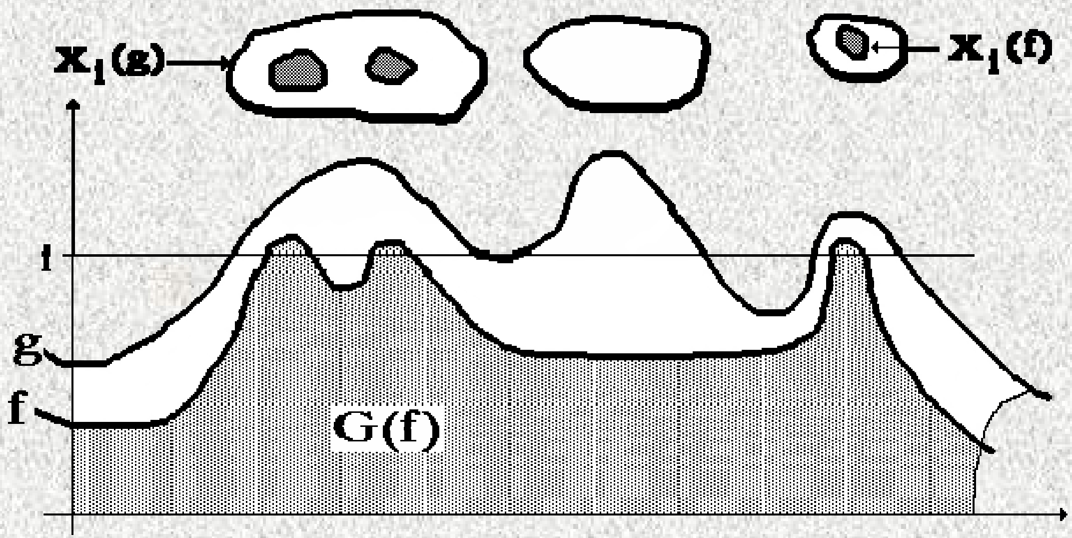
Numerical geodesy

The set geodesic transformations can be extended to numerical functions in two ways:

• Either through the sections of functions by applying set geodesic operators to these sections and by constructing new functions with the transformed sections

$$X_i(f) = \{x : f(x) \geq i\}$$

$$X_i(g) = \{x : g(x) \geq i\}$$



• Or by using the numerical functions to define general geodesic distances themselves used to define generalised geodesic operators

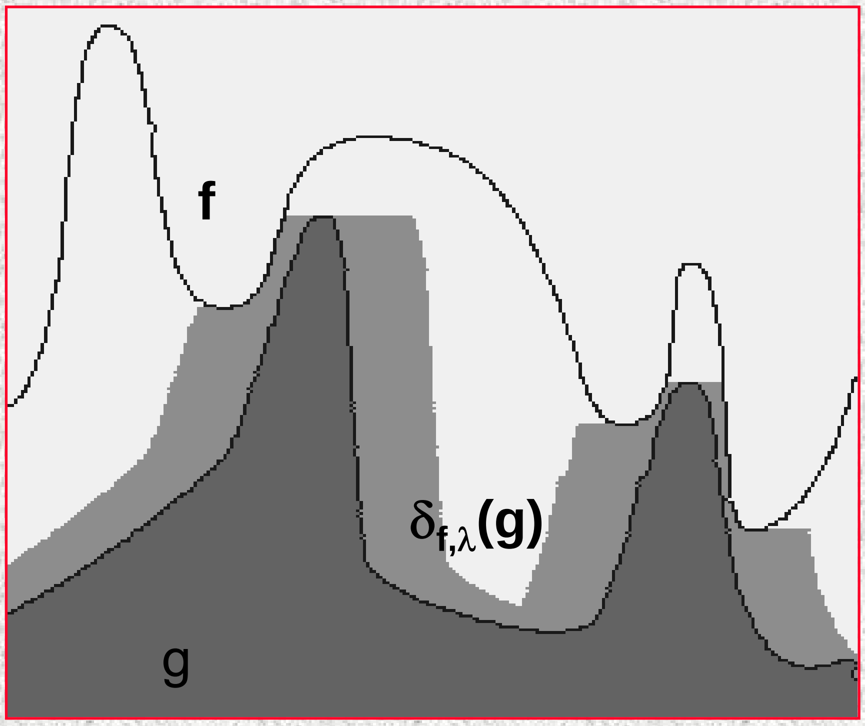
Numerical geodesic dilations

Definition

Let f and g two functions (greytone images), with $g \leq f$.

Consider sections at height h of f and g

The set geodesic dilation of size λ of each section at height h of g inside the corresponding section of f defines on g a dilation $\delta_{f,\lambda}(g)$.



The set under the graph of $\delta_{f,\lambda}(g)$ is made of the points of the set under the graph of f which are linked to g by an horizontal path of length $\leq \lambda$.

The digital version of this operator uses the elementary geodesic dilation:

$$\delta_f(g) = \delta(g) \wedge f$$

Iterated n times:

$$\delta_{f,n}(g) = \delta_f\,(\delta_f\ \ldots\ (\delta_f\,(g))).$$

Numerical geodesic erosions

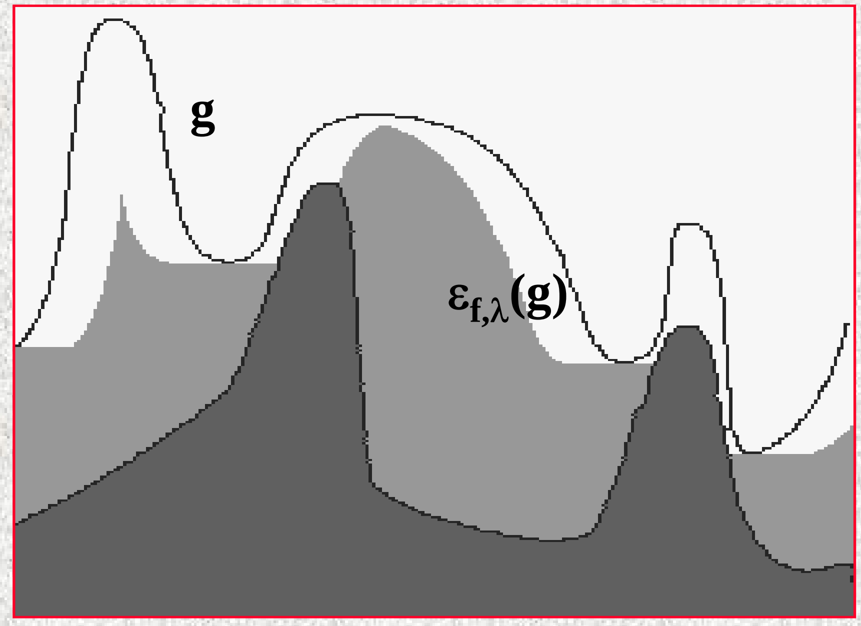
The numerical geodesic erosion of f by g, with $g \geq f$, proceeds from the geodesic dilation from the duality generated by the rotation around a revolving value m:

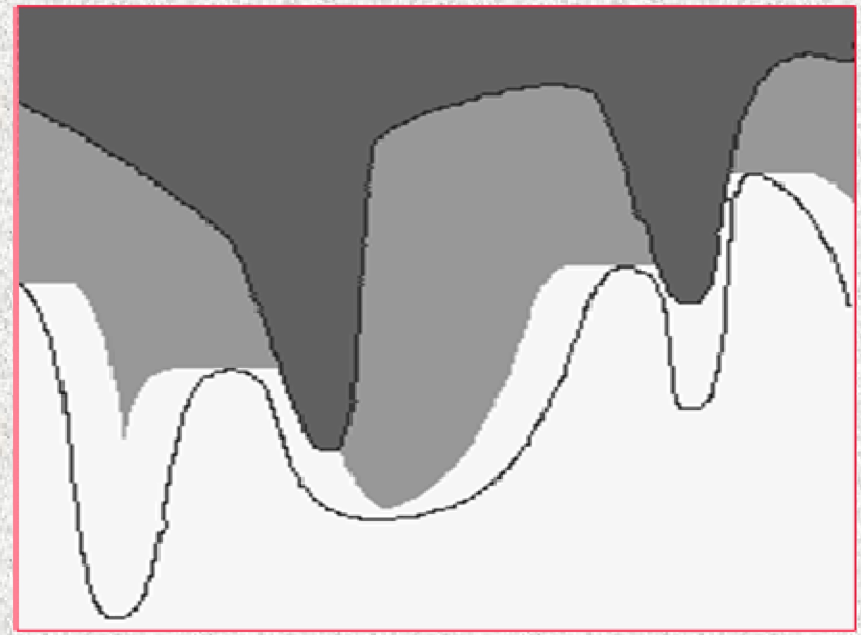
$$\varepsilon_{f,\lambda}(g) = m - \delta_{m-f,\lambda}(m - g)$$

The elementary digital geodesic erosion can be written as:

$$\varepsilon_f(g) = m - \{\delta(m - g) \wedge (m - f)\}$$
$$= \{m - \delta(m - g)\} \vee \{m - (m-f)\}$$
$$\varepsilon_f(g) = \varepsilon(g) \vee f$$

- This duality is different from the duality by complementation (set transformations)
- The result does not depend on the revolving value m



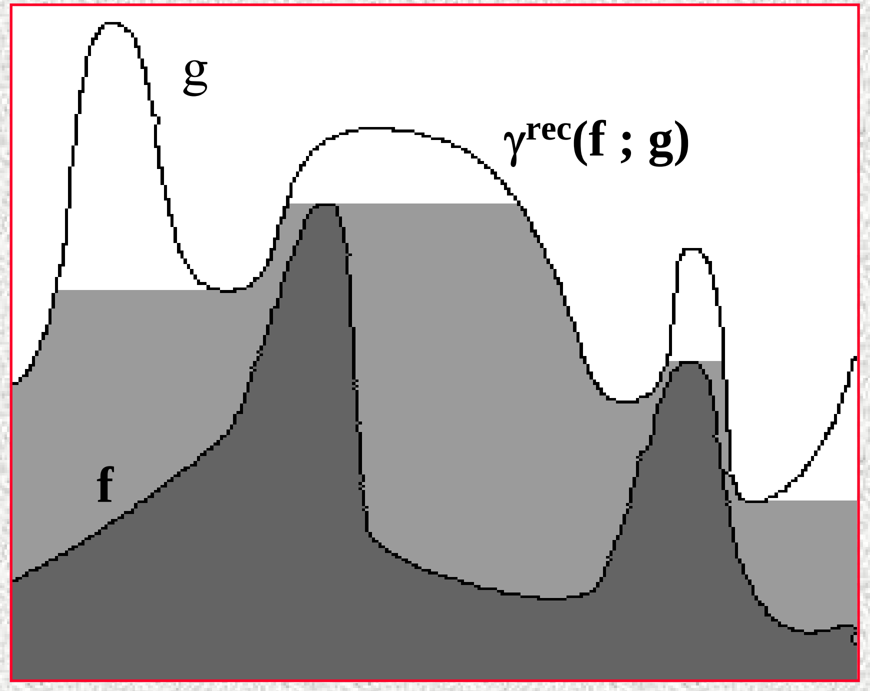


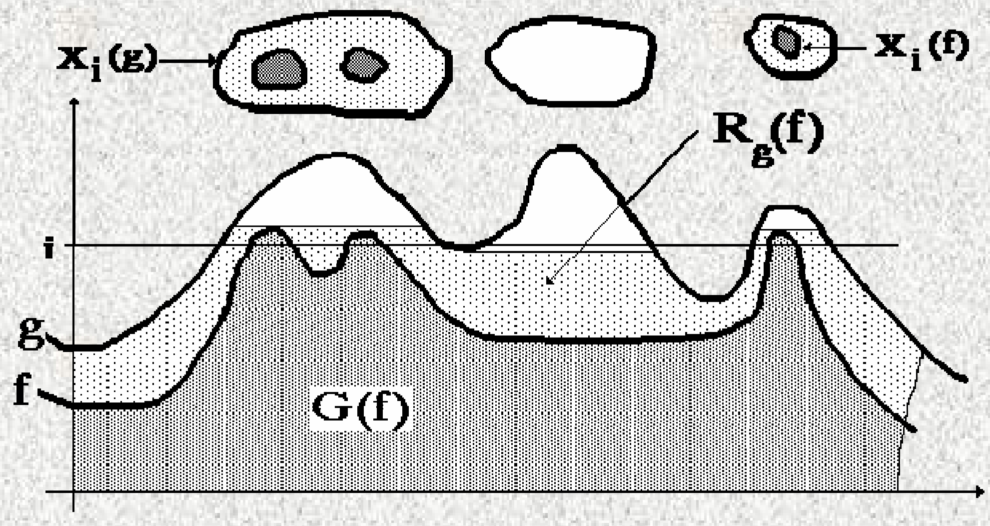
Numerical geodesic reconstruction

- The geodesic reconstruction of g from f is the supremum of the geodesic dilations of f under g. It is denoted $R_g(f)$:

$$R_g(f) = \vee \{ \delta_{g,\lambda}(f) \ , \ \lambda > 0 \}$$

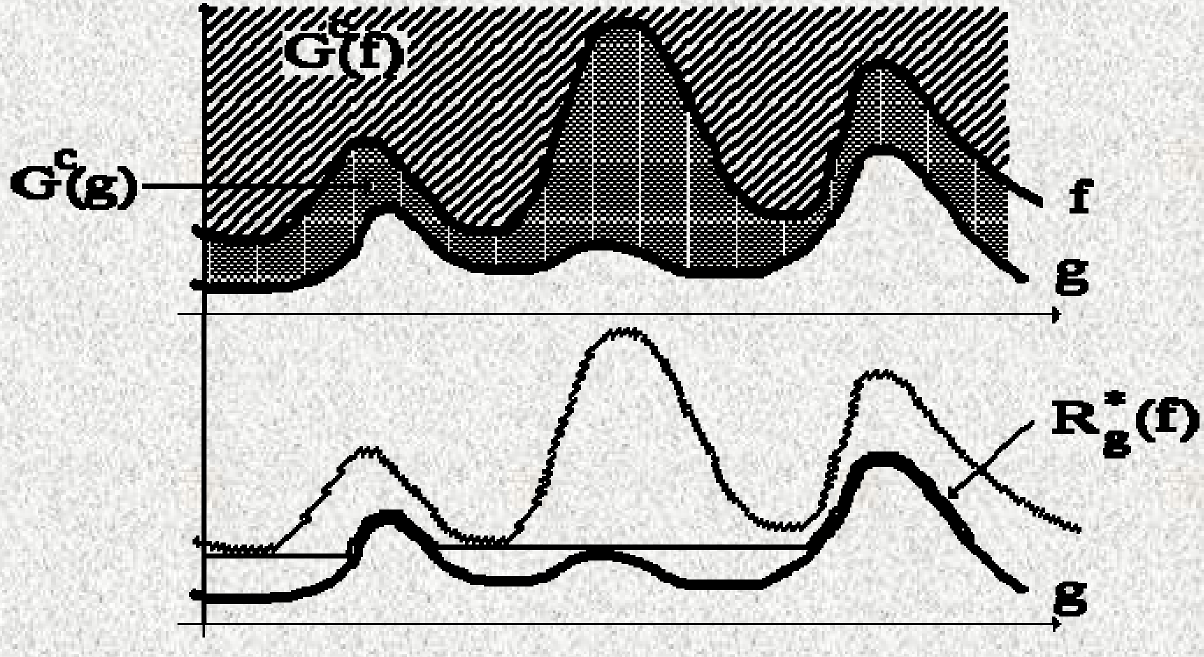
- This transformation and its dual one are very important machine-tools in mathematical morphology.





Each section at level i of the reconstruction is equal to the binary reconstruction of the section of g at the same level by the corresponding section of f.

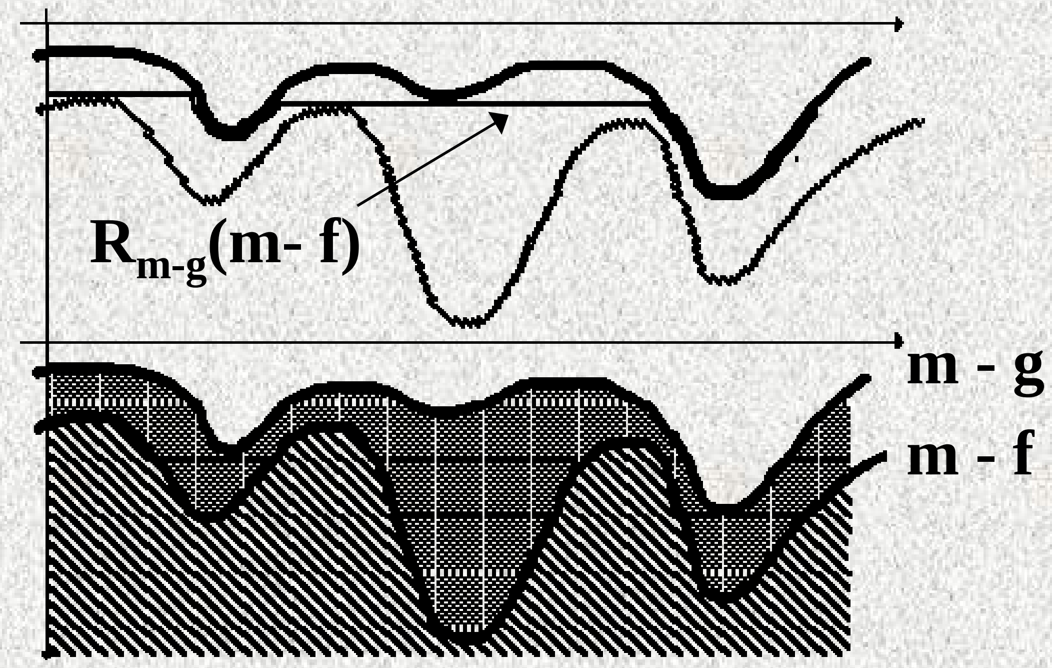
Dual reconstruction



The dual reconstruction $R^*_g(f)$ of g by f is the infimum of the geodesic erosions of f over g

This duality is identical to the one used in the geodesic erosion (rotation around a revolving value m):

$R^*_g(f) = m - R_{m-g}(m- f)$



Opening by erosion-reconstruction

Preserving the contours

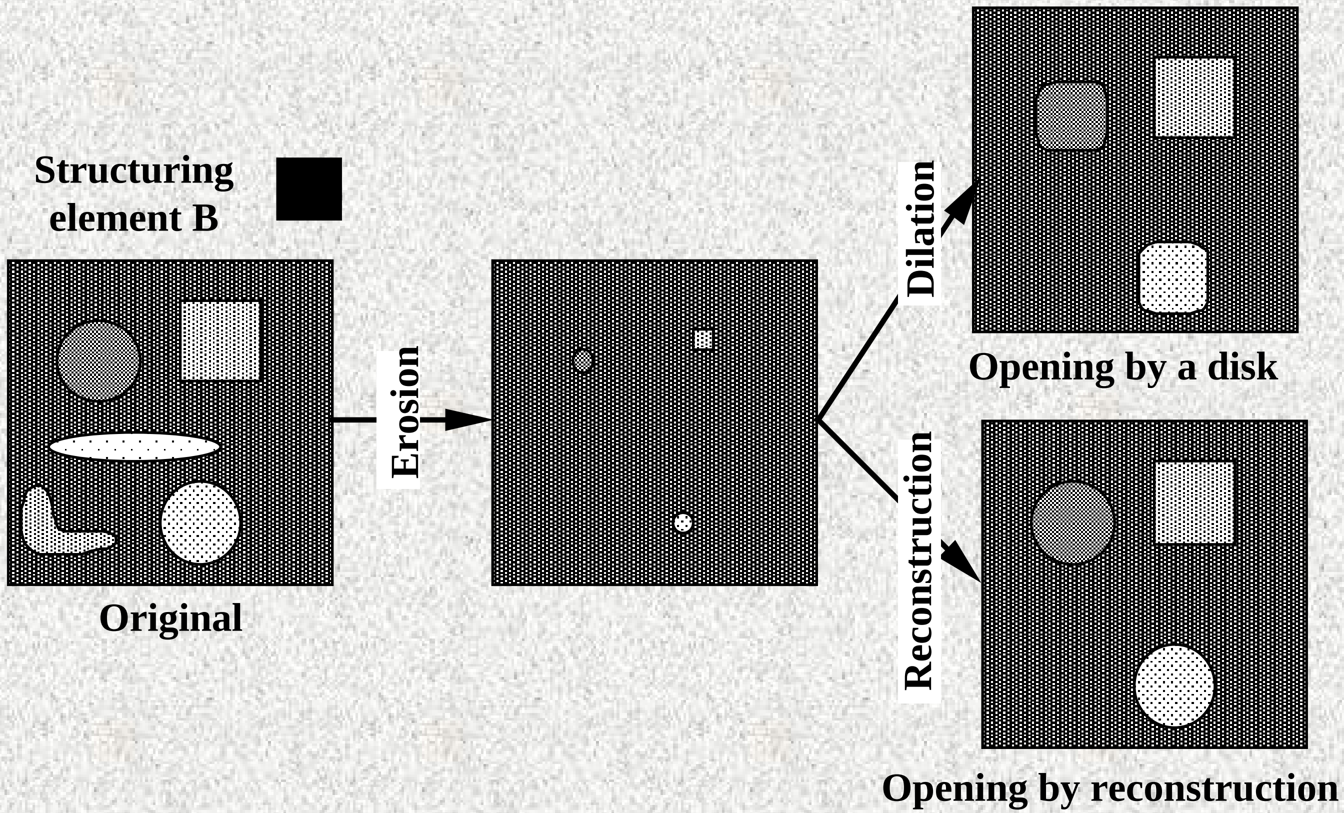
Whereas the classical opening modifies contours, this transformation efficiently and precisely rebuilds the contours of objects which have not been entirely removed by the erosion.

Algorithm

- The original image is the geodesic space.

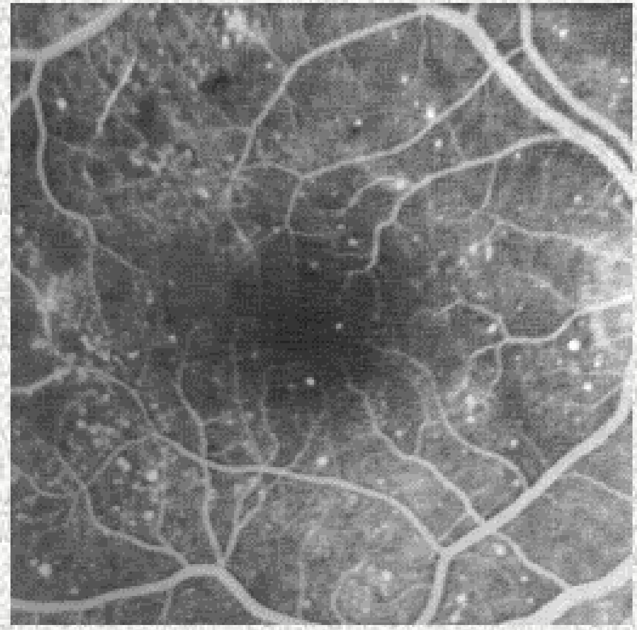
- The marker is the euclidean erosion of the original image.

$$R_f\left[\varepsilon_B(f)\right]$$

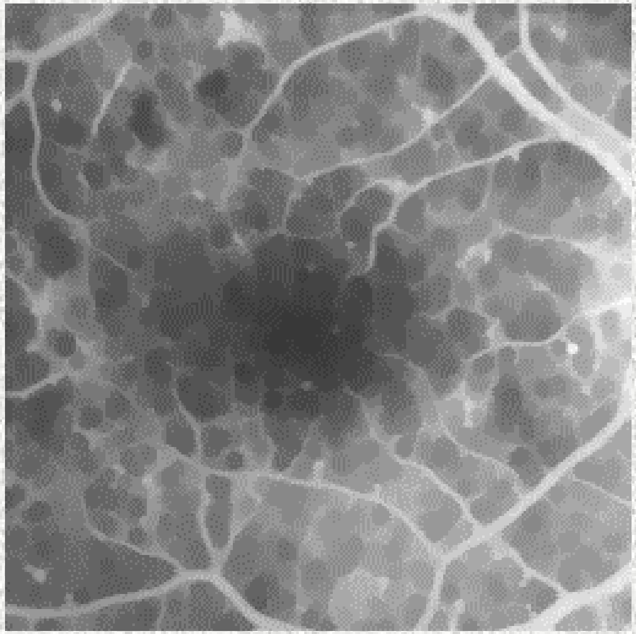


Application to retinopathy

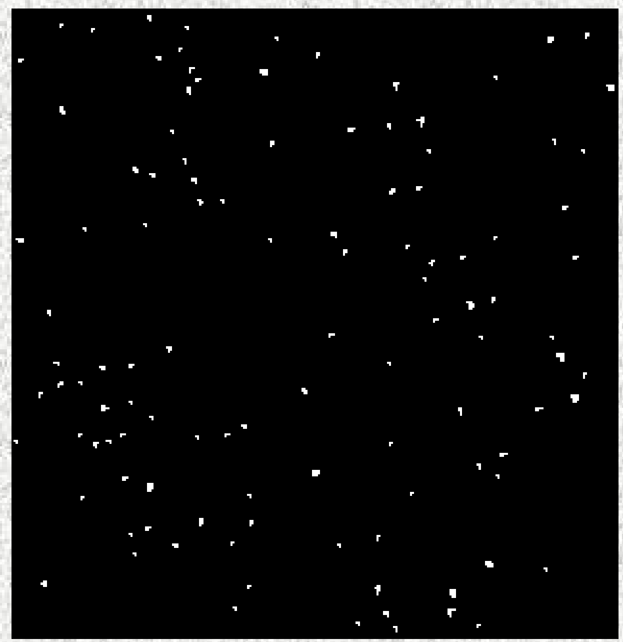
The operation aims at locating and extracting retinal aneurisms. Reconstruction operations insure that only small isolated peaks are suppressed.



Initial image



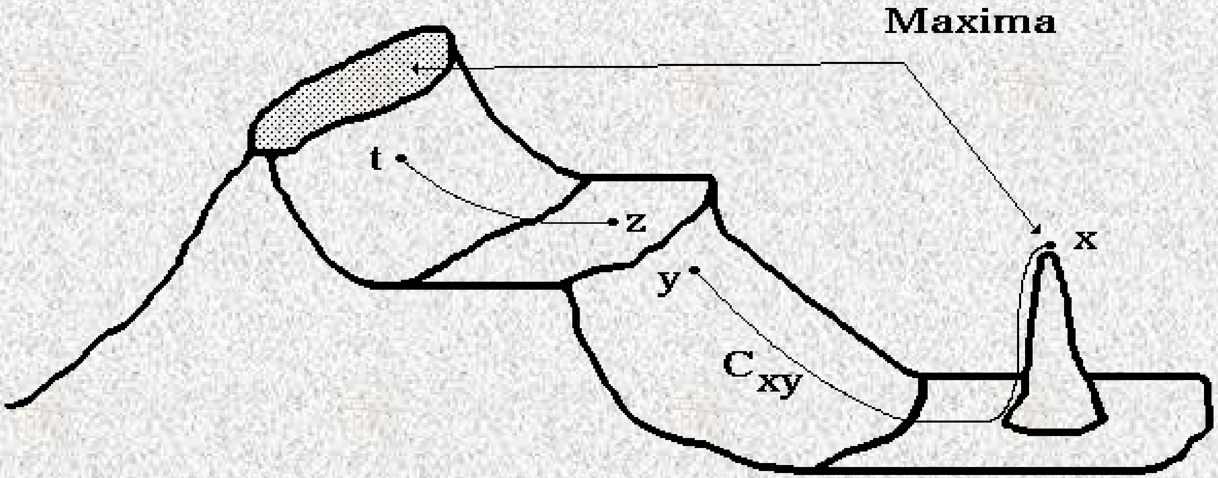
Opening by erosion- reconstruction



difference between the two images and thresholding

Maxima of a function

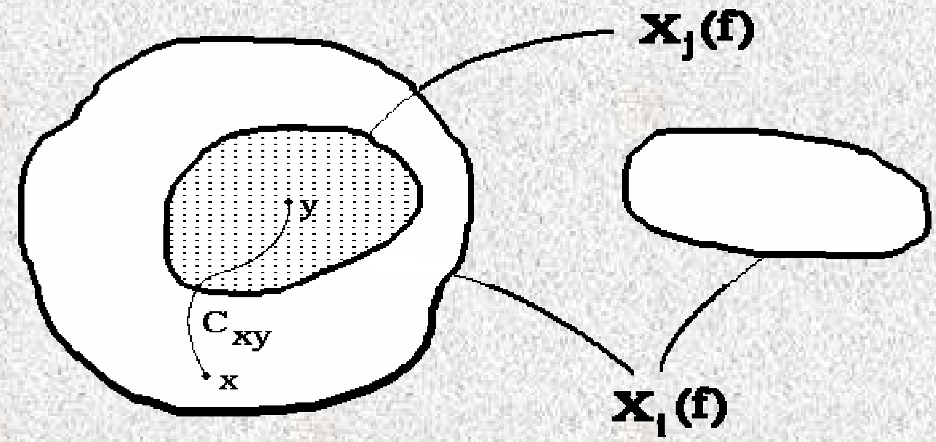
a **maximum of a fun**ction f (or regional maximum) is a summit of the topographic surface, that is a connected region (but not necessarily reduced to a point) from where it is not possible, starting from any point of this region, to reach a point of the topographic surface at a higher altitude by a never descending path.

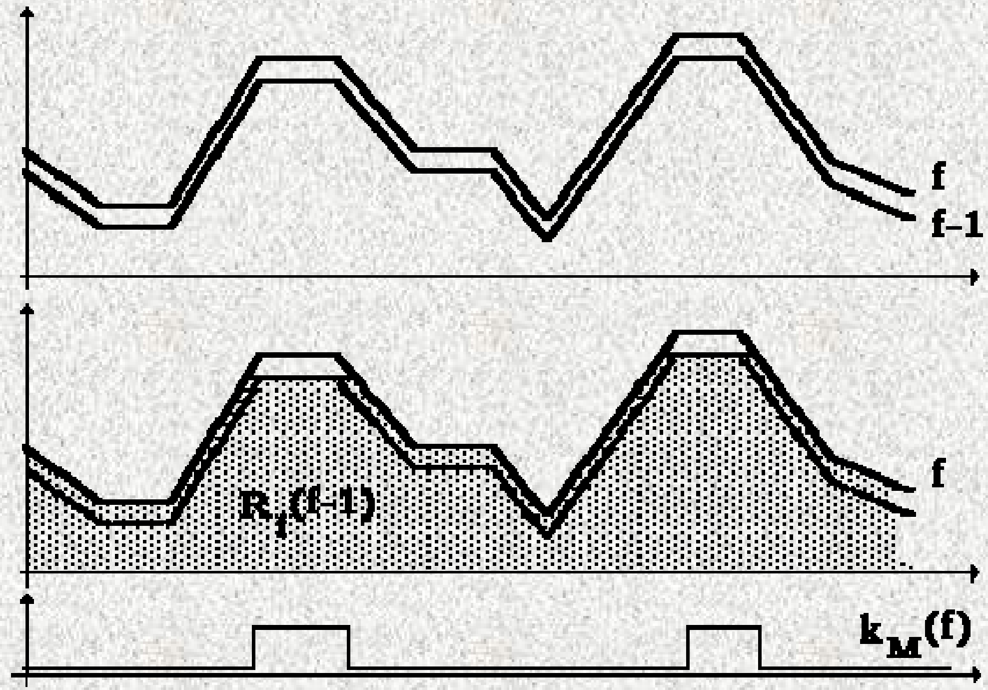


- The point x is a maximum (to reach y, the path C_{xy} contains descending portions)
- The points z, y, t do not belong to maxima

Maxima detection

A maximum of the function f at altitude i is the connected component of the section $X_i(f)$ of f which does not contain any connected component of any section $X_j(f)$ where j > i.





Writing j=i+1, it can be shown that the indicator function $k_M(f)$ of the maxima M of f is equal to:

$$k_M(f) = f - R_f(f-1)$$

The maxima are the residues of the geodesic reconstruction of f by f -1

A similar definition and construction (using the dual reconstruction) are available for the minima m of f:

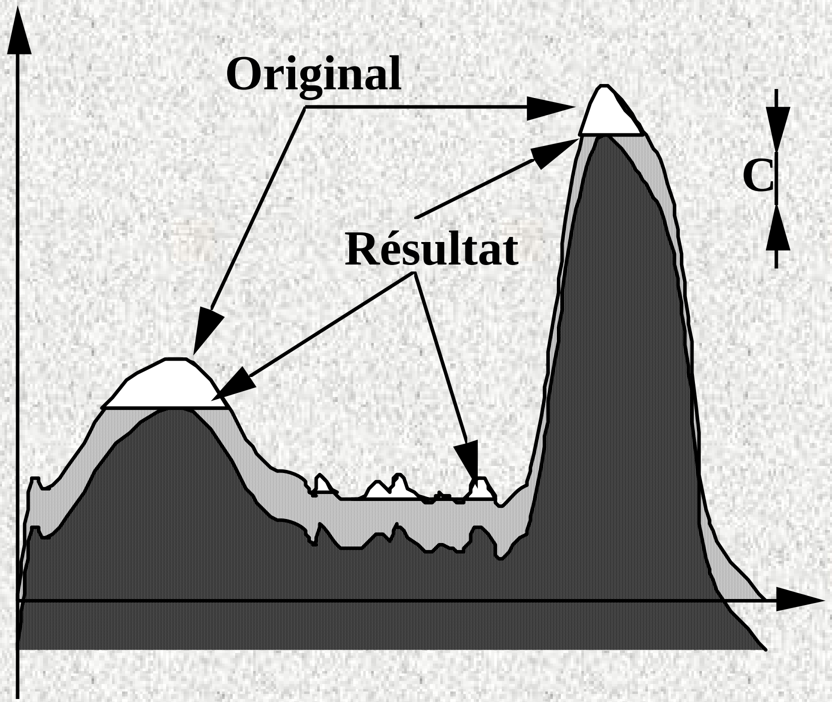
$$k_m(f) = R_f^*(f+1) - f$$

Extended maxima

The maxima extraction can be restricted to those which correspond to peaks (or domes) whose height is at least equal to c. The descending paths coming from these maxima have a height at least equal to c. Therefore, these maxima can be obtained by reconstructing the initial function f with the function f-c.

- We perform the reconstruction $R_f(f-c)$
- The difference f – $R_f(f-c)$ produces the extended maxima of height c of f
- The maxima $M[R_f(f-c)]$ can be extracted
- The initial maxima $M_c(f)$ of height c of f are equal to:

$$M_c(f) = M(f) \cap M[R_f(f-c)]$$

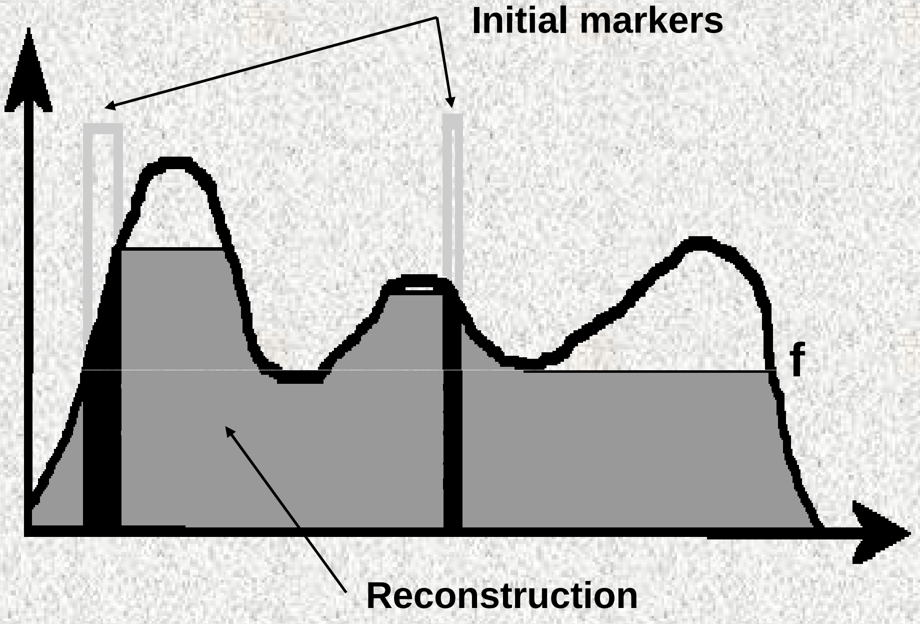


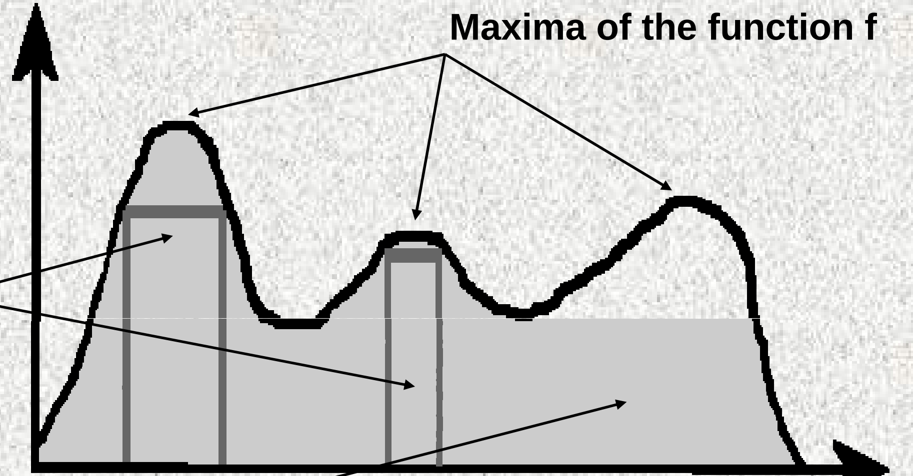
Reconstructing a function from markers

- A function can be rebuilt from any marker set.

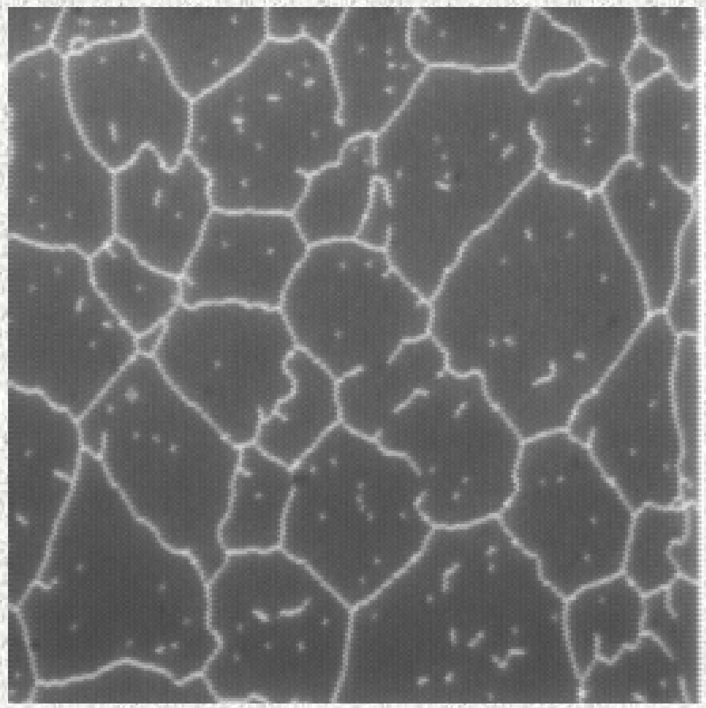
- This operation is an homotopy modification: the marked domes only are (partly) kept.

- This reconstruction is partial. A complete reconstruction of the domes which have not been totally erased can be performed by the previously described algorithm.

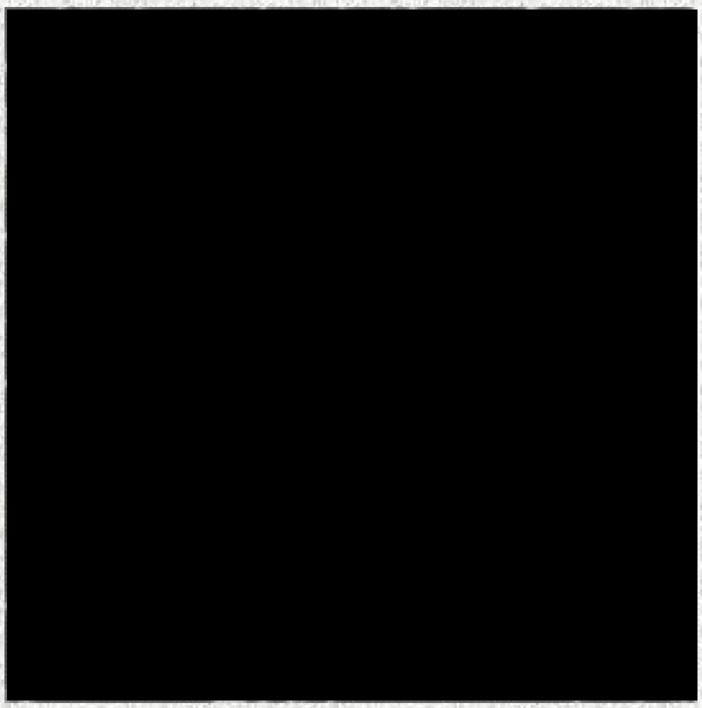




Application: isolated components removal



Alumine grains with inclusions



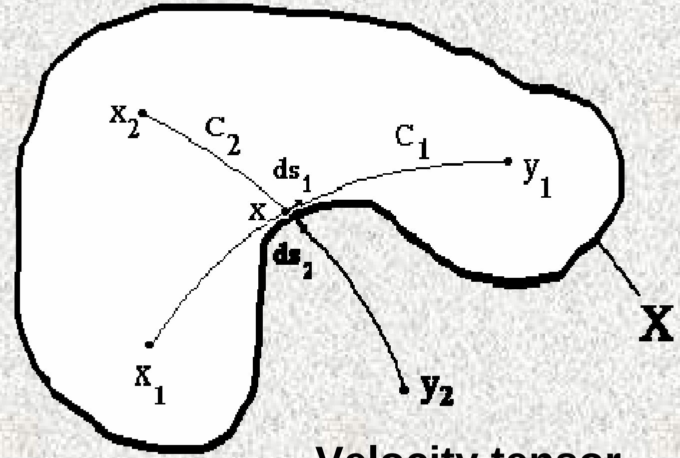
Reconstruction from the edges of the image field

Generalised geodesic distance

The geodesic distance between x and y is equal to the length of the minimal path C_{xy} linking these two points. This length can also be expressed in time for travelling from x to y.

$$L(C_{xy}) = \int_{C_{xy}} ds = v \int_{C_{xy}} dt = vT(C_{xy})$$

When the speed v is constant, the path travelling time $T(C_{xy})$ can be used to measure its length.

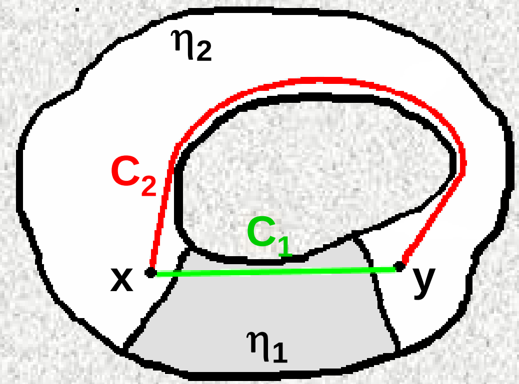


Velocity tensor

If the speed is not constant but replaced by a velocity field ω, the length of a path C_{xy} will then be defined as its travelling time.

$$T(C_{xy}) = \int_{C_{xy}} \frac{ds}{\omega} = \int_{C_{xy}} \eta \, ds$$

The inverse $\eta = 1/\omega$ of the speed is called refringence. Given this refringence field allows to compute the travelling time of any path and therefore to define the generalised geodesic distance between two points x and y as the minimal travelling time between these two points.



C_2 is the shortest path

Digital generalised geodesic distance

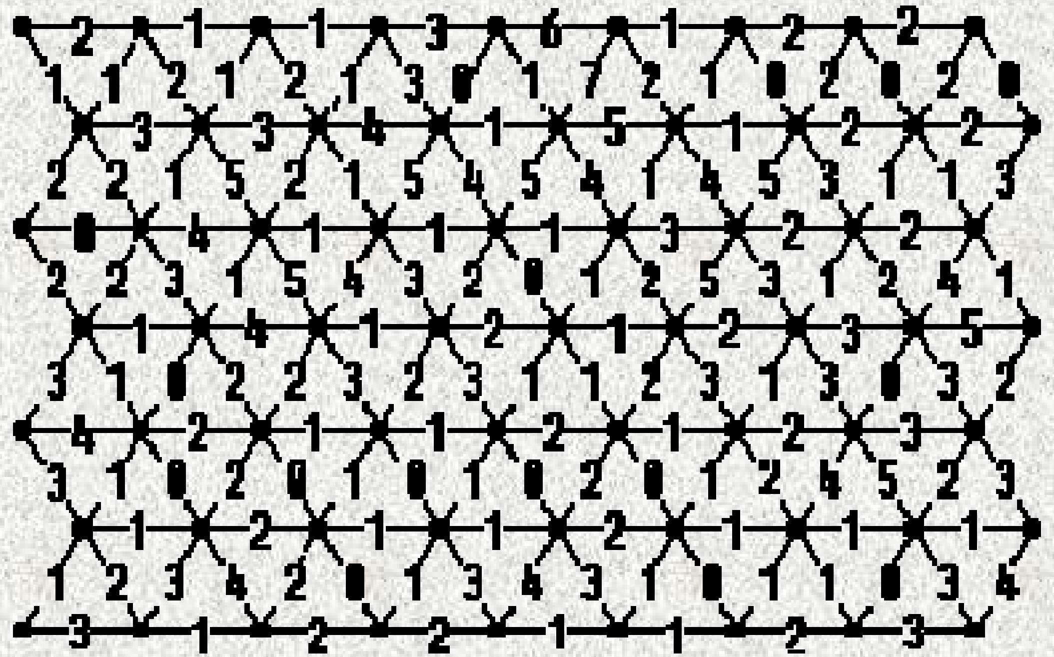
- In a digital space, the refringence field is a valued graph
- This field can be « derived » from any function (image) f:

$$\eta(x,y) = |f(x) - f(y)|$$

- The distance may possibly be replaced by a pseudometric (zero refringence)
- This distance can be algebric

 (non symmetrical)

Example of refringence graph on an hexagonal grid



Generalised geodesic dilation

•The time n dilation of a set Y in a refringence field η is an iterative process

•This dilation is obtained by n dilations of size 1

<u>Elementary dilation algorithm</u>

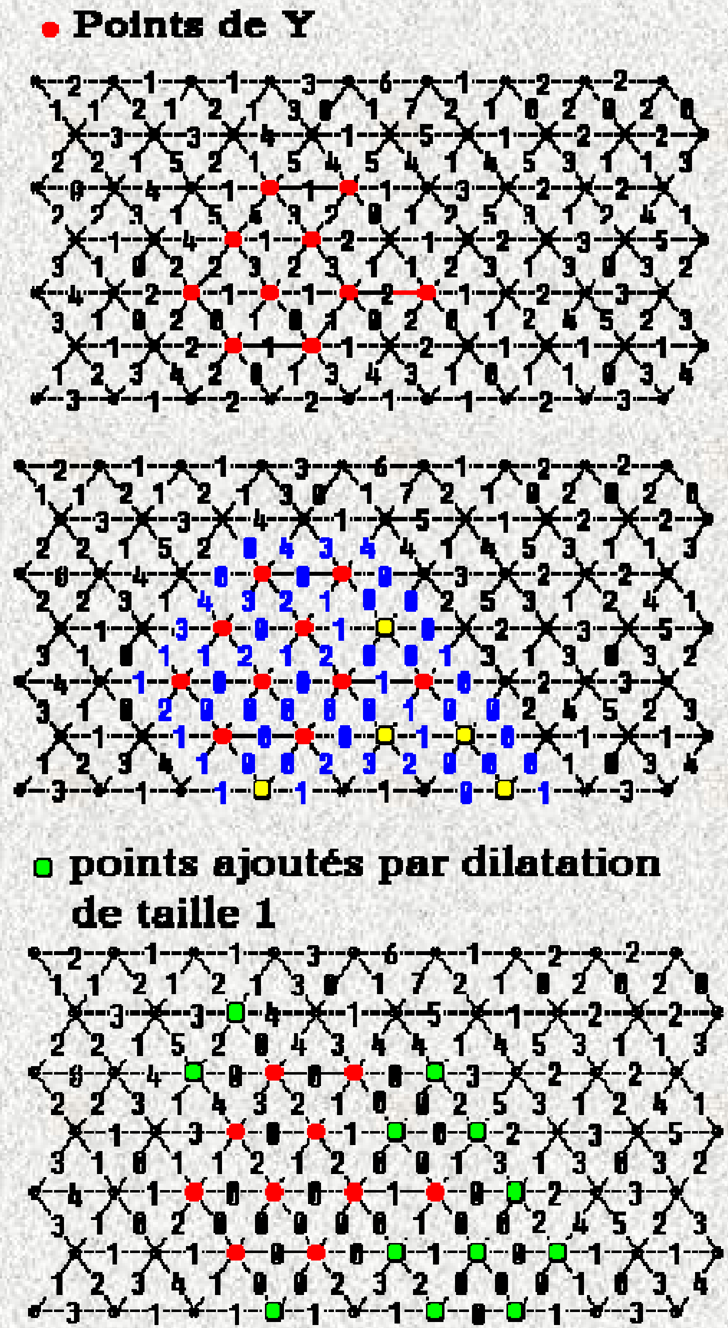
-Points at zero distance of Y are added to Y

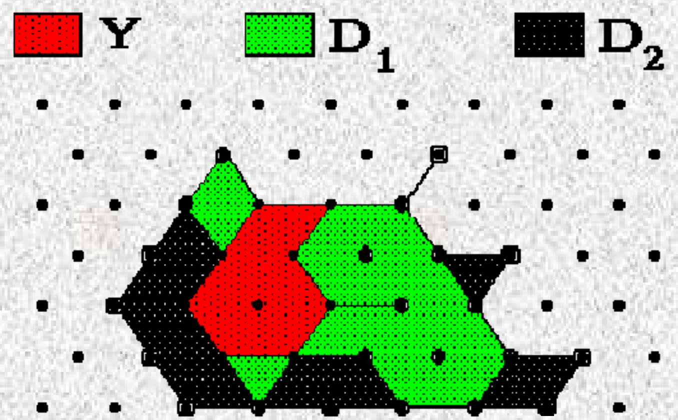
-Refringence of the graph edges connected to Y is lowered of 1

-Points at distance of Y equal to 0 are added to the set

→ *Dilation of size 1*

The procedure is iterated with the new values of the refringence field

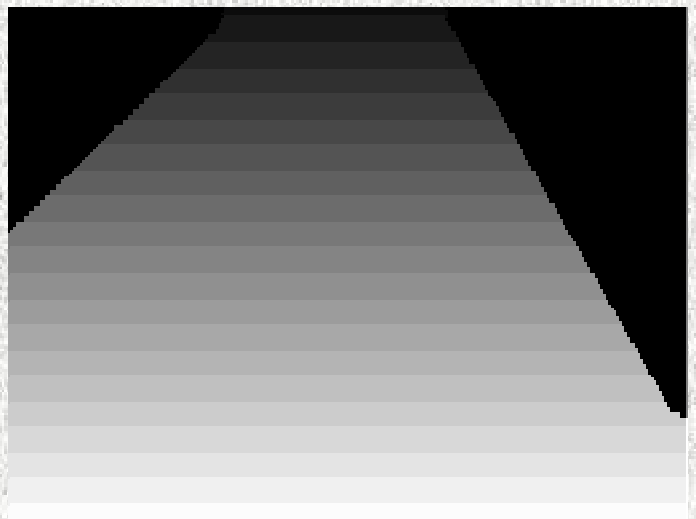




Application to traffic analysis

The generalised geodesic transformations allow to define operators which take into account the perspective, according to the real size of objects in the scene

Traffic image



Scale factor used to define the refringence field

Opening of size 1 meter (ground distance)