From Markov Switching to Change Points, and Back

Ethan Goolish

1. Markov Chain Time Series

Let X be a Markov Chain with k states: $S_1, ..., S_k$. It is known that we can associate with X a $(k \times k)$ transition matrix P such that given $X_t = S_i$ (meaning chain X is in state i at time t), we have that P_{ij} denotes the probability that $X_{t+1} = S_j$. However, if at time t, the state of X_t is not directly observable, but rather X_t emits some signal y_t that probabilistically depends upon the (unobservable) state of X_t, we call X a Hidden Markov Model. This is the case that will we will concern ourselves with first. In specific, we associate with each state S_i a mean μ_i along with a fixed variance σ^2 so that if our chain is in state S_i at time t, we can model our emitted signal as $y_t = \mu_i + \epsilon$, where $\epsilon \overset{i.i.d.}{\sim} N(0, \sigma^2)$. With this, we have our Markov Chain Time Series model: Begin in a state S_1. Using our above signal model, we obtain signal $y_1 = \mu_1 + \epsilon$. Then, we transition to a new state S_j with probability P_{1j}. We obtain our next signal $y_2 = \mu_j + \epsilon$ before transitioning to our next state $S_{j'}$ with probability $P_{jj'}$, and so forth. Thus given a $(k \times k)$ transition matrix P, a $(k \times 1)$ vector of state means S, and a variation σ^2, we can create a time series $Y = (y_1, y_2, ..., y_n)$ of length n. Our question is this: given a time series created in the above manner but with unknown parameters P and S, can we recover our transition probabilities and means?

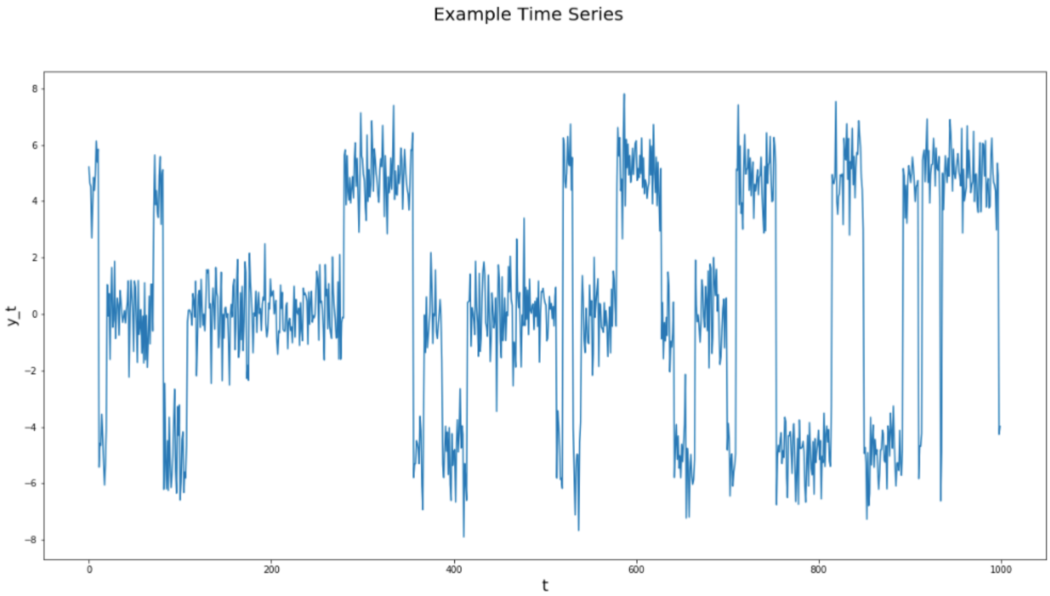
2. Process

To do so, we suggest the follow procedure. In the basic case, we will assume the number of states k is known. Recall all we are given is a time series $Y = (y_1, ..., y_n)$ of length n where each y_i has associated with it some unknown state S_j, and is drawn probabilistically as $y_i = \mu_j + \epsilon$. For example, for if we took $k = 3$ and the following parameters:

$$S = \begin{bmatrix} 5 & 0 & -5 \end{bmatrix} \qquad \sigma^2 = 1 \qquad P = \begin{pmatrix} 0.95 & 0.02 & 0.03 \\ 0.01 & 0.95 & 0.04 \\ 0.03 & 0.03 & 0.94 \end{pmatrix}$$

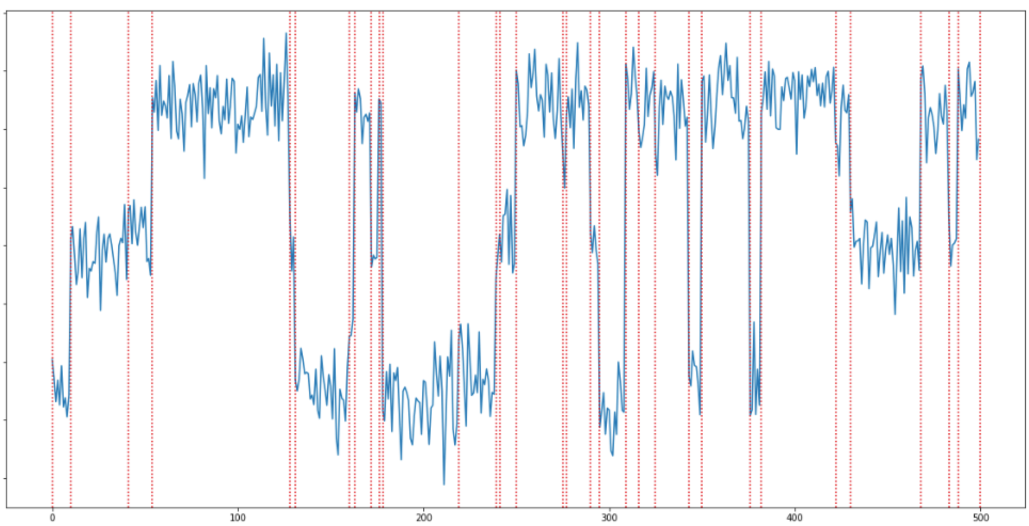
Key words and phrases. Markov Switching, Change Points, MCMC.

we could have the following series:

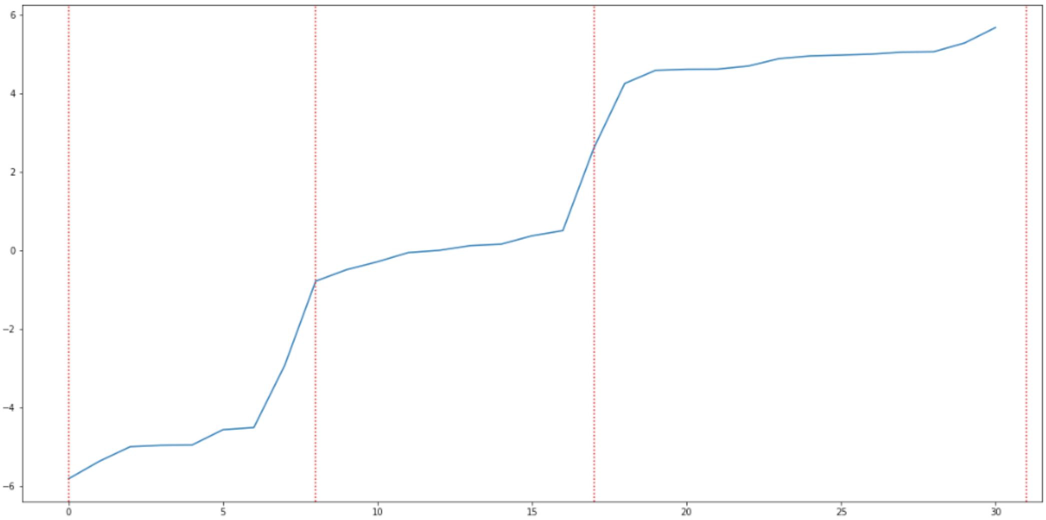


Our suggested method takes advantage of the ECP change point package (TODO: cite). Within this package, we can use the *e_divisive* call to divide a time series Y into segments based on its change points. The method takes several parameters including Y, the time series in question, k, the number of estimated change points, *alpha*, the exponentiation for the distance matrix, and *min_size*, the minimum number of points allowed between change point estimations. For our method, we want to overpredict the number of change points to ensure all change points between one state S_i and another state S_j are found, and then merge the extraneous splits that separate two segments that should belong to the same state. Currently, we estimate k by assuming the transition probabilities to be no lower than 0.75, thus assuming the average time before transition to be approximately $1/(1-0.75) = 4$. Then we take $n/4$, where n is the total length of Y, to be a rough approximation of the number of change points, and then add $n/100$ more in order to over-estimate k appropriately. We also take *alpha* to be 2, and take *min_size* to be the smallest allowed (2) to allowed for the finest granularity. With this set of parameters, we expect a call to *e_divisive* to return segmentation similar to the following:

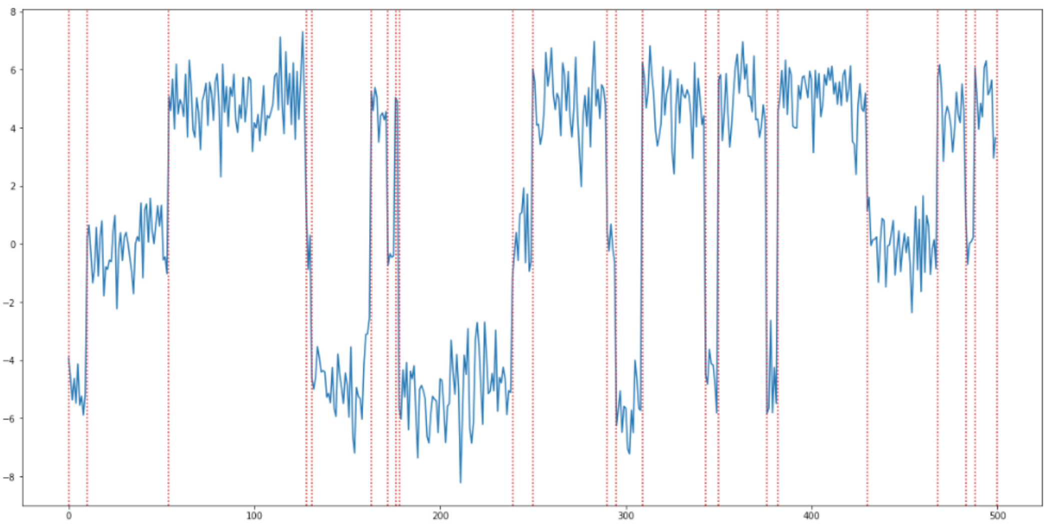
```
Estimated Number of Change Points: 30.0
Minimum Size before Change: 2
Time:  1.7300300598144531
Found Change Points: 30
```



Note for the most part that while the state transitions are marked by *e_divisive*, we have some excess segmentation, such as the third mark which incorrectly divides two segments from the same state S_1, the state corresponding to $\mu_1 = 0$. Our next goal is to remove such incorrect splitting. To do this, begin by finding the mean g_i of each guessed segment G_i (which was returned by *e_divisive*). Then, sort the list of G_i incrementally by their respective g_i. We can then run *e_divisive* for a second time on our sorted G_i to group our means into estimates for which state each segment guess was in. In other words, using *e_divisive* on a sorted list of g_i groups each G_i into a guess for the segment label $\hat{S}_i$. If the number of states k is known, we can use $k-1$ as the parameter for the number of change points in this second use of *e_divisive*. To continue the above example, here we plot the sorted means g_i, and then segment using the second *e_divisive* call. The segments G_i corresponding to points in the first segment will be estimated to be in state $\hat{S}_0$, the segments corresponding to points in the second segment will be estimated to be in state $\hat{S}_1$, and the segments corresponding to points in the third segment will be estimated to be in state $\hat{S}_2$:



After assigning each G_i to the appropriate $\hat{S}_i$, we then do the merging – we can iterate through our original (unsorted) list of G_i, and if some G_i and G_{i+1} have the same assigned state label $\hat{S}_i$, then we propose a new segment G'_i such that G'_i contains the points that used to be in G_i and G_{i+1}. After doing this, we get this new estimation of the change points, which we can see is much more accurate:



Now with our fixed segmentation, we estimate our transition probabilities $P_{i,j}$ as well as our state means μ_i. To do so, we first calculate the average stay time for each state label $\hat{S}_i$. By this we mean we take each updated segment G_i and observe its length. We groupby the segment labels $\hat{S}_i$ and take the average to find the average time of stay for a given label $\hat{S}_i$, call it a_i. Then note that

$$a_i = \frac{1}{1 - P_{i,i}} \iff P_{i,i} = \frac{a_i - 1}{a_i}$$

and thus we can estimate our staying probabilities $P_{i,i}$. We can also estimate our off-diagonal probabilities $P_{i,j}$: We first count the number of times we transition from a state i to every other state j where $j \neq i$, call it $c_{i,j}$. Then, we know that $P_{i,j}$ is proportional to the number of times we transition from i to j over the number of times we transition from i to any other state j'. Since we know the sum across a row $\sum_j P_{i,j} = 1$, we can say

$$P_{i,j} = \frac{c_{i,j}(1 - P_{i,i})}{\sum_j c_{i,j}}$$

Alternatively, we can take a prior α_i so that $c'_{i,j} = c_{i,j} + \alpha_i$. In this basic example we take $\alpha_i = 0$, but other natural choices include $\alpha_i = 1$, $\alpha_i = 1/2$, or $\alpha_i = 1/K$ (where K is our number of states). In total, this gives us our full probability matrix P.

Then, to estimate $S = \{\mu_1, ..., \mu_k\}$, we simply groupby our updated segment labels $\hat{S}_i$ and take the mean to find each $\hat{\mu_i}$. Doing the above on our example dataset above yields us:

$$\hat{S} = \begin{bmatrix} -5.04134908 & -0.03278774 & 4.91458622 \end{bmatrix} \qquad \hat{P} = \begin{pmatrix} 0.95384615 & 0.01538462 & 0.03076923 \\ 0.01818182 & 0.93636364 & 0.04545455 \\ 0.01298077 & 0.02163462 & 0.96538462 \end{pmatrix}$$

yielding us an accurate recovering of our original parameters.

Department of Statistical Science, Cornell University, Ithaca, New York, 14853
E-mail address: efg36@cornell.edu