Math Olympiad Problems

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Last update: September 9, 2024

Here are some of the problems that I have composed for various math olympiads.

Currently only the problems from International Mathematical Tournament Of Towns and Caucasus Mathematical Olympiad are included.

Some problems are provided with answers and solutions that are written in blue. For the solutions there were used the following sources:
- International Mathematical Tournament Of Towns
- International Mathematical Tournament Of Towns in Toronto
- problems.ru
- Caucasus Mathematical Olympiad

Contents

1 Arithmetic problems

Problem 1. (International Mathematical Tournament Of Towns, 2012)
Five students have the first names Clark, Donald, Jack, Robin and Steve, and have the last names (in a different order) Clarkson, Donaldson, Jackson, Robinson and Stevenson. It is known that:
Clark is 1 year older than Clarkson,
Donald is 2 years older than Donaldson,
Jack is 3 years older than Jackson,
Robin is 4 years older than Robinson.
Who is older, Steve or Stevenson and what is the difference in their ages?

Solution. The sum of ages of Clark, Donald, Jack, Robin and Steve is equal to the sum of ages of Clarkson, Donaldson, Jackson, Robinson and Stevenson. Hence Stevenson is older than Steve, and the difference is $1 + 2 + 3 + 4 = 10$ years.

Problem 2. (International Mathematical Tournament Of Towns, 2014)
During Christmas party Santa handed out to the children 47 chocolates and 74 marmalades. Each girl got 1 more chocolate than each boy but each boy got 1 more marmalade than each girl. What was the number of the children?

Answer: 11 or 121.
Solution. Each child got the same number of treats and the total number of treats is $74 + 47 = 121$. Therefore there could be either (a) 11 children, or (b) 121, or (c) just 1 child, and each child got 11, 1, or 121 treat respectively.
In case (a) let x denote the number of boys and c the number of chocolates each girl got. Then $(c - 1)x + c(11 - x) = 47$ or $11c = 47 + x$. The only integer solution with $0 \le x \le 11$ is $x = 8$, $c = 5$ (so, 8 boys, 3 girls). In case (b) each boy got just 1 marmalade, and each girl got just 1 chocolate (so, 74 boys and 47 girls). Case (c) is impossible.

Problem 3. (Caucasus Mathematical Olympiad, 2020)
Using one magic nut, Wicked Witch can either turn a flea into a beetle or a spider into a bug; but using one magic acorn, she can either turn a flea into a spider or a beetle into a bug. In the evening Wicked Witch had spent 20 magic nuts and 23 magic acorns. By a sequence of these actions, the number of beetles increased by 5. Determine what was the change in the number of spiders.

Answer. The number of spiders increased by 8.
Solution. When using a magic nut either the number b of bugs increases by 1 or the number s of spiders decreases by 1, so the difference $b-s$ increases by 1. When using a magic acorn vice versa: either the number b of bugs decreases by 1 or the number s of spiders increases by 1, so the difference $b-s$ decreases by 1. Due to these operations, the difference $b - s$ increased by 20 and decreased by 23, therefore in total it decreased by 3. Since b increased by 5, then s increased by $5 + 3 = 8$.

Problem 4. (International Mathematical Tournament Of Towns, 2015)
Three players play the game "rock-paper-scissors". In every round, each player simultaneously shows one of these shapes. Rock beats scissors, scissors beat paper, while paper beats rock. If in a round exactly two distinct shapes are shown (and thus one of them is shown twice) then 1 point is added to the score of the player(s) who showed the winning shape, otherwise no point is added. After several rounds it occurred that each shape had been shown the same number of times. Prove that the total sum of points at this moment was a multiple of 3.

Problem 5. (International Mathematical Tournament Of Towns, 2014)
There are 99 sticks of lengths 1, 2, 3, ..., 99. Is it possible to arrange them in a shape of a rectangle?

Problem 6. (International Mathematical Tournament Of Towns, 2016)
Twenty children stand in a circle (both boys and girls are present). For each boy, his clockwise neighbour is in a blue T-shirt, and for each girl, her counterclockwise neighbour is in a red T-shirt. Is it possible to determine the precise number of boys in the circle?

Problem 7. (International Mathematical Tournament Of Towns, 2014)
On a circular road there are 25 police posts equally distant. Every policeman (one at each post) has a badge with a unique number, from 1 to 25. The policemen are ordered to switch their posts so that the numbers on the badges would be in the consecutive order, from 1 to 25 clockwise. If the total sum of distances walked by the policemen along the road is minimal possible, prove that one of them remains at his initial position.

2 Arithmetic problems in tables

Problem 8. (International Mathematical Tournament Of Towns, 2022)
Is it possible to arrange 36 distinct numbers in the cells of a 6×6 table, so that in each 1×5 rectangle (both vertical and horizontal) the sum of the numbers equals 2022 or 2023?

Problem 9. (International Mathematical Tournament Of Towns, 2018)
In each cell of a 4×4 square there is an integer number. The sum of the numbers in each column and each row is the same. Seven of the numbers are known, while the rest are hidden (see figure).

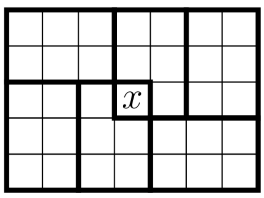
1	?	?	2
?	4	5	?
?	6	7	?
3	?	?	?

Is it possible to uniquely determine
a) at least one of the hidden numbers;
b) at least two of the hidden numbers?

Problem 10. (International Mathematical Tournament Of Towns, 2014)
The entries of a 5×7 table are filled with numbers so that in each 2×3 rectangle (vertical or horizontal) the sum of numbers is 0. For 100 dollars Peter may choose any single entry and learn the number in it. What is the least amount of dollars he should spend in order to learn the total sum of numbers in the table for sure?

Answer. 100 dollars.

Solution. Let S be a total sum of the numbers in the table. Let Peter divide the table into 6 rectangles as shown on the picture (two rectangles overlap on a marked entry). Then $S = 0 \cdot 5 - x$ where x is the value in the marked entry he pays for.

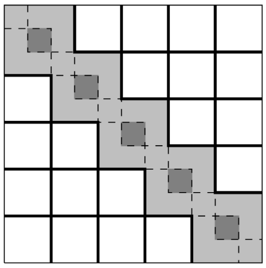


On the other hand, table filled with numbers $(-a)$ and a in a checkerboard pattern satisfies the condition. Hence sum $S = a$ can take any value, so Peter needs to know at least one number.

Problem 11. (International Mathematical Tournament Of Towns, 2020)
What is the maximum number of distinct integers in a row such that the sum of any 11 consequent integers is either 100 or 101?

Problem 12. (International Mathematical Tournament Of Towns, 2012)
Some cells of a 11×11 table are filled with pluses. It is known that the total number of pluses in the given table and in any of its 2×2 sub-tables is even. Prove that the total number of pluses on the main diagonal of the given table is also even. (2×2 sub-table consists of four adjacent cells, four cells around a common vertex.)

Solution. Let us split the given square into 2×2 squares and a grey diagonal part as shown in the picture. Since the given 11×11 square as well as any 2×2 square contains an even number of pluses, the diagonal part contains an even number of pluses.



We can obtain the main diagonal from the diagonal part by excluding two diagonal rows of 2×2 squares shown in dashed and including the double sum of every second square in the main diagonal (shown in dark grey). Since the number of pluses in the part we include and in the part we exclude is even, the number of pluses in the main diagonal is also even.

Problem 13. (Caucasus Mathematical Olympiad, 2020)
Peter wrote 100 distinct integers on a board. Basil needs to fill the cells of a table 100×100 with integers so that the sum in each rectangle 1×3 (either vertical, or horizontal) is equal to one of the numbers written on the board. Find the greatest n such that, regardless of numbers written by Peter, Basil can fill the table so that it would contain each of numbers $1, 2, \ldots, n$ at least once (and possibly some other integers).

Answer. 6.

Solution. Suppose that Peter wrote out the numbers $a_1, a_2, \ldots, a_{100}$. An example can be obtained by repeating the table 3×3 located below. The sum of the numbers in every rectangle 1×3 is a_1.

1	5	$a_1 - 6$
6	$a_1 - 8$	2
$a_1 - 7$	3	4

Suppose all Peter's numbers are divisible by $k > 18$. We will consider an arbitrary table filled with numbers satisfying the condition, and will show that it can not contain all the numbers from 1 to 7 simultaneously.

Let us consider an arbitrary rectangle 1×4; suppose it is filled with the numbers a, b, c, d. The sums $a + b + c$ and $b + c + d$ should be equal to some of the Peter's numbers, thus they both are divisible by k. Then their difference $a - d$ is also divisible by k. So, the numbers a and d written in the first and the last cells of this rectangle are congruent modulo k.

Let us paint the table in 9 colors in the following order:

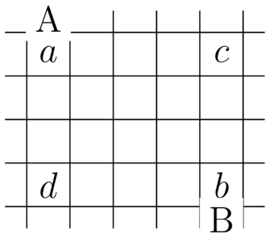
c_1	c_2	c_3	c_1	c_2	c_3	...
c_4	c_5	c_6	c_4	c_5	c_6	...
c_7	c_8	c_9	c_7	c_8	c_9	...
c_1	c_2	c_3	c_1	c_2	c_3	...
c_4	c_5	c_6	c_4	c_5	c_6	...
c_7	c_8	c_9	c_7	c_8	c_9	...
...	...	...	...	...	...	...

Consider an arbitrary color c_i. The two numbers in any two "adjacent" cells of this color are congruent modulo k, because they occupy the first cell and the last cell of some rectangle 1×4. Between any two cells of color c_i one can move making every step in one of the "adjacent" cells of the same color, consequently all the numbers of color c_i are congruent modulo k.

So, each number in the table is congruent modulo k to one of the numbers in the top left square 3×3. We will show that it is impossible to fill it with numbers in such a way that it contains every remainder from 1 to 7 modulo k. Let us assume the opposite. Suppose that numbers with remainders from 1 to 7 occupy some 7 cells of the square 3×3. Then some 3 of them form a rectangle 1×3 and their sum should be divisible by k. But sum of this three remainders is greater than 0 and not greater than $5 + 6 + 7 = 18$, hence less than k. So it is not divisible by k; contradiction.

3 Matchings

Problem 14. (International Mathematical Tournament Of Towns, 2013)
Eight rooks are placed on a 8×8 chessboard, so that no two rooks attack one another. All squares of the board are divided between the rooks as follows. A square where a rook is placed belongs to it. If a square is attacked by two rooks then it belongs to the nearest rook; in case these two rooks are equidistant from this square each of them possesses a half of the square. Prove that every rook possesses the equal area.



Solution. Observe that a rook attacks 15 squares in total, 7 squares in a column and 7 squares in a row where it stands plus a square where it stands. Let us denote a square and rook that stands on it by the same letter, correspondingly small and capital. Let rook A stand on square a. Consider a square c in the same row with square a. It is attacked by another rook B which stands on square b in the same column with c. Rook B will also attack a square d which is in the same column with a. The squares a, b, c, d are the corners of a rectangle. If it is a square then each rook A and B gets a half of c and d. Otherwise, one of the squares completely belongs to rook A, and another to rook B. Consequently, each rook possesses 8 squares in total: the square it stands on and a half of the remaining 14 squares. The statement holds for every rook.

Problem 15. (International Mathematical Tournament Of Towns, 2013)
Eight rooks are placed on a chessboard so that no two rooks attack each other. Prove that one can always move all rooks, each by a move of a knight so that in the final position no two rooks attack each other as well. (In intermediate positions several rooks can share the same square.)

Solution. Observe that condition "no two rooks attack one another" means exactly that
(a) Each horizontal has 1 rook,
(b) Each vertical has 1 rook.
Break movement into two steps:
Step 1: Rooks from verticals 1,2,5,6 move 2 squares right – to verticals 3,4,7,8 respectively; rooks from verticals 3,4,7,8 move 2 squares left – to verticals 1,2,5,6 respectively. Obviously both conditions (a), (b) remains fulfilled.
Step 2: Rooks from horizontals 1,3,5,7 move 1 square up – to horizontals 2,4,7,8; rooks from horizontals 2,4,7,8 move 1 square down – to horizontals 1,3,5,7 respectively. Obviously both conditions (a), (b) remains fulfilled.
As a result each rook made a knight's move.

Problem 16. (Caucasus Mathematical Olympiad, 2020)
All vertices of a regular 100-gon are colored in 10 colors. Prove that there exist 4 vertices of the given 100-gon which are the vertices of a rectangle and which are colored in at most 2 colors.

Solution. Let us inscribe this polygon into the circle. Consider each of the 50 diameters formed by the vertices of the polygon. Note that if there are two diameters among them for which the vertex color set matches, then the four ends of these diameters form a rectangle that matches the condition. There are $10 \cdot 9 : 2 = 45$ options to paint the ends of the diameter in two different colors (without taking into account the order). Hence if a suitable rectangle has not yet been found, there will be at least five diameters whose ends are of the same color. Choose any two, and their ends form a suitable rectangle.

Problem 17. (International Mathematical Tournament Of Towns, 2014)
Inspector Gadget has 36 stones with masses 1 gram, 2 grams, ..., 36 grams. Doctor Claw has a superglue such that one drop of it glues two stones together (thus two drops glue 3 stones together and so on). Doctor Claw wants to glue some stones so that in obtained set Inspector Gadget cannot choose one or more stones with the total mass 37 grams. Find the least number of drops needed for Doctor Claw to fulfil his task.

Answer: 9.

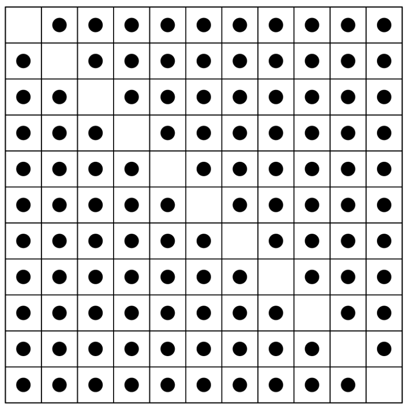
Solution. (a) Among the given stones there are 18 stones with odd masses which could be split into 9 pairs. To glue stones in pairs Doctor Claw needs 9 drops. In new group of stones there is no stone with odd weight. Therefore, Inspector Gadget cannot fulfil his task.

(b) Let us split all stones into 18 pairs so that in each pair a total weight of stones is 37. Then Doctor Claw needs to “spoil” at least one stone in each pair which is impossible with less than 9 drops.

Problem 18. (International Mathematical Tournament Of Towns, 2018)
Pete is placing 500 kings on a 100×50 board so that none of them attacks one another. Basil is placing 500 kings on white cells of a 100×100 chessboard so that none of them attacks one another. Who has more ways to place the kings?

Problem 19. (International Mathematical Tournament Of Towns, 2013)
On a table, there are 11 piles of ten stones each. Pete and Basil play the following game. In turns they take 1, 2 or 3 stones at a time: Pete takes stones from any single pile while Basil takes stones from different piles but no more than one from each. Pete moves first. The player who cannot move, loses. Which of the players, Pete or Basil, has a winning strategy?

Solution. Let us position stones as on the picture so that piles correspond to columns. Peter must take several stones from one column and Basil must take them from different columns. Basil's strategy is to make moves symmetric to those of Peter with respect to empty diagonal. Since a row symmetric to a column has no common stones with it, Basil can each time restore the broken symmetry, so he always has a move. Since the number of stones is finite, eventually Peter loses.



4 Counting

Problem 20. (International Mathematical Tournament Of Towns, 2013)
Twenty children, ten boys and ten girls, are standing in a line. Each boy counted the number of children standing to the right of him. Each girl counted the number of children standing to the left of her. Prove that the sums of numbers counted by the boys and the girls are the same.

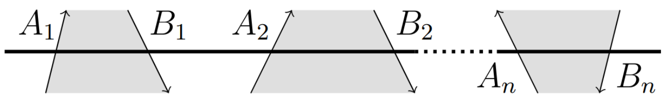
Solution 1. Assume that the children in a line stay to the right of the first person. Let a boy on the k-th position count the number $20 - k$ while a girl on the n-th position count the number $n - 1$. Therefore the total sums of numbers obtained by boys and girls are $200 - S_b$ and $S_g - 10$ respectively, where S_b is the sum of boys' positions and S_g is the sum of girls' positions. It remains to check that $200 - S_b = S_g - 10$. The latter follows from $S_b + S_g = 1 + 2 + \cdots + 20 = 210$.

Solution 2. Let B and G be the sums counted by boys and girls respectively. Note that if a boy and a girl interchange their places in the line, both sums will increase or decrease on the same amount. Therefore the difference between B and G is always the same. However, in situation when ten girls are followed by ten boys it is obvious that both sums are the same.

Problem 21. (International Mathematical Tournament Of Towns, 2013)
A boy and a girl were sitting on a long bench. Then twenty more children one after another came to sit on the bench, each taking a place between already sitting children. Let us call a girl *brave* if she sat down between two boys, and let us call a boy *brave* if he sat down between two girls. It happened, that in the end all girls and boys were sitting in the alternating order. Is it possible to uniquely determine the number of brave children?

Solution. Divide the bench into segments occupied by boys or girls only. These segments alternate. Notice that if a not brave child comes to the bench then the number of segments does not change. If a brave child comes to the bench then the number of segments increases by 2. Initially there were two segments. In the end there were 22 segments. Therefore, the number of brave children is $(22 - 2) : 2 = 10$.

Problem 22. (International Mathematical Tournament Of Towns, 2015)
Area 51 has the shape of a non-convex polygon. It is protected by a chain fence along its perimeter and is surrounded by a minefield so that a spy can only move along the fence. The spy went around the Area once so that the Area was always on his right. A straight power line with 36 poles crosses this area so that some of the poles are inside the Area, and some are outside it. Each time the spy crossed the power line, he counted the poles to the left of him (he could see all the poles). Having passed along the whole fence, the spy had counted 2015 poles in total. Find the number of poles inside the fence.



Answer. 1.

Solution. Let A_1, B_1, A_2, B_2, $\ldots$, A_n, B_n be consecutive points where the power line enters and exits the Area; a Spy goes along the fence so that the Area is on his right. Let us orient the line so that when it enters the Area the Spy goes "up" and when it exits the Area, the Spy goes "down" (see the figure). Then passing through A_k and B_k (A_k and B_k are not necessary consecutive for the Spy) the Spy counts all poles to the left from A_k and all poles to the right of B_k, therefore he counts $36 - a_k$ poles (skipping a_k poles between A_k and B_k). Then coming back to the point he started the Spy counts $36n - x = 2015$ poles where $x = a_1 + a_2 + \cdots + a_n$ $(0 \leq x \leq 36)$. Since $2016 = 2015 + 1$ is divisible by 36 this equation has an unique solution $x = 1$.

Problem 23. (International Mathematical Tournament Of Towns, 2014)
During his last school year, Andrew recorded his marks in maths. He called his upcoming mark (2, 3, 4, or 5) *unexpected* if until this moment it appeared less often than any other possible mark. (For instance, if

he had marks 3,4,2,5,5,5,2,3,4,3 on his list then unexpected marks would be the first 5 and the second 4). It happened that at the end of the year Andrew had on his record list forty marks and each possible mark was repeated exactly 10 times (the order of marks is unknown). Is it possible to determine the number of unexpected marks?

Problem 24. (International Mathematical Tournament Of Towns, 2014)
Originally there was a pile of silver coins on a table. One can either add a gold coin and record the number of silver coins on the first list or remove a silver coin and record the number of gold coins on the second list. It happened that after several such operations only gold coins remained on the table. Prove that at that moment the sums of the numbers on the two lists were equal.

5 Processes

Problem 25. (International Mathematical Tournament Of Towns, 2017)
Ten children of distinct height stand in a circle. Sometimes, one of them moves to a new place in the circle between two children. The children want to be arranged as soon as possible by increasing height clockwise (from the lowest to the highest child). What is the minimal number of moves sufficient for this regardless of initial arrangement of children?

Problem 26. (International Mathematical Tournament Of Towns, 2019)
a) There is a row of 100 cells each containing a token. For 1 dollar it is allowed to interchange two neighbouring tokens. Also it is allowed to interchange with no charge any two tokens such that there are exactly 3 tokens between them. What is the minimum price for arranging all the tokens in the reverse order?
b) There is a row of 100 cells each containing a token. For 1 dollar it is allowed to interchange two neighbouring tokens. Also it is allowed to interchange with no charge any two tokens such that there are exactly 4 tokens between them. What is the minimum price for arranging all the tokens in the reverse order?

a) **Answer:** 50 dollars.
Solution. *Lower bound.* Note that the free (chargeless) operation does not change the parity of the numbers of places of the tokens. As each token needs to change the parity of the place where it stands, we need at least 100 such changes, which means at least 50 nonfree operations.
Algorithm. Let us color the places in the row into four colors: $abcdabcdabcd \dots abcd$. As the free operation can interchange two tokens in the consecutive cells of the same color, it is possible by using it several times to arrange the tokens on the cells of the same color in any order we wish. Let us interchange the tokens in all pairs bc and all pairs da for 49 dollars. Now let us move the tokens 1 and 100 to the adjacent positions using the free operations and interchange them for the remaining 1 dollar. Now all the tokens that were standing on the color a are standing on the color d, and similarly for other colors. Thus we may rearrange them in the reverse order using the free operations.
b) **Answer:** 61 dollar.
Solution. Note that the free (chargeless) operation does not change the residue of the token modulo 5. Let us color the places in the row into 5 colors corresponding to their residue modulo 5, for simplicity let us name those color simply as 0, 1, 2, 3, 4. Note that any tokens that stay on cells of the same color may be rearranged in any order by using the free operations, thus we may think of the nonfree operation as if it just traded a pair of counters between two adjacent colors. In this context we may reformulate the problem as follows: there are 5 piles of counters in a circle, one may interchange two counters between two adjacent piles for 1 dollar. What is the minimum cost for exchanging all the counters between 1 and 0 and between 2 and 4?
Lower bound. If every token from pile 0 travelled to pile 1, they all were used in at least 1 nonfree operation. The same holds for the tokens from pile 1. Now as every token from pile 2 travelled to pile 4, each of them was used in at least 2 nonfree operations, the same stays for counters from pile 4. As each operation works with two counters simultaneously, we may find that the minimum number of nonfree operations is $(20+20+40+40) : 2 = 60$. However if it were possible to perform exactly 60 operations then each token from pile 2 passed through pile 3, thus at least one token originally from pile 3 also was used in the operations and so there were more operations than 60.
Algorithm. After we have reformulated the problem, the algorithm is easy. Let us exchange the tokens between 0 and 1 directly for 20 dollars. Let us select a token X in the pile 3. Then we trade the token X for some token a_1 from pile 2, now we trade a_1 for some token b_1 in pile 4, now b_1 is in pile 3, so we may trade it for some token a_2 in pile 2 and so on. In the end we will have b_2 in pile 3, so we will interchange it with X. This part of the algorithm costs 41 dollar.

Problem 27. (International Mathematical Tournament Of Towns, 2016)
A hundred bear-cubs picked up berries in a forest. The youngest bear-cub got one berry, the second youngest got 2 berries, the third youngest got 4 berries, and so on; the eldest cub got 2^{99} berries. They meet a fox

who suggests to divide the berries "fairly". The fox chooses two bear-cubs and divides their berries equally between them, but if one berry is left over then the fox eats it. The fox proceeds in such a way until all bear-cubs have the same number of berries. What is the least possible number of berries that fox can leave for cubs?

Problem 28. (Caucasus Mathematical Olympiad, 2021)
4 tokens are placed in the plane. If the tokens are now at the vertices of a convex quadrilateral P, then the following move could be performed: choose one of the tokens and shift it in the direction perpendicular to the diagonal of P not containing this token; while shifting tokens it is prohibited to get three collinear tokens. Suppose that initially tokens were at the vertices of a rectangle Π, and after a number of moves tokens were at the vertices of one another rectangle Π' such that Π' is similar to Π but not equal to Π. Prove that Π is a square.

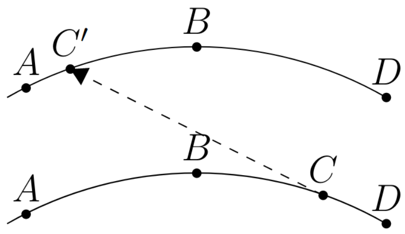
Solution. Let $ABCD$ be the quadrilateral with its vertices at tokens. If we shift, say, A along the perpendicular to BD, then $AB^2 - DA^2$ is invariant. Hence, under any operation $f(ABCD) = AB^2 - BC^2 + CD^2 - DA^2$ is invariant. If $A'B'C'D'$ is similar to $ABCD$ with ratio k, $f(A'B'C'D') = \pm k^2 \cdot f(ABCD)$. Therefore, under conditions of the problem, we have $f(\Pi) = 0$. To complete the solution, it suffices to show that for Π which is not a square, we have $f(\Pi) \neq 0$.

Remark. Note that it is well known that $AB^2 - BC^2 + CD^2 - DA^2 = 0$ is equivalent to $AC \perp BD$. Instead of invariant f one can use some other versions of invariant, like $\overrightarrow{AC} \cdot \overrightarrow{BD}$.

Problem 29. (International Mathematical Tournament Of Towns, 2014)
Points $A_1, A_2, \ldots, A_{10}$ are marked on a circle clockwise. It is known that these points can be divided into pairs of points symmetric with respect to the centre of the circle. Initially at each marked point there was a grasshopper. Every minute one of the grasshoppers jumps over its neighbour along the circle so that the resulting distance between them doesn't change. It is not allowed to jump over any other grasshopper and to land at a point already occupied. It occurred that at some moment nine grasshoppers were found at points $A_1, A_2, \ldots, A_9$ and the tenth grasshopper was on arc $A_9A_{10}A_1$. Is it necessarily true that this grasshopper was exactly at point A_{10}?

Answer. Yes.

Solution. 10 grasshoppers divide circle into 10 arcs. Let us paint alternatively black and white. Originally sums of the lengths of white and black arcs are equal because for any white arc an arc which is symmetric to it with respect to the center is black and conversely for any black arc an arc which is symmetric to it with respect to the center is white.

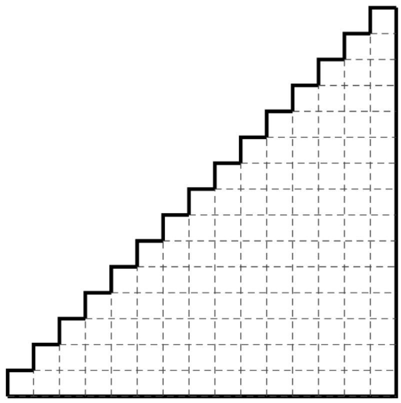


It follows from the figure that the grasshopper's jump does not change these sums. Indeed, sum of arcs AC' and BD equals to the sum of arcs AB and CD. In the final configuration we know 4 black arcs and know where the fifth is located and therefore position of 10-th grasshopper is defined uniquely. On the other hand, A_{10} satisfies to the sums of black and white arcs are equal condition.

6 Grid paper

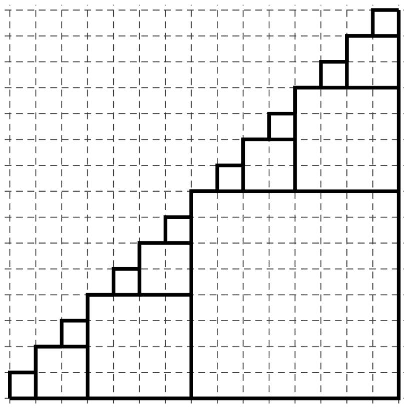
Problem 30. (International Mathematical Tournament Of Towns, 2016)
Pete has drawn a polygon consisting of 100 cells on a grid paper. This polygon can be dissected along grid lines both into 2 congruent polygons and into 25 congruent polygons. Is it always true that this polygon can be also dissected along grid lines into 50 congruent polygons?

Problem 31. (International Mathematical Tournament Of Towns, 2015)

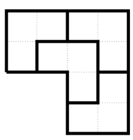


A figure, given on the grid, consists of a 15-step staircase and horizontal and vertical bases (see the figure). What is the least number of squares one can split this figure into? (Splitting is allowed only along the grid).

Solution. Note that each step's corner belongs to some square and no two corners belong to the same square. Therefore the number of squares is no less than 15. Example that splitting the figure into 15 squares can be achieved:



Problem 32. (International Mathematical Tournament Of Towns, 2015)
A grid polygon is called *amazing* if it is not a rectangle and several its copies can form a polygon similar to it. For instance, a corner consisting of three cells is an amazing polygon (see the figure).



a) Find an amazing polygon consisting of 4 cells.
b) Determine all $n > 4$ such that there exists an amazing polygon consisting of n cells.

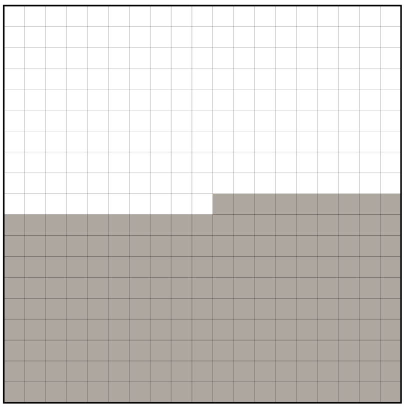
Problem 33. (International Mathematical Tournament Of Towns, 2016)
On a checkered square 10×10 the cells of the upper left 5×5 square are black and all the other cells are white. What is the maximal n such that the original square can be dissected (along the borders of the cells) into n polygons such that in each of them the number of black cells is three times less than the number of white cells? (The polygons need not be congruent or even equal in area.)

Problem 34. (International Mathematical Tournament Of Towns, 2017)
Each cell of a square 1000×1000 table contains a number. It is known that the sum of the numbers in each rectangle of area S with sides along the borders of cells, contained in the table, is the same. Find all values of S which guarantee that all the numbers in the table are equal.

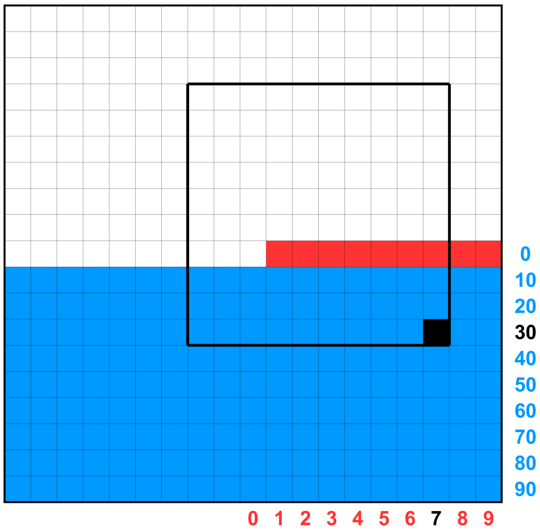
Problem 35. (International Mathematical Tournament Of Towns, 2017)
Pete colored each cell of a 1000×1000 grid square by one of ten colors. He also created a grid polygon F consisting of 10 cells such that no matter how you place it on the square along grid lines, it would cover all ten different colors. Is it necessary for F to be a rectangle?

Problem 36. (International Mathematical Tournament Of Towns, 2013)
There is a 19×19 board. Is it possible to mark some 1×1 squares so that each of 10×10 squares contain different number of marked squares?

Answer. Yes, it is possible.

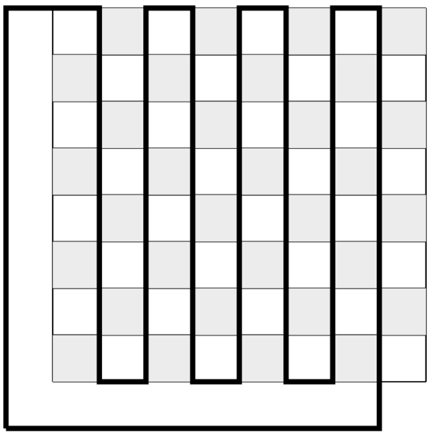


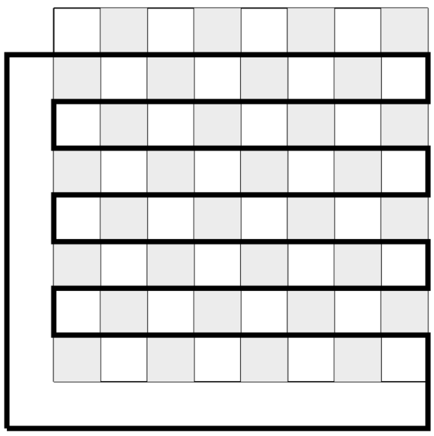
Solution. Let us mark every cell in each of nine bottom rows and all cells in central row to the right of it. Consider a 10×10 square at the top left position. It has no marked cells. Let us move this square to the right, one column at time. In this way, each new 10×10 square will have one more marked cell than the previous one. Therefore we get squares with 0, 1,. . . , 9 marked cells. Now, move each of these ten squares down one row at time. It is easy to see that each new 10×10 square contains 10 more marked cells than the square one position above it. In this way, we get squares with every number of marked cells from 0 to 99.



For example, a square with 37 marked cells.

Problem 37. (International Mathematical Tournament Of Towns, 2013)
There is a 8×8 table, drawn in a plane and painted in a chess board fashion. Peter mentally chooses a square and an interior point in it. Basil can draws any polygon (without self-intersections) in the plane and ask Peter whether the chosen point is inside or outside this polygon. What is the minimal number of questions sufficient to determine whether the chosen point is black or white?





Solution. One question is not enough because a polygon containing all white points and no black point has to be self-intersecting. However two questions are enough: if a point belongs to just one polygon then it is white, and if a point belongs to both or none then it is black.

7 Graphs

Problem 38. (International Mathematical Tournament Of Towns, 2018)
At each vertex of a polyhedron, exactly three faces meet. Each face of this polyhedron is coloured red, yellow or blue. The vertices, where the faces of all three colours meet, are called multicoloured. Prove that the number of multicoloured vertices is even. *(Egor Bakaev, Alexandr Gribalko, Inessa Raskina)*

Problem 39. (International Mathematical Tournament Of Towns, 2015)
In a country there are 100 cities. Every two cities are connected by direct flight (in both directions). Each flight costs a positive (not necessarily integer) number of doubloons. The flights in both directions between two given cities are of the same cost. The average cost of a flight is 1 doubloon. A traveller plans to visit any m cities for m flights, starting and ending at his native city (which is one of these m cities). Can the traveller always fulfil his plans given that he can spend at most m doubloons if
a) $m = 99$;
b) $m = 100$?

Problem 40. (International Mathematical Tournament Of Towns, 2017)
A grasshopper can jump along a checkered strip for 8, 9 or 10 cells in any direction. A positive integer number n is called *jumpable* if the grasshopper can start from some cell of a strip of length n and visit every cell exactly once. Find at least one non-jumpable number $n > 50$.

8 Digits

Problem 41. (International Mathematical Tournament Of Towns, 2015)
Is it true that every positive integer can be multiplied by one of integers 1, 2, 3, 4 or 5 so that the resulting number starts with 1?

Problem 42. (International Mathematical Tournament Of Towns, 2015)
a) The integers x, x^2 and x^3 begin with the same digit. Does it imply that this digit is 1?
b) The same question for the integers x, x^2, x^3, ..., x^{2015}.

Answer. No.
Solution. a) *Example:* $x = 99, x^2 = 99^2 = 9801, x^3 = 99^3 = 970299$.
b) Obviously there exists such positive integer k that $10^{-k} \leq 1 - \sqrt[2015]{0,9}$. Then inequality

$$0,9 \leq (1 - 10^{-k})^{2015} \leq (1 - 10^{-k})^n < 1$$

holds for all $n \leq 2015$. Multiplying it by 10^{kn}, we get

$$0,9 \cdot 10^{kn} \leq (10^k - 1)^n < 10^{kn}.$$

So for $x = 10^k - 1$ all the numbers of the form x^n where $n = 1, 2, \ldots, 2015$ are beginning with 9.
Remark. $k = 5$ already fits, because according to Bernoulli inequality

$$0,9 \leq 1 - n \cdot 10^{-5} \leq (1 - 10^{-5})^n$$

for $n \leq 10000$.

Problem 43. (International Mathematical Tournament Of Towns, 2022)
Does there exist a positive integer number that can be represented as the product of two numeric palindromes in more than 100 ways? (A numeric palindrome is a positive integer number that is read the same from left to right as from right to left).

Problem 44. (International Mathematical Tournament Of Towns, 2013)
There is a positive integer A. Two operations are allowed: increasing this number by 9 and deleting a digit equal to 1 from any position. Is it always possible to obtain $A + 1$ by applying these operations several times? (If leading digit 1 is deleted, all leading zeros are deleted as well.)

Answer: Yes.
Solution. Given the number $A+1$ create a “new number” which starts with eight “1”s followed by the number $A + 1$. Note that the new number and the number A have the same remainders when divided by 9. Therefore given the number A one can get the number $A + 1$ by adding “9”s to A until one obtains the “new number”. Then one removes eight leading “1”s.

9 Geometry. Isosceles triangle, right-angled triangle

Problem 45. (International Mathematical Tournament Of Towns, 2016)
a) Four points are marked on a line, and one point is marked outside the line. Among 6 triangles with vertices at these points, what is the greatest possible number of isosceles triangles?
b) Hundred points are marked on a line, and one more is marked outside the line. Among the triangles with vertices at these points, what is the greatest possible number of isosceles triangles?

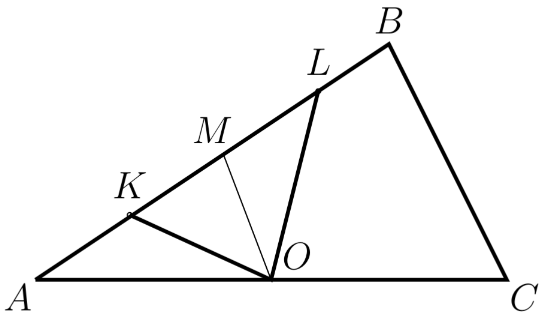
Problem 46. (International Mathematical Tournament Of Towns, 2018)
Let K be a point on the hypotenuse AB of a right triangle ABC, and L be a point on the side AC, such that $AK = AC$ and $BK = LC$. Let M be the intersection point of the segments BL and CK. Prove that the triangle CLM is isosceles.

Problem 47. (International Mathematical Tournament Of Towns, 2017)
a) Given a 10-gon (not necessary convex), draw circles with its sides as diameters. Is it possible that all these circles pass through a point which is not a vertex of this 10-gon?
b) Solve the same problem for a 11-gon.

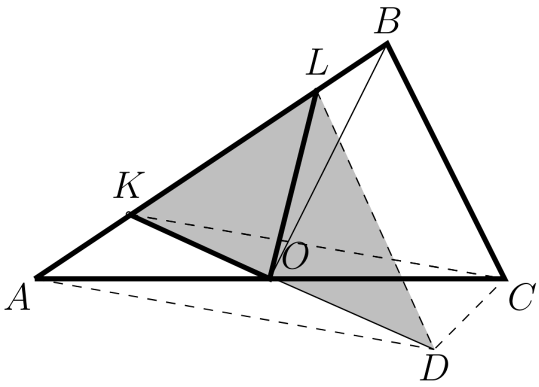
Problem 48. (International Mathematical Tournament Of Towns, 2016)
A point inside a convex quadrilateral is connected with all the vertices and with four more points such that each side contains one of them. The quadrilateral dissects into eight triangles with equal radii of circumcircles. Prove that the original quadrilateral is cyclic.

Problem 49. (Caucasus Mathematical Olympiad, 2021)
In a triangle ABC let K be a point on the median BM such that $CK = CM$. It appears that $\angle CBM = 2\angle ABM$. Prove that $BC = MK$.

Problem 50. (International Mathematical Tournament Of Towns, 2015)
Points K and L are marked on side AB of triangle ABC so that $KL = BC$ and $AK = LB$. Given that O is the midpoint of side AC, prove that $\angle KOL = 90°$.



Solution 1. Let M be a midpoint of AB. Then $MO = 1/2BC = 1/2KL = KM = ML$. Therefore, points K, M, and O belong to a circle with radius KM and centre at M. Since KL is a diameter of this circle, $\angle KOL = 90°$.



Solution 2. Let us construct parallelogram $AKCD$. Then O, the midpoint of AC is the point of intersection of its diagonals and therefore $KO = OD$. Quadrilateral $LBCD$ is also a parallelogram. Since $KL = BC = LD$, triangle KLD is isosceles and its median LO is also an altitude. Hence $\angle KOL = 90°$.

Problem 51. (International Mathematical Tournament Of Towns, 2019)
A point K is marked inside an isosceles triangle ABC so that $CK = AB = BC$ and $\angle KAC = 30°$. Find the angle AKB.

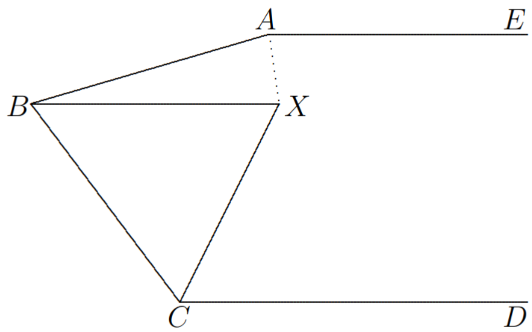
Problem 52. (International Mathematical Tournament Of Towns, 2022)
Consider an acute non-isosceles triangle. In a single step it is allowed to cut any one of available triangles into two triangles along its median. Is it possible that after a finite number of cuttings all triangles will be isosceles?

10 Geometry. Parallelogram

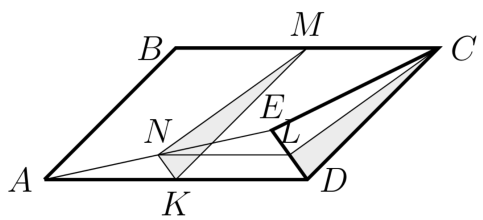
Problem 53. (International Mathematical Tournament Of Towns, 2016)
Suppose $ABCD$ is a parallelogram. Let K be a point such that $AK = BD$ and point M be the midpoint of CK. Prove that $\angle BMD = 90°$.

Problem 54. (International Mathematical Tournament Of Towns, 2019)
Let $ABCDE$ be a convex pentagon such that $AE \parallel CD$ and $AB = BC$. Let K be the intersection of the bisectors of the angles A and C. Prove that $BK \parallel AE$.

Solution. Let the line through B parallel to AE intersect the bisector of $\angle C$ at the point X. Then BX is parallel to CD so that $\angle XCD = \angle CXB$. Hence $\angle BCX = \angle BXC$ so that $BC = BX$. Since $BA = BC$, we have $BA = BX$ so that $\angle BAX = \angle BXA$. Since BX is parallel to AE, $\angle BXA = \angle XAE$. It follows that BX is the bisector of $\angle A$, so that X and K coincide. Hence BK is parallel to AE.



Problem 55. (International Mathematical Tournament Of Towns, 2015)
A point is chosen inside a parallelogram $ABCD$ so that $CD = CE$. Prove that the segment DE is perpendicular to the segment connecting the midpoints of the segments AE and BC.



Solution. Denote by M, N, K and L the midpoints of BC, AE, AD and ED respectively. Since triangle ECD is isosceles, the median CL is also an altitude and therefore $\angle CLD = 90°$. Since NK is the midline of triangle AED, NK is parallel to ED and $NK = LD$. Then triangles MKN and CDL are congruent. ($NK = LD$, $MK = CD$ and $\angle MKN = \angle CDL$, as angles between parallel sides). Hence, $\angle MNK = 90°$ implying that ED is perpendicular to MN.

Problem 56. (International Mathematical Tournament Of Towns, 2013)
Let ABC be an isosceles triangle. Suppose that points K and L are chosen on lateral sides AB and AC respectively so that $AK = CL$ and $\angle ALK + \angle LKB = 60°$. Prove that $KL = BC$.

Solution. Consider a parallelogram $BCLM$. Triangles AKL and BMK are equal: $BM = LC = AK$, $BK = AL$, $\angle KBM = \angle A$. Hence in triangle LKM: $KL = KM$, $\angle LKM = \angle BKM + \angle LKB = \angle ALK + \angle LKB = 60°$. Therefore this triangle is equilateral and $KL = ML = BC$.

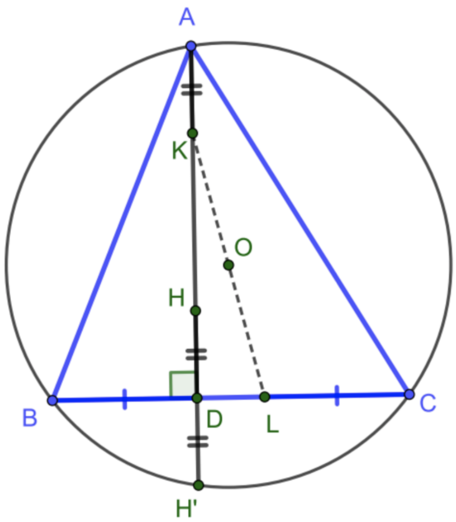
Problem 57. (Caucasus Mathematical Olympiad, 2019)
Points A' and B' lie inside the parallelogram $ABCD$ and points C' and D' lie outside of it, so that all sides

of 8-gon $AA'BB'CC'DD'$ are equal. Prove that A', B', C', D' are concyclic.

11 Geometry. Orthocenter and circumcenter

Problem 58. (International Mathematical Tournament Of Towns, 2020)
Let $ABCD$ be a rhombus, let $APQC$ be a parallelogram such that the point B lies inside it and the side AP is equal to the side of the rhombus. Prove that B is the orthocenter of the triangle DPQ.

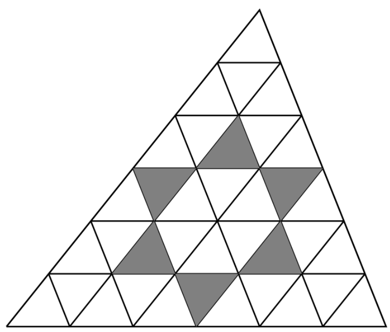
Problem 59. (Caucasus Mathematical Olympiad, 2021)
An acute triangle ABC is given. Let AD be its altitude, let H and O be its orthocenter and its circumcenter, respectively. Let K be the point on the segment AH with $AK = HD$; let L be the point on the segment CD with $CL = DB$. Prove that line KL passes through O.



Solution. Let us show that O is the midpoint of the hypothenuse KL in rectangular triangle KDL. It is sufficient to show that O lies on the perpendicular bisectors of DL and DK. The perpendicular bisector to DL coincides to the perpendicular bisector of BC, hence O lies on it. Further, let H' be the reflection of H in BC. It is known that H' lies on the circle (ABC). And, since $H'D = HD = AK$, the perpendicular bisector of DK coincides with the perpendicular bisector to AH', hence O belongs to it.

Remark. The fact that O lies on the perpendicular bisector of DK, could be proved in many ways, e.g., using the known fact that the distance from O to BC equals to $AH/2$.

Problem 60. (International Mathematical Tournament Of Towns, 2016)



An arbitrary triangle is dissected into congruent triangles by lines parallel to its sides (as is shown in the picture). Prove that the orthocenters of six painted triangles are concyclic.

Problem 61. (International Mathematical Tournament Of Towns, 2018)
Let O be the center of the circumscribed circle of the triangle ABC. Let AH be the altitude in this triangle,

and let P be the base of the perpendicular drawn from point A to the line CO. Prove that the line HP passes through the midpoint of the side AB.

Problem 62. (International Mathematical Tournament Of Towns, 2015)
Suppose that ABC and ABD are right-angled triangles with common hypotenuse AB (D and C are on the same side of line AB). If $AC = BC$ and DK is a bisector of angle ADB, prove that the circumcenter of triangle ACK belongs to line AD. *(Egor Bakaev, Alexandr Zimin)*

Problem 63. (International Mathematical Tournament Of Towns, 2014)
Points K and L are marked on sides AB and BC of square $ABCD$ respectively so that $KB = LC$. Let P be a point of intersection of segments AL and CK. Prove that segments DP and KL are perpendicular.

12 Geometry. Inscribed angles

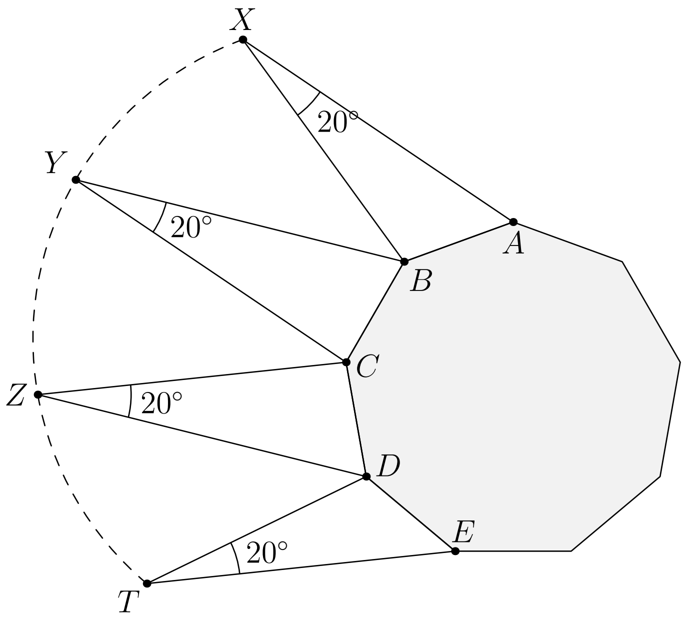
Problem 64. (International Mathematical Tournament Of Towns, 2018)
Is it possible to place a line segment inside a regular pentagon so that it would be seen under the same angle from each vertex of the pentagon? (From a point A a segment XY is seen under angle XAY.) *(Egor Bakaev, Sergey Dvoryaninov)*

Problem 65. (International Mathematical Tournament Of Towns, 2017)
Six circles of radius 1 with centers in the vertices of a regular hexagon are drawn, so that the center O of the hexagon lies inside all six circles. An angle with angular measure α and vertex O cuts out six arcs in these circles. Prove that the sum of the sizes of these arcs is equal to 6α.

Problem 66. (International Mathematical Tournament Of Towns, 2020)
There were ten points $X_1, \ldots, X_{10}$ on a line in this particular order. Pete constructed an isosceles triangle on each segment X_1X_2, X_2X_3, $\ldots$, X_9X_{10} as a base with the angle α at its apex. It so happened that all the apexes of those triangles lie on a common semicircle with diameter X_1X_{10}. Find α.

Problem 67. (International Mathematical Tournament Of Towns, 2017)
Given two concentric circumferences and a point A inside the inner circumference. The angle of a size α with vertex A cuts an arc on each circumference. Prove that if the arc of the outer circumference has the angular size α then the arc of the inner circumference also has the angular size α.

Problem 68. (International Mathematical Tournament Of Towns, 2022)
On the sides of a regular 9-gon $ABCDEFGHI$ the triangles XAB, YBC, ZCD and TDE are constructed outside the 9-gon. Given that the angles X, Y, Z, T of these triangles are $20°$ each. Among the angles XAB, YBC, ZCD, and TDE each next angle is $20°$ greater than the one listed before it. Prove that the points X, Y, Z, T lie on the same circle.



Problem 69. (International Mathematical Tournament Of Towns, 2019)
a) Two equal non-intersecting wooden disks, one gray and one black, are glued to a plane. A triangle with one gray side and one black side can be moved along the plane so that the disks remain outside the triangle, while the colored sides of the triangle are tangent to the disks of the same color (the tangency points are not the vertices). Prove that the line that contains the bisector of the angle between the gray and black sides always passes through some fixed point of the plane. *(Egor Bakaev, Pavel Kozhevnikov, Vladimir Rastorguev)*

b) Two not necessarily equal non-intersecting wooden disks, one gray and one black, are glued to a plane. An infinite angle with one gray side and one black side can be moved along the plane so that the disks remain outside the angle, while the colored sides of the angle are tangent to the disks of the same color (the tangency points are not the vertices). Prove that it is possible to draw a ray in the angle, starting from the vertex of the angle and such that no matter how the angle is positioned, the ray passes through some fixed point of the plane. *(Egor Bakaev, Ilya Bogdanov, Pavel Kozhevnikov, Vladimir Rastorguev)*

13 Geometry. Symmetry and rotation

Problem 70. (International Mathematical Tournament Of Towns, 2015)
In a right-angled triangle ABC ($\angle C = 90°$) points K, L and M are chosen on sides AC, BC and AB respectively so that $AK = BL = a$, $KM = LM = b$ and $\angle KML = 90°$. Prove that $a = b$.

Problem 71. (International Mathematical Tournament Of Towns, 2014)
In a right-angled triangle, two equal circles are constructed so that they touch one another and each one touches hypotenuse and one leg. Consider a segment connecting the points of tangency of the circles and the hypotenuse. Prove that the midpoint of this segment belongs to bisector of right angle of the triangle.

Problem 72. (Caucasus Mathematical Olympiad, 2020)
An equilateral triangle ABC is given. Points K and N lie in the segment AB, a point L lies in the segment AC, and a point M lies in the segment BC so that $CL = AK$, $CM = BN$, $ML = KN$. Prove that $KL \parallel MN$.

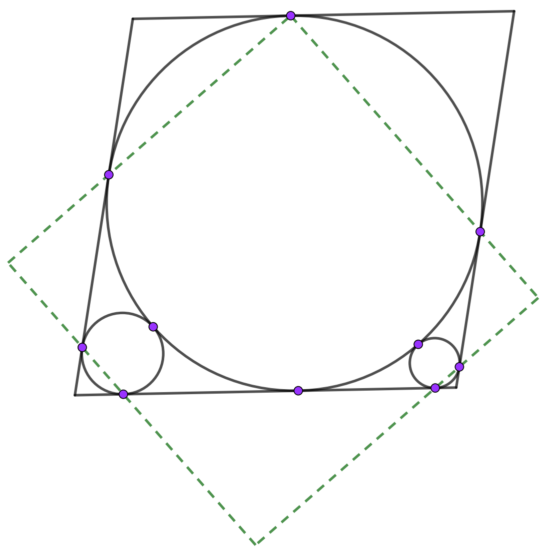
Problem 73. (International Mathematical Tournament Of Towns, 2013)
On an initially colourless plane three points are chosen and marked in red, blue and yellow. At each step two points marked in different colours are chosen. Then one more point is painted in the third colour so that these three points form a regular triangle with the vertices coloured clockwise in "red, blue, yellow". A point already marked may be marked again so that it may have several colours. Prove that for any number of moves all the points containing the same colour lie on the same line.

Solution. In what follows, *simple rotation* would mean "rotation by 60° clockwise". Denote the given points by R, B, and Y. Construct a point R' corresponding to B and Y, and a point B' corresponding to Y and R. Then the simple rotation about Y transforms segment $R'R$ into BB'. (If at least one of these segments degenerates into a point then the triangle RBY is regular and its vertices are listed clockwise; in this case no new points arise.) Thus line RR' is transformed into a line BB' by a simple rotation about their common point O. Let us call the first line *red*, the second line *blue*, and the line obtained by simple rotation of blue line about O *yellow*. Observe that a simple rotation about any point R_1 on the red line transforms blue line into yellow line because the distances from R_1 to these lines are equal. Consequently, if we construct a point Y_1 corresponding to two arbitrary points B_1 and R_1 on blue and red lines respectively, it gets on yellow line. Similarly, given two points on lines of different colours, a point constructed according to the rule will be on the line of the third colour.

14 Geometry. Similarity and trigonometry

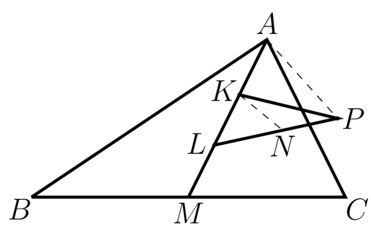
Problem 74. (International Mathematical Tournament Of Towns, 2014)
There are N right-angled triangles. In every given triangle Adam chose a leg and calculated the sum of the lengths of the selected legs. Then he found the total sum of the lengths of the remaining legs. Finally, he found the total sum of the hypotenuses. Given that these three numbers create a right-angled triangle, prove that in every given triangle the ratio of the greater leg to the smaller leg is the same. Consider the cases:
a) $N = 2$;
b) $N \geq 2$ is any positive integer.

Problem 75. (International Mathematical Tournament Of Towns, 2022)



A big circle is inscribed in a rhombus, each of two smaller circles touches two sides of the rhombus and the big circle as shown in the Figure on the right. Prove that the four dashed lines spanning the points where the circles touch the rhombus as shown in the Figure make up a square.

Problem 76. (International Mathematical Tournament Of Towns, 2015)
Points K and L are marked on the median AM of triangle ABC, so that $AK = KL = LM$. Point P is chosen so that triangles KPL and ABC are similar (the corresponding vertices are listed in the same order). Given that points P and C are on the same side of line AM, prove that point P lies on line AC.

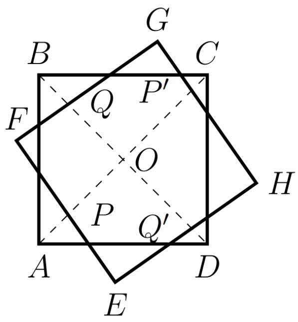


Let KN be median of triangle LKP. Since triangles ABC and KPL are similar, triangles LKN and CAM are similar as well. If $\angle CAM = \alpha$, then $\angle LKN = \alpha$. On the other hand, triangles LKN and LAP are similar ($KN \parallel AP$ as a midline of triangle LAP). Therefore, $\angle LAP = \angle LKN = \alpha$. Hence $\angle LAC = \angle LAP$ and therefore P lies on the line AC.

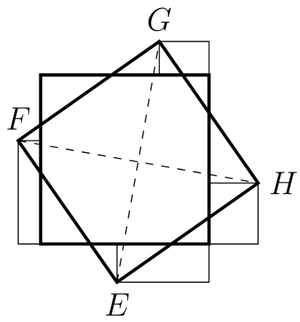
Problem 77. (International Mathematical Tournament Of Towns, 2014)
A square table is covered with a square cloth (may be of a different size) without folds and wrinkles. All corners of the table are left uncovered and all four hanging parts are triangular. Given that two adjacent hanging parts are equal prove that two other parts are also equal.

Solution 1. Let $ABCD$ be a cloth and $EFGH$ be a table (see Figure (a)). We see four hanging parts of the cloth and four triangular parts of the table which are not covered. Observe that all eight triangles are similar. Let us draw diagonals in $ABCD$. Observe that they are bisectors of the corresponding angles. Observe that since angles between AC and $FG \parallel EH$ and BD and $FE \parallel EH$ are equal and distances between two pairs of parallel lines are also equal then $QQ' = PP'$.

If triangles A and B are equal then their bisectors AP and BQ are equal and since $AO = BO = CO = DO$ we see that $PO = QO$. But then $P'O = Q'O$ and $P'C = Q'D$. Then triangles C and D are also equal.



(a)



(b)

Solution 2 (see Figure (b)). We define the *weight* of the hanging triangle as its height dropped from the right corner. Obviously all hanging parts are similar. Therefore parts are equal if and only if their heights are equal. Therefore it is sufficient to prove that the the sums of wights of opposite parts are equal. Adding to these sums the side of the table we get projection of diagonal FH to the "horizontal" side of the the table and of diagonal EG to the "vertical" side of the the table. Since diagonals are equal and orthogonal and the sides of the table are orthogonal, we conclude that projections are equal.

Problem 78. (Caucasus Mathematical Olympiad, 2019)
Given a triangle ABC with $BC = a$, $CA = b$, $AB = c$, $\angle BAC = \alpha$, $\angle CBA = \beta, \angle ACB = \gamma$. Prove that

$$a\sin(\beta-\gamma) + b\sin(\gamma-\alpha) + c\sin(\alpha-\beta) = 0.$$

Solution. Since $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$ (where R is the circumradius), we have

$$2Ra\sin(\beta-\gamma) = 2Ra\sin\beta\cos\gamma - 2Ra\sin\gamma\cos\beta = ab\cos\gamma - ac\cos\beta.$$

Using similar equalities, obtain

$$2R(a\sin(\beta-\gamma)+b\sin(\gamma-\alpha)+c\sin(\alpha-\beta)) = (ab\cos\gamma-ac\cos\beta)+(bc\cos\alpha-ba\cos\gamma)+(ca\cos\beta-cb\cos\alpha) = 0.$$

15 Geometry. Examples and counterexamples

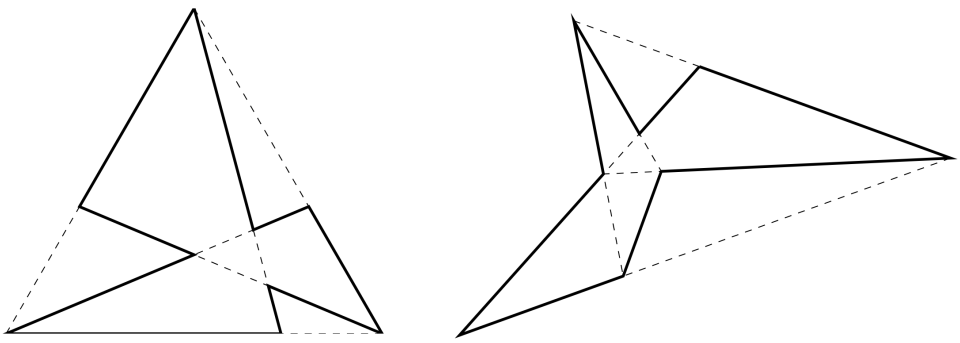
Problem 79. (International Mathematical Tournament Of Towns, 2015)
Is it possible to paint six face of a cube into three colours so that each colour is present, but from any position one can see at most two colours?

Answer. Yes, it is possible.
Solution. Colour two opposite faces of the cube in red and blue, while the other faces in green. From any position one can not see red and blue faces at the same time.

Problem 80. (International Mathematical Tournament Of Towns, 2015)
A rectangle is split into equal non-isosceles right-angled triangles (without gaps or overlaps). Is it true that any such arrangement contains a rectangle made of two such triangles?

Problem 81. (International Mathematical Tournament Of Towns, 2017)
Given are two coins of radius 1 cm, two coins of radius 2 cm, and two coins of radius 3 cm. You may place two of the coins on a table so that they are tangent, and add other coins, one at a time, so that a newly placed coin is tangent to at least two of previously placed ones. A new coin cannot lie over an old one. Is it possible to place several coins on the table so that the centers of three of them are collinear for sure?

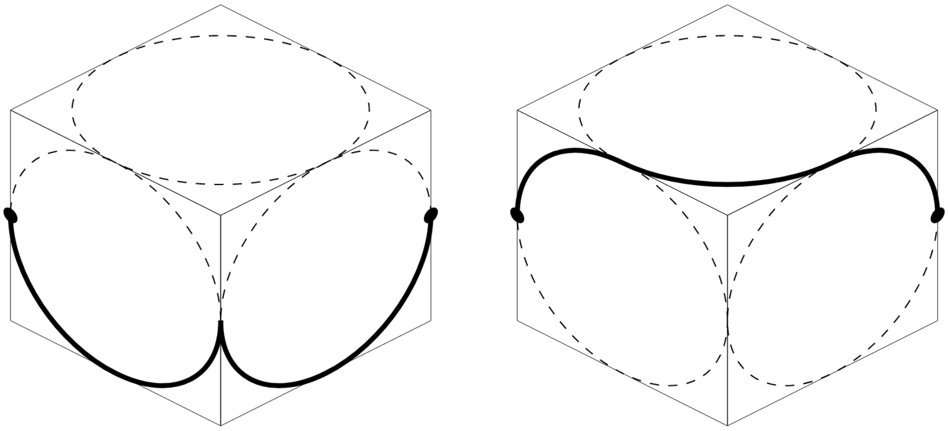
Problem 82. (International Mathematical Tournament Of Towns, 2015)
For each side of some polygon, the line containing it contains at least one more vertex of this polygon. Is it possible that the number of vertices of this polygon is
a) ≤ 9?
b) ≤ 8?


(a) (b)

Answer. Yes.
Solution. See the figures.

Problem 83. (International Mathematical Tournament Of Towns, 2013)
A spacecraft landed on an asteroid. It is known that the asteroid is either a ball or a cube. The rover started its route at the landing site and finished it at the point symmetric to the landing site with respect to the center of the asteroid. On its way, the rover transmitted its spatial coordinates to the spacecraft on the landing site so that the trajectory of the rover movement was known. Can it happen that this information is not sufficient to determine whether the asteroid is a ball or a cube?

Solution. Consider a sphere of radius r and a surface of cube with the side a with the same center. Observe that if $a = \sqrt{2}r$ the sphere touches each edge at its midpoint and therefore it intersects each face of the cube along circle of radius $r/\sqrt{2}$ in its center like on the figure below (we draw only three visible faces):



Then any path consisting of arcs of these circles belongs to both sphere and the surface of the cube and one can connect two symmetric points marked on the figure by such path. Therefore it can happen that such information is not sufficient to determine whether the asteroid is a ball or a cube.