

Adaptive Constructive Solid Geometry with constant evaluation complexity for modeling implicitly defined complex objects

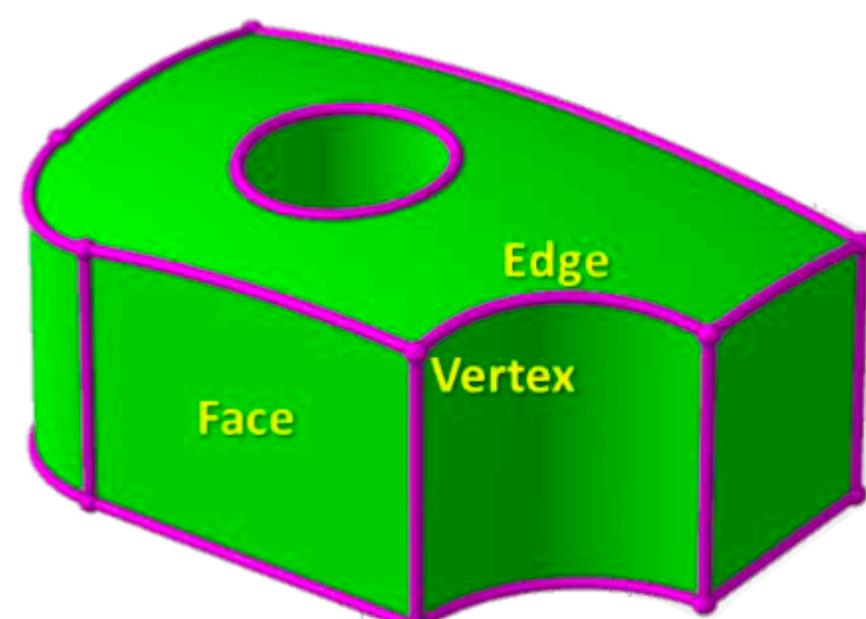
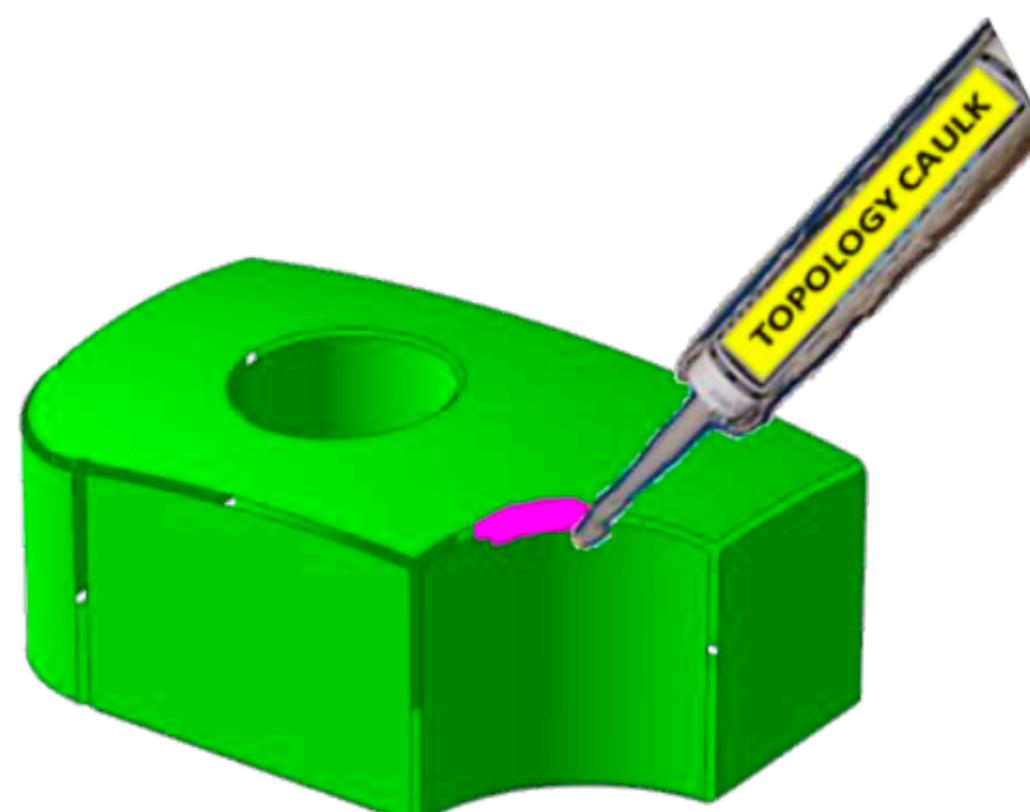
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MSc Program: Applied Computational Mechanics

Background

Object representation in computer graphics and design



Boundary Representation (**B**-rep)

external skin using faces, edges, and vertices

faces are "**glued**" together by topology information describing connectivity

+ straightforward and intuitive approach

+ efficient local changes

- high memory requirements

- homogeneous objects

- possible ill-defined behavior in geometric operations

Function Representation (**F**-rep) or implicit modeling

implicit functions $F(x,y,z)$

no explicit topology - objects can have any topology and change freely

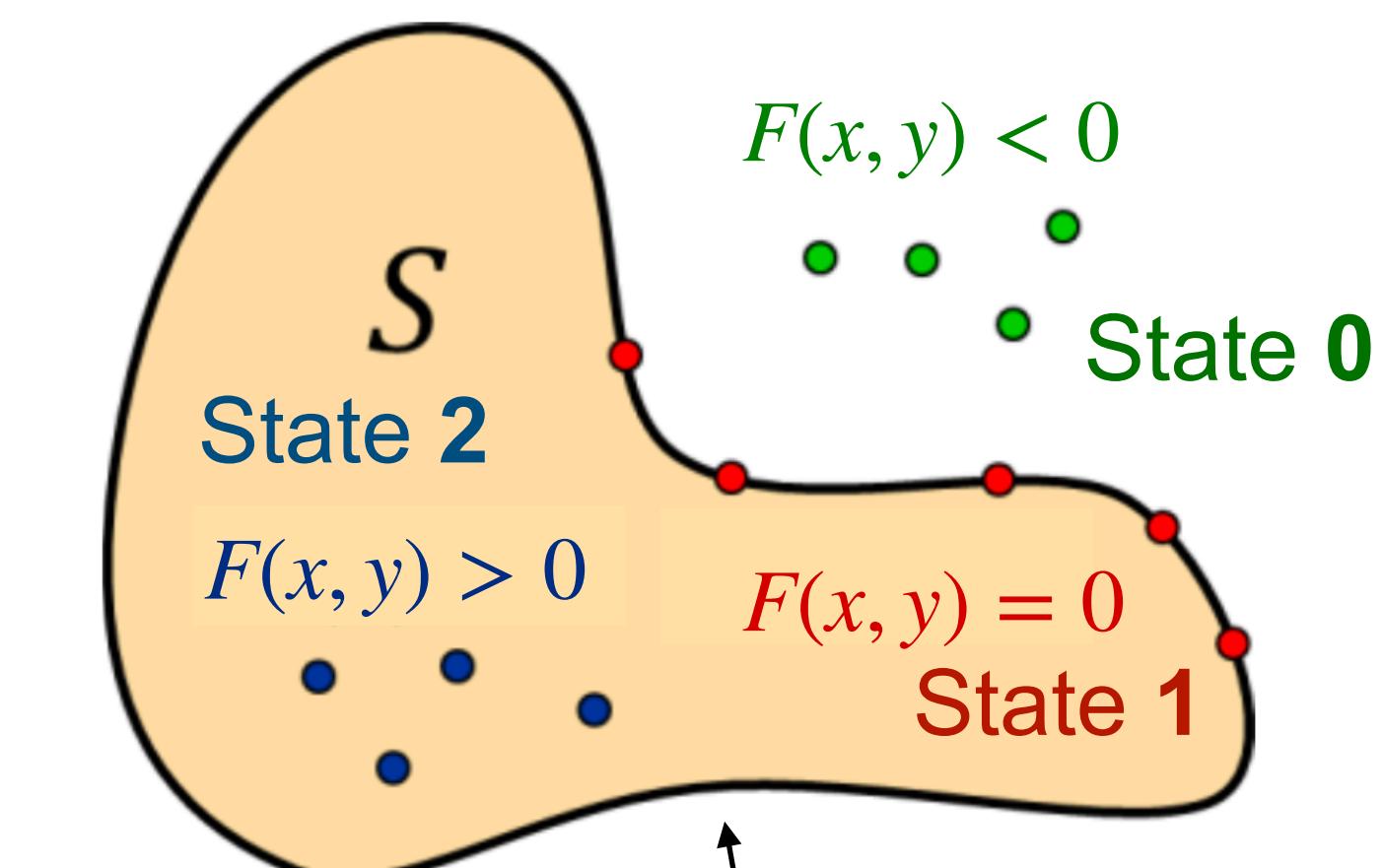
- complicated representation even for simple objects

- local changes affect entire function

+ low memory footprint

+ heterogenous objects

+ well-defined geometric operations



Only sign is important!

Background

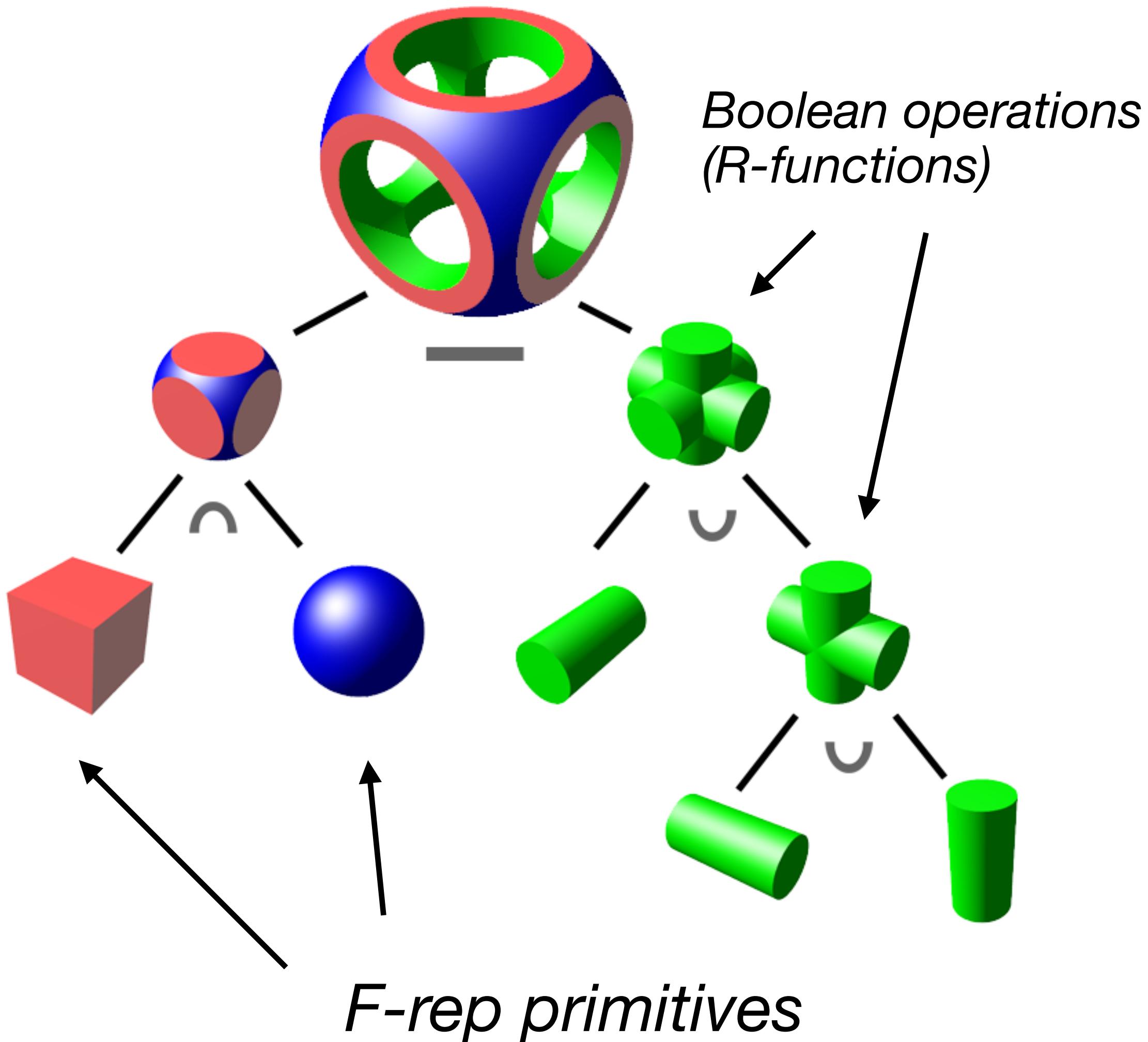
Constructive Solid Geometry (CSG)

Allows a modeler to create a complex object by using *Boolean operators* to combine simpler objects (*primitives*)

Easy to classify arbitrary points as being either inside or outside the shape (*Point Membership Classification*)

Exact Boolean operations on geometric objects

Evaluation complexity: $\mathcal{O}(N)$, where N is the number of nodes in CSG tree

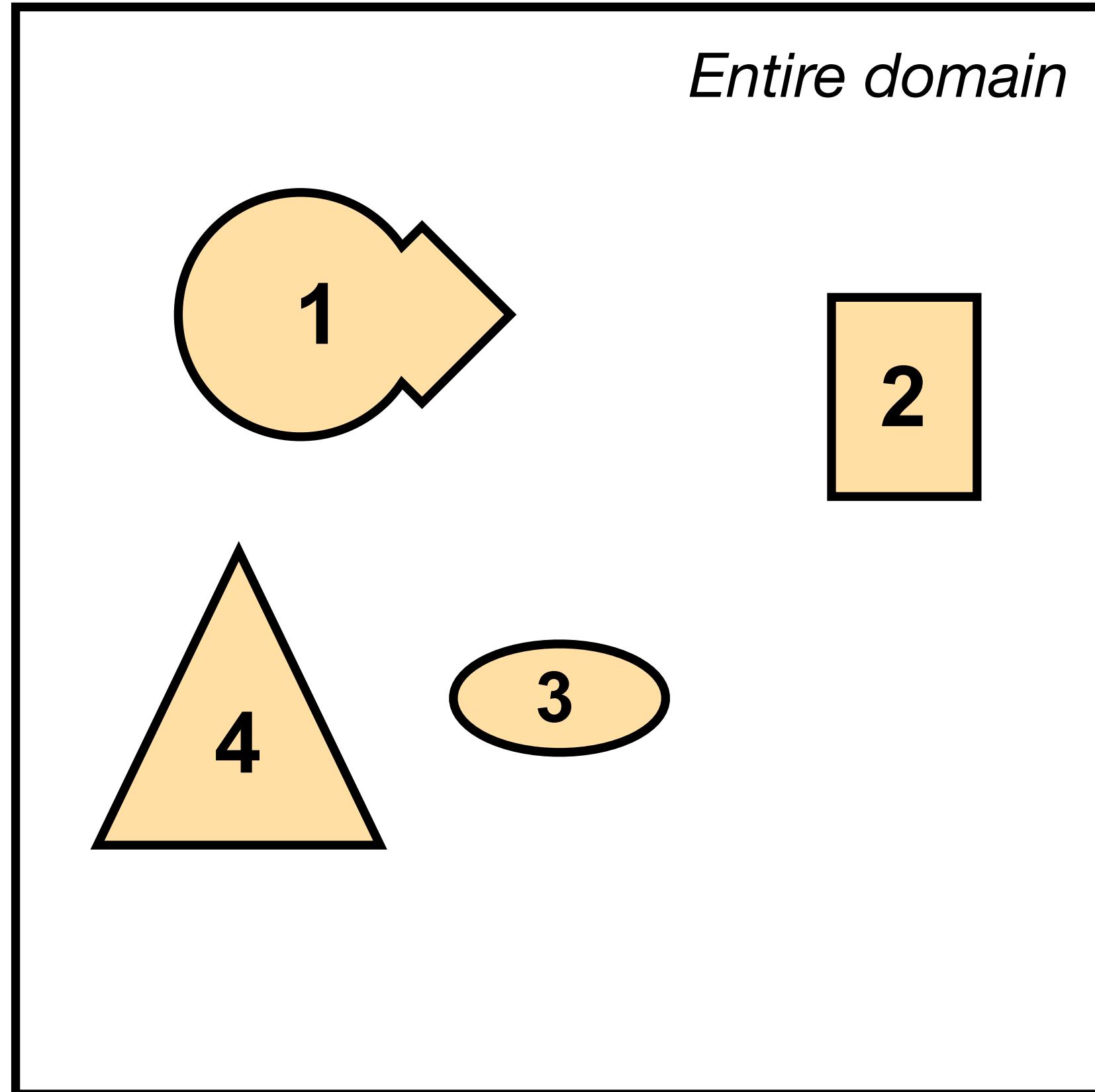


Overall Aim

To develop a comprehensive mathematical framework based on adaptive Constructive Solid Geometry for modeling complex implicitly defined objects

Spatially Adaptive F-rep

Challenges to be addressed: linear complexity of CSG tree

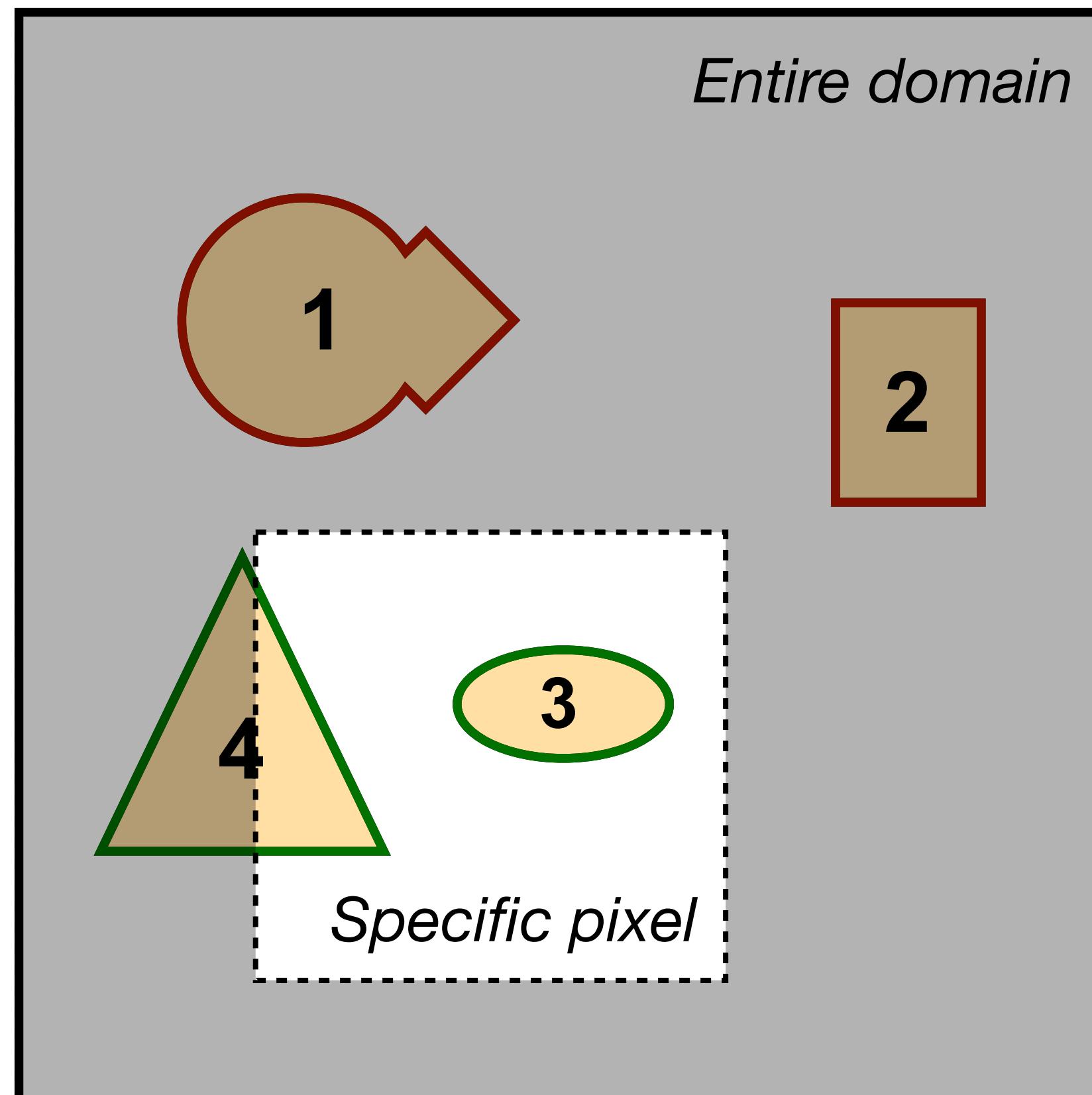


1. Storage of CSG tree of F-rep primitives

Infix and reverse Polish notations (RPN)

Spatially Adaptive F-rep

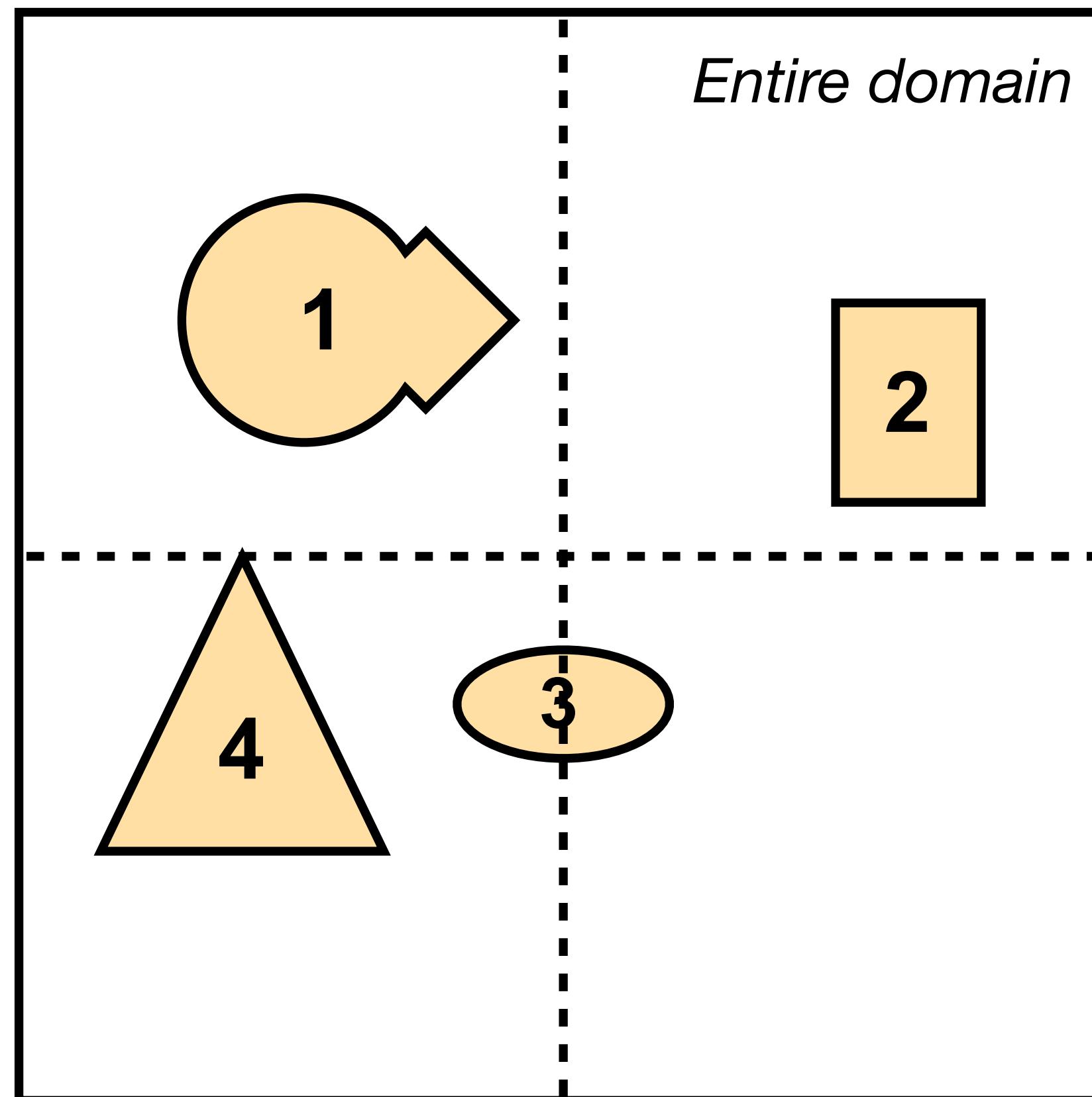
Challenges to be addressed: linear complexity of CSG tree



1. **Storage of CSG tree of F-rep primitives**
Infix and reverse Polish notations (RPN)
2. **Pruning (simplification) CSG tree for specific voxel / pixel (3D / 2D space volume).**
Range evaluation using interval analysis

Spatially Adaptive F-rep

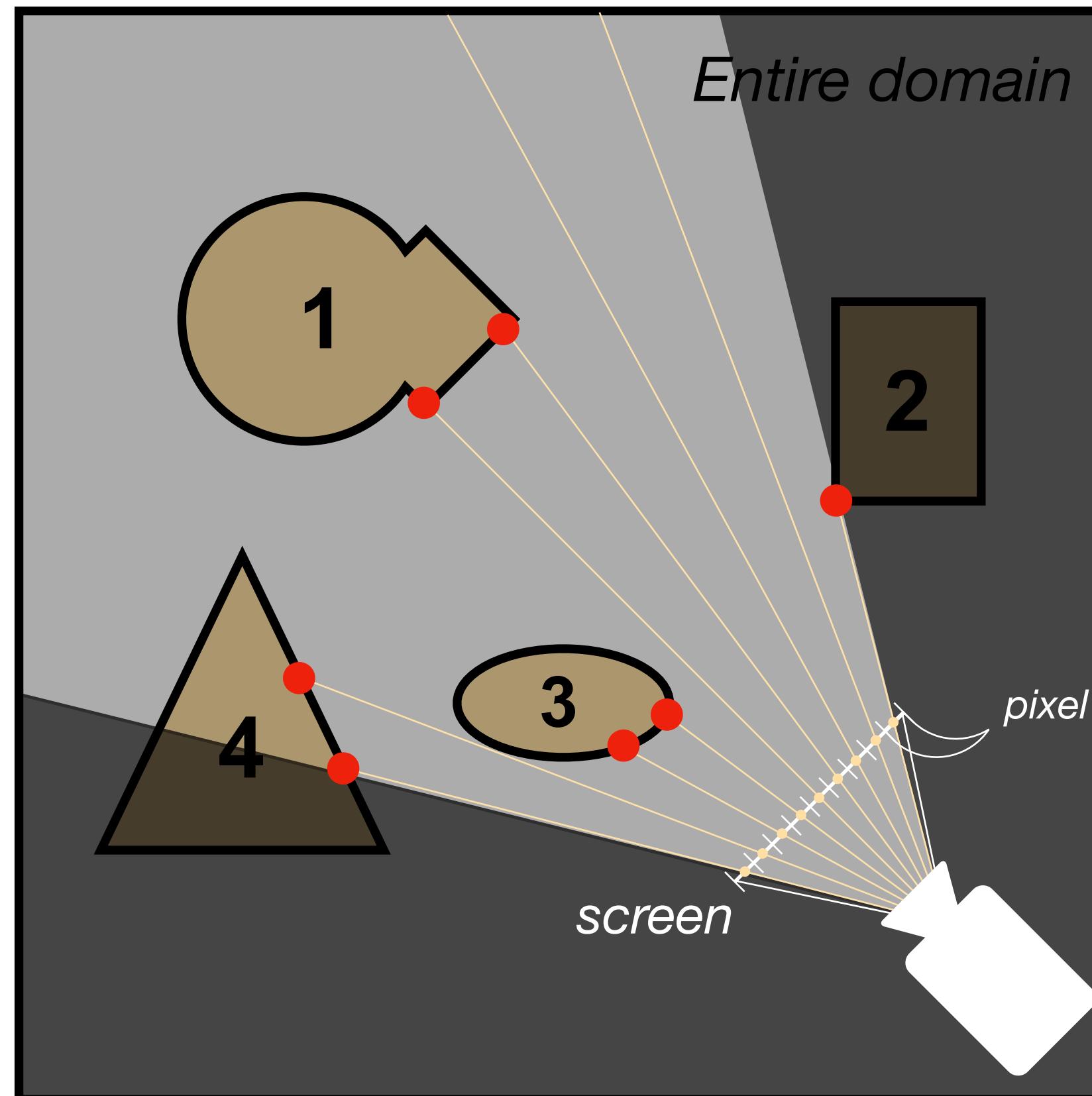
Challenges to be addressed: linear complexity of CSG tree



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Range evaluation using interval analysis
3. **Spatial sampling algorithm**
Sparse Voxel Octree (SVO)

Spatially Adaptive F-rep

Challenges to be addressed: linear complexity of CSG tree



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3. **Spatial sampling algorithm**
Sparse Voxel Octree (SVO)
4. **Rendering algorithm for visualization**
Ray-casting through SVO

Specific Objectives

1. *To design and implement* efficient algorithm for pruning complex F-rep scenes within localized spatial regions
2. *To develop* hierarchical data structure and algorithms for optimal storage and evaluation of compressed F-rep scene
3. *To design and implement* adaptive spatial sampling algorithm enabling constant evaluation complexity using Sparse Voxel Octree
4. *To develop* a robust ray-casting algorithm for rendering F-rep objects
5. *To conduct* performance evaluation of the proposed methodology

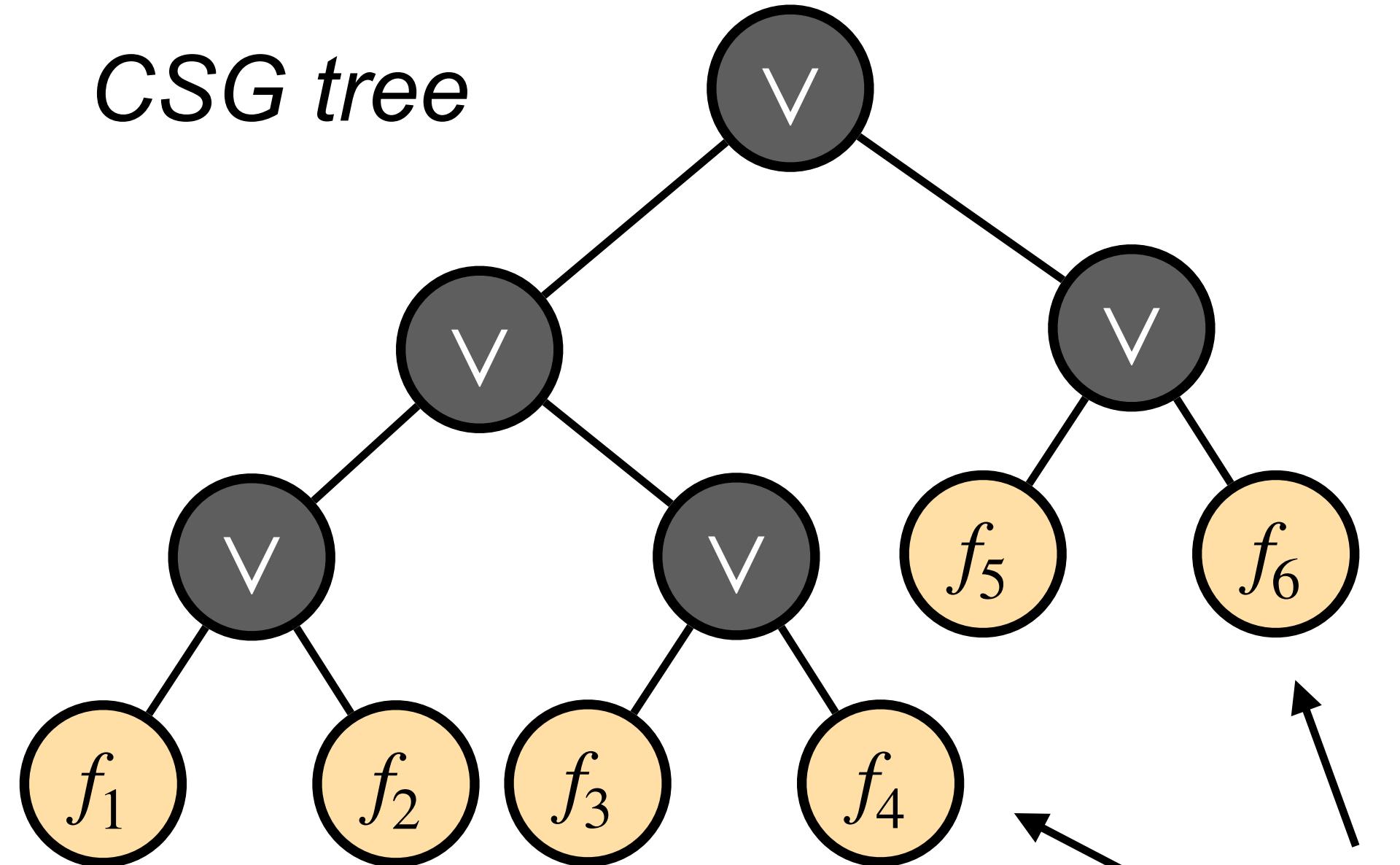
Composite F-rep as CSG tree

Data structures to store objects in F-rep

Infix notation (expression tree)

$$((f_1 \vee f_2) \vee (f_3 \vee f_4)) \vee (f_5 \vee f_6)$$

CSG tree

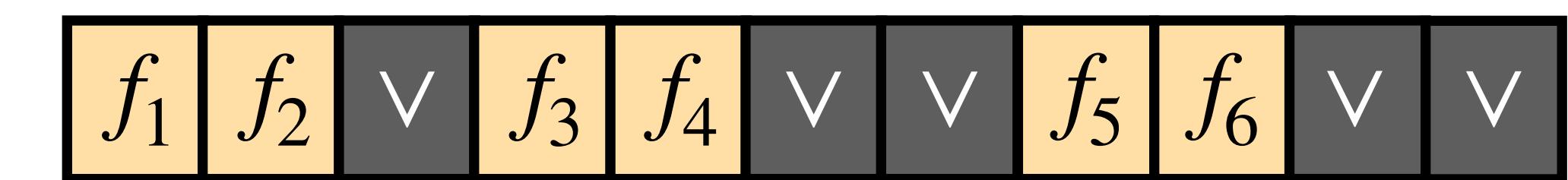


Reverse Polish notation (RPN)

$$f_1 f_2 \vee f_3 f_4 \vee \vee f_5 f_6 \vee \vee$$

Shunting-yard algorithm

Linearized CSG tree



Array with $O(1)$ access to elements

F-rep primitives

Pruning linearized CSG tree

Pruning of specific voxel domain

For each node i in the linearized CSG tree:

1. If node i is an F-rep primitive

Compute its F-rep interval using interval-analysis (IA) techniques (Pasko)

Store the sign of resulting interval: +, −, or \pm

2. If node i is a Boolean operator

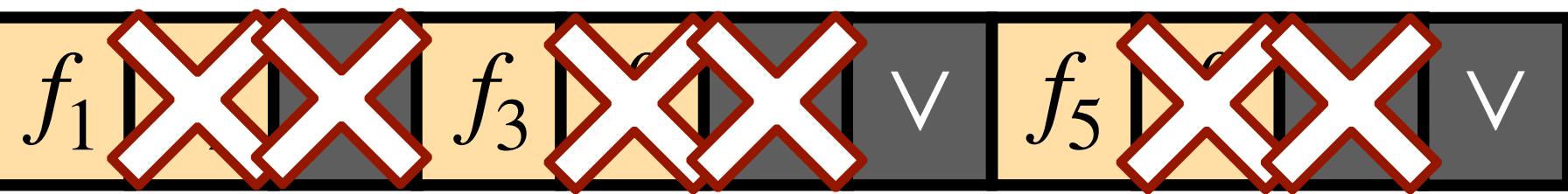
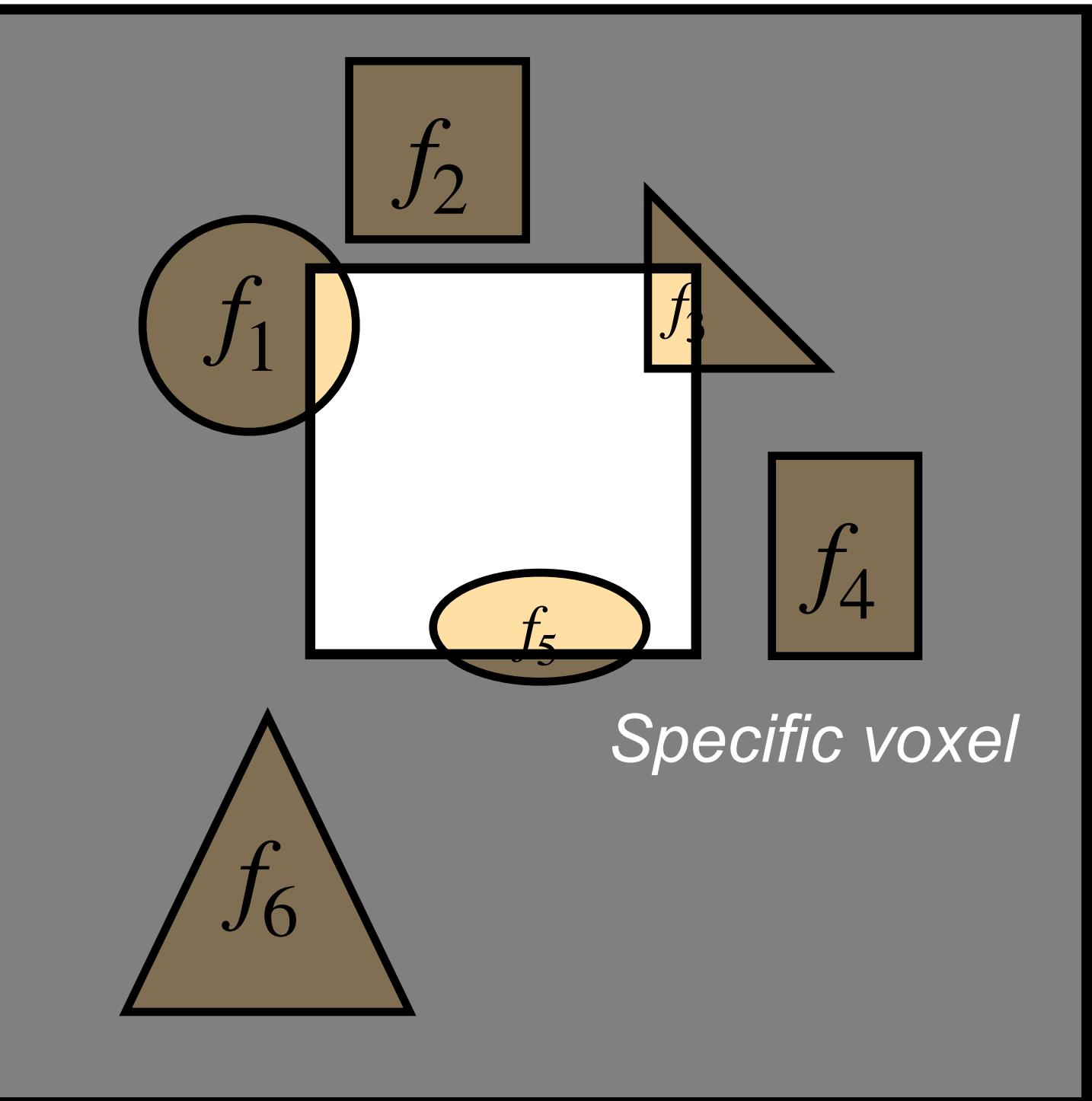
Check whether the node can be pruned (see *Table*).

If it *can* be pruned, recursively prune its subtree.

If it *cannot* be pruned, compute its interval with IA and store its sign.

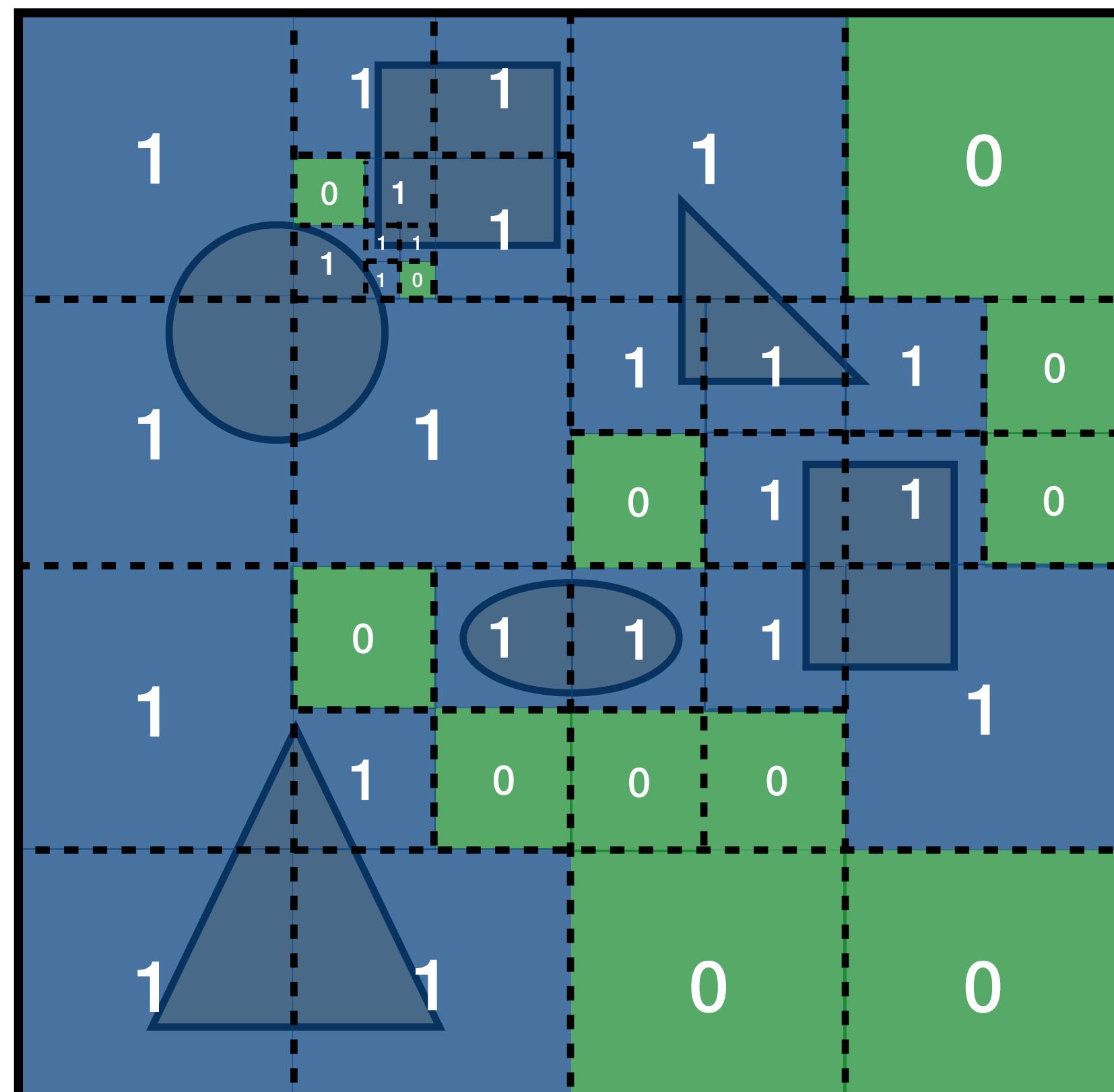
Result: A pruned F-rep represented as a bit-mask, bit-mask with 1 for *active* nodes and 0 for *pruned* nodes.

f_l	f_r	$f_l \vee f_r$	$f_l \wedge f_r$
+	+ , − , ±	f_l	f_r
+ , − , ±	+	f_r	f_l
−	+ , − , ±	f_r	f_l
+ , − , ±	−	f_l	f_r



Adaptive Spatial Decomposition

Sparse Voxel Octree (SVO)



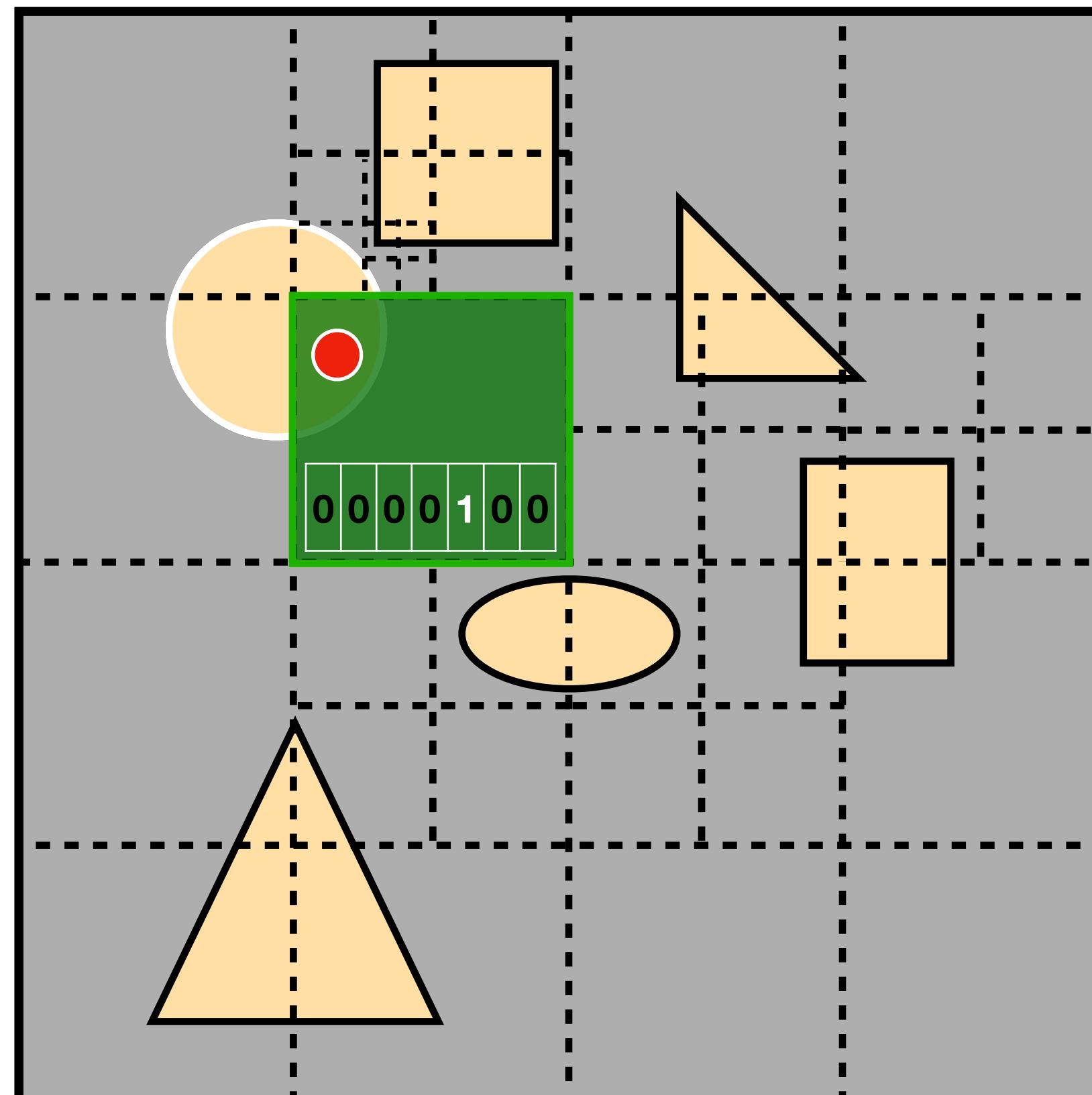
Implementation: octree structure

Subdivision process:

1. Prune composite F-rep (CSG tree) for region
2. Count active CSG tree nodes
3. Compare with threshold (e.g., 2)
4. Subdivide if needed ($n = 8$ for octree, $n = 4$ for quadtree)
5. Recursively prune for subregions

Result: voxel octree with pruned F-rep in each node (empty voxels - **green**, filled voxels - **blue**)

Complexity of spatially adaptive F-rep



Evaluation process:

1. Choose a point in the geometry domain
2. Determine SVO leaf node that the point belongs to

Complexity: $\mathcal{O}(K(L))$, where K is the level of SVO leaf node, and L is the threshold value

3. Get bit mask (compressed F-rep) from this node

Complexity: $\mathcal{O}(1)$

4. Evaluate pruned F-rep expression

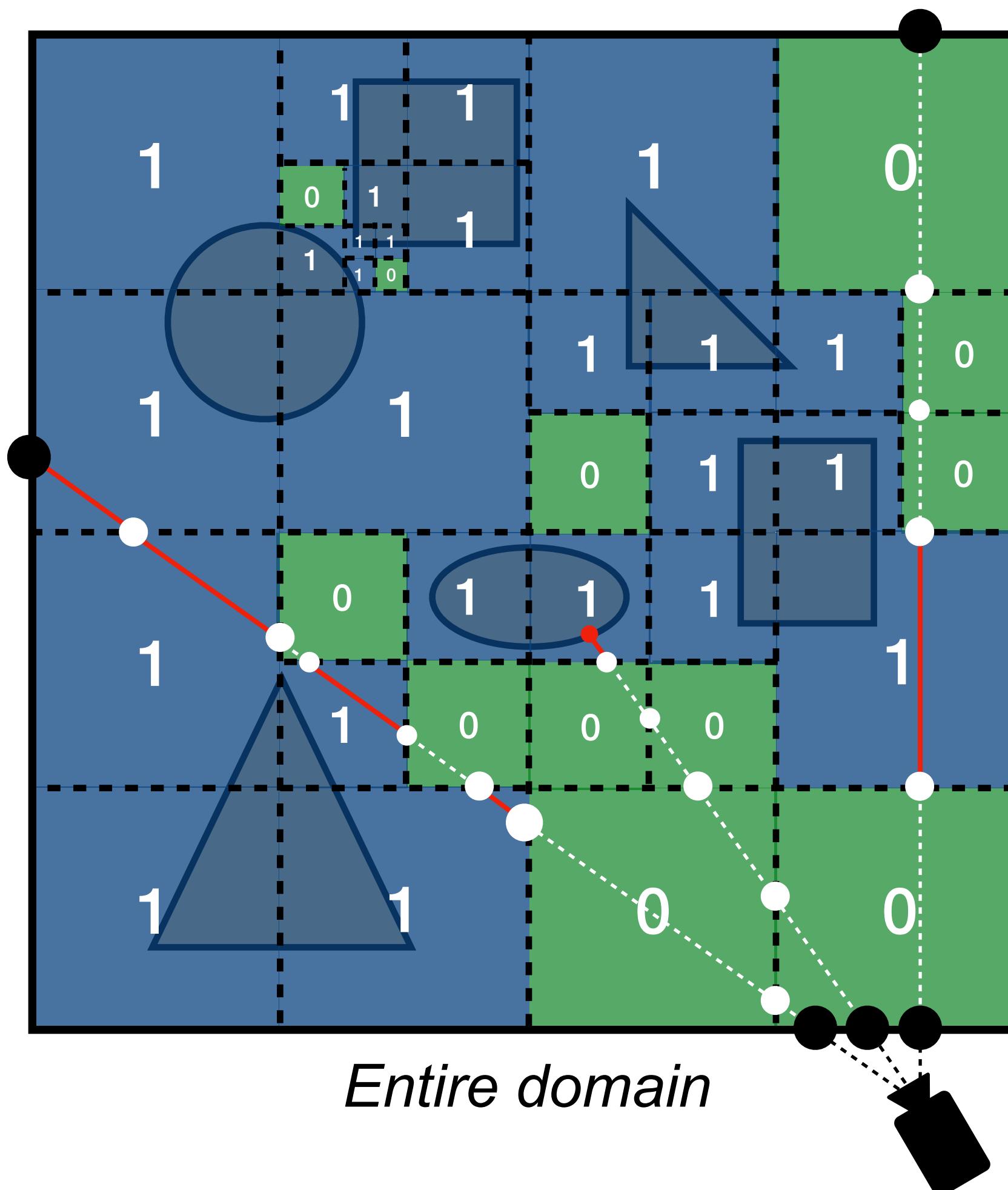
Complexity: $\mathcal{O}(L)$, where L is the threshold value

Total complexity: $C_1 \cdot K(L) + C_2 \cdot L + C_3$

In practice, for large L , $C_2 \cdot L \gg C_1 \cdot K(L) \implies$ total complexity is **constant**, controlled by threshold value

Rendering spatially adaptive F-rep

Ray-traversal through SVO



Implementation: ray-casting through voxel octree

Ray-traversal process:

1. Shoot a ray, specifying its origin and direction
2. Determine the voxels intersected by the ray
3. If a voxel is **empty**, skip it (*dotted line section*)
4. If a voxel is **filled**, attempt to find an intersection with the pruned F-rep (*red line section*)
5. If no intersection is found, proceed to the next voxel

Results

Developed core mathematical framework for an adaptive CSG kernel in C++.

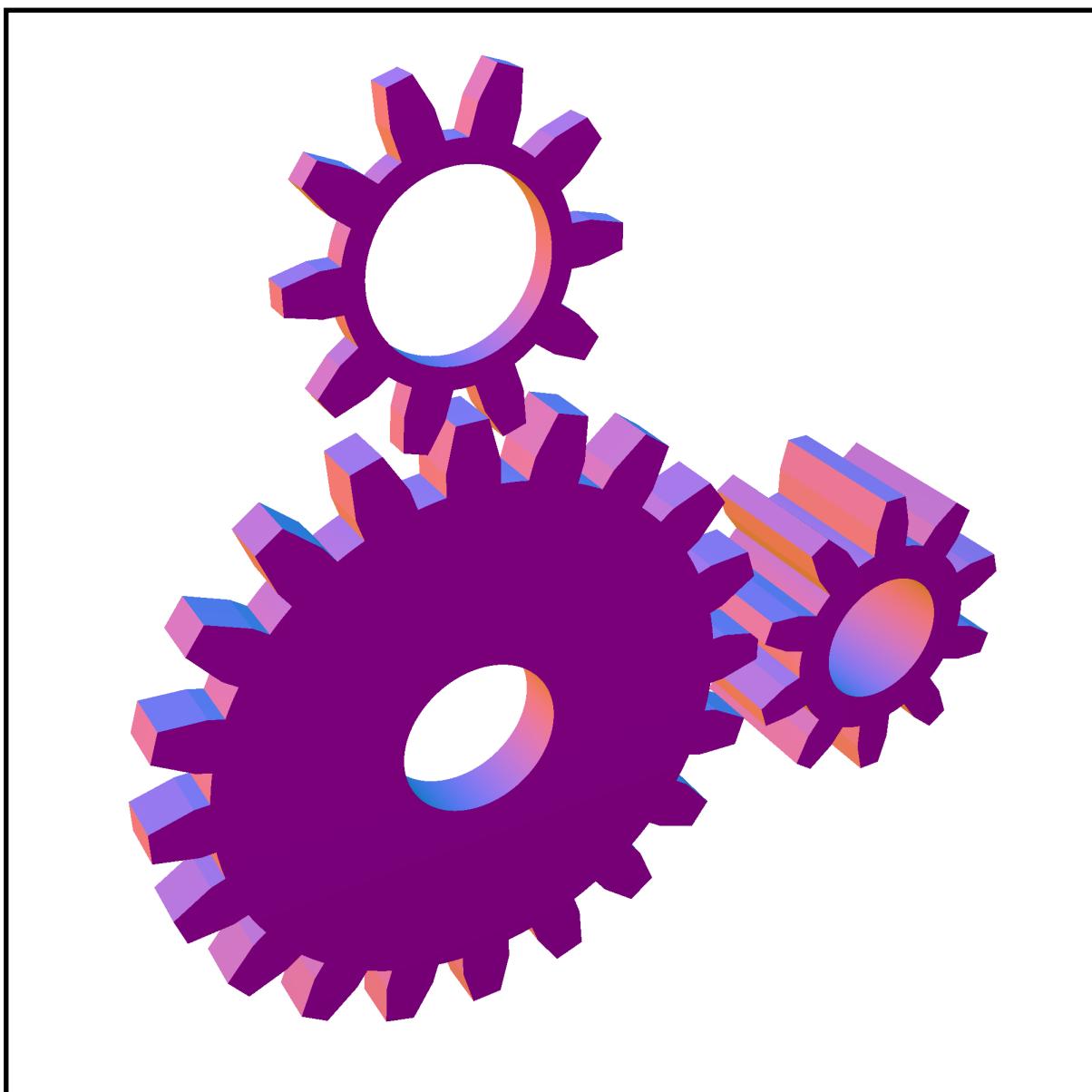
Specifically:

1. *Developed* a memory-efficient storage of CSG tree in reverse Polish notation, employing bitmasks.
2. *Designed* an algorithm that recursively prunes the CSG tree and builds a Sparse Voxel Octree (SVO) with a compressed F-rep in every node.
3. *Developed* robust ray-casting through SVO, based on an interval bisection method.
4. *Implemented* all algorithms in a prototype geometry kernel written entirely in C++.
5. *Compared* the proposed methodology with state-of-the-art approaches.

Results

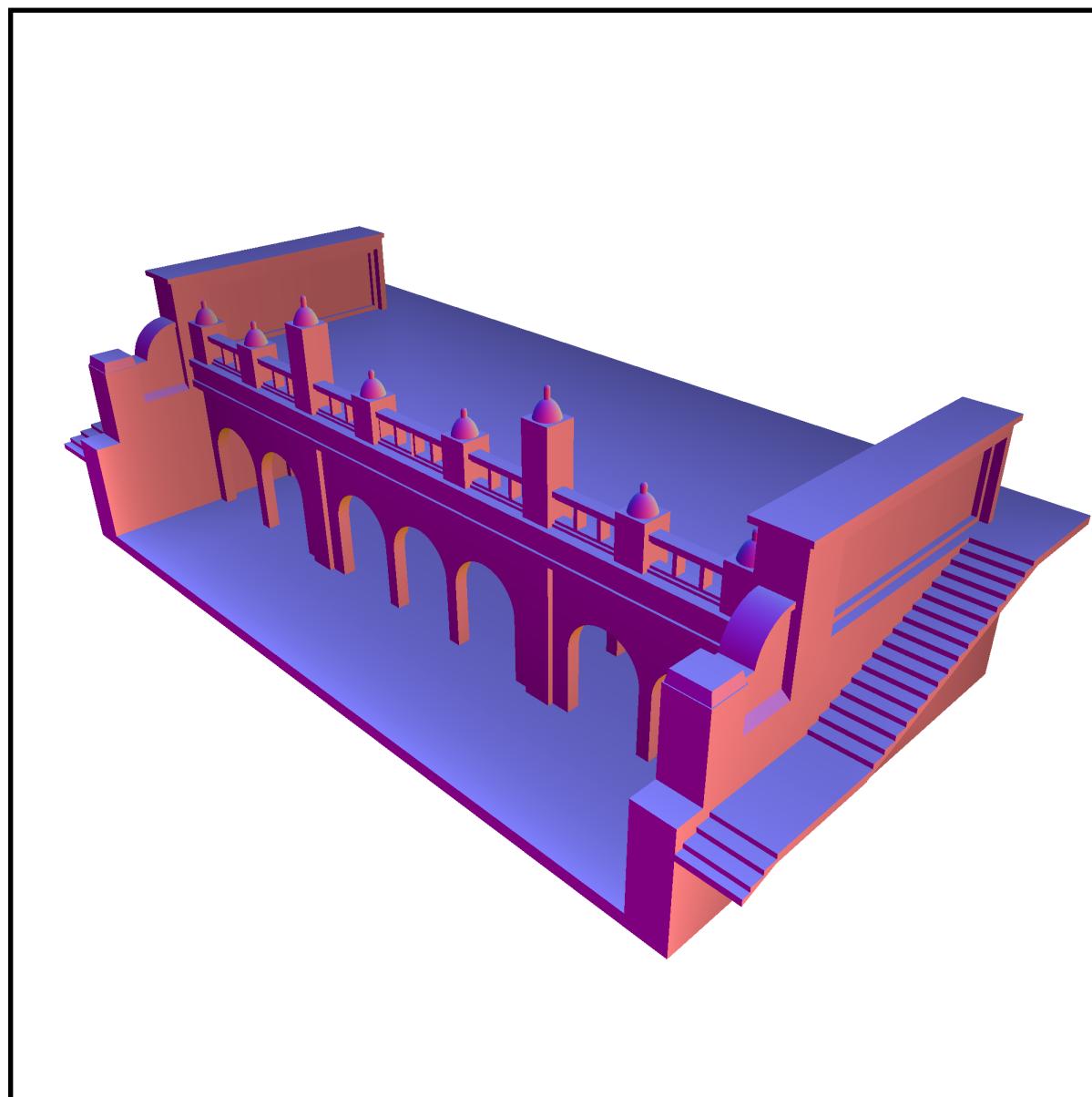
Complex F-rep 3D scenes for benchmarking framework

Gears



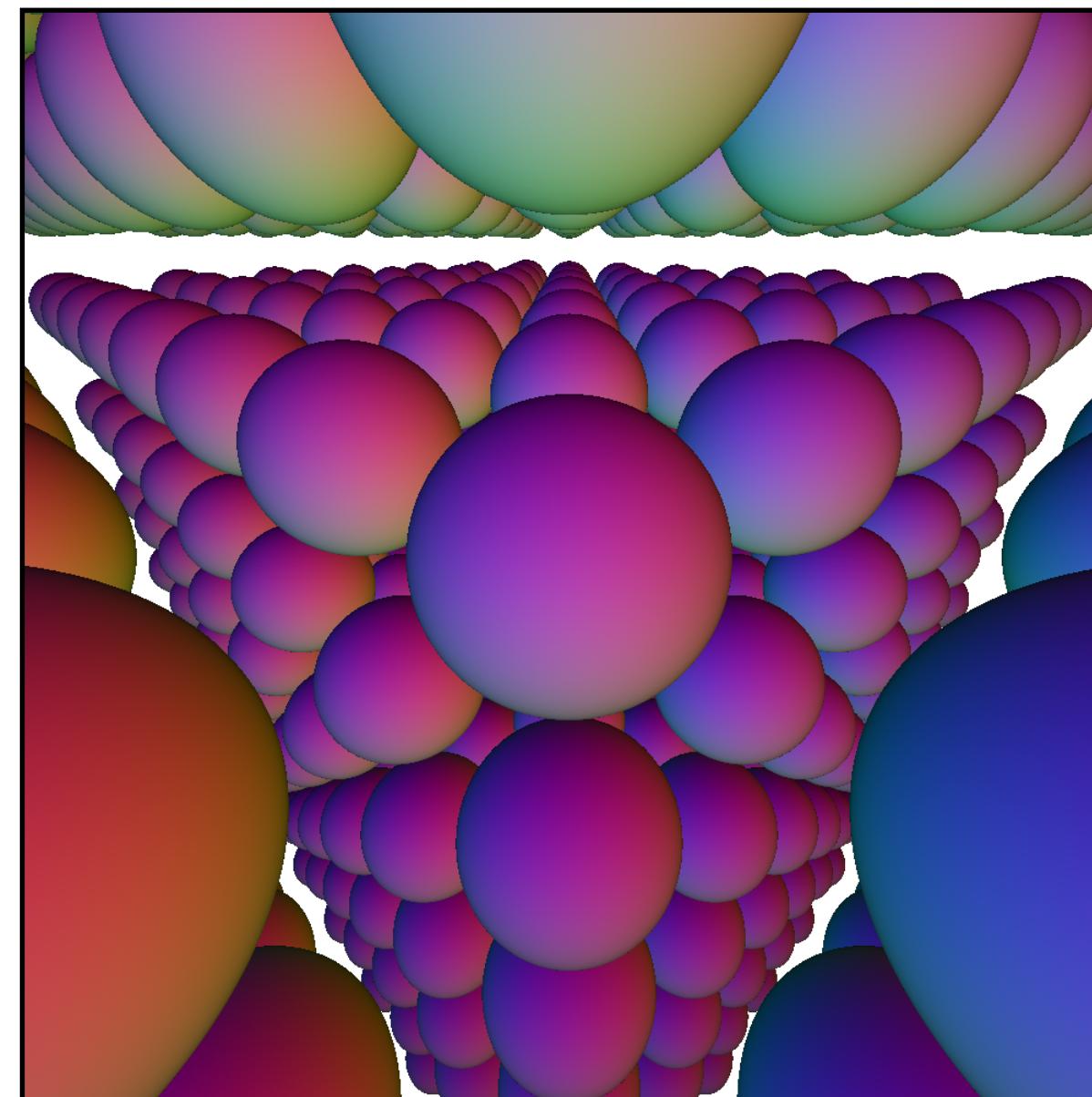
RPN length = **506**

Architecture



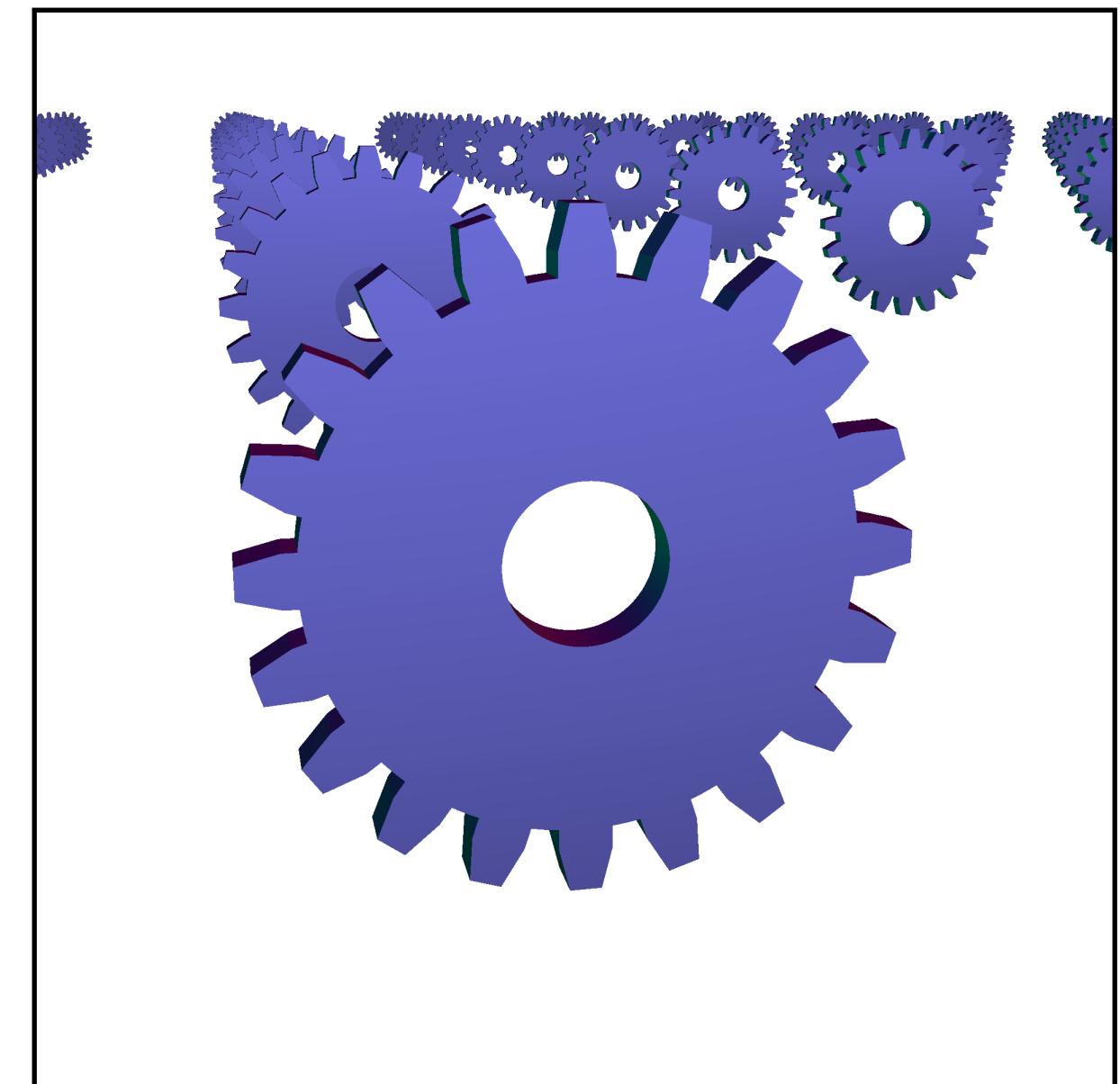
RPN length = **1431**

Many Spheres



RPN length = **5489**

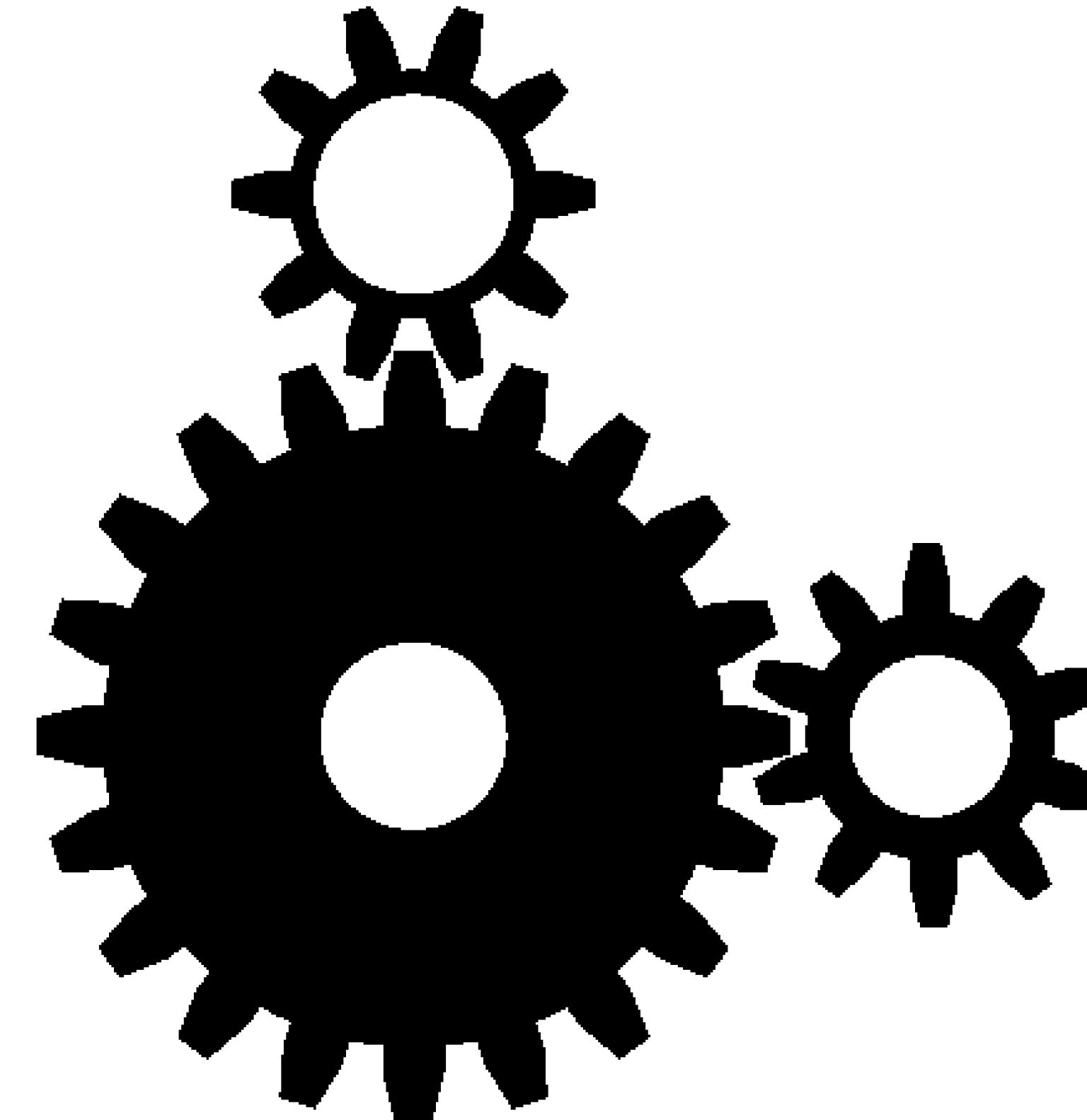
Many Gears



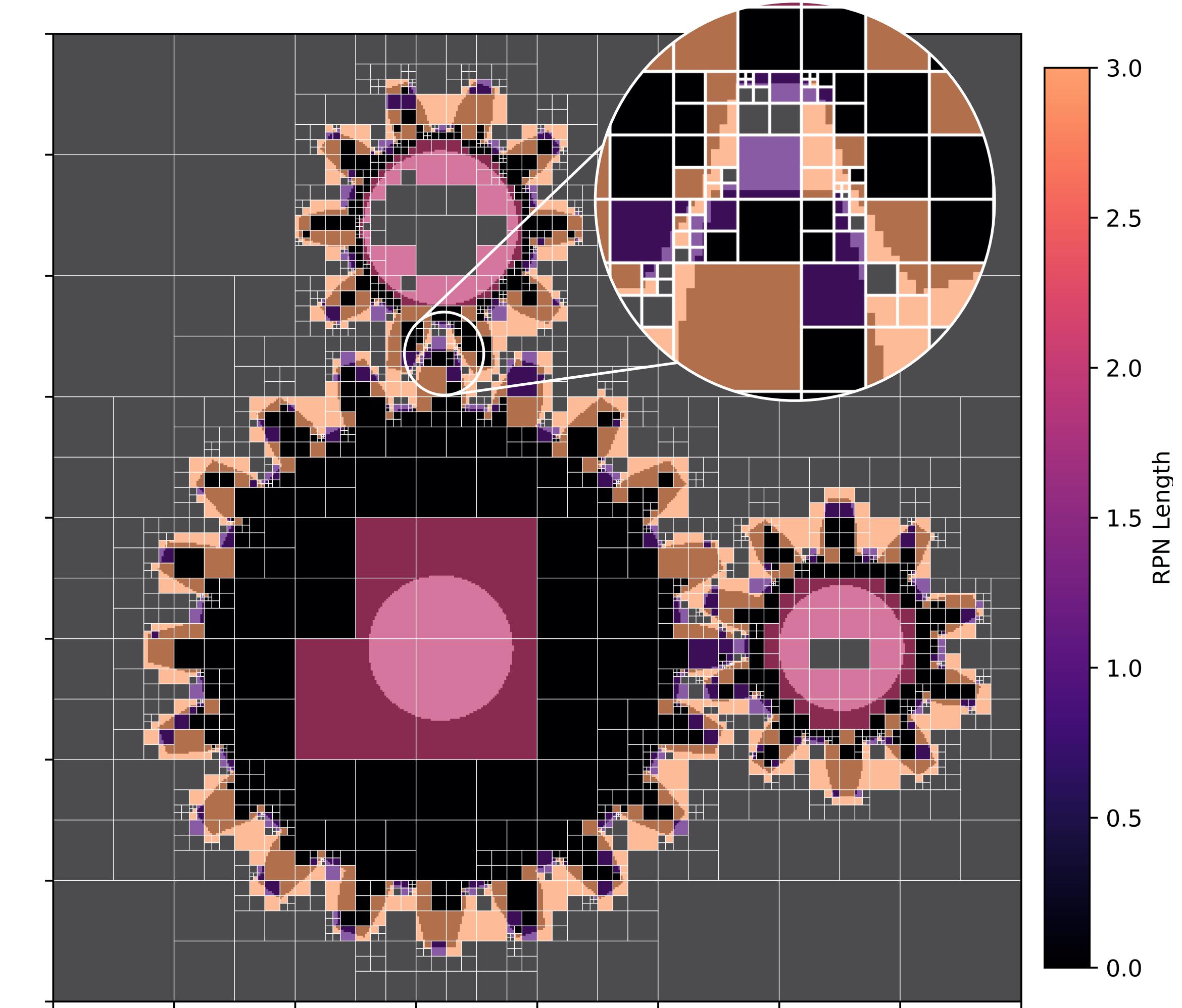
RPN length = **30377**

Results

SVO construction for 2D case



Original RPN length = **502**



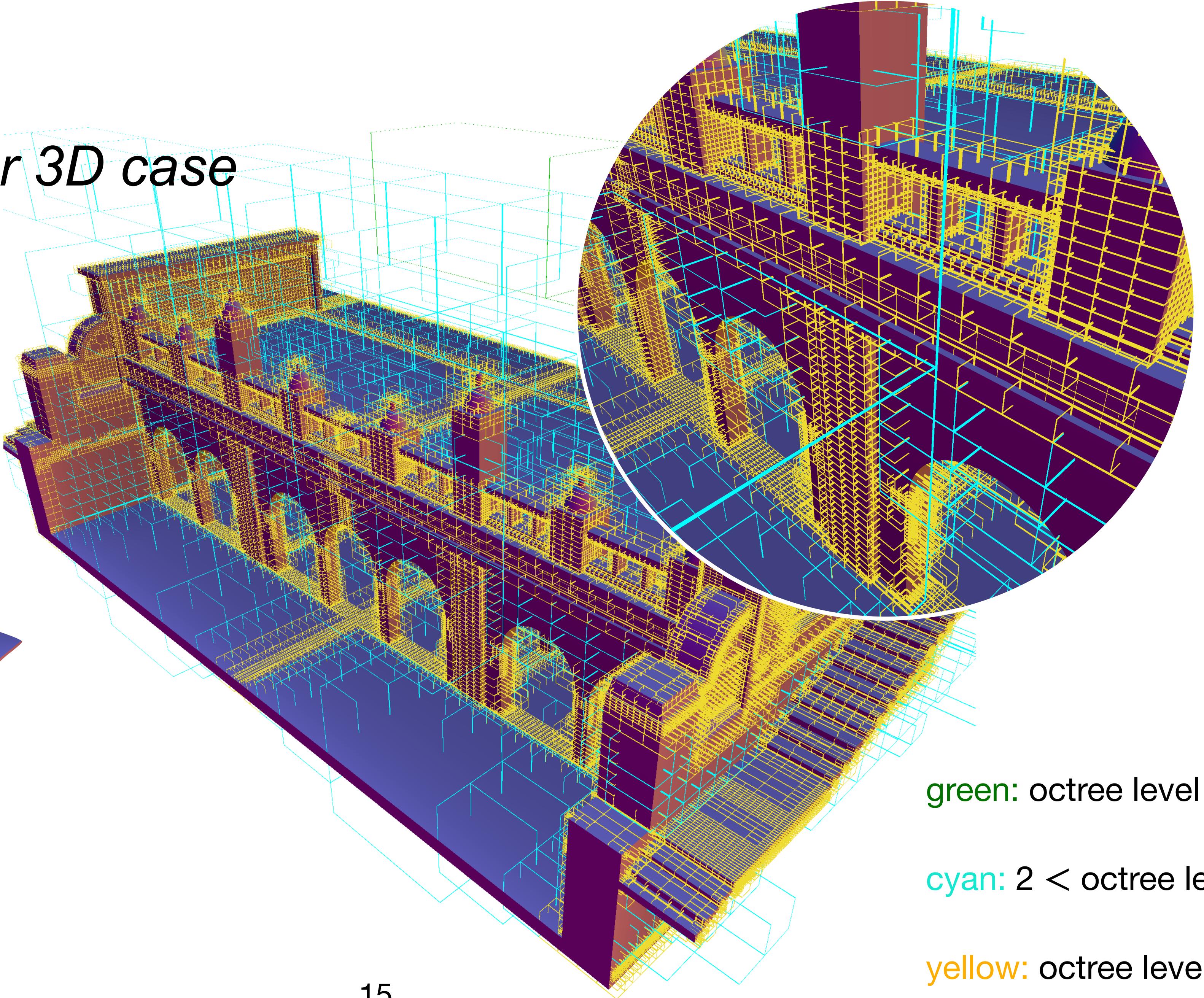
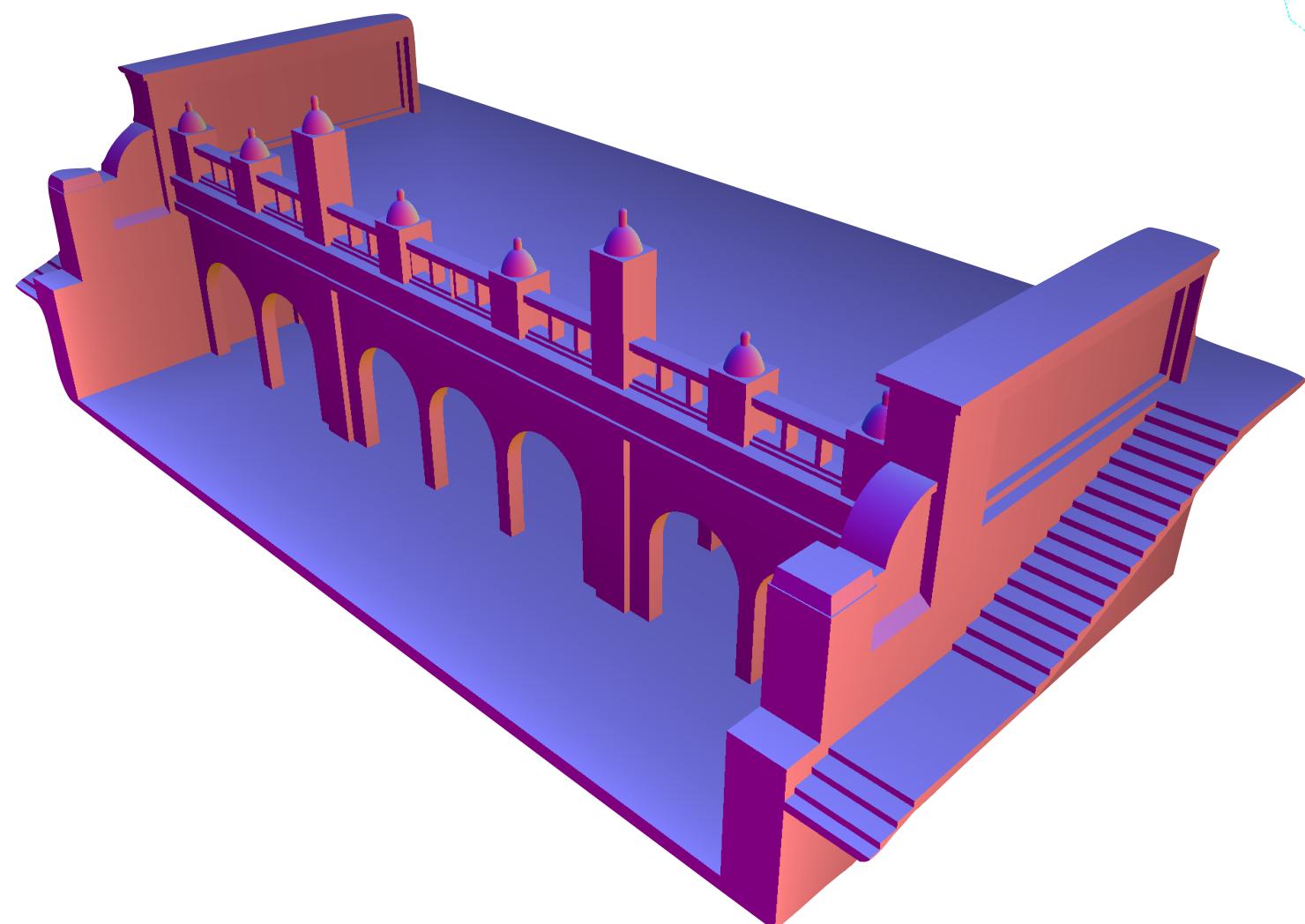
Max octree level = **10**

Threshold (max RPN length) = **3**

Skoltech

Results

SVO construction for 3D case



green: octree level ≤ 2

cyan: $2 < \text{octree level} \leq 5$

yellow: octree level > 5

Original RPN length = **1431**

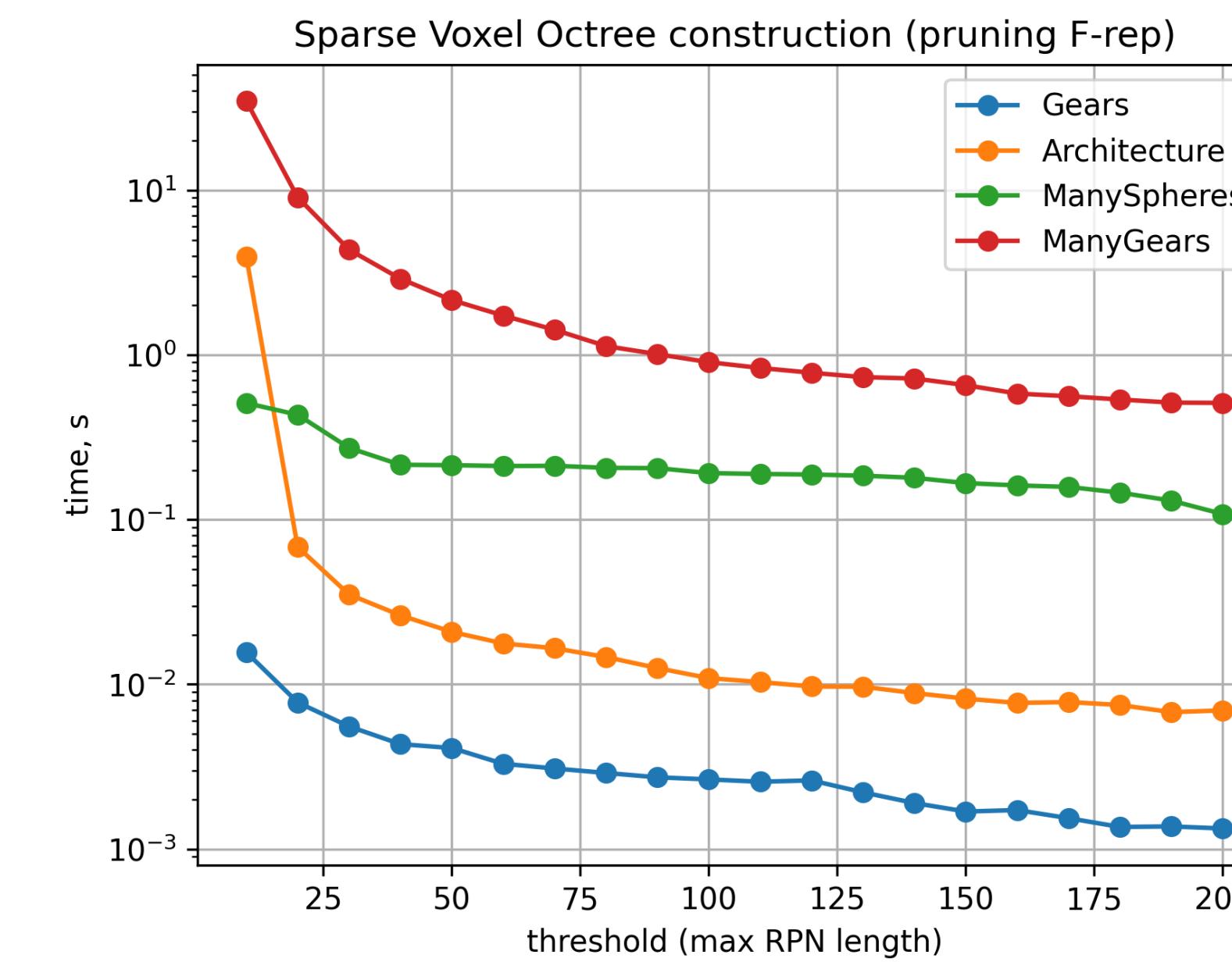
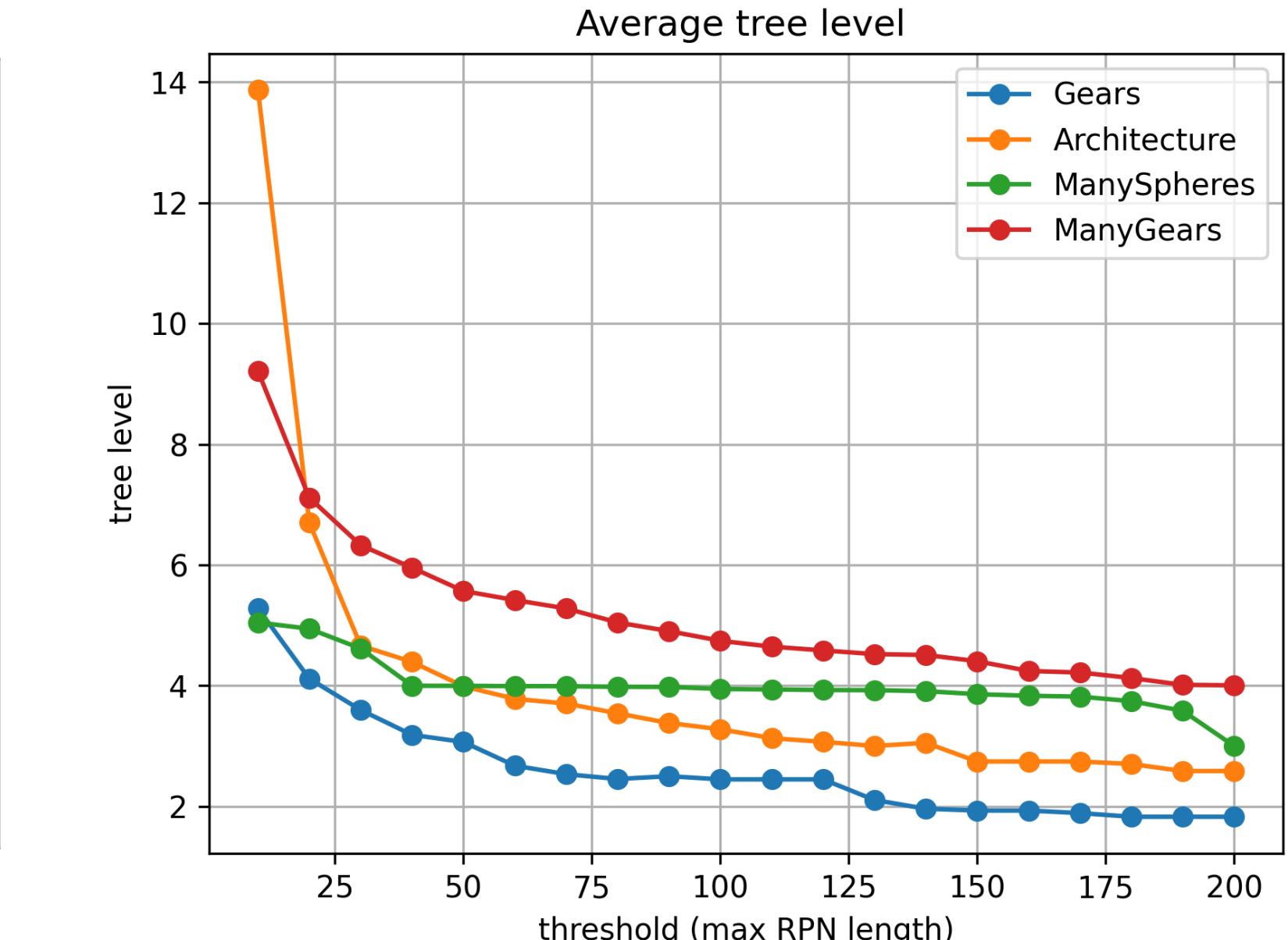
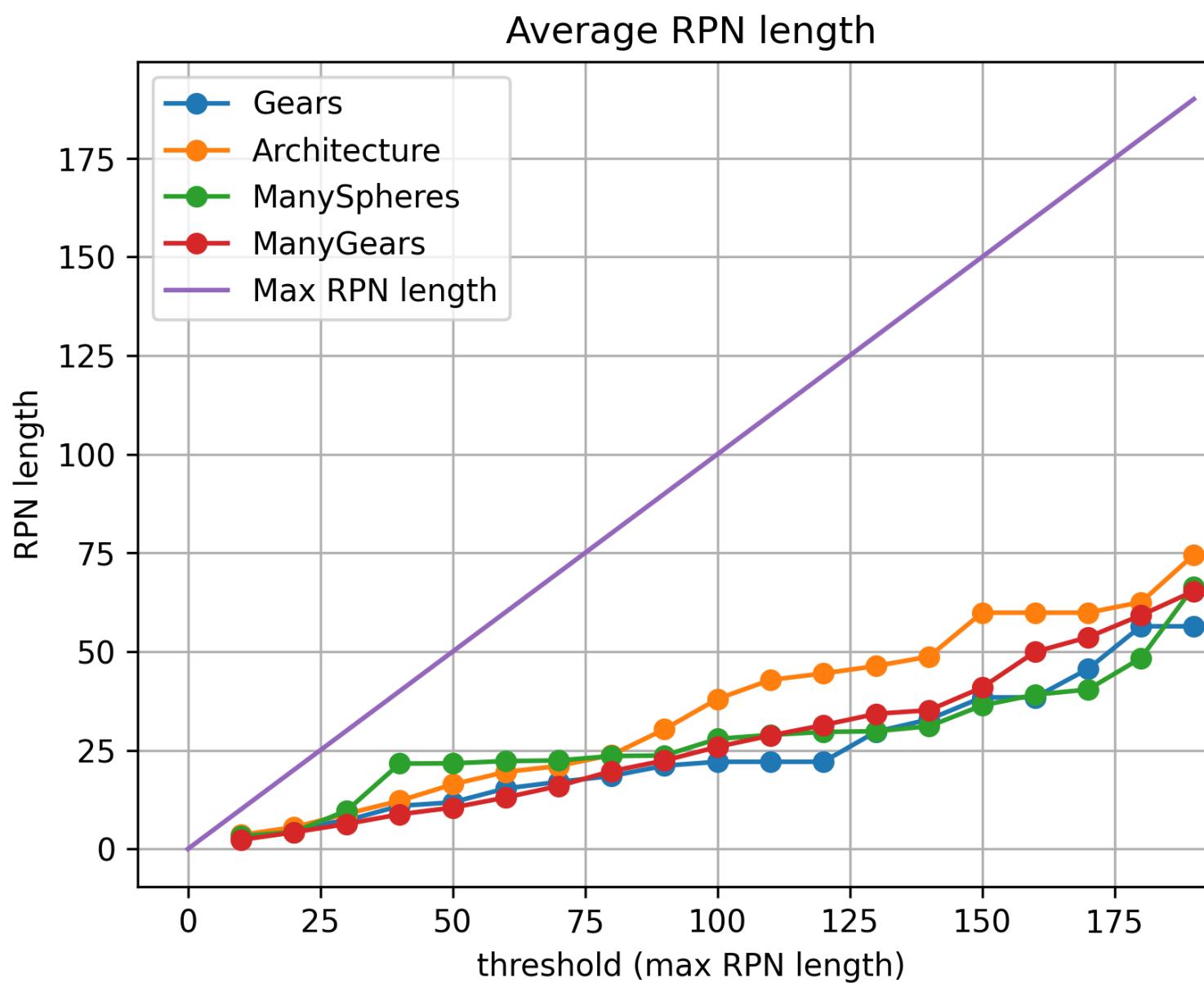
Results

Pruning results

Average RPN length is far below the threshold value because many SVO nodes are empty

For **small thresholds**, the average tree level rises sharply.

Pruning time grows exponentially as the threshold decreases: deeper trees mean more work, so run-time balloons for low thresholds.



Results

Evaluation speedup

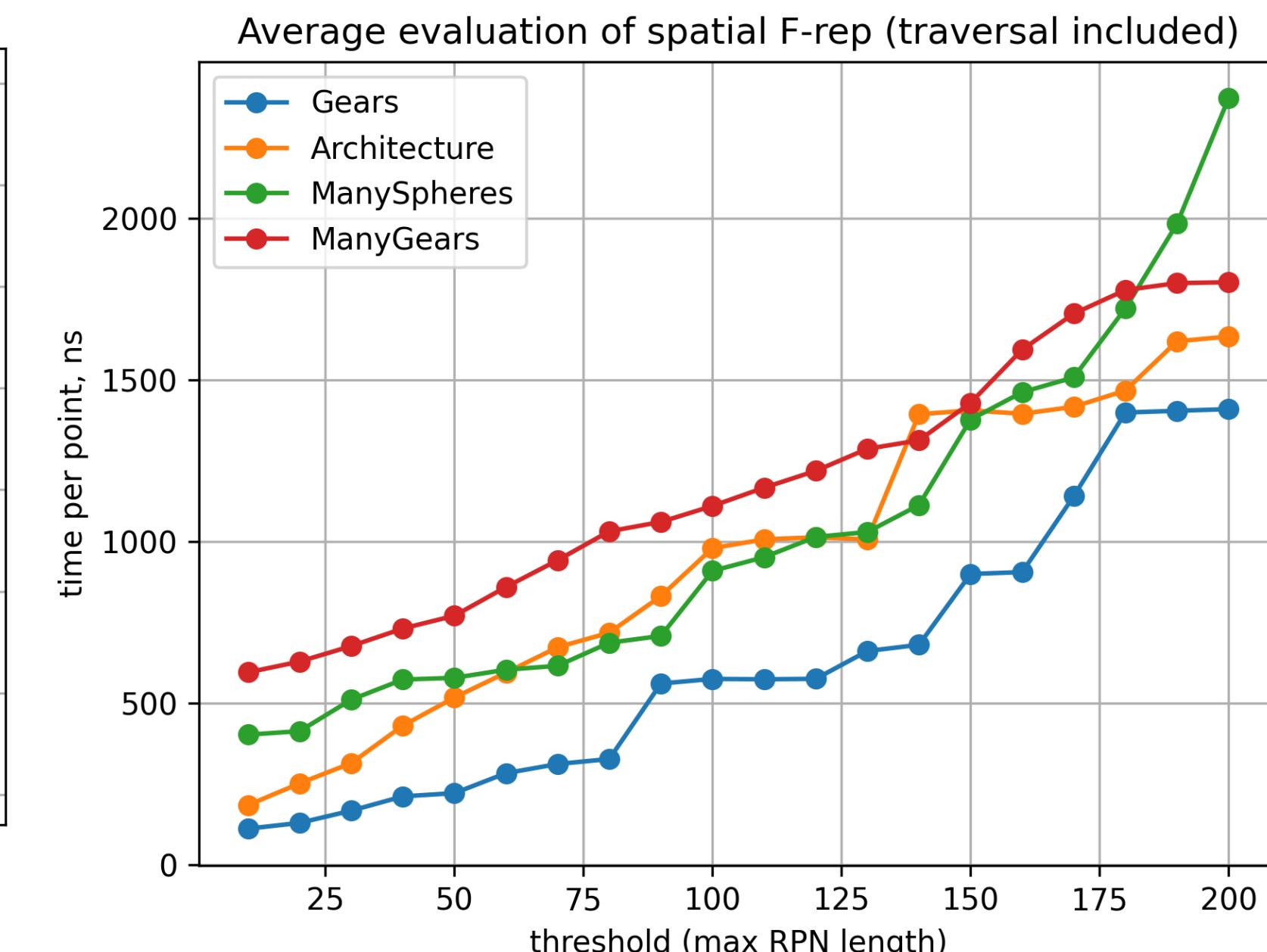
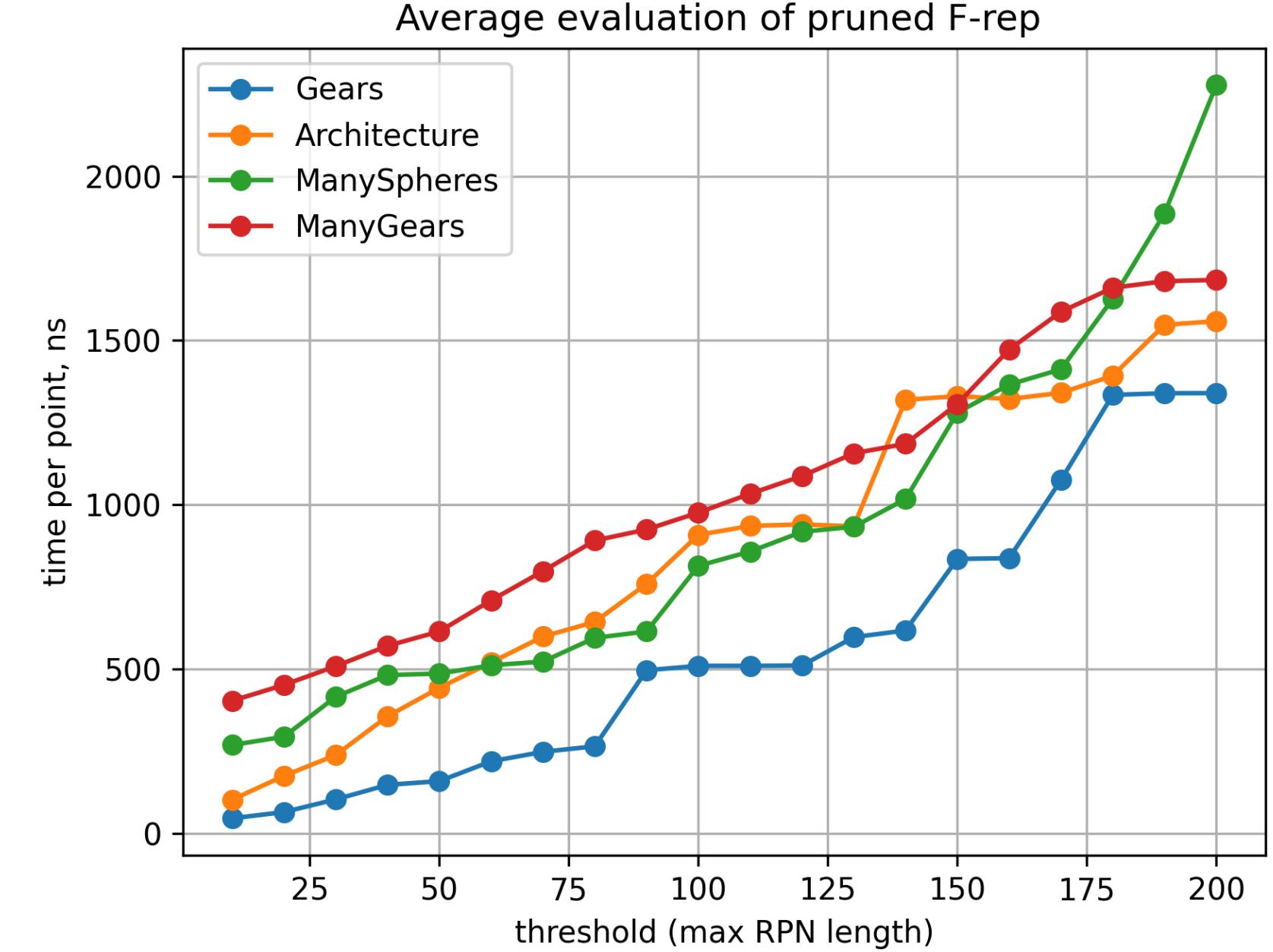
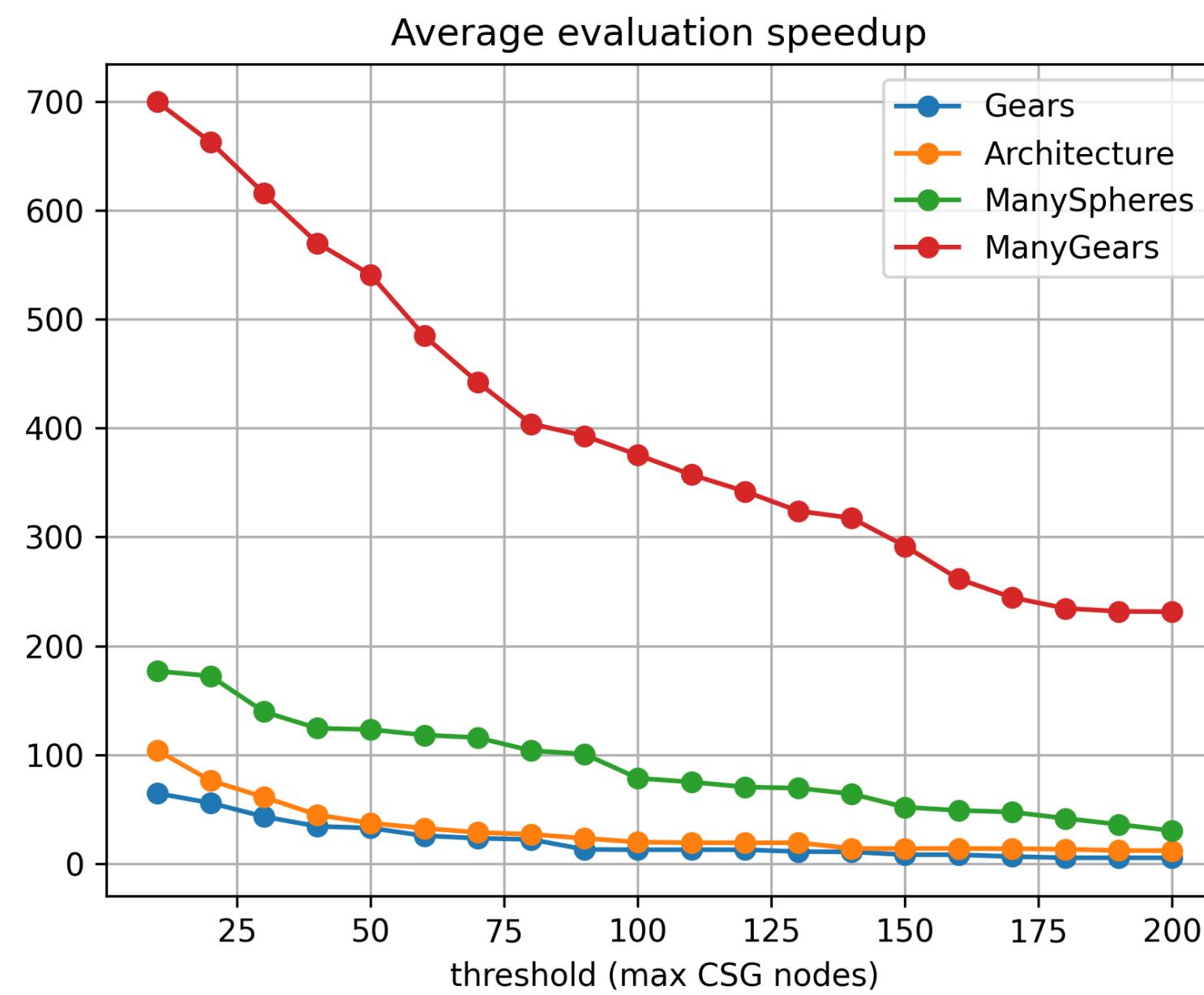
Evaluation of all benchmarking 3D scenes (SVO traversal is excluded) has **identical** complexity, despite the initial difference in model complexity.

The *ManyGears* scene achieves the greatest speedup, ranging from **200x** to **700x**

Even the simplest test case, *Gears*, attains a performance improvement of **4x** to **50x**.

Scene	Time per point, ns
Gears	6'000
Architecture	20'000
ManySpheres	75'000
ManyGears	409'000

Average evaluation of original (unpruned) F-rep



Results

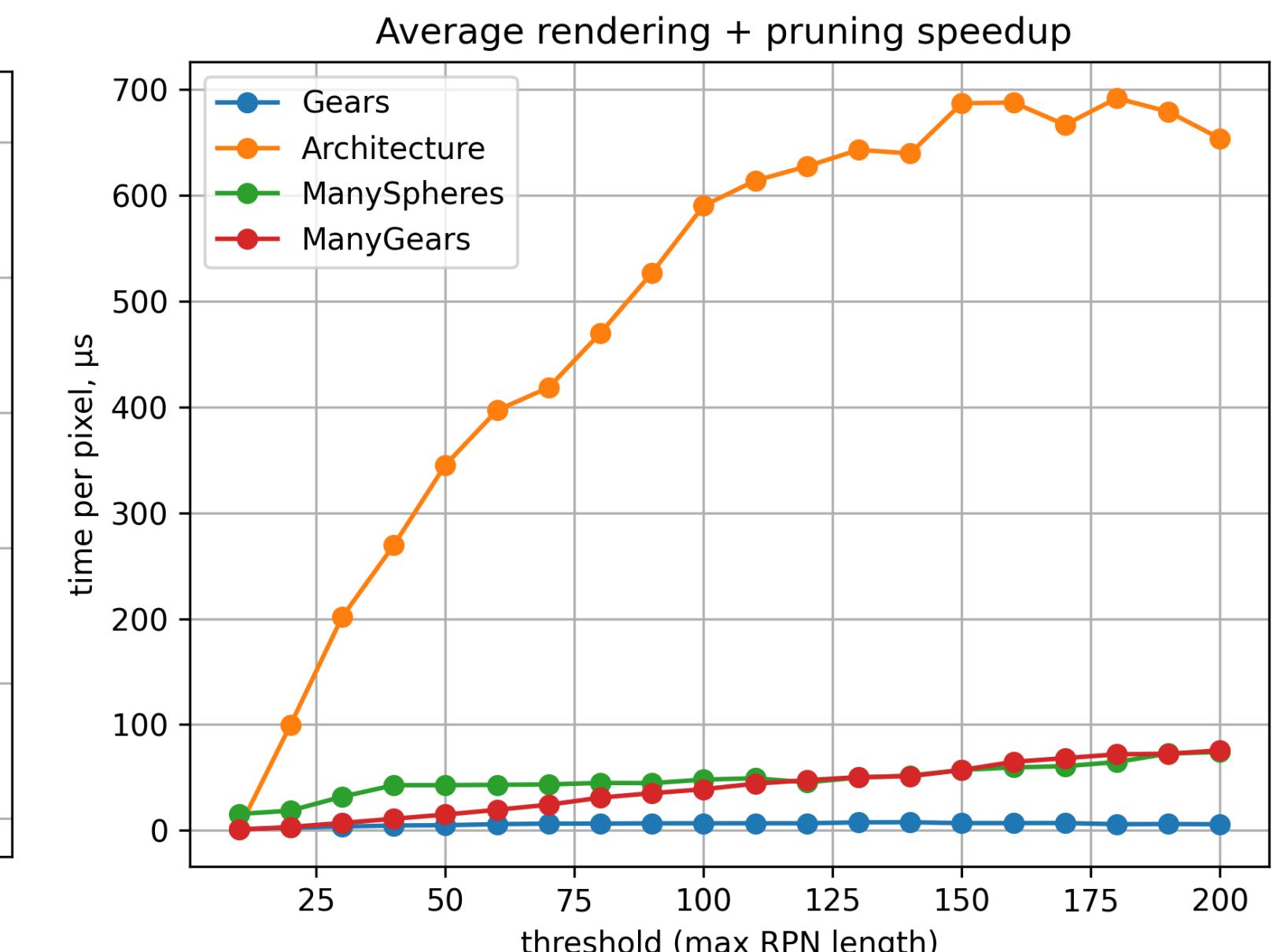
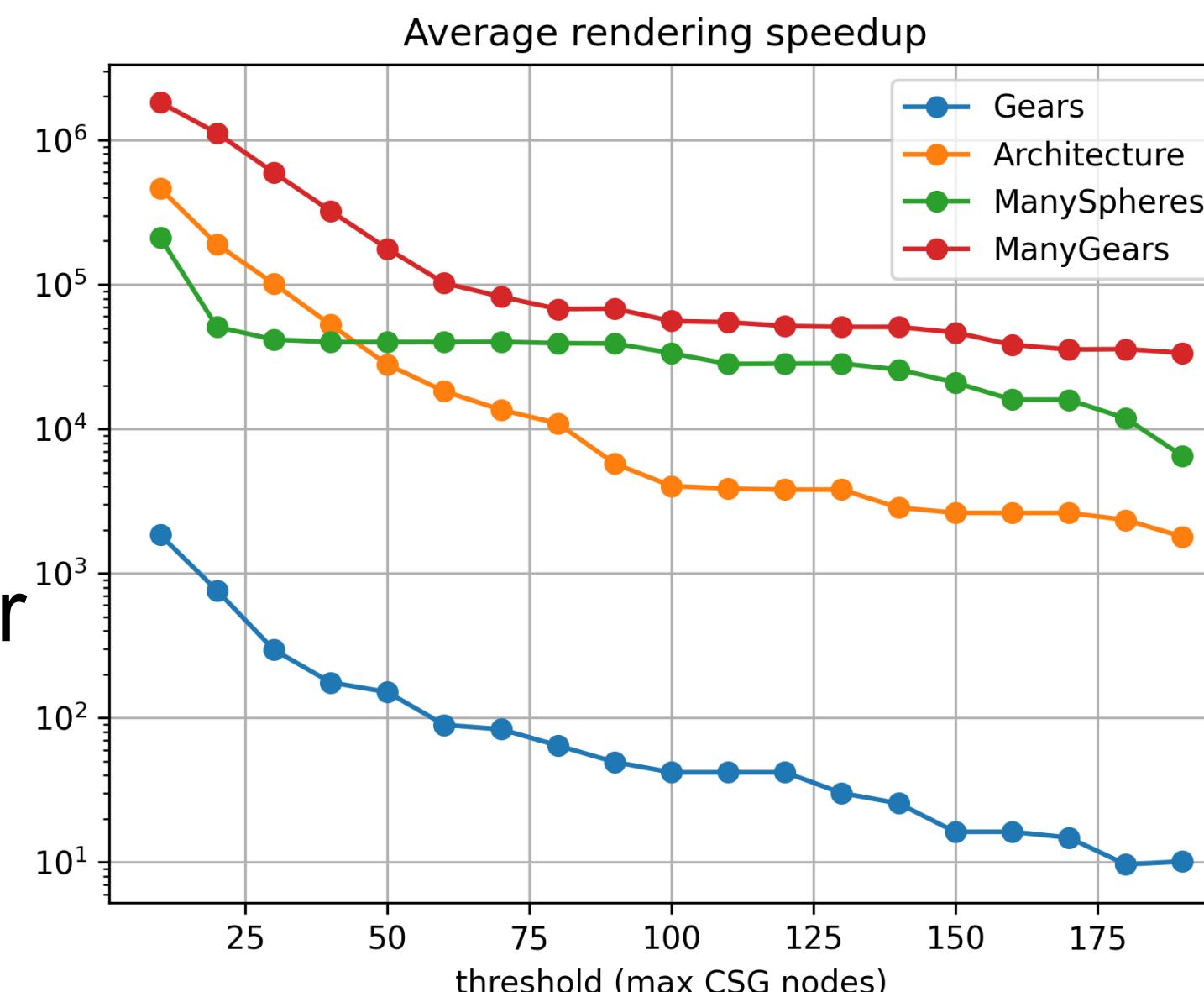
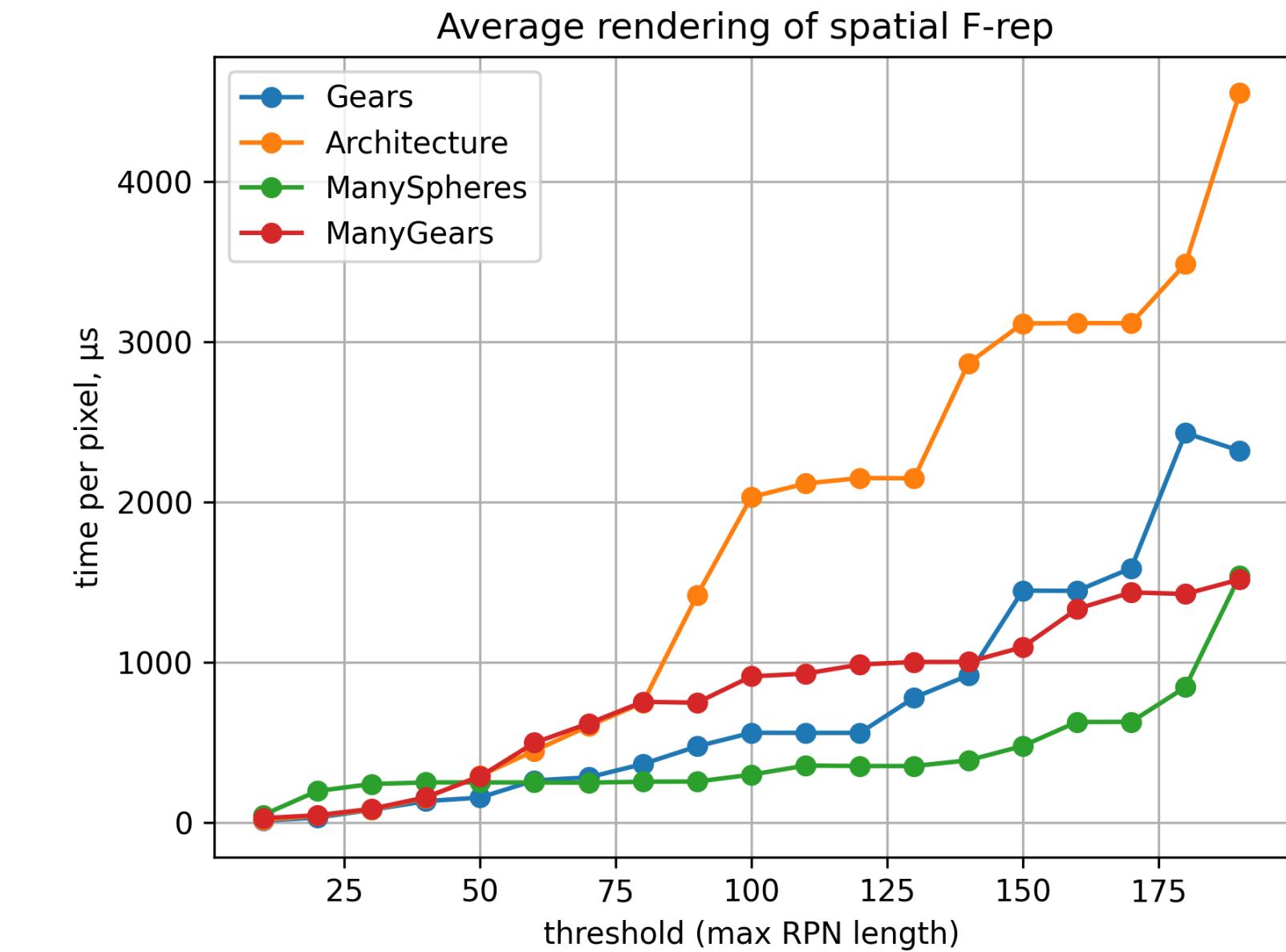
Rendering speedup

Rendering times are improved by one to six orders of magnitude ($\approx 10x$ to 10^6x)

This **significant** boost can be explained by the very poor performance of the interval bisection method for the original F-rep.

Additionally, computing pruning for every ray can boost ray-casting performance up to **700** times.

Scene	Time per pixel, μs
Gears	20'000
Architecture	3'990'000
ManySpheres	16'000'000
ManyGears	840'885'000
Average time of rendering original (unpruned) F-rep	

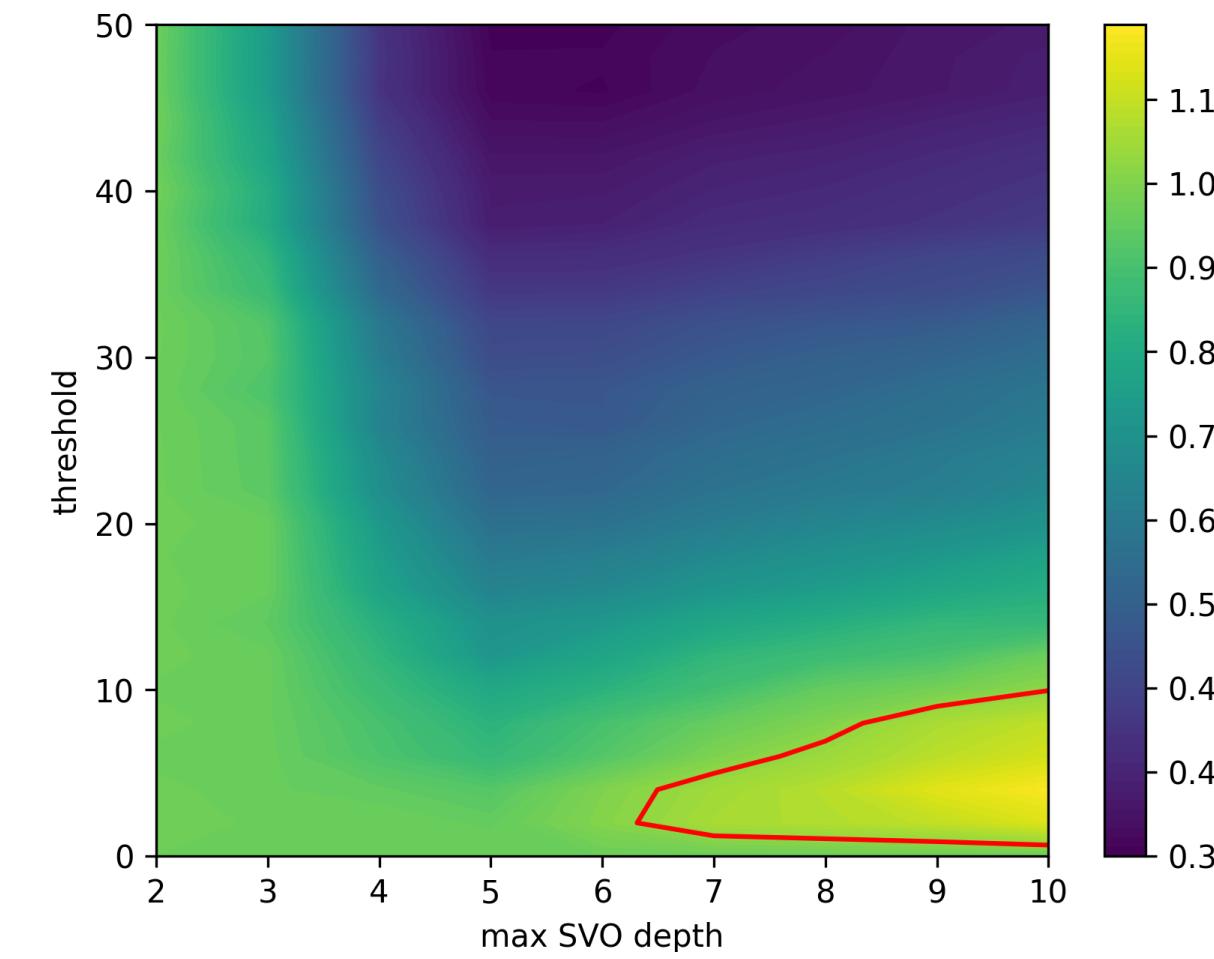


Results

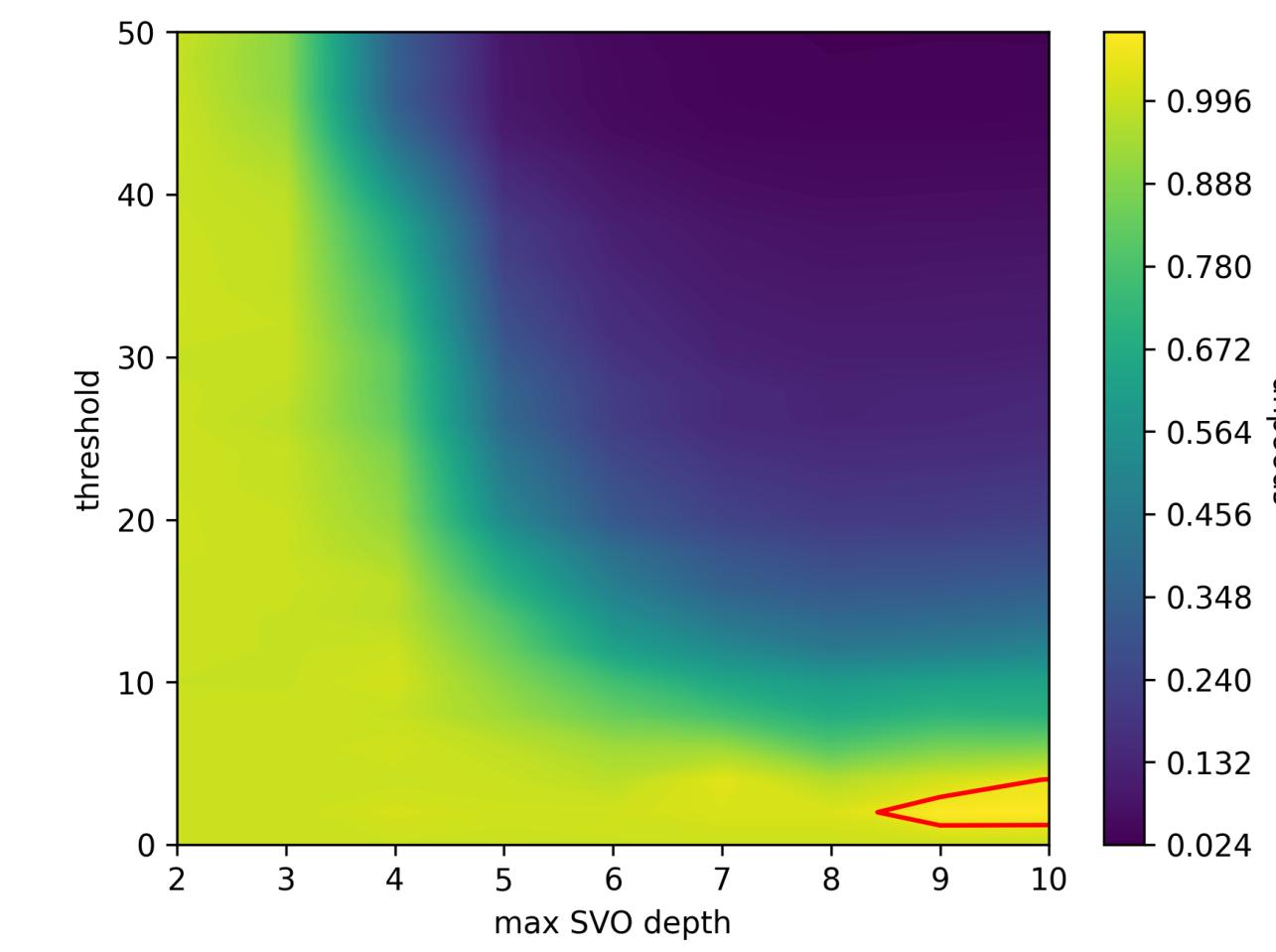
Comparison with state-of-the-art approaches

Architecture

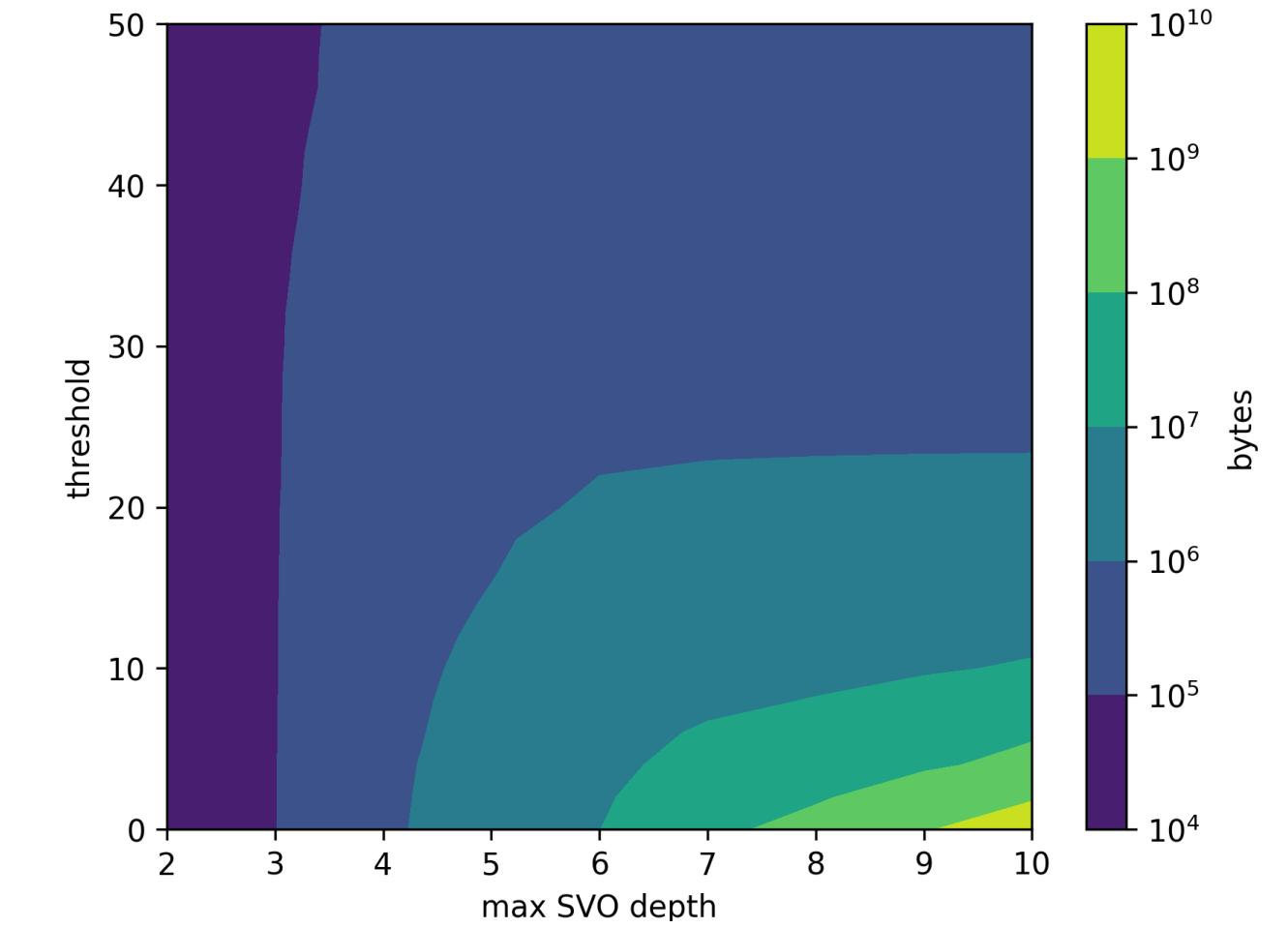
Evaluation speedup



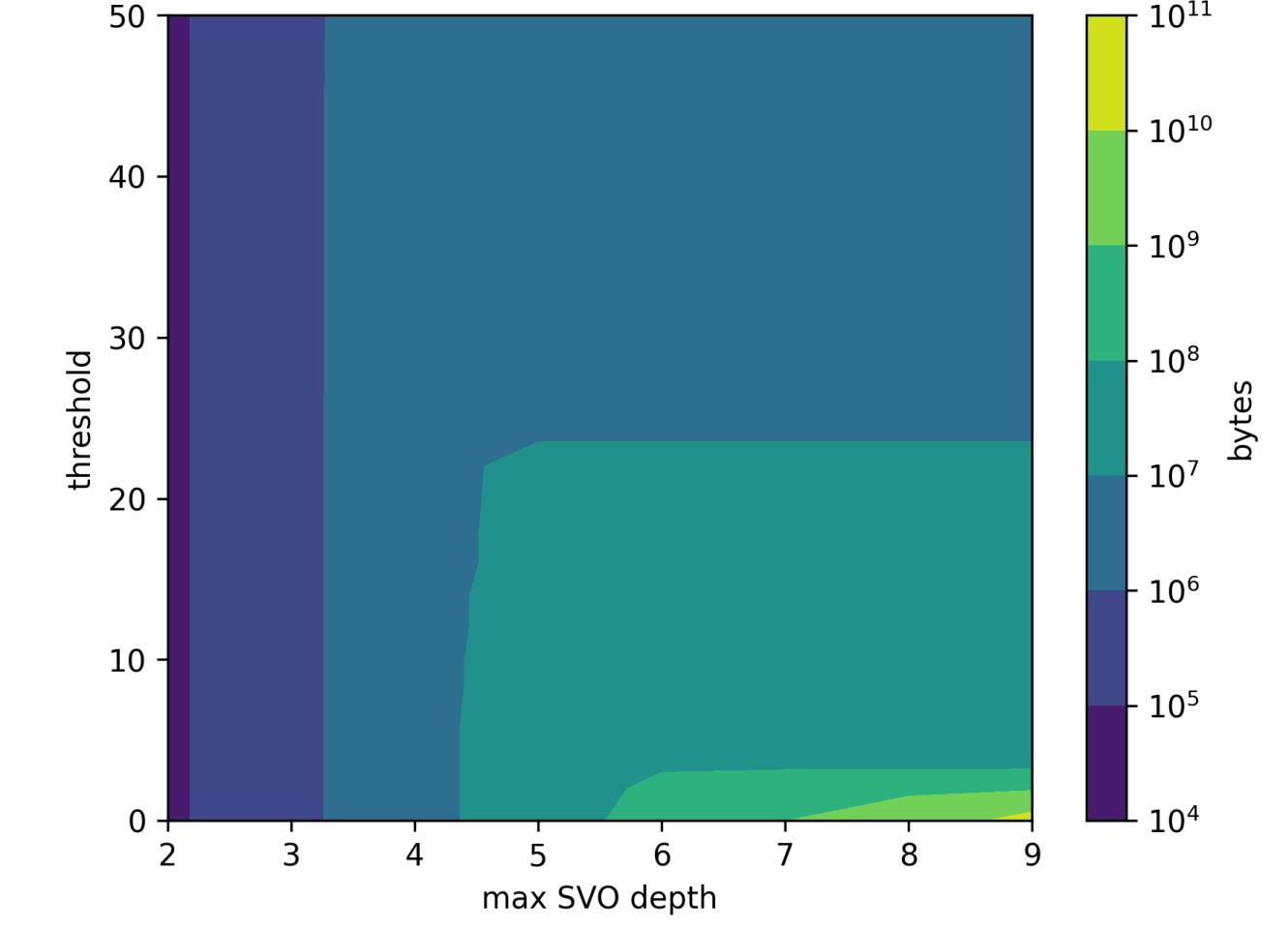
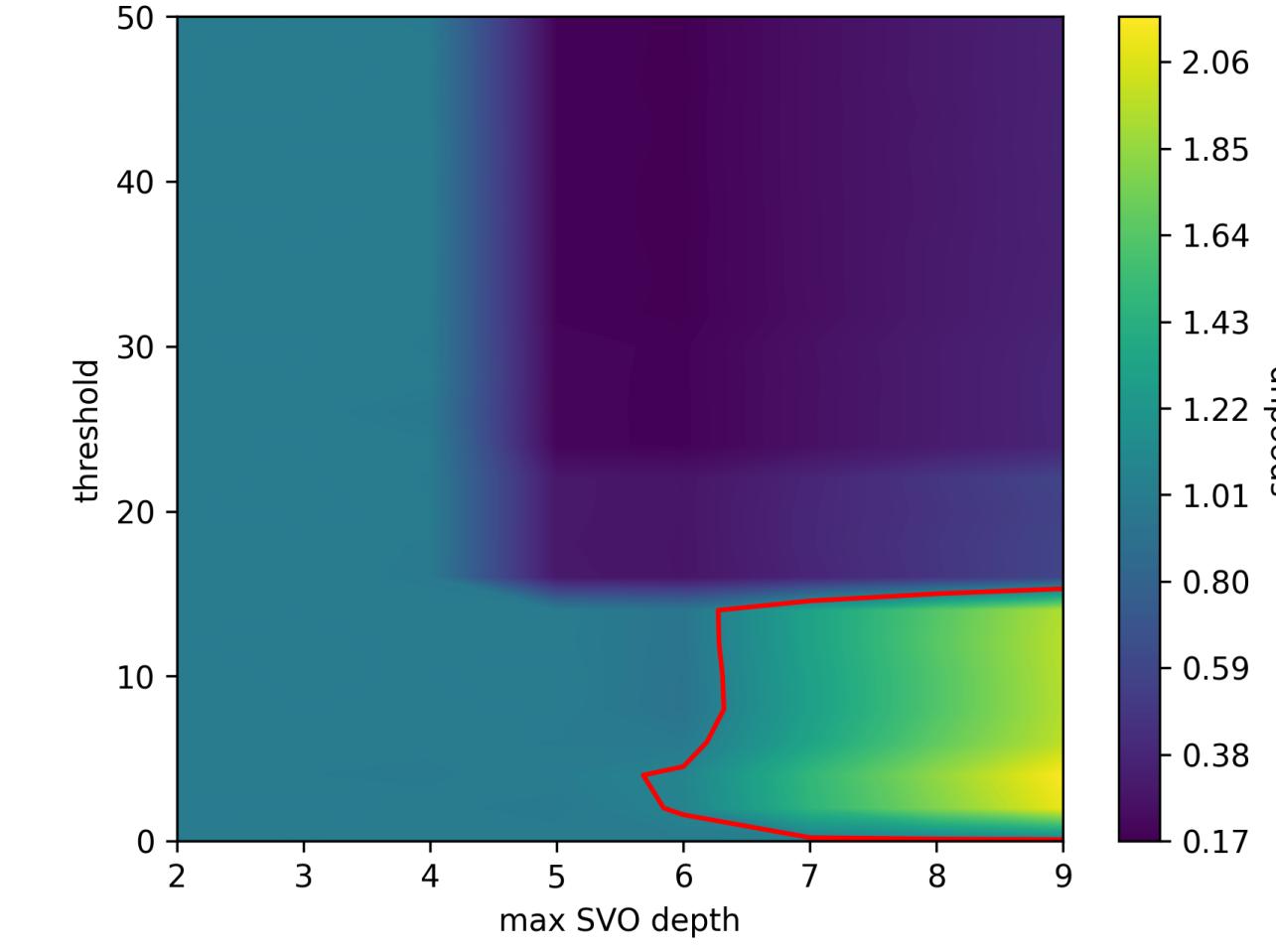
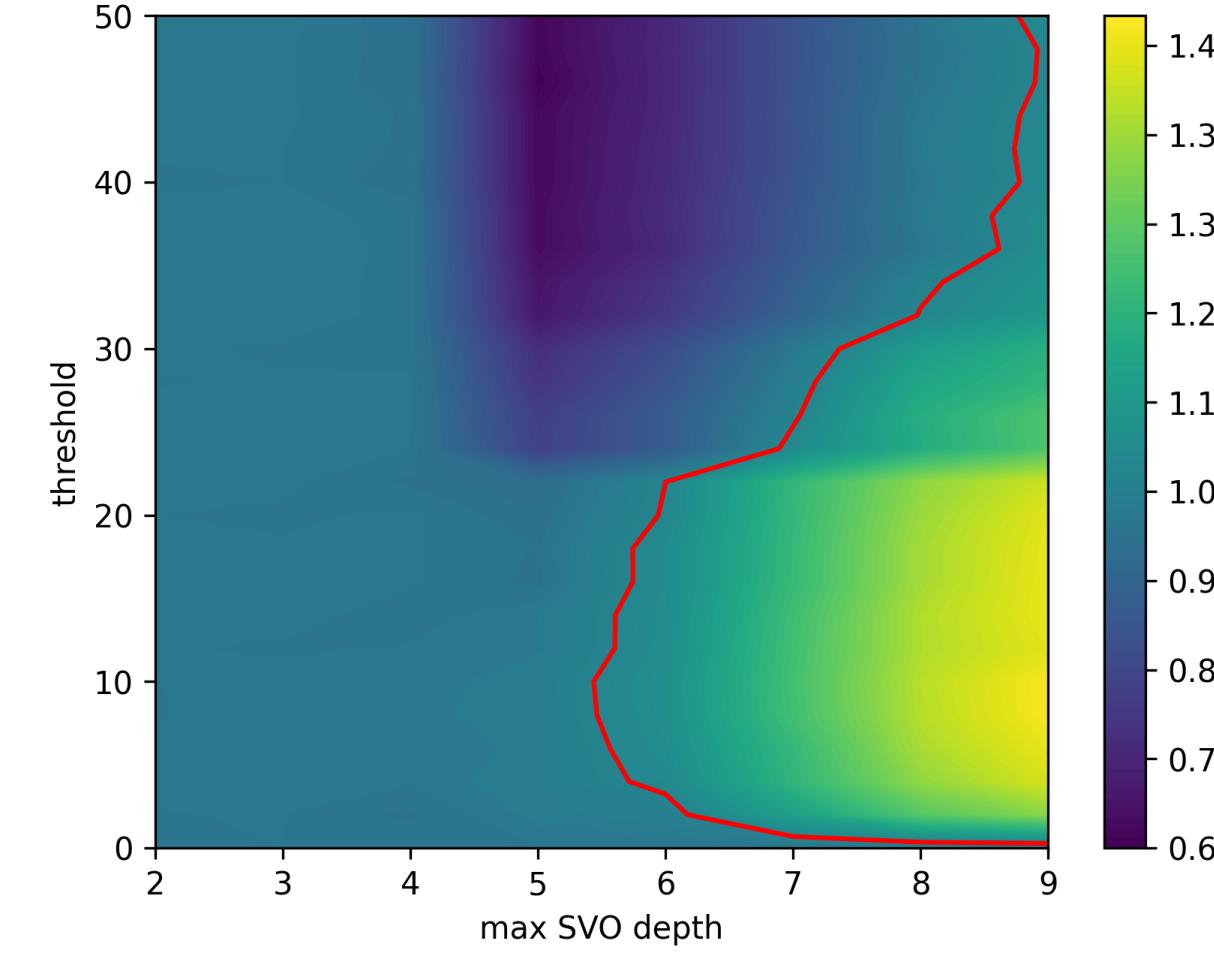
Rendering speedup



Memory consumption



Many Spheres



Conclusion

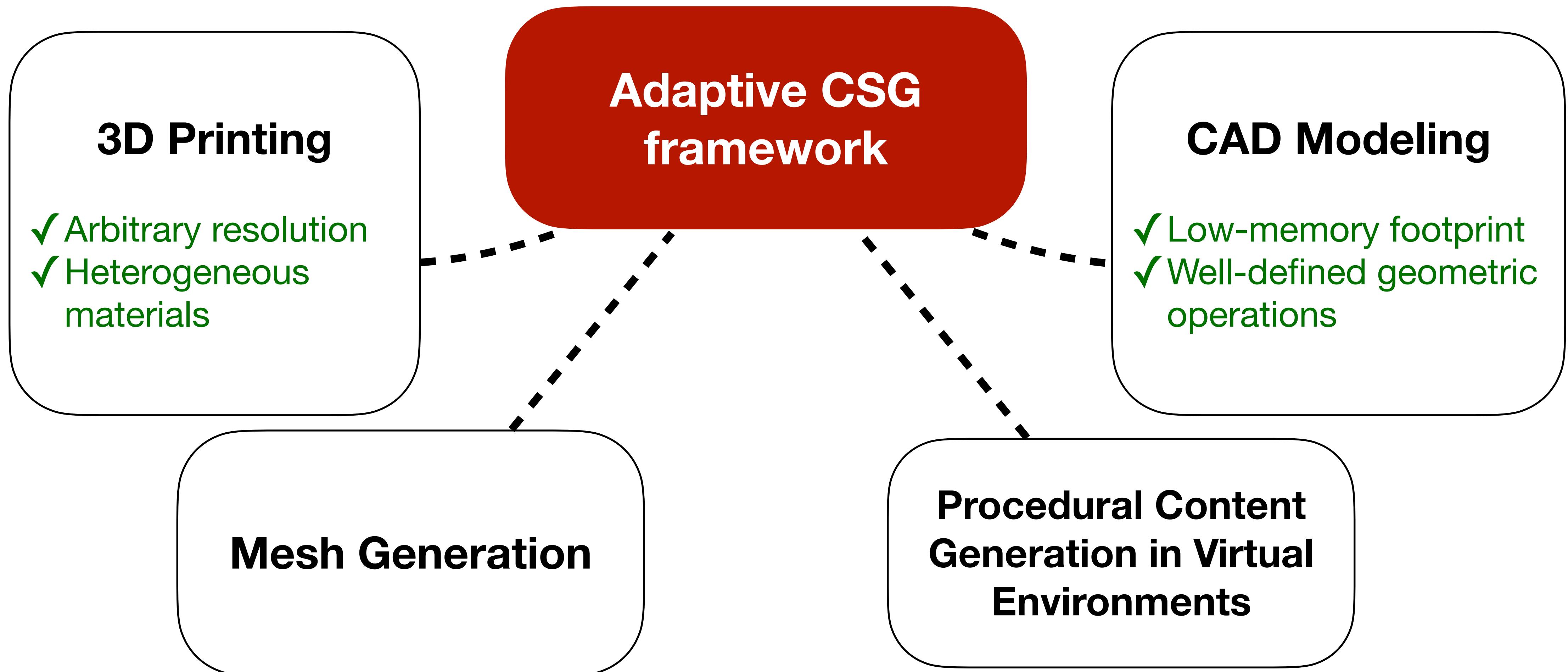
Core mathematical framework for adaptive CSG modeling with constant evaluation complexity was developed and tested on 3D scenes with varying levels of complexity.

Developed pruning algorithm showed a significant speedup in F-rep evaluation. The worst case speedup was **4x** for the *Gears* model, and the best case speedup was **700x** for the *Many Gears* model.

The developed rendering algorithm also showed a significant speedup in F-rep ray-casting. Even computing pruning for every ray can boost ray-casting performance up to **700x** times.

All these algorithms were implemented in a prototype geometry kernel and ray tracer in C++.

Future Applications



Thank you!

External Thesis Review

Questions

Can the author elaborate on how this threshold is chosen and whether it impacts the accuracy or generality of the results across varying model complexities? Can you suggest any preliminary recommendations for its value?

The threshold is chosen in range 5-200.

If at most N primitives have contact area, the threshold should be more than $2N-1$.

The preliminary recommendations are threshold in range 5-20.

External Thesis Review

Questions

Which real-world application domains or industries could benefit most from your constant-complexity CSG approach, and what would be required to integrate it into existing modeling or manufacturing workflows?

Real-world application domains: CAD/CAM and 3D design software (e.g., Adobe Project Neo, Womp).

For CSG modeling workflows the integration is straightforward.

Otherwise, integration depends on the implementation of existing workflows.

External Thesis Review

Questions

What are the main assumptions and limitations of your adaptive CSG method? For instance, are there specific cases (such as extremely complex geometries or degenerate configurations) where the constant-complexity guarantee might not hold or the algorithm could struggle?

Constant evaluation complexity is guaranteed in case when SVO time traversal is much less than CSG tree evaluation.

It might not hold in areas of contact between large number of primitives (see Architecture scene).

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