Dynamic Contracting with Multidimensional Screening

PRELIMINARY AND INCOMPLETE

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How to design an optimal contract under the long-term principal-agent interactions when the agent's type is described by more than one characteristic? I study a simple principal-agent model where a monopolist repeatedly sells two non-durable goods to a buyer. A buyer's type is two-dimensional private information that stochastically evolves over time according to a Markov process. I characterize the optimal contract and show how it is shaped by the history of the buyer's reports, cross-sectional distribution of the buyer's characteristics, and their persistence. In particular, I show that there exists a threshold on correlation between the subtypes such that (i) if correlation is above this threshold, then the quantity of a good does not depend on the report about the marginal valuation of another good, and (ii) if correlation is below this threshold, then the quantity of a good depends on the report about the marginal valuation of another good. The behavior of the optimal contract over time is shaped by persistence of the buyer's type. Furthermore, I apply this framework to the environment with optimal income taxation of couples. Extending the results from the principal-agent model, I show that the optimal tax schedule crucially depends on the interaction between the cross-sectional distribution of spousal types in the economy and the government's taste for redistribution. In addition, I obtain a generalization of the ABC-formula for the optimal marginal tax rates with multidimensional private information.

Keywords: dynamic contracting, multidimensional screening, persistent private information, optimal taxation

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1 Introduction

How to design an optimal contract under the long-term interactions between a principal and an agent when the agent's type is described by more than one characteristic? This environment has a broad range of applications in the real world: for instance, life insurance contracts and income taxation. Despite recent advances in contract theory, there are still a lot of challenging questions about the optimal dynamic contract design and multidimensional private information.

In this paper, I study a simple principal-agent model where a monopolist repeatedly sells two non-durable goods to a buyer. A buyer's type, that captures the preferences over the goods, is private information and has two dimensions. Moreover, it stochastically evolves over time according to a Markov process. Since the buyer's type is represented by a two-dimensional vector, I solve a multidimensional screening problem to characterize the optimal contract. This is a challenging task since the standard techniques, like the 'first-order approach', are, in general, not valid in this case. To the best of my knowledge, this is the first paper that embeds a multidimensional screening problem into dynamic context with persistent private information in an analytically tractable way. Despite simplicity of the model, it allows to get some interesting theoretical results and may serve as a useful benchmark for more complex models of multidimensional screening in dynamic settings.

The main results of the paper are as follows. First, the optimal contract is history-dependent and has infinite memory. In a given period of time, the optimal quantities depend on the full history of past buyer's reports about his type, the current report, and the cross-sectional distribution of the subtypes that capture the preferences over the goods. In particular, I show that there exists a threshold on correlation between the subtypes such that (i) if correlation is above this threshold, then the quantity of a good does not depend on the report about the marginal valuation of another good, and (ii) if correlation is below this threshold, then the quantity of a good depends on the report about the marginal valuation of another good. The behavior of the optimal contract over time is shaped by persistence of the buyer's type. In addition, I apply the considered framework to the problem of optimal income taxation of couples, and show how the cross-sectional distribution of spousal types, government's taste for redistribution, and persistence of the spousal types shape the optimal tax schedule. In particular, I obtain a generalization of the ABC-formula for the optimal marginal tax rates with multidimensional private information.

This paper contributes to three strands of literature. First, it contributes to the literature on multidimensional screening. The applications of this problem include, for example, optimal monopoly pricing, Armstrong (1996), Rochet and Choné (1998), Armstrong and Rochet (1999), or optimal taxation, Cremer et al. (2001), Lehmann et al. (2018), and Moser and Olea de Souza e Silva (2019). In related papers, Armstrong and Rochet (1999) and later Frankel (2014), for the optimal taxation setting, show that the correlation between types is an important sufficient statistic if one wants to determine what incentive constraints are binding. In the optimal taxation setting, I generalize their result by allowing for more general welfare weights. Carroll (2017) studies a robust version of the principal's problem where she the marginal distribution of each component of the agent's multidimensional type, but does not know the joint distribution. He shows that it is optimal to screen along each component separately. On a computational side, Judd et al. (2018) and Moser and Olea de Souza e Silva (2019) discuss the algorithms for solving the multidimensional screening problems. Similarly to Moser and Olea de Souza e Silva (2019), I also study multidimensional screening problem into dynamic context with persistent private information, but my framework allows to characterize the optimal schedule in an analytically tractable way.

Second, this paper contributes to the literature on dynamic contracting. I extend the results from Battaglini (2005) to the environment with multidimensional types. One of the first papers that extends the principal-agent model by adding stochastic types is Townsend (1982) who assumes the serially independent types. In a subsequent literature, featuring Baron and Besanko (1984), Rustichini and Wolinsky (1995), and Laffont and Tirole (1996), the models were extended to have persistent private information. In this framework, Battaglini (2007) characterizes the optimal renegotiation-proof contract. In a related paper, Williams (2011) considers a dynamic framework with a risk-averse agent, whose private information is persistent, that wants to borrow from a risk-neutral lender. Fu and Krishna (2019) characterize the optimal contract in a dynamic framework with a manager and an investor. Doval and Skreta (2020b) study the dynamic environment where the seller has a limited commitment and sells a durable good to a privately informed customer. They show that the posted prices constitute the optimal mechanism to sell this good. Another related paper, Doval and Skreta (2020a), also studies the mechanism design with limited commitment. The standard techniques that are applicable in a static environment, do not necessarily work in dynamic settings. Battaglini and Lamba (2019) discusses the limitations of the 'first-order approach' in dynamic models. The other closely related papers in dynamic contracting literature include Bloedel et al. (2020) and Krasikov and Lamba (2021).

Finally, this paper contributes to the optimal income taxation literature in the spirit of Mirrlees (1971). More precisely, my work is related to dynamic optimal taxation similarly to Battaglini and Coate (2008), Farhi and Werning (2013), and Golosov et al. (2016). A detailed discussion of this topic is provided by Stantcheva (2020). Similarly to Battaglini and Coate (2008), I consider risk neutral individuals, while the other cited papers assume more general preferences. Traditionally, design of the optimal individual tax schedule implies one dimension of private information, productivity of a person. However, if we study the optimal tax design for couples we have to deal with two-dimensional heterogeneity. The main feature of my paper is that I allow for this heterogeneity and study the multidimensional screening problem. On the theoretical side, my paper generalizes the results from Battaglini and Coate (2008) to the case of two-dimensional asymmetric information. I show how the interaction between the cross-sectional distribution of spousal types and the government's taste for redistribution shapes the optimal tax schedule. A celebrated negative jointness result from Kleven et al. (2009) is received under the assumption that the spousal types are independently distributed and that the government's taste for redistribution towards the couples where both spouses have low ability is high enough. In a different framework, using an equilibrium collective marriage market model, Gayle and Shephard (2019) also show that the optimal tax system features negative jointness. Rothschild and Scheuer (2013) and Kurnaz (2021) study the optimal tax problems in environments where agents have multidimensional characteristics. All these papers study the optimal taxation problem in static setting. In addition, my paper also contributes to the literature on the within-household inequality, see Blundell et al. (2005) and Lise and Seitz (2011) for a discussion of this topic. In particular, I study the optimal tax system design in an environment where the government cares not only about between- but also within-household inequality.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 contains the characterization of the optimal contract. In Section 4, I consider several extensions and applications, including negative cross-sectional correlation of types and optimal taxation framework. Finally, Section 5 concludes.

2 Model

2.1 Environment

Consider a model with two players, a buyer (consumer, he) and a seller (monopolist, she). The buyer repeatedly buys two non-durable goods from the seller. I assume that time is discrete and the relationship between the buyer and the seller lasts for $T \geq 2$ periods. Each period, a buyer's type is characterised by a two-dimensional vector (θ, φ). A buyer of type (θ, φ) enjoys a per-period utility $u\left(\theta, q^{\theta}\right) + v\left(\varphi, q^{\varphi}\right) - p$ where q^{θ} and q^{φ} stand for the number of units bought, p is a total price. The utility functions $u\left(\theta, q^{\theta}\right)$ and $v\left(\varphi, q^{\varphi}\right)$ are increasing and differentiable in both arguments with $u(\theta, 0) = v(\varphi, 0) = 0$, and concave in q^{θ} and q^{φ} correspondingly. In every period, the seller produces the goods with cost function $\mathcal{C}\left(q^{\theta}, q^{\varphi}\right) = c\left(q^{\theta}\right) + c\left(q^{\varphi}\right)$. The cost function is increasing, convex, and differentiable, with $c'(0) = 0$ and $\lim_{q \to \infty} c'(q) = \infty$. The per-period profit of the monopolist who sells quantities $\left(q^{\theta}, q^{\varphi}\right)$ to a buyer of type (θ, φ) is given by $p - c\left(q^{\theta}\right) - c\left(q^{\varphi}\right)$. Define the per-period surplus generated by a contract between the buyer and the seller:

$$s\left(\theta, \varphi, q^{\theta}, q^{\varphi}\right) = u\left(\theta, q^{\theta}\right) + v\left(\varphi, q^{\varphi}\right) - c\left(q^{\theta}\right) - c\left(q^{\varphi}\right) \tag{1}$$

The subtypes θ_t and φ_t evolve over time according to a Markov process. I assume that each period there are two possible realizations of each subtype: $\theta_t \in \Theta = \{\theta_L, \theta_H\}$ with $\theta_H > \theta_L$, and $\varphi_t \in \Phi = \{\varphi_L, \varphi_H\}$ with $\varphi_H > \varphi_L$. Hence there are four types, $(\theta_t, \varphi_t) \in \Theta \times \Phi$, with corresponding distribution $\psi\left(\theta_i, \varphi_j\right) \equiv \psi_{ij}$ where $i, j \in \{L, H\}$. The probability of reaching state k if the current state is l is given by $f^{\theta}\left(\theta_k | \theta_l\right) \equiv f^{\theta}_{kl}$ and $f^{\varphi}\left(\varphi_k | \varphi_l\right) \equiv f^{\varphi}_{kl}$. I assume that the subtypes are persistent, or positively correlated *over time*, i.e. $f^{\theta} \equiv f^{\theta}\left(\theta_H | \theta_H\right) \geq f^{\theta}\left(\theta_H | \theta_L\right)$ and $f^{\varphi} \equiv f^{\varphi}\left(\varphi_H | \varphi_H\right) \geq f^{\varphi}\left(\varphi_H | \varphi_L\right)$. I also assume that the subtypes are positively correlated in *cross-section*. In particular, following Armstrong and Rochet (1999), I define the covariance between types as follows:

$$\rho \equiv \psi_{HH}\psi_{LL} - \psi_{HL}\psi_{LH} \tag{2}$$

and assume $\rho \geq 0$. In Section 4.4, I discuss the case of $\rho < 0$.

In each period, the buyer observes the realization of his type. The seller, in turn, does not observe it, and can observe past allocations only. At date $t = 0$, the seller has a prior $\boldsymbol{\mu} =$

$\left(\boldsymbol{\mu^\theta}, \boldsymbol{\mu^\varphi}\right) = \left(\left(\mu_H^\theta, \mu_L^\theta\right), \left(\mu_H^\varphi, \mu_L^\varphi\right)\right)$ on the buyer's type. I assume that the prior has full support.

At date $t = 1$, the seller offers a supply contract to the buyer. The buyer can accept or reject it. If the buyer accepts the offer, he can leave the relationship at any date $t \geq 1$ if the expected continuation utility of the contract falls below the reservation utility $\underline{U} = 0$. The common discount factor is $\delta \in (0, 1)$. I assume that the seller commits to the offered contract.

Denote by $\left(\hat{\theta}_t, \hat{\varphi}_t\right)$ the buyer's type revealed at time t. Define a revelation history of a buyer at time t to be the sequence of his past and current type revelations, i.e. $\hat{\theta}^t = \{\hat{\theta}_1, ..., \hat{\theta}_t\}$ and $\hat{\varphi}^t = \{\hat{\varphi}_1, ..., \hat{\varphi}_t\}$. Alternatively, we can define it recursively: $\hat{\theta}^t = \{\hat{\theta}^{t-1}, \hat{\theta}_t\}$, $\hat{\theta}^0 = \varnothing$ and $\hat{\varphi}^t = \{\hat{\varphi}^{t-1}, \hat{\varphi}_t\}$, $\hat{\varphi}^0 = \varnothing$. Denote by $\hat{\Theta}^t$ and $\hat{\Phi}^t$ the sets of all possible revelation histories for subtypes θ and φ at time t. Denote by $\tilde{\Theta}^\tau$ (similarly, $\tilde{\Phi}^\tau$) the set of histories when $\hat{\theta}_t = \theta_L$ (similarly, $\hat{\varphi}_t = \varphi_L$), $\forall t = 1, ..., \tau$. In this environment, a form of the revelation principle, Myerson (1986), is valid, therefore, without loss of generality, I only consider contracts that in period t depend on the history of type revelations and the type revealed at date t. Formally, the contract can be written as

$$\langle \boldsymbol{p}, \boldsymbol{q^\theta}, \boldsymbol{q^\varphi} \rangle = \left\{ \left(p\left(\hat{\theta}^t, \hat{\varphi}^t\right), q^\theta\left(\hat{\theta}^t, \hat{\varphi}^t\right), q^\varphi\left(\hat{\theta}^t, \hat{\varphi}^t\right) \right) \right\}_{t=1}^T$$

A strategy for the seller consists of offering a direct mechanism $\langle \boldsymbol{p}, \boldsymbol{q^\theta}, \boldsymbol{q^\varphi} \rangle$ described above. In period t, the buyer knows his true type realizations for the periods up until the current one, i.e. $(\theta^t, \varphi^t) \in \Theta^t \times \Phi^t$, where (θ^t, φ^t) denotes the history of the true type realizations, Θ^t and Φ^t denote the sets of all possible true-type histories at time t. For a given contract, a strategy for the buyer is described by a function $\sigma^t(\cdot)$ that maps a history $\left\{ \left(\theta^{t-1}, \varphi^{t-1}\right), \left(\theta_t, \varphi_t\right), \left(\hat{\theta}_{t-1}, \hat{\varphi}_{t-1}\right) \right\}$ into a revealed type $\left(\hat{\theta}^t, \hat{\varphi}^t\right)$.

The seller's problem consists of choosing a contract $\langle \boldsymbol{p}, \boldsymbol{q^\theta}, \boldsymbol{q^\varphi} \rangle$ that maximizes the expected discounted profits subject to the incentive compatibility constraints and the individual rationality constraints. The expected discounted profits are given by

$$\Pi = \mathbb{E}_0 \sum_{t=1}^T \delta^{t-1} \left[p\left(\hat{\theta}^t, \hat{\varphi}^t\right) - c\left(q^\theta\left(\hat{\theta}^t, \hat{\varphi}^t\right)\right) - c\left(q^\varphi\left(\hat{\theta}^t, \hat{\varphi}^t\right)\right) \right] \tag{3}$$

where expectation is taken over the cross-section of the types and time.

The incentive compatibility constraints imply that, after any history, the buyer does not want to report a false type. Denote by $V\left(\hat{\theta}_i, \hat{\varphi}_j | \left(\hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right), (\theta_i, \varphi_j)\right)$ the expected utility of a buyer with type (θ_i, φ_j) who reports to be of type $\left(\hat{\theta}_i, \hat{\varphi}_j\right)$ at time t after history $\left(\hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right)$, and always reports his true type thereafter. Next, I denote by $V\left(\theta_i, \varphi_j | \hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right)$ the expected utility of a buyer with type (θ_i, φ_j) who truthfully reports his type at time t after history $\left(\hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right)$, and always reports his true type thereafter. Using the one-shot deviation principle, I describe the incentive compatibility constraints for type (θ_i, φ_j), after history $\left(\hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right)$ at time t as

$$V\left(\theta_i, \varphi_j | \hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right) \geq V\left(\hat{\theta}_i, \hat{\varphi}_j | \left(\hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right), (\theta_i, \varphi_j)\right) \tag{4}$$

$\forall\,(\theta_i, \varphi_j)$, $\left(\hat{\theta}_i, \hat{\varphi}_j\right)$, $\left(\hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right)$, $(i, j) \in \{L, H\}$. Denote the incentive compatibility constraint described in (4) by $IC_t\,(\theta_i, \varphi_j)$. Note that the buyer reports his type along two dimensions, and hence this is a multidimensional screening problem.

The individual rationality constraints imply that, after any history, the buyer receives at least his reservation utility $\underline{U} = 0$:

$$V\left(\theta_i, \varphi_j | \hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right) \geq 0 \tag{5}$$

$\forall\,(\theta_i, \varphi_j)$, $\left(\hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right)$, $(i, j) \in \{L, H\}$. Denote the individual rationality constraint described in (5) by $IR_t\,(\theta_i, \varphi_j)$. The contract that satisfies all the incentive compatibility and individual rationality constraints is said to be *implementable*.

To summarize, the seller chooses a contract $\langle \boldsymbol{p}, \boldsymbol{q}^{\boldsymbol{\theta}}, \boldsymbol{q}^{\boldsymbol{\varphi}} \rangle$ that maximizes (3) subject to $\boldsymbol{q}^{\boldsymbol{\theta}} \geq 0$, $\boldsymbol{q}^{\boldsymbol{\varphi}} \geq 0$, $IC_t\,(\theta_i, \varphi_j)$ described in (4), $IR_t\,(\theta_i, \varphi_j)$ described in (5), $\forall i, j \in \{L, H\}, t, \left(\hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right)$.

2.2 Multidimensional Screening

The standard approach to characterize the optimal contract in a similar setting but with one dimension of private information follows two steps. First, we solve a relaxed problem where keep the local downward incentive compatibility constraints for H-type and the individual rationality constraints for L-type only. Second, after solving this problem, we ex-post verify the remaining constraints. This 'first-order approach' is widely used in the literature, see Baron and Besanko (1984), Kapička (2013), Pavan et al. (2014), and Farhi and Werning (2013) among many other studies. Battaglini and Lamba (2019) discuss the applicability of the 'first-order approach' in various environments, and conclude that it can be problematic in the settings where expected continuation values are important relative to instant payoffs.

In this paper, I consider an environment with multidimensional screening where applicability of the 'first-order approach' can be problematic as well. See Moser and Olea de Souza e Silva (2019) for a discussion of the challenges associated with multidimensional screening. However, as shown by Armstrong and Rochet (1999), under some conditions on the cross-sectional distribution of types, the 'first-order approach' may be applicable to this class of problems. Furthermore, as I discuss later, these conditions are empirically plausible. In particular, we need to assume that subtypes θ and φ are positively correlated. The current buyer-seller environment can be applied to the joint life insurance contracts for spouses, hence positive correlation between θ and φ can be thoughts as positive correlation between health conditions of the spouses. Mapping our problem into the setting with the optimal taxation of couples, it again means that we assume positive assortative mating in earning ability between spouses. To tackle this problem, I combine the techniques from Armstrong and Rochet (1999) and Battaglini (2005).[1] First, I consider a problem with the downward incentive compatibility constraints only, i.e. when, after any history, HH-buyer pretends to be either LL-, LH-, or HL-buyer, and LH- or HL-buyer pretend to be LL-buyer. Hence I do not consider the upward incentive compatibility constraints as well as the constraints where HL-buyer pretends to be LH-buyer ans vice versa. Second, I show that the incentive compatibility constraints, corresponding to LH- and HL-buyer pretending to be LL-buyer, are always binding. Third, I show the conditions under which the incentive compatibility constraints, corresponding to HH-buyer pretending to be LL-, LH-, or HL-buyer, hold with equality. Throughout the proofs, I use the result from Battaglini (2005) that in a dynamic setting, although the constraints are not necessarily binding in every optimal scheme, it is without loss of generality to assume that constraints in the relaxed problem hold with equality. Finally, I show that the relaxed problem solves the full problem.

[1] If the buyer has type (θ_i, φ_j), I call him ij*-buyer*. For instance, the buyer of type (θ_H, φ_H) is called HH-buyer.

3 Optimal Contract

3.1 Characterization

I begin by stating a useful lemma.

Lemma 1. *Suppose that the menu $\langle \boldsymbol{p}, \boldsymbol{q}^{\boldsymbol{\theta}}, \boldsymbol{q}^{\varphi} \rangle$ solves the 'relaxed' problem. Then the incentive compatibility constraint corresponding to LH- and HL-buyer pretending to be LL-buyer at period $t = 1$ is binding.*

Proof. *See Appendix.*

Next, I refer to the the proposition formulated in Armstrong and Rochet (1999).

Proposition 1. *Consider a static model, i.e. $T = 1$. Suppose $\rho \geq 0$. Then there exists a threshold $\bar{\rho}$ given by*

$$\bar{\rho} = \frac{\psi_{HL}\psi_{LH}}{\psi_{LL}} \tag{6}$$

such that if (i) $\rho > \bar{\rho}$, then the incentive compatibility constraints corresponding to HH-buyer pretending to be HL-, LH-, and LL-buyer hold with equality, (ii) $\rho \in [0, \bar{\rho}]$, then the incentive compatibility constraints corresponding to HH-buyer pretending to be HL- and LH-buyer hold with equality.

Proof. *See Armstrong and Rochet (1999).*

In Section 4.1, I generalize Proposition 1 to an environment with optimal taxation and general government's tastes for redistribution. Next, consider the following lemma.

Lemma 2. *Suppose that the menu $\langle \boldsymbol{p}, \boldsymbol{q}^{\boldsymbol{\theta}}, \boldsymbol{q}^{\varphi} \rangle$ satisfies the constraints of the 'relaxed' problem. Then there exist a price schedule $\tilde{\boldsymbol{p}}$ such that $\langle \tilde{\boldsymbol{p}}, \boldsymbol{q}^{\boldsymbol{\theta}}, \boldsymbol{q}^{\varphi} \rangle$ (i) satisfies all the constraints of the 'relaxed' problem, (ii) delivers the same profits as $\langle \boldsymbol{p}, \boldsymbol{q}^{\boldsymbol{\theta}}, \boldsymbol{q}^{\varphi} \rangle$, (iii) satisfies with equality the incentive compatibility constraints corresponding to LH- and HL-buyer pretending to be LL-buyer and the individual rationality constraint for LL-buyer after any history, and (iv) satisfies with equality the incentive compatibility constraints corresponding to HH-couples pretending to be HL-, LH-, and LL-couples after any history if $\rho > \bar{\rho}$; satisfies with equality the incentive compatibility constraints corresponding to HH-couples pretending to be HL- and LH-couples after any history if $\rho \in [0, \bar{\rho}]$, where $\bar{\rho}$ is given in Proposition 1.*

Proof. *See Appendix.*

Thus, the downward incentive compatibility constraints and the individual rationality constraints for LL-buyer can be assumed to hold with equality after any history without loss of generality. Note that the configuration of the incentive compatibility constraints for HH-buyer depends on the cross-sectional distribution of types. The next proposition characterizes the optimal contract. For illustration, I assume the version of the model proposed by Mussa and Rosen (1978), where $u(\theta, q^\theta) = \theta q^\theta$, $v(\varphi, q^\varphi) = \varphi q^\varphi$, and $c(q) = q^2/2$. Hence θ and φ account for the marginal valuations of the goods.

Proposition 2.

1. *If a buyer ever revealed θ_H or φ_H in his history, then the optimal contract in period t is efficient and characterized by*

$$\tilde{q}^\theta\left(\hat{\theta}_t, \hat{\varphi}_t | \hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right) = \begin{cases} \theta_H & \text{if } \hat{\theta}_t = \theta_H, \forall t, \hat{\theta}^{t-1} \notin \tilde{\Theta}^{t-1} \\ \theta_L & \text{if } \hat{\theta}_t = \theta_L, \forall t, \hat{\theta}^{t-1} \notin \tilde{\Theta}^{t-1} \end{cases} \tag{7}$$

$$\tilde{q}^\varphi\left(\hat{\theta}_t, \hat{\varphi}_t | \hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right) = \begin{cases} \varphi_H & \text{if } \hat{\varphi}_t = \varphi_H, \forall t, \varphi^{t-1} \notin \tilde{\Phi}^{t-1} \\ \varphi_L & \text{if } \hat{\varphi}_t = \varphi_L, \forall t, \varphi^{t-1} \notin \tilde{\Phi}^{t-1} \end{cases} \tag{8}$$

2. *Suppose $\rho > \bar{\rho}$. In period $t = 1$, if a buyer reports θ_L or φ_L, then the optimal contract is characterized by*

$$\tilde{q}^\theta\left(\theta_L, \varphi_H\right) = \tilde{q}^\theta\left(\theta_L, \varphi_L\right) < \theta_L \tag{9}$$

$$\tilde{q}^\varphi\left(\theta_H, \varphi_L\right) = \tilde{q}^\varphi\left(\theta_L, \varphi_L\right) < \varphi_L \tag{10}$$

3. *Suppose $\rho \in [0, \bar{\rho}]$. In period $t = 1$, if a buyer reports θ_L or φ_L , then the optimal contract is characterized by*

$$\tilde{q}^\theta\left(\theta_L, \varphi_L\right) < \tilde{q}^\theta\left(\theta_L, \varphi_H\right) < \theta_L \tag{11}$$

$$\tilde{q}^\varphi\left(\theta_L, \varphi_L\right) < \tilde{q}^\varphi\left(\theta_H, \varphi_L\right) < \varphi_L \tag{12}$$

4. *The optimal contract in periods $t > 1$ satisfy*

$$\tilde{q}^\theta\left(\hat{\theta}^t, \hat{\varphi}^t\right) = \frac{2f^\theta - 1}{f^\theta} \tilde{q}^\theta\left(\hat{\theta}^{t-1}, \hat{\varphi}^{t-1}\right) \tag{13}$$

$$\tilde{q}^{\varphi}\left(\hat{\theta}^{t},\hat{\varphi}^{t}\right)=\frac{2f^{\varphi}-1}{f^{\varphi}}\tilde{q}^{\varphi}\left(\hat{\theta}^{t-1},\hat{\varphi}^{t-1}\right) \tag{14}$$

Proof. *See Appendix.*

Proposition 2 generalizes the results from Battaglini (2005) to the framework with multidimensional types. First, notice that the optimal contract is history-dependent and has infinite memory. Despite this, we can easily characterize it. If the buyer has ever reported θ_H (similarly, φ_H) in his history, then the quantity q^{θ} (similarly, q^{φ}) is at the efficient level and depends on the reported type in a given period. This is consistent with the so-called 'generalized no-distortion at the top principle' as proposed by Battaglini (2005). Next, if the buyer has always reported θ_L (similarly, φ_L), then the optimal contract is shaped by both the cross-sectional correlation between θ and φ as well as their persistence. In particular, if $\rho > \bar{\rho}$, then quantity q^{θ} (similarly, q^{φ}) does not depend on reported φ (similarly, θ). This result resembles the one established by Carroll (2017). In a multidimensional screening setting, he considers a robust version of the principal's problem, in which she knows the marginal distribution of each component of the agent's type, but does not know the joint distribution, and shows that under very general conditions it is optimal for the principal to screen along each component separately. In turn, if $\rho \in [0, \bar{\rho}]$, then quantity q^{θ} (similarly, q^{φ}) decreases in reported φ (similarly, θ). Finally, the optimal contract converges over time to an efficient contract along any history. This result corresponds to the 'vanishing distortion at the bottom principle' as proposed by Battaglini (2005). The speed of convergence depends on the persistence of private information.

3.2 Application: Joint Insurance Contracts

One of the natural candidates for application of my results is the design of joint life insurance contracts. These contracts offer coverage for two people for a single premium payment each month. Guner et al. (2018) show that there is positive assortative mating in spousal health. Assuming that θ and φ capture the health conditions of the spouses, and hence positively correlated, I can directly apply the results from Section 3.1. To the best of my knowledge, the joint life insurance contracts are almost unexplored in the literature, see Youn and Shemyakin (1999), Luciano et al. (2008), and Gourieroux and Lu (2015) for a discussion. In a related work, Hendel and Lizzeri (2003) use the data on life insurance contracts to study the properties of long-term contracts with the lack of commitment by buyers. Extend.

4 Extensions and Alternative Applications

The framework considered in Section 2 can be further extended and/or applied to the other environments. In particular, I pursue three directions. First, I discuss the optimal contract under the negative cross-sectional correlation between the subtypes. Second, I apply the techniques from Section 3 to characterize the solution to the optimal income taxation problem. These results can be applied, for instance, to the taxation of couples. Finally, within the optimal taxation framework, I study the setting where the government cares about both between- and within-family redistribution.

4.1 Optimal Taxation

How should the optimal income taxes for married couples be designed? Over the last decades, one of the stable features of the U.S. and many European economies is a presence of a positive assortative mating between spouses, i.e. people more likely match and marry partners with similar characteristics.[2] As a consequence, this gave rise to concerns that positive assortative mating can potentially be one of the driving forces of increasing between-household inequality.[3] Furthermore, another well-known observation is that the current tax systems in the U.S. and many European countries create significant disincentive effects for the married women's labor supply.[4] These concerns inevitably lead to the questions about the optimal income tax design for couples.

In this section, I apply our framework to study the optimal income taxation of couples in a dynamic Mirrlees setting. I study how the cross-sectional distribution of spousal types, government's taste for redistribution, and persistence of the spousal types shape the optimal tax schedule. Note that the solution to the monopoly pricing problem, considered earlier, is one point that corresponds to the maximum profit. In turn, in the optimal taxation setting, I characterize the part of Pareto frontier. Furthermore, the individual rationality constraint in the buyer-seller

[2] Discussion of this topic can be found in Schwartz (2010) for the case of earnings, Eika et al. (2019) for education.

[3] Using the data on household surveys from 34 countries, Fernandez et al. (2005) show that there is a positive relationship between sorting in skills and income inequality. However, Eika et al. (2019) argue that changes in educational assortative mating over time barely move the trends in household income inequality.

[4] See Bick and Fuchs-Schündeln (2017b), Bick and Fuchs-Schündeln (2017a), and Bick et al. (2019).

framework implies that we assume the Rawlsian welfare function, while in this section I allow for more general taste for redistribution.

Consider an economy populated by a continuum of couples. Each couple consists of two spouses—a male (denoted by m) and a female (denoted by f). Spouses differ in their abilities to produce. The ability of a male is $\theta \in \Theta = \{\theta_L, \theta_H\}$ and the ability of a female is $\varphi \in \Phi = \{\varphi_L, \varphi_H\}$ with $\theta_H > \theta_L > 0$ and $\varphi_H > \varphi_L > 0$. Let y_{ij}^k be the earnings or output of individual $k \in \{m, f\}$ where $i, j \in \{L, H\}$. Assume linear production technology, so that $y_{ij}^m = \theta n_{ij}^m$ and $y_{ij}^f = \varphi n_{ij}^f$ where n_{ij}^k are working hours. The rest of notation follows one from Section 2.

Assume the following per-period utility function of a couple:

$$U\left(c_t, y_t^m, y_t^f, \theta_t, \varphi_t\right) = c_t - \phi\left(\frac{y_t^m}{\theta_t}\right) - \phi\left(\frac{y_t^f}{\varphi_t}\right) \tag{15}$$

where $\phi(\cdot)$ is increasing, strictly convex, and twice continuously differentiable. Risk neutrality in preferences implies that the only source of distortions is the desire of the government to redistribute resources.

The government observes consumption and spousal outputs, but not their abilities. It evaluates total welfare using the weights $\lambda\left(\theta_i, \varphi_j\right)$ assigned to the couples that have type $\left(\theta_i, \varphi_j\right)$ in period 0, where $i, j \in \{L, H\}$. Normalize the weights such that

$$\lambda\left(\theta_i, \varphi_j\right) \equiv \frac{\omega_{ij}\psi_{ij}}{\sum_{k,l} \omega_{kl}\psi_{kl}} \tag{16}$$

With the risk-neutral agents, the utilitarian government will set all the marginal taxes to zero, and thus I assume that it has a taste for redistribution that is different from utilitarian. In what follows, I make the following assumptions.

Assumption 1. *The primitive welfare weights are non-negative, $\omega_{ij} \geq 0$, and satisfy the following conditions: (i) $\omega_{HL} = \omega_{LH} \equiv \tilde{\omega}$, (ii) $\omega_{LL} > 2\tilde{\omega}$, and (iii) $\tilde{\omega} \geq \omega_{HH}$.*

Assumption (i) accounts for 'anonymity'. Assumption (ii) implies that the government has a strong enough taste for redistribution towards LL-couples. This is equivalent to promising some level of reservation utility $\underline{U}$ to LL-buyer in Section 2. Assumption 1 implies that $\sum_{i,j} \omega_{ij}\psi_{ij} \equiv \mathbb{E}(\omega) > \tilde{\omega} \geq \omega_{HH}$.

An allocation in this economy is given by

$$\langle \boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f} \rangle = \left\{ \left(c_t\left(\theta^t, \varphi^t\right), y_t^m\left(\theta^t, \varphi^t\right), y_t^f\left(\theta^t, \varphi^t\right) \right) \right\}_{t=1}^{T}$$

To simplify notation, in this section I omit explicit dependence on the past history. Whenever it does not cause confusion, a notation $x_t^k(\theta, \varphi)$ denotes the value of a random variable x_t^k at a history $(\theta^{t-1}, \varphi^{t-1}, \theta_t, \varphi_t)$, and x_{t-1}^k denotes $x_{t-1}^k(\theta^{t-1}, \varphi^{t-1})$.

Define expected discounted utility of a couple:

$$V_t\left(\boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f}\right) = \mathbb{E}_t \left\{ \sum_{s=t}^{T} \delta^{s-t} \left[c_s(\theta, \varphi) - \phi\left(\frac{y_s^m(\theta, \varphi)}{\theta_s}\right) - \phi\left(\frac{y_s^f(\theta, \varphi)}{\varphi_s}\right) \right] |(\theta_t, \varphi_t) \right\} \tag{17}$$

An allocation is said to be *resource feasible* if it satisfies the aggregate resource constraint:

$$\sum_{t=0}^{T} \left(\frac{1}{R}\right)^t \mathbb{E}_0\left[c_t(\theta, \varphi) |\theta_0, \varphi_0\right] + G \leq \sum_{t=0}^{T} \left(\frac{1}{R}\right)^t \mathbb{E}_0\left[y_t^m(\theta, \varphi) + y_t^f(\theta, \varphi) |(\theta_0, \varphi_0)\right] \tag{18}$$

I study the partial equilibrium where the government can transfer aggregate resources across periods at a gross rate of return R. Note that the aggregate resource constraint is an additional constraint that we do not have in the monopoly pricing model.

An allocation is said to be *incentive compatible* if it satisfies the following set of incentive constraints for each couple's report σ^t, history (θ^t, φ^t), and t:

$$\begin{aligned} V_t\left(\boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f}\right) \geq c_t\left(\sigma^t(\theta, \varphi)\right) - \phi\left(\frac{y_t^m\left(\sigma^t(\theta, \varphi)\right)}{\theta_t}\right) - \phi\left(\frac{y_t^f\left(\sigma^t(\theta, \varphi)\right)}{\varphi_t}\right) + \\ \delta \mathbb{E}_t \left\{ V_{t+1}\left(\left(\boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f}\right), (\theta^{t-1}, \varphi^{t-1}), \sigma^t(\theta, \varphi), (\theta_{t+1}, \varphi_{t+1})\right) |(\theta_t, \varphi_t) = (\theta, \varphi) \right\} \end{aligned} \tag{19}$$

The government solves the following dynamic mechanism design problem:

$$\max_{\langle \boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f} \rangle} \sum_{i,j} \lambda(\theta_i, \varphi_j) V_0\left(\boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f}\right)$$

subject to the aggregate resource constraint (18) and the incentive compatibility constraints (19).

First, I state the proposition that generalizes Proposition 1 to the environment with more general welfare weights.

Proposition 3. *Consider a static model, i.e. $T = 1$. Suppose $\rho \geq 0$. Then there exists a threshold $\bar{\rho}$ given by*

$$\bar{\rho} = \frac{(\omega_{LL} + \omega_{HH} - 2\tilde{\omega})\,\psi_{HL}\psi_{LH}}{(\omega_{LL} - \omega_{HH})\,\psi_{LL} + (\tilde{\omega} - \omega_{HH})\,(\psi_{HL} + \psi_{LH})} \tag{20}$$

such that if $\rho > \bar{\rho}$, then the incentive compatibility constraints corresponding to HH-couples pretending to be HL-, LH-, and LL-couples hold with equality, (ii) $\rho \in [0, \bar{\rho}]$, then the incentive compatibility constraints corresponding to HH-couples pretending to be HL- and LH-couples hold with equality.

Proof. *See Appendix.*

Proposition 3 is a generalization of the result from Armstrong and Rochet (1999) and Frankel (2014). In these papers, the authors assume that $\omega_{LL} > \omega_{LH} = \omega_{HL} = \omega_{HH} \geq 0$. This results in the threshold $\bar{\rho} = \psi_{HL}\psi_{LH}/\psi_{LL}$.

I use Lemmas 1 and 2 from Section 3 to argue the following. Suppose that the allocation $\langle \boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f} \rangle$ satisfies the constraints of the relaxed problem. The there exist a consumption allocation $\tilde{\boldsymbol{c}}$ such that $\langle \tilde{\boldsymbol{c}}, \boldsymbol{y^m}, \boldsymbol{y^f} \rangle$ (i) satisfies all the constraints of the 'relaxed' problem, (ii) provides the same welfare as $\langle \boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f} \rangle$, (iii) satisfies with equality the incentive compatibility constraints corresponding to LH- and HL-couples pretending to be LL-couples after any history, and (iv) satisfies with equality the incentive compatibility constraints corresponding to HH-couples pretending to be HL-, LH-, and LL-couples after any history if $\rho > \bar{\rho}$; satisfies with equality the incentive compatibility constraints corresponding to HH-couples pretending to be HL- and LH-couples after any history if $\rho \in [0, \bar{\rho}]$, where $\bar{\rho}$ is given in Proposition 3. Next, referring to the arguments from Section 3, I argue that the solution to the relaxed problem solves the full government's problem, i.e. one with all the incentive compatibility constraints.

I turn to the optimal distortions. Define the optimal labor wedges τ^m_{ij} and τ^f_{ij} as

$$MRS^{m,ij}_{n,c} = (1 - \tau^m_{ij})\theta_i \tag{21}$$

$$MRS^{f,ij}_{n,c} = (1 - \tau^f_{ij})\varphi_j \tag{22}$$

Proposition 4 characterizes the optimal distortions.

Proposition 4.

1. *The optimal distortions for the individuals who ever had high ability in their history are zero:*

$$\frac{\tau_t^k(\theta,\varphi)}{1-\tau_t^k(\theta,\varphi)}=0 \qquad \forall t,\theta^t\notin\tilde{\Theta}^t,\varphi^t\notin\tilde{\Phi}^t,k\in\{m,f\} \tag{23}$$

2. *Suppose $\rho>\bar{\rho}$. Then the optimal distortions at $t=1$ for the low-ability spouses satisfy*

$$\frac{\tau_1^m(\theta_L,\varphi_L)}{1-\tau_1^m(\theta_L,\varphi_L)}=\frac{\tau_1^m(\theta_L,\varphi_H)}{1-\tau_1^m(\theta_L,\varphi_H)} \tag{24}$$

$$\frac{\tau_1^f(\theta_L,\varphi_L)}{1-\tau_1^f(\theta_L,\varphi_L)}=\frac{\tau_1^f(\theta_H,\varphi_L)}{1-\tau_1^f(\theta_H,\varphi_L)} \tag{25}$$

3. *Suppose $\rho\in[0,\bar{\rho}]$. Then the optimal distortions at $t=1$ for the low-ability spouses satisfy*

$$\frac{\tau_1^m(\theta_L,\varphi_L)}{1-\tau_1^m(\theta_L,\varphi_L)}>\frac{\tau_1^m(\theta_L,\varphi_H)}{1-\tau_1^m(\theta_L,\varphi_H)} \tag{26}$$

$$\frac{\tau_1^f(\theta_L,\varphi_L)}{1-\tau_1^f(\theta_L,\varphi_L)}>\frac{\tau_1^f(\theta_H,\varphi_L)}{1-\tau_1^f(\theta_H,\varphi_L)} \tag{27}$$

4. *The optimal distortions in periods $t>1$ satisfy*

$$\frac{\tau_t^m(\theta,\varphi)}{1-\tau_t^m(\theta,\varphi)}=\delta R\frac{2f^\theta-1}{f^\theta}\cdot\frac{\tau_{t-1}^m}{1-\tau_{t-1}^m} \tag{28}$$

$$\frac{\tau_t^f(\theta,\varphi)}{1-\tau_t^f(\theta,\varphi)}=\delta R\frac{2f^\varphi-1}{f^\varphi}\cdot\frac{\tau_{t-1}^f}{1-\tau_{t-1}^f} \tag{29}$$

Proof. *See Appendix.*

Proposition 4 generalizes the results from Battaglini and Coate (2008) to the framework with multidimensional types. The optimal taxes are history-dependent. In particular, if a person has ever had high ability, her/his earnings are undistorted and the optimal marginal tax rate for her/him is zero irrespective of the history of abilities of her/his spouse. In turn, positive distortions exist only for those who are currently and have always had low ability. What is crucially different from the individual optimal taxation literature, is that the intratemporal, or cross-sectional, component of the optimal distortions is significantly richer because of interdependency between the spousal types.

The 'no distortion at the top' result (23) is by construction because I only have two types, and it also crucially depends on the assumption of risk neutrality because in all the periods the individuals have the same marginal utility of consumption equal to 1. In the case of general $u(c)$, the government's generalized welfare weights in period t depend on the marginal utility of consumption in that period, and we would not expect that the optimal distortions for the individuals who ever had high ability in their history are set to zero.

If $\rho > \bar{\rho}$, (24) and (25) show that in period $t = 1$ the optimal distortions for low-ability individuals whose spouses also have low ability are equal to the optimal distortions for low-ability individuals with high-ability spouses. Hence I say that there is *separability* in the marginal tax rates, i.e. earnings and a marginal tax rate for an individual are determined by her/his type and are not affected by the type of her/his spouse.

If $\rho \in [0, \bar{\rho}]$, (26) and (27) show that in period $t = 1$ the optimal distortions for low-ability individuals whose spouses also have low ability are greater than the optimal distortions for low-ability individuals with high-ability spouses. Hence I say that there is *negative jointness* in the marginal tax rates, i.e. the optimal distortion of an individual decreases in the earnings of her/his spouse. This tax schedule is proposed to be optimal in Kleven et al. (2009) and Gayle and Shephard (2019).

In Figure 1, to compare the conclusions about the optimal tax schedule, I compare the threshold on ρ from Armstrong and Rochet (1999) and Frankel (2014) with the threshold from Proposition 3 assuming $\tilde{\omega} > \omega_{HH} = 0$.

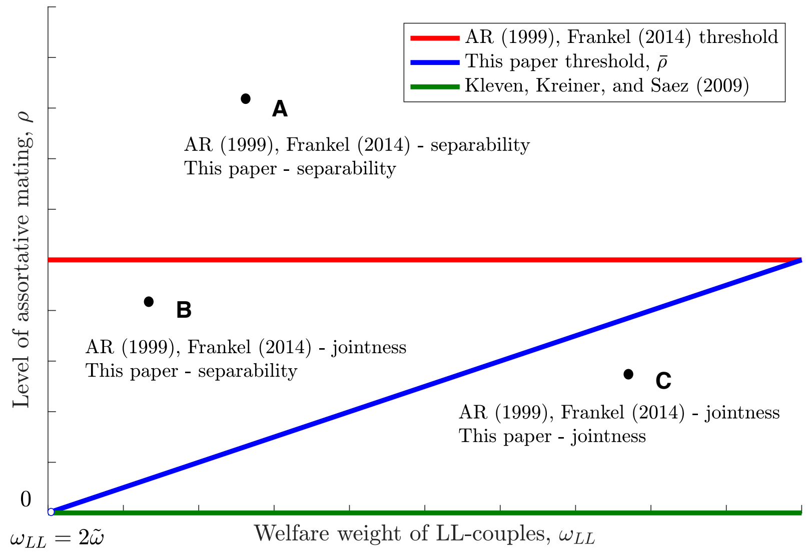

Figure 1: Thresholds for the measure of assortative mating, ρ, and the optimal tax systems.

Figure 1 illustrates several interesting features. First, the threshold from Armstrong and Rochet (1999) and Frankel (2014), the red line, is weakly greater than the threshold from Proposition 1, the blue line. The only case, when they coincide, corresponds to $\omega_{LH} = \omega_{HL} = \omega_{HH}$. Second, as $\omega_{LL} \to 2\tilde{\omega}$, the threshold from Proposition 3 goes to zero. Third, it turns out the government's taste for redistribution have important implication for the optimal tax schedule. Consider three countries—A, B, and C. From Figure 1 we observe that the assortative mating in country A is above both thresholds for any configuration of the welfare weights that we allow. Hence in this case my conclusion coincides with those from Armstrong and Rochet (1999) and Frankel (2014): the optimal tax system in country A should feature separability. We also come to a similar conclusion about country C where the assortative mating is below both thresholds. However, turning to country B, we can see that restricting the welfare weights to $\omega_{LH} = \omega_{HL} = \omega_{HH}$, we should conclude that the optimal tax system should feature negative jointness. However, with more general welfare weights we conclude that it should be separable like in country A. I also show how my results correspond to Kleven et al. (2009) who assume $\rho = 0$. In their paper, the government maximizes the sum of increasing and concave transformations $\Psi(\cdot)$ of the couples' utilities with $\Psi'(\cdot)$ strictly convex. Note that this is exactly the assumption about the welfare weights that I use. This simple example illustrates the importance of both distribution of spousal types in the economy and the government's taste for redistribution.

Finally, the result about the dynamics of the optimal distortions over time is fully consistent with the individual taxation papers by Battaglini and Coate (2008), Farhi and Werning (2013), and Golosov et al. (2016) for the case of risk neutrality. The size of the optimal distortions converges to zero over time since I assume $f^{\theta} \in (0.5, 1)$ and $f^{\varphi} \in (0.5, 1)$.

For those individuals whose earnings are distorted, the optimal marginal taxes can be described as a sum of two terms, as in Golosov et al. (2016): an intratemporal (cross-sectional) component and an intertemporal (time-series) component. Since the individuals are risk neutral, they do not need insurance against the life-cycle shocks and thus the intratemporal components are equal to zero in all periods $t > 1$. However, as I show, at period $t = 1$, the intratemporal component crucially depends on the level of assortative mating in the economy and the government's taste for redistribution. In turn, the intertemporal component is zero at period $t = 1$, and positive in subsequent periods. To conclude this section, I want to map the results from Proposition 4 with the results from the new dynamic public finance literature.

Assume that disutility of labor takes the following form:

$$\phi(n) = \frac{n^{1+1/\eta}}{1+1/\eta} \tag{30}$$

With this functional form, η is the Frisch elasticity of labor supply. I want to compare the optimal distortions from Proposition 4 with the results from Diamond (1998) and Saez (2001), or, in the dynamic context, Golosov et al. (2016). In particular, applying equation (17) from their paper to the risk neutral case, Golosov et al. (2016) get that in the first period the optimal labor distortion is

$$\frac{\tau_1(\theta)}{1-\tau_1(\theta)} = \underbrace{\frac{1+1/\eta}{\theta f_1(\theta)}}_{A_1(\theta)\frac{B_1(\theta)}{1-F_1(\theta)}} \underbrace{\int_{\theta}^{\infty} (1-\alpha(x))\, f_1(x)dx}_{(1-F_1(\theta))\, C_1(\theta)} \equiv A_1(\theta)B_1(\theta)C_1(\theta)$$

In this paper, if $\rho \geq 0$, the optimal labor distortions in period $t = 1$, or the intratemporal component, are given by

$$\frac{\tau_1^m(\theta_L, \varphi)}{1 - \tau_1^m(\theta_L, \varphi)} = \frac{1 - \left(\frac{\theta_L}{\theta_H}\right)^{1+1/\eta}}{\psi_{LH} + \psi_{LL}} \sum_{s=L,H} \psi_{Hs} \left(1 - \frac{\omega_{Hs}}{\sum_{ij} \omega_{ij}\psi_{ij}}\right) + J^m(\varphi) \cdot \mathbb{I}\{\rho \in [0, \bar{\rho}]\} \quad (31)$$

$$\frac{\tau_1^f(\theta, \varphi_L)}{1 - \tau_1^f(\theta, \varphi_L)} = \frac{1 - \left(\frac{\varphi_L}{\varphi_H}\right)^{1+1/\eta}}{\psi_{HL} + \psi_{LL}} \sum_{s=L,H} \psi_{sH} \left(1 - \frac{\omega_{sH}}{\sum_{ij} \omega_{ij}\psi_{ij}}\right) + J^f(\theta) \cdot \mathbb{I}\{\rho \in [0, \bar{\rho}]\} \quad (32)$$

where $\mathbb{I}\{\cdot\}$ is an indicator function that takes value 1 if $\rho \in [0, \bar{\rho}]$ and 0 otherwise, and J^m and J^f are the terms that capture jointness. I intentionally emphasize that these terms depend on the spousal types. We can clearly see that (31) and (32) closely correspond to the well-known ABC-formula for the optimal marginal tax rates that I state above, but also capture the possibility of interdependence between the types. Hence I obtain a generalization of the ABC-formula for the case with multidimensional private information.

In period $t = 1$, the intertemporal component is zero. However, for $t > 1$, Golosov et al. (2016) show that in the risk neutral case:

$$\frac{\tau_t(\theta)}{1 - \tau_t(\theta)} = \delta R \upsilon \frac{\tau_{t-1}}{1 - \tau_{t-1}}$$

where υ measures the persistence of ability shocks. This is the intertemporal component of optimal labor distortions. If $\upsilon = 0$, i.e. there is no persistence in the types, then $\tau_t(\theta) = 0$, $\forall t$. If $\upsilon \in (0, 1)$, then the size of distortions converges to zero over time. Finally, if $\upsilon = 1$, i.e. the types are constant, the distortions are constant over time as well. Note that this is exactly what equations (28) and (29) show.

4.2 Taxation of Couples: Within-Family Redistribution

Recent work emphasize that within-household inequality can account for a sizeable part of the observed inequality between individuals. Using the data for the United Kingdom for 1968-2001, Lise and Seitz (2011) show that ignoring consumption inequality within households underestimates the initial level of cross-sectional consumption inequality by 50 percent. There are very few papers on optimal taxation that allow for within-household redistribution. One of notable examples is Gayle and Shephard (2019). My framework is flexible enough to consider this issue too. Suppose that the government cares both about between- and within-household redistribution. Denote by $\kappa_{ij} \in [0,1]$ the welfare weight that the government assigns to the male in ij-couple, $i,j \in \{L,H\}$. Next, denote by $\xi_{ij} \in [0,1]$ the male's consumption share.

Then, the expected utility takes the following form:

$$V_t\left(\boldsymbol{c},\boldsymbol{y^m},\boldsymbol{y^f}\right) = \mathbb{E}_t \Bigg\{ \sum_{s=t}^{T} \delta^{s-t} \Bigg[\kappa_{ij} \left(\xi_{ij} \cdot c_s\left(\theta,\varphi\right) - \phi\left(\frac{y_s^m\left(\theta,\varphi\right)}{\theta_s}\right)\right) + \\ (1-\kappa_{ij}) \left((1-\xi_{ij}) \cdot c_s\left(\theta,\varphi\right) - \phi\left(\frac{y_s^f\left(\theta,\varphi\right)}{\varphi_s}\right)\right) |(\theta_t,\varphi_t) \Bigg\} \tag{33}$$

In this setting, the Lagrange multipliers can be decomposed into two terms: one accounts for between-household redistribution (one that we have in the previous section) and another one—for within-household redistribution. The threshold for assortative mating is now a function of not only cross-sectional distribution of couples and between-household taste for redistribution, but also within-household taste for redistribution.

To illustrate, first, consider the case when husbands and wives split consumption equally, i.e. $\xi_{ij} = 1/2$, $i,j \in \{L,H\}$. Under separability in the marginal tax rates, the Lagrange multipliers corresponding to the incentive constraints for HH-couples that want to mimic HL- and LH-couples in $t=1$:

$$\gamma_2 = \underbrace{\frac{1}{2} \cdot \frac{\psi_{HL}}{\psi_{HL}+\psi_{LL}} \cdot \left[\frac{\omega_{LL}\psi_{LL}+\omega_{HL}\psi_{HL}}{\sum_{s,r}\omega_{sr}\psi_{sr}} - \psi_{HL} - \psi_{LL}\right]}_{\text{between-household redistribution}} +$$

$$\underbrace{\frac{\phi'\left(\frac{y_L^f}{\varphi_L}\right)/\varphi_L}{\phi'\left(\frac{y_L^f}{\varphi_L}\right)/\varphi_L - \phi'\left(\frac{y_L^f}{\varphi_H}\right)/\varphi_H} \left[\psi_{LL}\left(\frac{1}{2}-\kappa_{HL}\right)\frac{\omega_{HL}\psi_{HL}}{\sum_{s,r}\omega_{sr}\psi_{sr}} + \psi_{HL}\left(\kappa_{LL}-\frac{1}{2}\right)\frac{\omega_{LL}\psi_{LL}}{\sum_{s,r}\omega_{sr}\psi_{sr}}\right]}_{\text{within-household redistribution}}$$

$$\gamma_3 = \underbrace{\frac{1}{2} \cdot \frac{\psi_{LH}}{\psi_{LH}+\psi_{LL}} \cdot \left[\frac{\omega_{LL}\psi_{LL}+\omega_{LH}\psi_{LH}}{\sum_{s,r}\omega_{sr}\psi_{sr}} - \psi_{LH} - \psi_{LL}\right]}_{\text{between-household redistribution}} +$$

$$\underbrace{\frac{\phi'\left(\frac{y_L^m}{\theta_L}\right)/\theta_L}{\phi'\left(\frac{y_L^m}{\theta_L}\right)/\theta_L-\phi'\left(\frac{y_L^m}{\theta_H}\right)/\theta_H}\left[\psi_{LL}\left(\frac{1}{2}-\kappa_{LH}\right)\frac{\omega_{LH}\psi_{LH}}{\sum_{s,r}\omega_{sr}\psi_{sr}}+\psi_{LH}\left(\kappa_{LL}-\frac{1}{2}\right)\frac{\omega_{LL}\psi_{LL}}{\sum_{s,r}\omega_{sr}\psi_{sr}}\right]}_{\text{within-household redistribution}}$$

Note that if the government assigns equal welfare weights for each spouse, $\kappa_{ij}=1/2$, then the second term in both equations is equal to zero, and we are back to the original model with no within-household redistribution.

Finish.

4.3 Renegotiation-Proofness

4.4 Negative Cross-Sectional Correlation

Add this subsection.

5 Conclusion

In this paper, I study a simple principal-agent model where a monopolist repeatedly sells two non-durable goods to a buyer. A two-dimensional buyer's type is private information and stochastically evolves over time according to a Markov process. I characterize the optimal contract in this environment. I show that it is history-dependent and has infinite memory. In each period of time, the optimal quantities depend on the full history of past buyer's reports about his type, the current report, and the cross-sectional distribution of the subtypes that capture the preferences over the goods. In particular, I show that there exists a threshold on correlation between the subtypes that determines whether the quantity of a good depends on the report about the marginal valuation of another good or not. The behavior of the optimal contract over time is shaped by persistence of the buyer's type. In addition, I apply the principal-agent framework to the problem of optimal income taxation of couples, and show how the cross-sectional distribution of spousal types, government's taste for redistribution, and persistence of the spousal types shape the optimal tax schedule. I obtain a generalization of the ABC-formula for the optimal marginal tax rates with multidimensional private information. Summing it up, the results of this paper can be applied to various settings, including the joint insurance contracts and taxation of couples.

To the best of my knowledge, this is the first paper that embeds a multidimensional screening problem into dynamic context with persistent private information in an analytically tractable way. Despite its simplicity, it allows to get some interesting theoretical results and may serve as a benchmark for more complex models of multidimensional screening in dynamic settings.

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Appendix

Proof of Lemma 1

Proof of Lemma 2

Proof of Proposition 2

Proof of Proposition 3

Denote $c(\theta_i, \varphi_j) \equiv c_{ij}$, $y^m(\theta_i, \varphi_j) \equiv y^m_{ij}$, $y^f(\theta_i, \varphi_j) \equiv y^f_{ij}$, and $\lambda(\theta_i, \varphi_j) \equiv \lambda_{ij}$. Since the model is static, I omit the time indices. The government solves the following problem:

$$\begin{aligned}\max_{\langle \boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f} \rangle} \lambda_{HH}\left[c_{HH} - \phi\left(\frac{y^m_{HH}}{\theta_H}\right) - \phi\left(\frac{y^f_{HH}}{\varphi_H}\right)\right] + \lambda_{HL}\left[c_{HL} - \phi\left(\frac{y^m_{HL}}{\theta_H}\right) - \phi\left(\frac{y^f_{HL}}{\varphi_L}\right)\right] + \\ \lambda_{LH}\left[c_{LH} - \phi\left(\frac{y^m_{LH}}{\theta_L}\right) - \phi\left(\frac{y^f_{LH}}{\varphi_H}\right)\right] + \lambda_{LL}\left[c_{LL} - \phi\left(\frac{y^m_{LL}}{\theta_L}\right) - \phi\left(\frac{y^f_{LL}}{\varphi_L}\right)\right]\end{aligned} \tag{A.1}$$

$$\begin{aligned}\text{s.t.} \quad \psi_{HH}\left[y^m_{HH} + y^f_{HH} - c_{HH}\right] + \psi_{HL}\left[y^m_{HL} + y^f_{HL} - c_{HL}\right] + \\ \psi_{LH}\left[y^m_{LH} + y^f_{LH} - c_{LH}\right] + \psi_{LL}\left[y^m_{LL} + y^f_{LL} - c_{LL}\right] - G \geq 0\end{aligned} \tag{A.2}$$

$$c_{HH} - \phi\left(\frac{y^m_{HH}}{\theta_H}\right) - \phi\left(\frac{y^f_{HH}}{\varphi_H}\right) \geq c_{LL} - \phi\left(\frac{y^m_{LL}}{\theta_H}\right) - \phi\left(\frac{y^f_{LL}}{\varphi_H}\right) \tag{A.3}$$

$$c_{HH} - \phi\left(\frac{y^m_{HH}}{\theta_H}\right) - \phi\left(\frac{y^f_{HH}}{\varphi_H}\right) \geq c_{HL} - \phi\left(\frac{y^m_{HL}}{\theta_H}\right) - \phi\left(\frac{y^f_{HL}}{\varphi_H}\right) \tag{A.4}$$

$$c_{HH} - \phi\left(\frac{y^m_{HH}}{\theta_H}\right) - \phi\left(\frac{y^f_{HH}}{\varphi_H}\right) \geq c_{LH} - \phi\left(\frac{y^m_{LH}}{\theta_H}\right) - \phi\left(\frac{y^f_{LH}}{\varphi_H}\right) \tag{A.5}$$

$$c_{HL} - \phi\left(\frac{y^m_{HL}}{\theta_H}\right) - \phi\left(\frac{y^f_{HL}}{\varphi_L}\right) \geq c_{LL} - \phi\left(\frac{y^m_{LL}}{\theta_H}\right) - \phi\left(\frac{y^f_{LL}}{\varphi_L}\right) \tag{A.6}$$

$$c_{LH} - \phi\left(\frac{y^m_{LH}}{\theta_L}\right) - \phi\left(\frac{y^f_{LH}}{\varphi_H}\right) \geq c_{LL} - \phi\left(\frac{y^m_{LL}}{\theta_L}\right) - \phi\left(\frac{y^f_{LL}}{\varphi_H}\right) \tag{A.7}$$

where (A.2) is the aggregate resource constraint, (A.3)-(A.7) is the set of the incentive compatibility constraints.

First, notice that (A.6) and (A.7) hold with equalities. In what follows, I prove that (A.6) holds with equality, and the proof for (A.7) follows similar arguments. Consider a contract $\langle \boldsymbol{c}, \boldsymbol{y^m}, \boldsymbol{y^f} \rangle$ that solves the government's problem. Suppose that (A.6) holds with strict inequality. Consider an alternative contract $\langle \tilde{\boldsymbol{c}}, \tilde{\boldsymbol{y}}^{\boldsymbol{m}}, \tilde{\boldsymbol{y}}^{\boldsymbol{f}} \rangle$ such that $\tilde{\boldsymbol{y}}^{\boldsymbol{m}} = \boldsymbol{y}^{\boldsymbol{m}}$, $\tilde{\boldsymbol{y}}^{\boldsymbol{f}} = \boldsymbol{y}^{\boldsymbol{f}}$, and

$$(\tilde{c}_{HH}, \tilde{c}_{HL}, \tilde{c}_{LH}, \tilde{c}_{LL}) = (c_{HH} + \varepsilon, c_{HL} - \delta, c_{LH} + \varepsilon, c_{LL} + \varepsilon)$$

with $\varepsilon > 0$ and $\delta > 0$ small enough such that (A.6) is still satisfied. Choose $\delta = (1 - \psi_{HL})\,\varepsilon/\psi_{HL}$, so that the aggregate resource constraint is also satisfied. The change in welfare is given by

$$\Delta W = (\lambda_{HH} + \lambda_{LH} + \lambda_{LL})\,\varepsilon - \lambda_{HL}\frac{(1 - \psi_{HL})\,\varepsilon}{\psi_{HL}} = (\lambda_{HH} + \lambda_{HL} + \lambda_{LH} + \lambda_{LL})\,\varepsilon - \frac{\lambda_{HL}}{\psi_{HL}}\varepsilon = \left[1 - \frac{\lambda_{HL}}{\psi_{HL}}\right]\varepsilon = \left[1 - \frac{\tilde{\omega}}{\sum_{i,j}\omega_{ij}\psi_{ij}}\right]\varepsilon > 0$$

where I use normalization $\sum_{i,j} \lambda_{ij} = 1$ in the third equality, and definition of λ_{ij} from (16) in the fourth equality. By Assumption 1, $\tilde{\omega} < \sum_{i,j} \omega_{ij}\psi_{ij} \equiv \mathbb{E}(\omega)$, a new contract delivers strictly greater welfare. This is a contradiction to the fact that the original contract is a solution to the problem. Hence the incentive compatibility constraint (A.6) holds with equality. Q.E.D.

It is convenient to change the variables. Denote $U_{ij} \equiv c_{ij} - \phi\left(\frac{y^m_{ij}}{\theta_i}\right) - \phi\left(\frac{y^f_{ij}}{\varphi_j}\right)$, and rewrite the government's problem:

$$\max_{\langle \boldsymbol{U}, \boldsymbol{y}^{\boldsymbol{m}}, \boldsymbol{y}^{\boldsymbol{f}} \rangle} \lambda_{HH} U_{HH} + \lambda_{HL}\left[U_{LL} + \phi\left(\frac{y^m_{LL}}{\theta_L}\right) - \phi\left(\frac{y^m_{LL}}{\theta_H}\right)\right] + \lambda_{LH}\left[U_{LL} + \phi\left(\frac{y^f_{LL}}{\varphi_L}\right) - \phi\left(\frac{y^f_{LL}}{\varphi_H}\right)\right] + \lambda_{LL} U_{LL}$$

$$\begin{aligned}
\text{s.t.} \quad & \psi_{HH}\left[y^m_{HH} + y^f_{HH} - \phi\left(\frac{y^m_{HH}}{\theta_H}\right) - \phi\left(\frac{y^f_{HH}}{\varphi_H}\right) - U_{HH}\right] + \\
& \psi_{HL}\left[y^m_{HL} + y^f_{HL} - \phi\left(\frac{y^m_{HL}}{\theta_H}\right) - \phi\left(\frac{y^f_{HL}}{\varphi_L}\right) - U_{LL} - \phi\left(\frac{y^m_{LL}}{\theta_L}\right) + \phi\left(\frac{y^m_{LL}}{\theta_H}\right)\right] + \\
& \psi_{LH}\left[y^m_{LH} + y^f_{LH} - \phi\left(\frac{y^m_{LH}}{\theta_L}\right) - \phi\left(\frac{y^f_{LH}}{\varphi_H}\right) - U_{LL} - \phi\left(\frac{y^f_{LL}}{\varphi_L}\right) + \phi\left(\frac{y^f_{LL}}{\varphi_H}\right)\right] + \\
& \psi_{LL}\left[y^m_{LL} + y^f_{LL} - \phi\left(\frac{y^m_{LL}}{\theta_L}\right) - \phi\left(\frac{y^f_{LL}}{\varphi_L}\right) - U_{LL}\right] - G \geq 0
\end{aligned}$$

$$U_{HH} \geq U_{LL} + \phi\left(\frac{y^m_{LL}}{\theta_L}\right) - \phi\left(\frac{y^m_{LL}}{\theta_H}\right) + \phi\left(\frac{y^f_{LL}}{\varphi_L}\right) - \phi\left(\frac{y^f_{LL}}{\varphi_H}\right) \tag{A.8}$$

$$U_{HH} \geq U_{LL} + \phi\left(\frac{y^m_{LL}}{\theta_L}\right) - \phi\left(\frac{y^m_{LL}}{\theta_H}\right) + \phi\left(\frac{y^f_{HL}}{\varphi_L}\right) - \phi\left(\frac{y^f_{HL}}{\varphi_H}\right) \tag{A.9}$$

$$U_{HH} \geq U_{LL} + \phi\left(\frac{y^m_{LH}}{\theta_L}\right) - \phi\left(\frac{y^m_{LH}}{\theta_H}\right) + \phi\left(\frac{y^f_{LL}}{\varphi_L}\right) - \phi\left(\frac{y^f_{LL}}{\varphi_H}\right) \tag{A.10}$$

where I use $U_{HL} = U_{LL} + \phi\left(\frac{y^m_{LL}}{\theta_L}\right) - \phi\left(\frac{y^m_{LL}}{\theta_H}\right)$ and $U_{LH} = U_{LL} + \phi\left(\frac{y^f_{LL}}{\varphi_L}\right) - \phi\left(\frac{y^f_{LL}}{\varphi_H}\right)$ that follow from (A.6) and (A.7) holding with equalities.

Denote by ζ the Lagrange multiplier corresponding to the aggregate resource constraint. Denote by γ_1, γ_2, and γ_3 the Lagrange multipliers corresponding to the incentive compatibility constraints (A.8), (A.9), and (A.10) correspondingly. I obtain the following first-order conditions:

$$[U_{HH}] \qquad \lambda_{HH} - \psi_{HH}\zeta + \gamma_1 + \gamma_2 + \gamma_3 = 0$$

$$[U_{LL}] \qquad \lambda_{HL} + \lambda_{LH} + \lambda_{LL} - (\psi_{HL} + \psi_{LH} + \psi_{LL})\,\zeta - \gamma_1 - \gamma_2 - \gamma_3 = 0$$

$$[y^m_{HH}] \qquad \psi_{HH}\zeta\left[1 - \frac{1}{\theta_H}\phi'\left(\frac{y^m_{HH}}{\theta_H}\right)\right] = 0$$

$$[y^f_{HH}] \qquad \psi_{HH}\zeta\left[1 - \frac{1}{\varphi_H}\phi'\left(\frac{y^f_{HH}}{\varphi_H}\right)\right] = 0$$

$$[y^m_{HL}] \qquad \psi_{HL}\zeta\left[1 - \frac{1}{\theta_H}\phi'\left(\frac{y^m_{HL}}{\theta_H}\right)\right] = 0$$

$$[y^f_{HL}] \qquad \psi_{HL}\zeta\left[1 - \frac{1}{\varphi_L}\phi'\left(\frac{y^f_{HL}}{\varphi_L}\right)\right] - \gamma_2\left[\frac{1}{\varphi_L}\phi'\left(\frac{y^f_{HL}}{\varphi_L}\right) - \frac{1}{\varphi_H}\phi'\left(\frac{y^f_{HL}}{\varphi_H}\right)\right] = 0$$

$$[y^m_{LH}] \qquad \psi_{LH}\zeta\left[1 - \frac{1}{\theta_L}\phi'\left(\frac{y^m_{LH}}{\theta_L}\right)\right] - \gamma_3\left[\frac{1}{\theta_L}\phi'\left(\frac{y^m_{LH}}{\theta_L}\right) - \frac{1}{\theta_H}\phi'\left(\frac{y^m_{LH}}{\theta_H}\right)\right] = 0$$

$$[y^f_{LH}] \qquad \psi_{LH}\zeta\left[1 - \frac{1}{\varphi_H}\phi'\left(\frac{y^f_{LH}}{\varphi_H}\right)\right] = 0$$

$$[y^m_{LL}] \qquad \psi_{LL}\zeta\left[1 - \frac{1}{\theta_L}\phi'\left(\frac{y^m_{LL}}{\theta_L}\right)\right] + (\lambda_{HL} - \zeta\psi_{HL} - \gamma_1 - \gamma_2)\left[\frac{1}{\theta_L}\phi'\left(\frac{y^m_{LL}}{\theta_L}\right) - \frac{1}{\theta_H}\phi'\left(\frac{y^m_{LL}}{\theta_H}\right)\right] = 0$$

$$[y^f_{LL}] \qquad \psi_{LL}\zeta\left[1 - \frac{1}{\varphi_L}\phi'\left(\frac{y^f_{LL}}{\varphi_L}\right)\right] + (\lambda_{LH} - \zeta\psi_{LH} - \gamma_1 - \gamma_3)\left[\frac{1}{\varphi_L}\phi'\left(\frac{y^f_{LL}}{\varphi_L}\right) - \frac{1}{\varphi_H}\phi'\left(\frac{y^f_{LL}}{\varphi_H}\right)\right] = 0$$

First, from the first-order conditions for U_{HH} and U_{LL}, we obtain $\zeta = 1$. Next, from the first-order conditions for y^m_{HH}, y^f_{HH}, y^m_{HL}, and y^f_{LH}, we obtain:

$$\phi'\left(\frac{y^m_{HH}}{\theta_H}\right) = \phi'\left(\frac{y^m_{HL}}{\theta_H}\right) = \theta_H \tag{A.11}$$

$$\phi'\left(\frac{y^f_{HH}}{\varphi_H}\right) = \phi'\left(\frac{y^f_{LH}}{\varphi_H}\right) = \varphi_H \tag{A.12}$$

or, alternatively,

$$y^m_{HH} = y^m_{HL} = \theta_H \left(\phi'\right)^{-1}\left(\theta_H\right) \equiv y^m_H \tag{A.13}$$

$$y^f_{HH} = y^f_{LH} = \varphi_H \left(\phi'\right)^{-1}\left(\varphi_H\right) \equiv y^f_H \tag{A.14}$$

Next, following the procedure from Armstrong and Rochet (1999), consider two cases. First, $\gamma_1 > 0$, $\gamma_2 > 0$, and $\gamma_3 > 0$. Second, $\gamma_1 = 0$, $\gamma_2 > 0$, and $\gamma_3 > 0$.

Case $\gamma_1 > 0$, $\gamma_2 > 0$, *and* $\gamma_3 > 0$

We have incentive compatibility constraints (A.8)-(A.10) holding with equality. From these equalities, we obtain that $y^m_{LH} = y^m_{LL} \equiv \tilde{y}^m_L$ and $y^f_{HL} = y^f_{LL} \equiv \tilde{y}^f_L$. Using this result together with $\zeta = 1$, from the first-order conditions for y^f_{HL} and y^f_{LL} we obtain

$$\frac{\psi_{HL}}{\psi_{LL}} = \frac{\gamma_2}{\gamma_1 + \gamma_3 - \lambda_{LH} - \psi_{LH}}$$

Inserting the first-order condition for U_{LL}, we get

$$\frac{\psi_{HL}}{\psi_{LL}} = \frac{\gamma_2}{\lambda_{HL} + \lambda_{LL} - \psi_{HL} - \psi_{LL} - \gamma_2}$$

Finally, we solve for γ_2 verify that $\gamma_2 > 0$:

$$\gamma_2 = \frac{\psi_{HL}}{\psi_{HL} + \psi_{LL}}\left(\lambda_{HL} - \psi_{HL} + \lambda_{LL} - \psi_{LL}\right) =$$
$$\frac{\psi_{HL}}{\psi_{HL} + \psi_{LL}}\left[\left(\omega_{LL} - \tilde{\omega}\right)\psi_{LH}\psi_{LL} + \left(\omega_{LL} - \omega_{HH}\right)\psi_{HH}\psi_{LL} + \left(\tilde{\omega} - \omega_{HH}\right)\psi_{HH}\psi_{HL}\right] > 0 \tag{A.15}$$

Following the similar steps, we obtain

$$\gamma_3 = \frac{\psi_{LH}}{\psi_{LH} + \psi_{LL}} \left(\lambda_{LH} - \psi_{LH} + \lambda_{LL} - \psi_{LL}\right) =$$
$$\frac{\psi_{LH}}{\psi_{LH} + \psi_{LL}} \left[\left(\omega_{LL} - \tilde{\omega}\right) \psi_{HL}\psi_{LL} + \left(\omega_{LL} - \omega_{HH}\right) \psi_{HH}\psi_{LL} + \left(\tilde{\omega} - \omega_{HH}\right) \psi_{HH}\psi_{LH}\right] > 0 \quad \text{(A.16)}$$

Inserting (A.15) and (A.16) into the first-order condition for U_{HH}, we get

$$\gamma_1 = \psi_{HH} - \lambda_{HH} - \frac{\psi_{HL}}{\psi_{HL} + \psi_{LL}} \left(\lambda_{HL} - \psi_{HL} + \lambda_{LL} - \psi_{LL}\right) - \frac{\psi_{LH}}{\psi_{LH} + \psi_{LL}} \left(\lambda_{LH} - \psi_{LH} + \lambda_{LL} - \psi_{LL}\right)$$

After doing some algebra and using the definition of ρ from (2), we obtain

$$\gamma_1 = \frac{\pi_{LL}}{\left(\pi_{HL} + \pi_{LL}\right)\left(\pi_{LH} + \pi_{LL}\right)} \cdot$$
$$\left\{\left[\left(\omega_{LL} - \omega_{HH}\right) \psi_{LL} + \left(\tilde{\omega} - \omega_{HH}\right)\left(\psi_{HL} + \psi_{LH}\right)\right] \rho - \left(\omega_{LL} + \omega_{HH} - 2\tilde{\omega}\right) \psi_{HL}\psi_{LH}\right\} \quad \text{(A.17)}$$

It follows from (A.17) that $\gamma_1 > 0$ if

$$\rho > \frac{\left(\omega_{LL} + \omega_{HH} - 2\tilde{\omega}\right) \psi_{HL}\psi_{LH}}{\left(\omega_{LL} - \omega_{HH}\right) \psi_{LL} + \left(\tilde{\omega} - \omega_{HH}\right)\left(\psi_{HL} + \psi_{LH}\right)} \equiv \bar{\rho} > 0 \quad \text{(A.18)}$$

where the last inequality follows from Assumption 1.

Summing up, the incentive compatibility constraints (A.8)-(A.10) hold with equality if $\rho > \bar{\rho}$ where $\bar{\rho}$ is defined in (A.18).

Case $\gamma_1 = 0$, $\gamma_2 > 0$, *and* $\gamma_3 > 0$

We have incentive compatibility constraints (A.9) and (A.10) holding with equality. Incentive compatibility constraint (A.8) holds with strict inequality. FINISH