Question 1. Linear Neural Field.

1.1 We are given the following linear neural field

$$\tau \dot{u}(x,t) = -u(x,t) + \int_{-\infty}^{+\infty} w(x-x')u(x',t)dx' + s(x,t) \tag{1}$$

We assume that the input signal is constant over time and is given by

$$s(x) = \exp\left(-\frac{x^2}{4d^2}\right)/(2d\sqrt{\pi}) \tag{2}$$

and that the interaction kernel is given by the Gabor funciton

$$w(x) = a\left(\exp\left(-\frac{x^2}{4b^2}\right)\cos(k_0 x)\right)/(b\sqrt{\pi}) \tag{3}$$

Note that equation 1 is a partial integro-differential equation. To solve it, we begin by transforming it in x to the frequency domain using the Fourier Transform $\mathcal{F}$

$$\tau\frac{d\widetilde{u}(\omega,t)}{dt} = -\widetilde{u}(\omega,t) + \widetilde{w}(\omega)\widetilde{u}(\omega,t) + \widetilde{s}(\omega)$$

and after rearranging we get

$$\tau\frac{d\widetilde{u}(\omega,t)}{dt} = (-1+\widetilde{w}(\omega))\widetilde{u}(\omega,t) + \widetilde{s}(\omega)$$

This is now a linear inhomogeneous ODE in the Fourier domain. If we further assume that the system has a stable solution that does not depend on time, we get

$$\frac{d\widetilde{u}(\omega)}{dt} = 0 \implies (-1+\widetilde{w}(\omega))\widetilde{u}(\omega) + \widetilde{s}(\omega) = 0$$

Therefore, we obtain the solution in the Fourier domain

$$\widetilde{u}(\omega,\infty) = \frac{\widetilde{s}(\omega)}{1-\widetilde{w}(\omega)}$$

1.2 Note that both the interaction kernel and input terms are Gaussians, and hence their Fourier Transforms are

$$\widetilde{s}(\omega) = \mathcal{F}[s](\omega) = \frac{\exp(-1/4d^2\omega^2)}{2\sqrt{2\pi}}$$

$$\begin{aligned}
\widetilde{w}(\omega) = \mathcal{F}[w](\omega) &= a\sqrt{2}\exp\left(-\frac{1}{2b^2}\right)\int_{-\infty}^{+\infty}\exp\left(-\frac{x^2}{b\sqrt{2\pi}}\right)e^{-i\omega x}dx\int_{-\infty}^{+\infty}\cos(k_0x)e^{-i\omega x}dx \\
&= a\sqrt{2}\exp\left(-\frac{1}{2b^2}\right)\exp\left(\frac{-b^2\omega^2}{2}\right)\int_{-\infty}^{+\infty}\frac{e^{ik_0x}+e^{-ik_0x}}{2}e^{-i\omega x}dx \\
&= a\sqrt{2}\exp\left(-\frac{1}{2b^2}-\frac{b^2\omega^2}{2}\right)\left[\frac{1}{2}\delta(k_0-\omega)+\frac{1}{2}\delta(k_0+\omega)\right]
\end{aligned}$$

1.3 Now, to transform the solution back to the spatial domain we apply the inverse Fourier Transform $\mathcal{F}^{-1}$

$$u(x,\infty) = \int_{-\infty}^{+\infty} \frac{\widetilde{s}(\omega)}{1-\widetilde{w}(\omega)} e^{i\omega x} d\omega$$

1.4 Now, we simulate the neural field equation by approximating the integral as a Riemann sum

$$\int_A^B f(x)dx \approx \sum_{i=1}^{N} f(x_i)\Delta x$$

with $x_i = A + i\Delta x$ and $\Delta x = (B-A)/N$. Thus, we now have

$$\tau \dot{u}(x,t) = -u(x,t) + \sum_{i=1}^{N} w(x-x_i')u(x_i',t)\Delta x' + s(x,t)$$

We also need to make use of the Forward Euler method to approximate the derivative. Therefore, we finally obtain

$$u(x,t+\Delta t) = u(x,t) + \frac{\Delta t}{\tau}\left(-u(x,t) + \sum_{i=1}^{N} w(x-x_i')u(x_i',t)\Delta x' + s(x,t)\right)$$

The simulation parameters are as follows

- $A = 10$
- $B = -10$
- $N \geq 200$
- $\tau = 10$
- $a = 1$
- $b = 0.6$
- $d = 2$
- $k_0 = 4$

1.5 We can define a Green's function $g(x,t)$ for the neural field that describes its response to a delta input signal of the form $s(x,t) = \delta(x)\delta(t)$. If the function is know, then the response of the filed to $s(x,t)$ is characterised by

$$u(x,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x-x',t-t')s(x',t')dx'dt'$$

Note that we can apply a 2D Fourier Transform to the above equation (i.e. transform both space and time) to obtain $\mathcal{F}[g](k,\omega)$. First, we transorm the spatial domain (using the convolution theorem)

$$\widetilde{u}(k,t) = \int_{-\infty}^{+\infty} \widetilde{g}(k,t-t')\widetilde{s}(k,t')dt'$$

Now, we transform the temporal domain, again applying the convolution theorem

$$\mathcal{F}[\tilde{u}](k,\omega) = \mathcal{F}[\tilde{g}](k,\omega)\mathcal{F}[\tilde{s}](k,\omega)$$

Hence, the 2D Fourier Transform of the Green's function of our neural field is

$$\mathcal{F}[\tilde{g}](k,\omega) = \frac{\mathcal{F}[\tilde{u}](k,\omega)}{\mathcal{F}[\tilde{s}](k,\omega)}$$

1.6 We now assume a different interaction kernel $w(x)$ given by

$$w(x) = e^{-c|x|}\mathrm{sign}(x)$$

where $c > 0$ and a time-dependent stimulus of the form

$$s(x,t) = c\exp\left(-\frac{(x-vt)^2}{4d_1^2}\right)/(2d_1\sqrt{\pi})$$

where v is the stimulus peak travelling speed.