Question 1. Linear Dynamical System.

We have the following linear dynamical system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{s}(t)$$

where

$$A = \begin{pmatrix} -0.5 & -0.5 & 0 \\ -0.5 & -0.5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

1.1 Assuming $\mathbf{s}(t) = 0$, we need to compute and sketch the solutions for the initial conditions

$$\mathbf{x}_{0,1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{0,2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{0,3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_{0,4} = \begin{pmatrix} 0 \\ 0 \\ 10^{-6} \end{pmatrix}$$

Note that the initial value problem has a solution of the form

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

For non-singular $\mathbf{A}$, we can rewrite it as follows

$$\mathbf{x}(t) = \mathbf{Q}e^{\mathbf{\Lambda}t}\mathbf{Q}^{-1}\mathbf{x}_0$$

where $\mathbf{Q}$ is a matrix of eigenvectors and $\mathbf{\Lambda}$ is a daigonal matrix with non-zero entries representing corresponding eigenvalues. To solve this, we have to first find the eigenvalues and eigenvectors of $\mathbf{A}$. Hence, we start by solving the characteristic equation

$$\det(A - I\lambda) = 0$$

$$\begin{vmatrix} -0.5-\lambda & -0.5 & 0 \\ -0.5 & -0.5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \quad \Longleftrightarrow \quad -\lambda^3 + \lambda^2 + 2\lambda = 0$$

We find that $\mathbf{A}$ has three distinct eigenvalues, namely, $\lambda_1 = 0$, $\lambda_2 = 2$, $\lambda_3 = -1$, with the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Now, we have

$$\mathbf{Q} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{\Lambda} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and we can thus write our solution as follows

$$\mathbf{x}(t) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{-1} \mathbf{x}_0$$

$$\implies \mathbf{x}(t) = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -0.5 & 0.5 & 0 \\ 0 & 0 & e^{2t} \\ 0.5e^{-t} & 0 & 0 \end{pmatrix} \mathbf{x}_0$$

$$\implies \mathbf{x}(t) = \begin{pmatrix} 0.5 + 0.5e^{-t} & -0.5 & 0 \\ -0.5 + 0.5e^{-t} & 0.5 & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \mathbf{x}_0$$

And in a non-matrix form this becomes

$$\mathbf{x}(t) = \begin{pmatrix} (0.5 + 0.5e^{-t})x_1 - 0.5x_2 \\ (-0.5 + 0.5e^{-t})x_1 + 0.5x_2 \\ e^{2t}x_3 \end{pmatrix}$$

Finally, for the aforementioned initial conditions, we have

$$\mathbf{x}_{0,1}(t) = \begin{pmatrix} 0.5 + 0.5e^{-t} & -0.5 & 0 \\ -0.5 + 0.5e^{-t} & 0.5 & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5e^{-t} \\ 0.5e^{-t} \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{0,2}(t) = \begin{pmatrix} 0.5 + 0.5e^{-t} & -0.5 & 0 \\ -0.5 + 0.5e^{-t} & 0.5 & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 + 0.5e^{-t} \\ -0.5 + 0.5e^{-t} \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{0,3}(t) = \begin{pmatrix} 0.5 + 0.5e^{-t} & -0.5 & 0 \\ -0.5 + 0.5e^{-t} & 0.5 & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 0.5 \\ 0 \end{pmatrix}$$

$$\mathbf{x}_{0,4}(t) = \begin{pmatrix} 0.5 + 0.5e^{-t} & -0.5 & 0 \\ -0.5 + 0.5e^{-t} & 0.5 & 0 \\ 0 & 0 & e^{2t} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 10^{-6} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10^{-6}e^{t} \end{pmatrix}$$

And the phase diagrams for these initial conditions appear in Figures 1-4.

1.2 The eigenvalues and eigenvectors of $\mathbf{A}$ were computed in 1.1. Eigenvalue sign tells us about the direction and speed of the dynamics evolution, and the eigenvectors tell us in which direction the flow will be linear, i.e. only affected by the dynamics matrix as a scalar multiplication.

1.3 The projections of $\mathbf{f}(x) = \mathbf{A}\mathbf{x}$ appear in Figures 5-7. The dynamics observed in Figure 5 can be explained using the fact that the eigenvalue associated with eigenvector $\mathbf{v}_1$ is zero. Therefore, we see an invariant manifold along the direction of this eigenvector, with particles attracted towards it. As for Figures 6 and 7, there is a saddle point at $(0, 0)$, for there is one stable and one unstable direction.

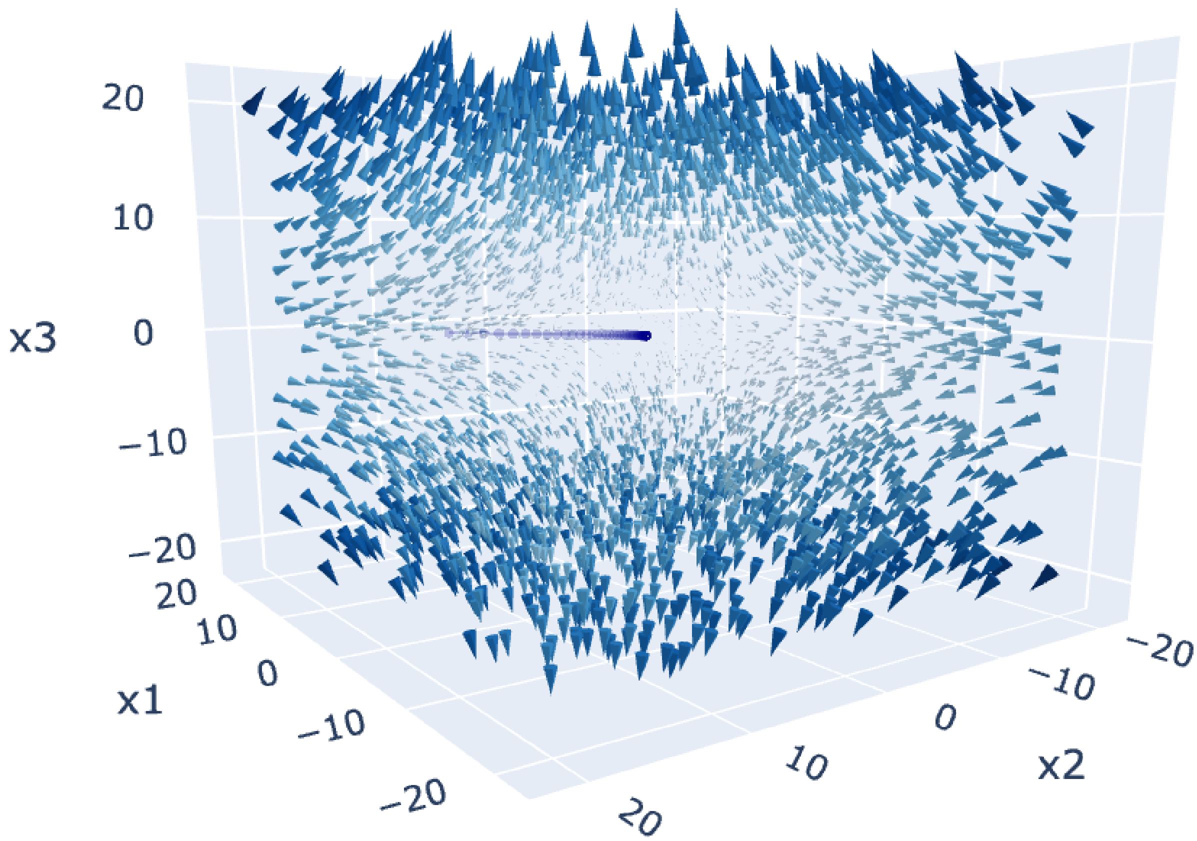

Figure 1: 3D dynamics of $\mathbf{x}(t)$ following the initial condition $\mathbf{x}_{0,1}$ for $t \in [-3, 3]$.

1.4 Now we want to project the vector field of the dynamics onto the subspace defined by basis

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Recall the formula for projecting vector $\mathbf{x}$ onto a line spanned by vector $\mathbf{u}$

$$\text{proj}_{\mathbf{u}}(\mathbf{x}) = \frac{\mathbf{u} \cdot \mathbf{x}}{\|\mathbf{u}\|}\mathbf{u}$$

Therefore, to project vector $\mathbf{x}$ onto a subspace V spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, we need to do the following

$$\text{proj}_V(\mathbf{x}) = \frac{\mathbf{e}_1 \cdot \mathbf{x}}{\|\mathbf{e}_1\|}\mathbf{e}_1 + \frac{\mathbf{e}_2 \cdot \mathbf{x}}{\|\mathbf{e}_2\|}\mathbf{e}_2$$

And since $\mathbf{e}_1$ and $\mathbf{e}_2$ form the basis, we know they are unit vectors, and hence we are left with

$$\text{proj}_V(\mathbf{x}) = (\mathbf{e}_1 \cdot \mathbf{x})\,\mathbf{e}_1 + (\mathbf{e}_2 \cdot \mathbf{x})\,\mathbf{e}_2$$

1.5 Now we want to compute the stationary solution for $(t \to \infty)$ with a constant input

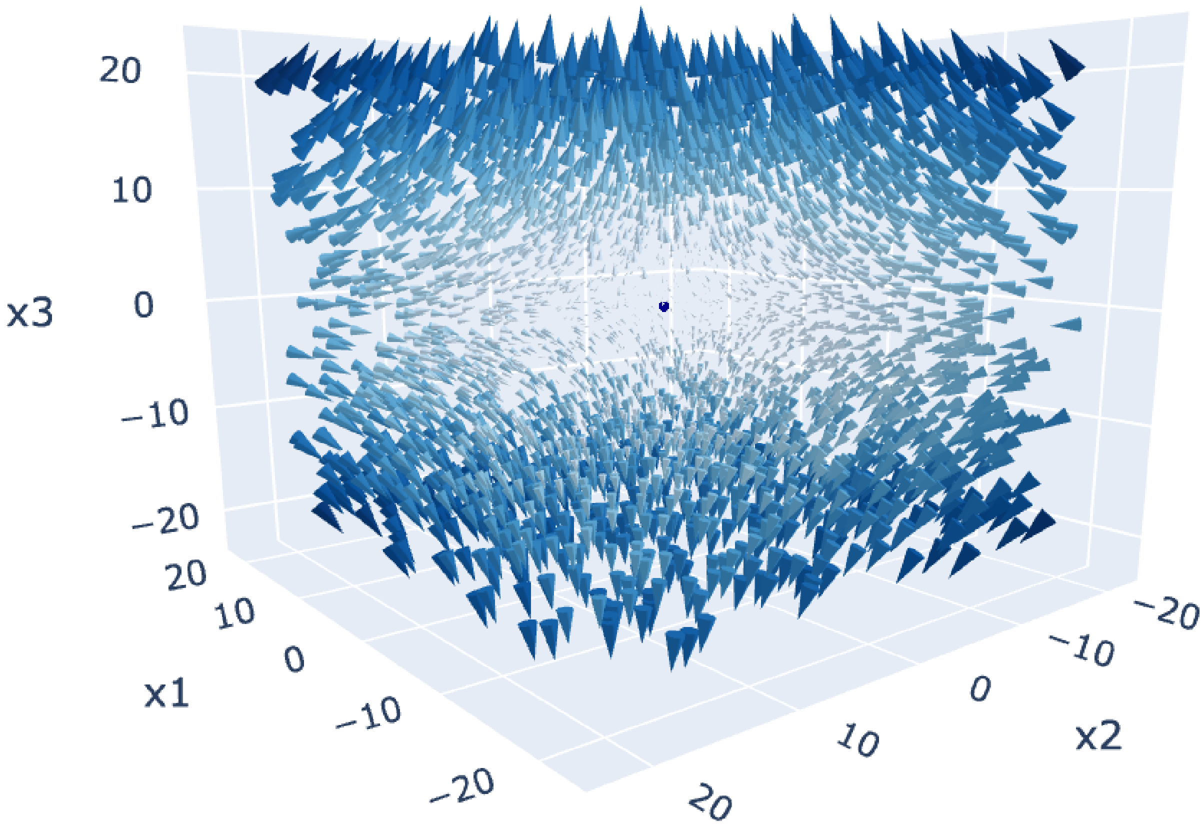

Figure 2: 3D dynamics of $\mathbf{x}(t)$ following the initial condition $\mathbf{x}_{0,2}$.

$\mathbf{s} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and the initial condition $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. To find the solution, we will use the undetermined coefficients methods to solve this system of inhomogeneous DEQs. Recall that our general solution to the homogeneous system was

$$\mathbf{x}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}$$

We need to find a particular solution $\mathbf{x}_p(t)$, so let

$$\mathbf{x}_p(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \text{(since our input is constant)}$$

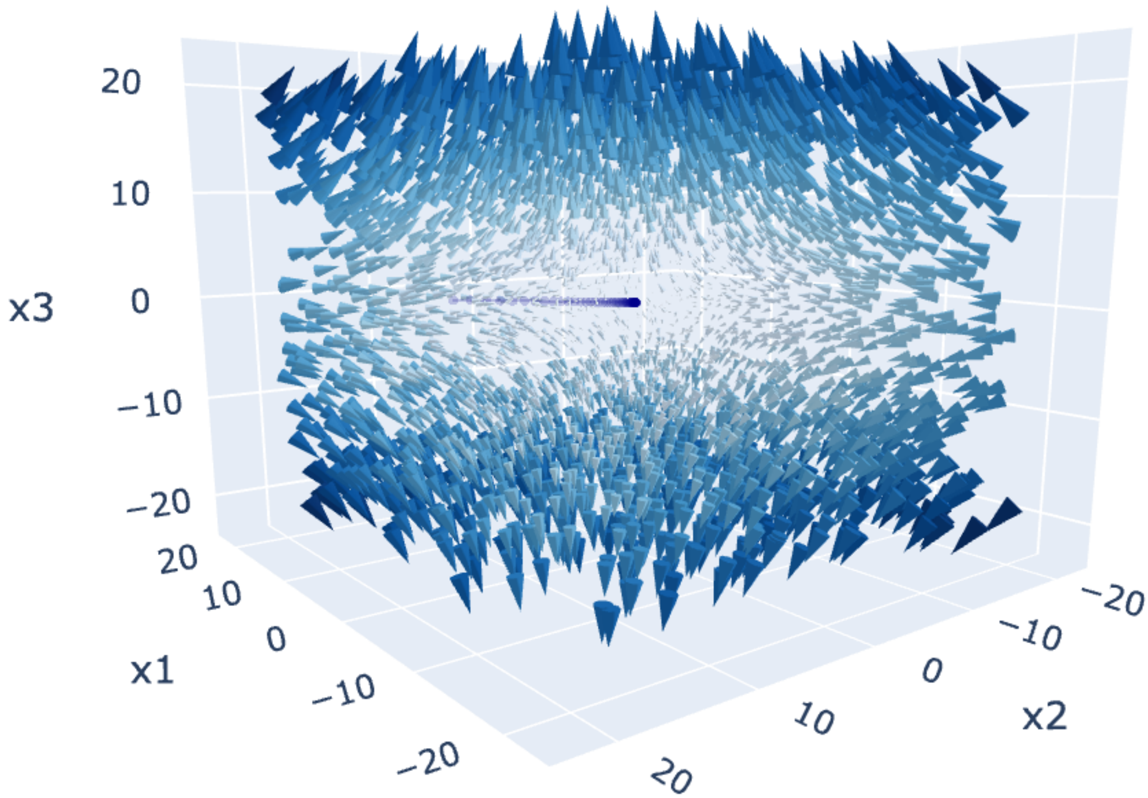

Figure 3: 3D dynamics of $\mathbf{x}(t)$ following the initial condition $\mathbf{x}_{0,3}$ for $t \in [-3, 3]$.

And now we plug this into our orginal system to find x_1, x_2, and x_3

$$\begin{aligned}
\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} &= \mathbf{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \mathbf{s} \\
\implies \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -0.5 & -0.5 & 0 \\ -0.5 & -0.5 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \\
\implies \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} -0.5x_1 - 0.5x_2 + 1 \\ -0.5x_1 - 0.5x_2 + 2 \\ 2x_3 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
& 2 = x_1 + x_2 \\
\implies & 4 = x_1 + x_2 \\
& 0 = x_3
\end{aligned}$$

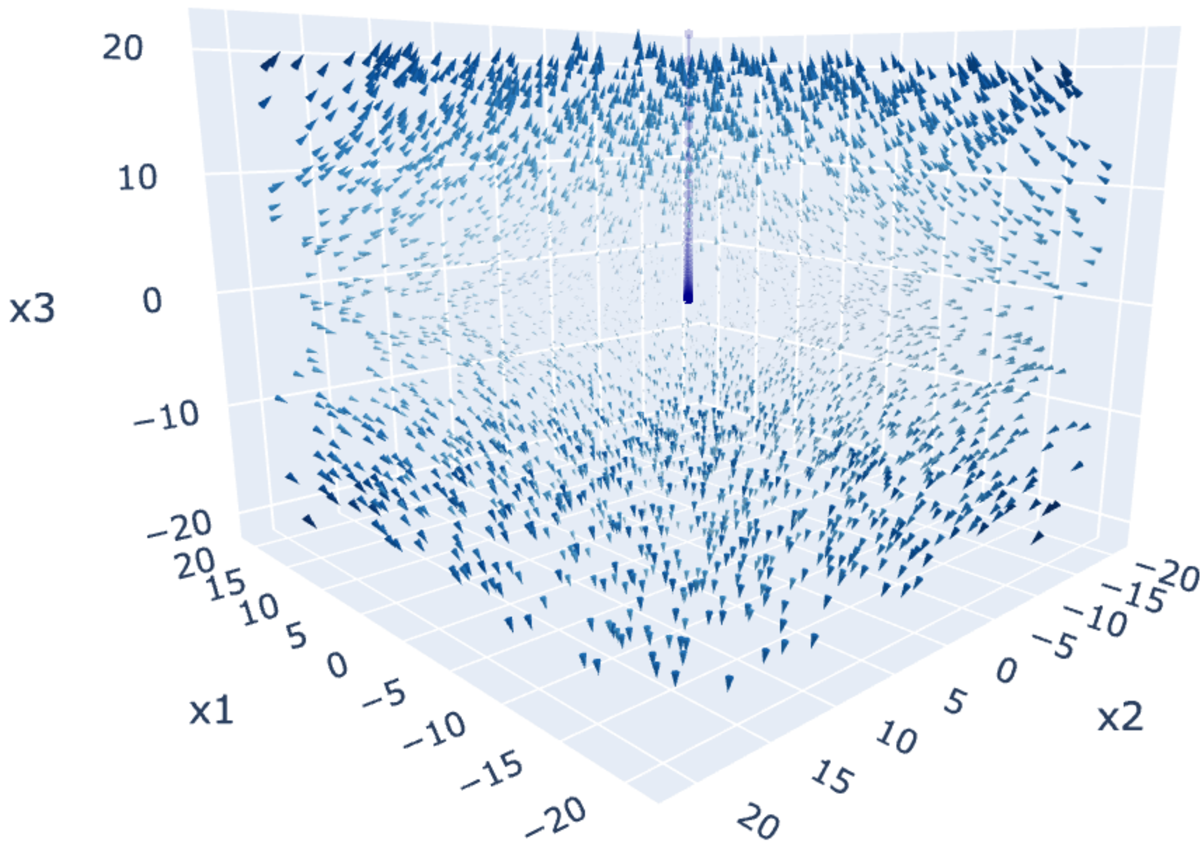

Figure 4: 3D dynamics of $\mathbf{x}(t)$ following the initial condition $\mathbf{x}_{0,4}$ for $t \in [-17, 17]$.

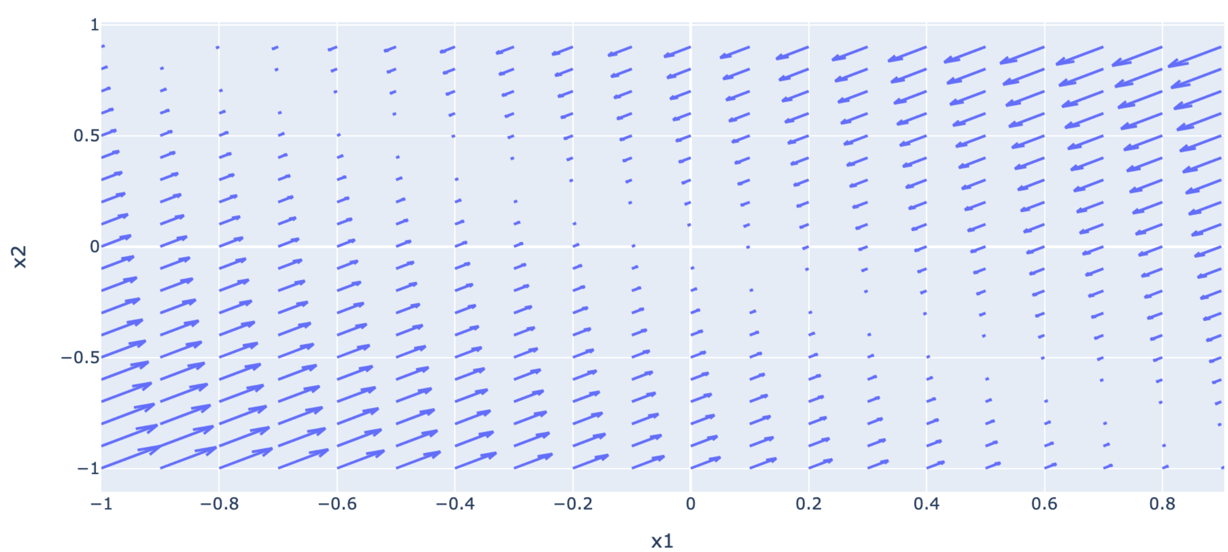

Figure 5: Projection of $\mathbf{f}(x) = \mathbf{A}\mathbf{x}$ onto a plane defined by $x_3 = 0$.

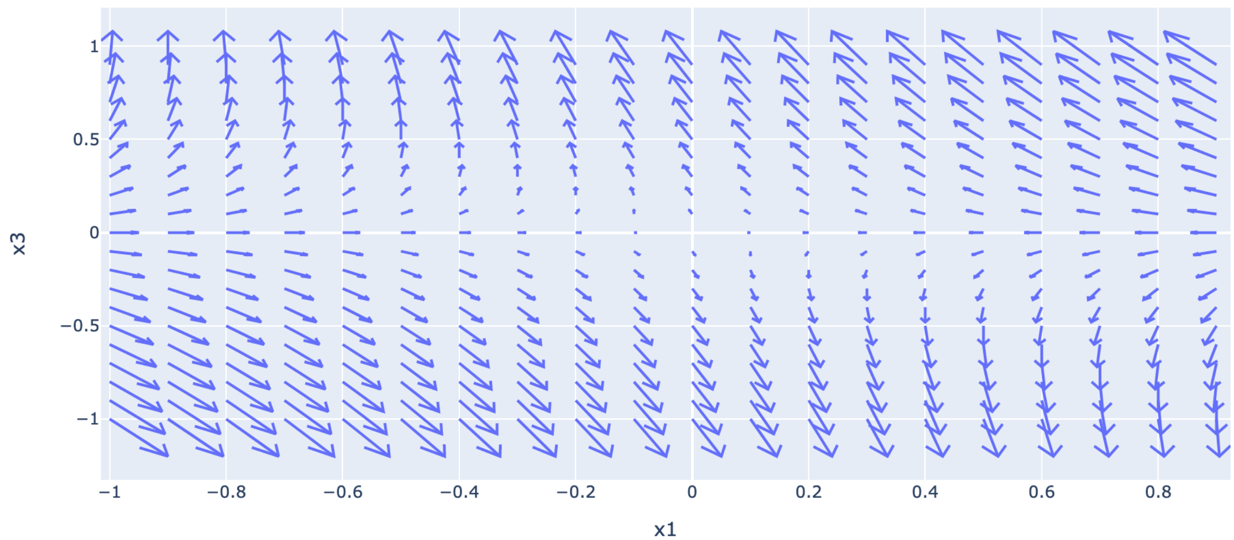

Figure 6: Projection of $\mathbf{f}(x) = \mathbf{Ax}$ onto a plane defined by $x_2 = 0$.

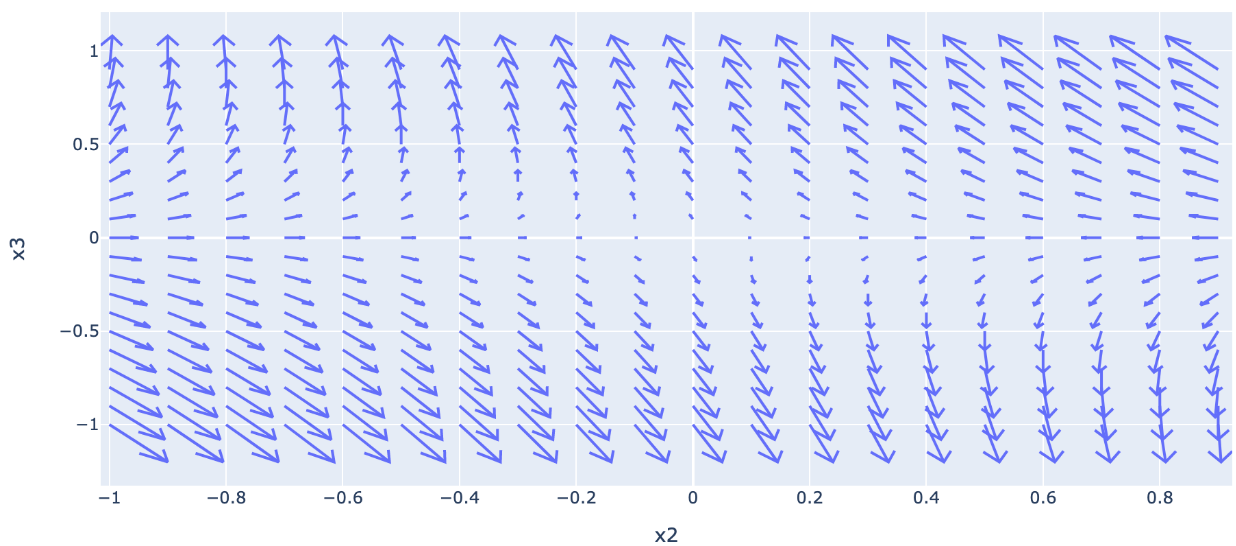

Figure 7: Projection of $\mathbf{f}(x) = \mathbf{Ax}$ onto a plane defined by $x_1 = 0$.