Question 1. Nonlinear network with two divisive inhibitory neurons.

1.1 We have a network of two nonlinear neurons with divisive feedback given by the following system of differential equations

$$\tau \dot{u}_1(t) = -u_1(t) + \frac{s_1}{1 + u_2(t)}$$
$$\tau \dot{u}_2(t) = -u_2(t) + \frac{s_2}{1 + u_1(t)}$$

where s_1 and s_2 are two nonnegative inputs, respectively. We are required to prove that when the initial condition $\mathbf{u(0)}$ lies in the first quadrant of the state space with u_1, $u_2 \geq 0$, the system stays in the first quadrant forever.

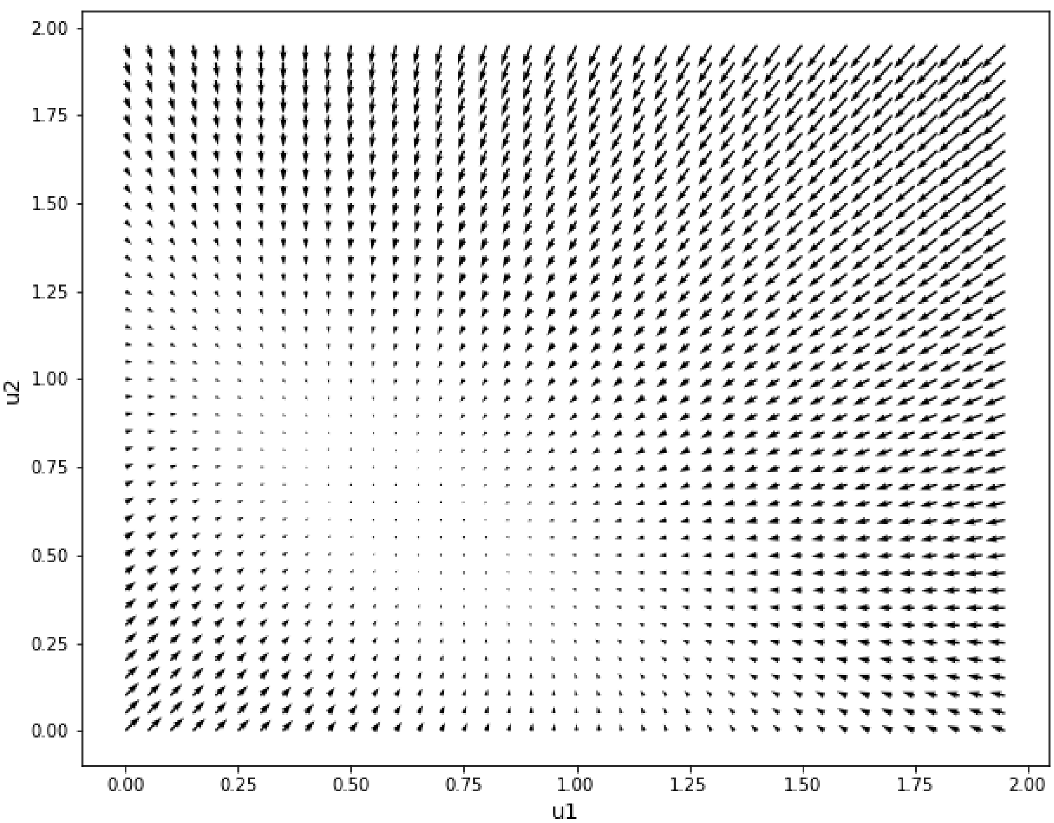

Figure 1: Vector field dynamics for the given nonlinear dynamical system for $u_1, u_2 \geq 0$.

The dynamics of the vector field for $s_1 = 1$, $s_2 = 1$, and $\tau = 1\,\text{ms}$ appears in Figure 1. Note that in order for the system to stay in the first quadrant, whenever u_1 or u_2 is at the boundary, its derivative has to be nonnegative. Thus, we have

$$u_1 = 0 \implies \frac{du_1}{dt} \geq 0 \Leftrightarrow u_2 > -1$$

$$u_2 = 0 \implies \frac{du_2}{dt} \geq 0 \Leftrightarrow u_1 > -1$$

1.2 To compute the fixed points for an input of the form $s_1 = s_2 \geq 0$, we can plot the isoclines

$$\frac{du_1}{dt} = 0 \implies u_1 = \frac{s_1}{1+u_2}$$
$$\frac{du_2}{dt} = 0 \implies u_2 = \frac{s_2}{1+u_1}$$

A stationary point is then given by the intersection of these two isoclines

$$u_1 = \frac{s_1(1+u_1)}{1+u_1+s_2} \Leftrightarrow u_1^2 + u_1(s_2 - s_1 + 1) - s_1 = 0$$

Since we are given that $s_1 = s_2$, this simplifies to

$$u_1^2 + u_1 - s_1 = 0$$

And we therefore find that

$$u_1 = \frac{-1+\sqrt{1+4s_1}}{2}$$
$$u_2 = \frac{2s_2}{1+\sqrt{1+4s_1}}$$

The two isoclines are plotted in Figure 2.
For $s_1 = s_2 = 0$ the fixed point appears at $u_1 = u_2 = 0$ and for $s_1 = s_2 = 3/4$ it is at $u_1 = u_2 = 0.5$.

1.3 Recall that for the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$$

points with $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ are called fixed points, for the dynamics at these points does not change. Also recall that the linear dynamical system

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t)$$

where

$$\mathbf{A} = \frac{\partial \mathbf{f}(\mathbf{u}_0)}{\partial \mathbf{u}}$$

is called the linearised dynamics at the point $\mathbf{u}_0$. For the system given, the linearised dynamics at $\mathbf{u}_0$ is of the form

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\bigg|_{\mathbf{u}_0} = \frac{1}{\tau}\begin{pmatrix} -1 & -\frac{s_1}{(1+u_{0,2})^2} \\ -\frac{s_2}{(1+u_{0,1})^2} & -1 \end{pmatrix}$$

After eigendecomposition, the eigenvectors of the linearised dynamics matrix are

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

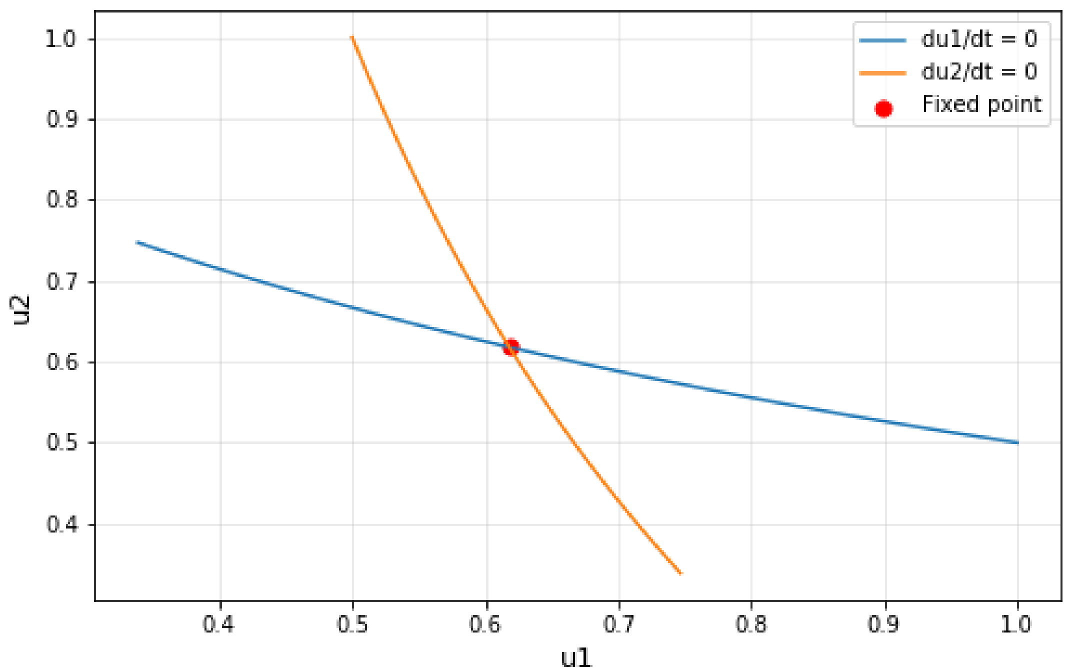

Figure 2: Computed fixed point given by the intersection of the two isoclines for $s_1 = s_2 = 1$.

and the corresponding eigenvalues are

$$\lambda_1 = \frac{-s\tau - u_0^2 - 2u_0 - 1}{\tau^2(u_0+1)^2}$$
$$\lambda_2 = \frac{s\tau - u_0^2 - 2u_0 - 1}{\tau^2(u_0+1)^2}$$

where $s = s_1 = s_2$ and $u_0 = u_{0,1} = u_{0,2}$. To analyse the stability of the system we need to fix τ and consider inputs of varying strength. For example, consider $\tau = 1\,\text{ms}$ and $s = 1$. Then the fixed point coordinates are $u_0 = 0.618$ and the linearised dynamics matrix is

$$\mathbf{A} = \begin{pmatrix} -1 & -0.381966 \\ -0.381966 & -1 \end{pmatrix}$$

The eigenvlues for this dynamics are

$$\lambda_1 = \frac{-2 - \sqrt{0.58359}}{2} = -1.38196,\ \lambda_2 = \frac{-2 + \sqrt{0.58359}}{2} = -0.6180$$

Both of the eigenvalues are negative, and hence the fixed point is stable.

1.4 Phase portraits for two different inputs appear in Figures 3 & 4.

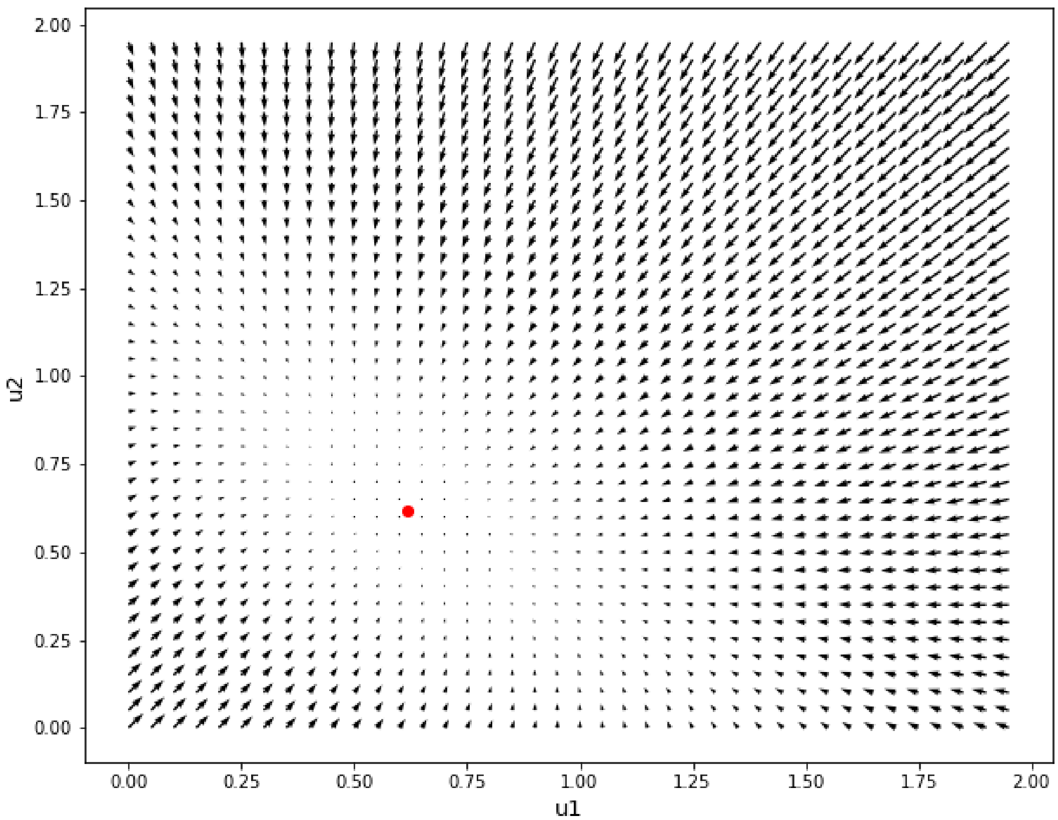

Figure 3: Phase portrait for $s = 1$. The fixed point is shown in red.

1.5 We are required to derive a Lyapunov function for the given system. Let us denote

$$F(u_1, u_2) = \frac{1}{\tau}\left(-u_1 + \frac{s_1}{1+u_2}\right)$$

$$G(u_1, u_2) = \frac{1}{\tau}\left(-u_2 + \frac{s_2}{1+u_1}\right)$$

Then, since both F and G are C^1 and have a finite number of joint zeroes, then by the Lyapunov theorem the following function with $|\epsilon| < 1$

$$E(\mathbf{u}) = \frac{1}{2}F^2(\mathbf{u}) + \epsilon F(\mathbf{u})G(\mathbf{u}) + \frac{1}{2}G^2(\mathbf{u})$$

is positive definite in regions around each zero. The resulting Lyapunov function for

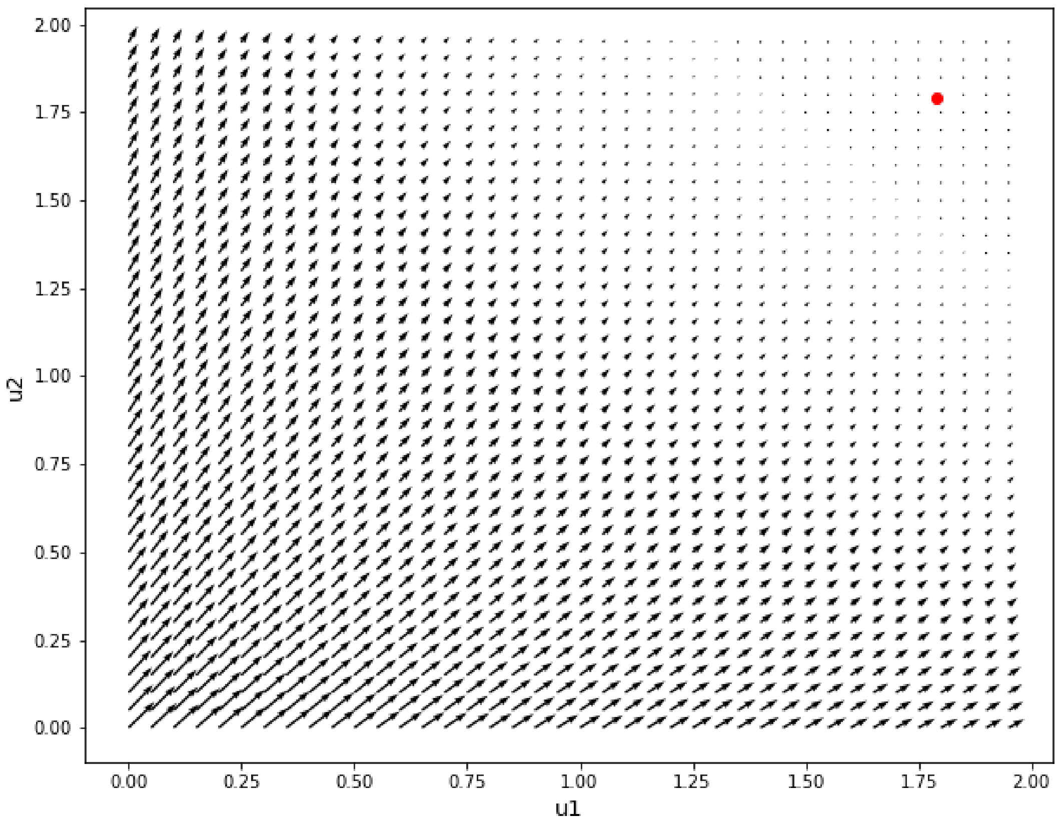

Figure 4: Phase portrait for $s = 5$. The fixed point is shown in red.

$|\epsilon| = 0$ is

$$\begin{aligned} E(u_1, u_2) &= \frac{1}{2}F^2(u_1, u_2) + \frac{1}{2}G^2(u_1, u_2) \\ &= \frac{1}{2\tau}\left(\left(-u_1 + \frac{s_1}{1+u_2}\right)^2 + \left(-u_2 + \frac{s_2}{1+u_1}\right)^2\right) \end{aligned}$$

The derivative of Lyapunov function is of the form

$$\dot{E}(u_1, u_2) = F^2\frac{\partial F}{\partial u_1} + FG\left(\frac{\partial F}{\partial u_2} + \frac{\partial G}{\partial u_1}\right) + G^2\frac{\partial G}{\partial u_2}$$

Hence, if we let $a = c = \frac{\partial F}{\partial u_1} = \frac{\partial G}{\partial u_2} = -\frac{1}{\tau}$ then, if the derivative $\dot{E}(u_0, u_1)$ is negative definite in the regions surrounding the fixed points, then the function $E(\mathbf{u})$ is truly Lyapunov. Thus, we check the corollary condition

$$|b| = \left|\frac{\partial F}{\partial u_2} + \frac{\partial G}{\partial u_1}\right| = \frac{1}{\tau}\left| - \frac{s}{(1+u_2)^2} - \frac{s}{(1+u_1)^2}\right| < \frac{2}{\tau}$$

which simplifies to

$$\frac{s}{(1+u_2)^2} + \frac{s}{(1+u_1)^2} < 2$$

The condition is fulfilled when

$$\begin{aligned}(1+u_0)^2 &> s \\ \implies 1 + 2u_0 + u_0^2 &> s \\ \implies u_0^2 + 2u_0 + (1-s) &> 0\end{aligned}$$

where $u_0 = u_1 = u_2$. Therefore, we have

$$u_0 = \frac{-2 \pm \sqrt{3-s}}{2}$$

For $s = 3/4$, we find

$$u_0 = -1.75;\ -0.25$$

Thus, the function is Lyapunov for $s = 3/4$ when $u_0 \in (-0.25, +\infty)$.

Also observe that the left hand side of the condition is at maximum when $u_1 = u_2 = 0$. Hence, for the condition to be fulfilled, we require $s < 1$.

Question 2. Simple autoassociative memory.

2.1 Assume an autoassociative memory network given by

$$\tau\dot{\mathbf{u}}(t) = -\mathbf{u}(t) + [\mathbf{M}\mathbf{u}(t)]_+$$

where matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{pmatrix} 1 & -0.1 & -0.1 \\ -0.1 & 1 & -0.1 \\ -0.1 & -0.1 & 1 \end{pmatrix}$$

We are required to find an equation which determines the stationary points of the network. Similarly to Question 1, the stationary points are described by the intersections of isoclines. Therefore, we solve

$$\frac{du_1}{dt} = 0 \implies u_1 = [u_1 - 0.1u_2 - 0.1u_3]_+$$
$$\frac{du_2}{dt} = 0 \implies u_2 = [-0.1u_1 + u_2 - 0.1u_3]_+$$
$$\frac{du_3}{dt} = 0 \implies u_3 = [-0.1u_1 - 0.1u_2 + u_3]_+$$