Question 1. Linear Neural Field.

1.1 We are given the following linear neural field

$$\tau \dot{u}(x,t) = -u(x,t) + \int_{-\infty}^{\infty} w(x-x')u(x',t)dx' + s(x,t) \tag{1}$$

We assume that the input signal is constant over time and is given by

$$s(x) = \exp\left(-\frac{x^2}{4d^2}\right)/(2d\sqrt{\pi}) \tag{2}$$

and that the interaction kernel is given by the Gabor funciton

$$w(x) = a\left(\exp\left(-\frac{x^2}{4b^2}\right)\cos(k_0 x)\right)/(b\sqrt{\pi}) \tag{3}$$

Note that equation 1 is a partial integro-differential equation. To solve it, we begin by transforming it in x to the frequency domain using the Fourier Transform $\mathcal{F}$

$$\tau \frac{d\widetilde{u}(\omega,t)}{dt} = -\widetilde{u}(\omega,t) + \widetilde{w}(\omega)\widetilde{u}(\omega,t) + \widetilde{s}(\omega)$$

and after rearranging we get

$$\tau \frac{d\widetilde{u}(\omega,t)}{dt} = (-1 + \widetilde{w}(\omega))\widetilde{u}(\omega,t) + \widetilde{s}(\omega)$$

This is now a linear inhomogeneous ODE in the Fourier domain. If we further assume that the system has a stable solution that does not depend on time, we get

$$\frac{d\widetilde{u}(\omega)}{dt} = 0 \implies (-1 + \widetilde{w}(\omega))\widetilde{u}(\omega) + \widetilde{s}(\omega) = 0$$

Therefore, we obtain the solution in the Fourier domain

$$\widetilde{u}(\omega,\infty) = \frac{\widetilde{s}(\omega)}{1 - \widetilde{w}(\omega)}$$

1.2 Note that both the interaction kernel and input terms are Gaussians, and hence their Fourier Transforms are

$$\widetilde{s}(\omega) = \mathcal{F}[s](\omega) = \frac{\exp(-1/4d^2\omega^2)}{2\sqrt{2\pi}}$$

$$\widetilde{w}(\omega) = \mathcal{F}[w](\omega) = \frac{a}{b\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{4b^2}\right)\cos(k_0 x)e^{-i\omega x}dx$$

$$= \frac{a}{b\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{4b^2}\right)\frac{e^{ik_0 x} + e^{-ik_0 x}}{2}e^{-i\omega x}dx$$

1.3 Now, to transform the solution back to the spatial domain we apply the inverse Fourier Transform $\mathcal{F}^{-1}$

$$u(x, \infty) = \mathcal{F}^{-1}[u](\omega, \infty) = \int_{-\infty}^{\infty} \frac{\widetilde{s}(\omega)}{1 - \widetilde{w}(\omega)} e^{i\omega x} d\omega$$

1.4 Now, we simulate the neural field equation by approximating the integral as a Riemann sum

$$\int_{A}^{B} f(x) dx \approx \sum_{i=0}^{N} f(x_i) \Delta x$$

with $x_i = A + i\Delta x$ and $\Delta x = (B - A)/N$. Thus, we now have

$$\tau \dot{u}(x, t) = -u(x, t) + \sum_{i=0}^{N} \left[w(x - x_i) u(x_i, t) \Delta x \right] + s(x, t)$$

We also need to make use of the Forward Euler method to approximate the derivative. Therefore, we finally obtain

$$u(x, t + \Delta t) = u(x, t) + \frac{\Delta t}{\tau} \left(-u(x, t) + \sum_{i=0}^{N} \left[w(x - x_i) u(x_i, t) \Delta x \right] + s(x, t) \right)$$

The simulation parameters are as follows

- $A = 10$
- $B = -10$
- $N \geq 200$
- $\tau = 10$
- $a = 1$
- $b = 0.6$
- $d = 2$
- $k_0 = 4$

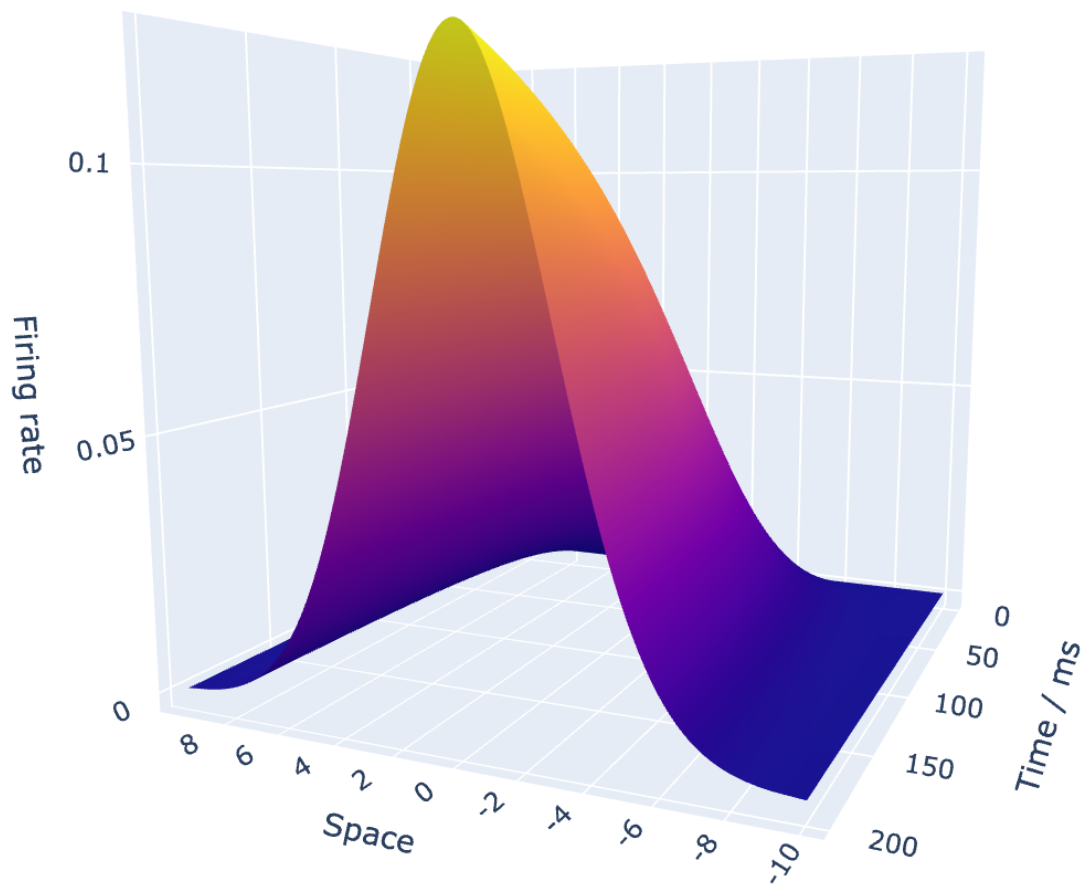The result of the simulation appears in Figure 1.

Figure 1: Dynamics of the linear neural field defined by equation 1.

We can also add a small amount of Gaussian noise to the input signal. This result is shown in Figure 2 for $\sigma^2(s(x)) = 0.01^2$.
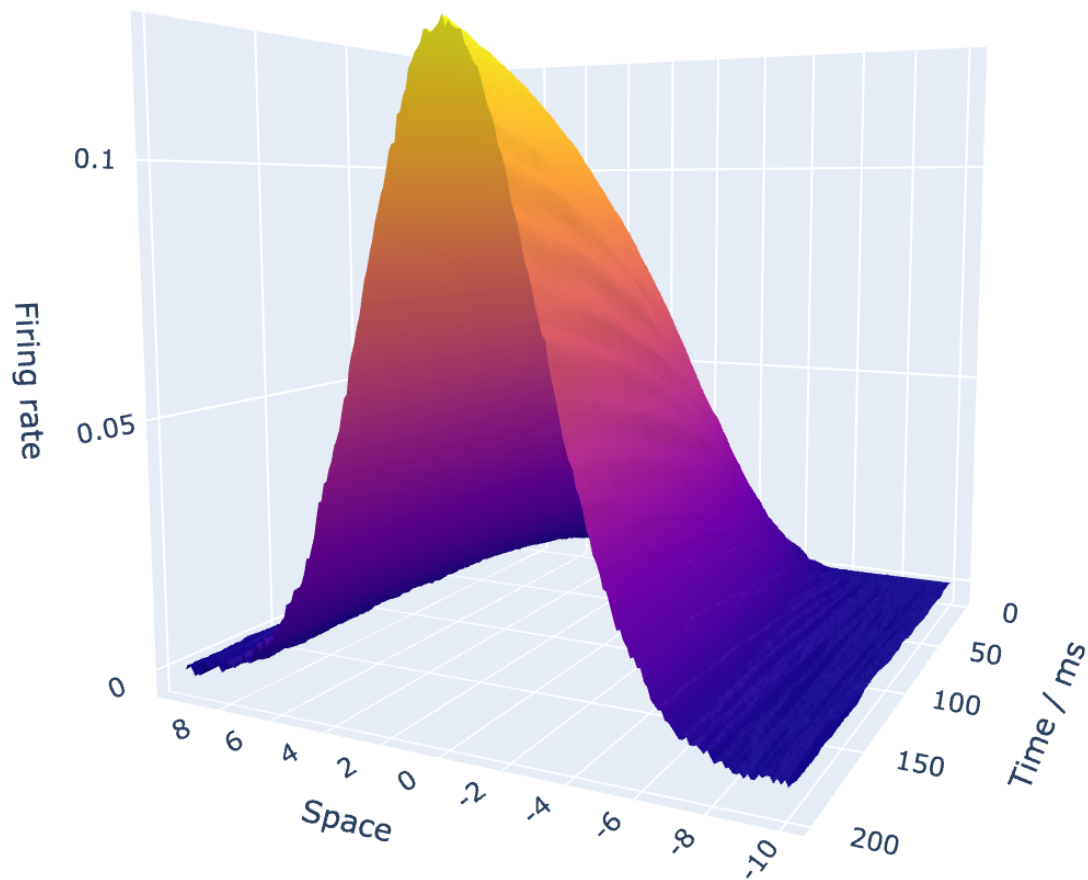
Figure 2: Same as in Fig. 1 but with added Gaussian noise.

Now, let us change the interaction kernel parameter a keeping the same noise level as before. The results for $a = 0.7$ and $a = 1.5$ appear in Figures 3 & 4, repsectively.
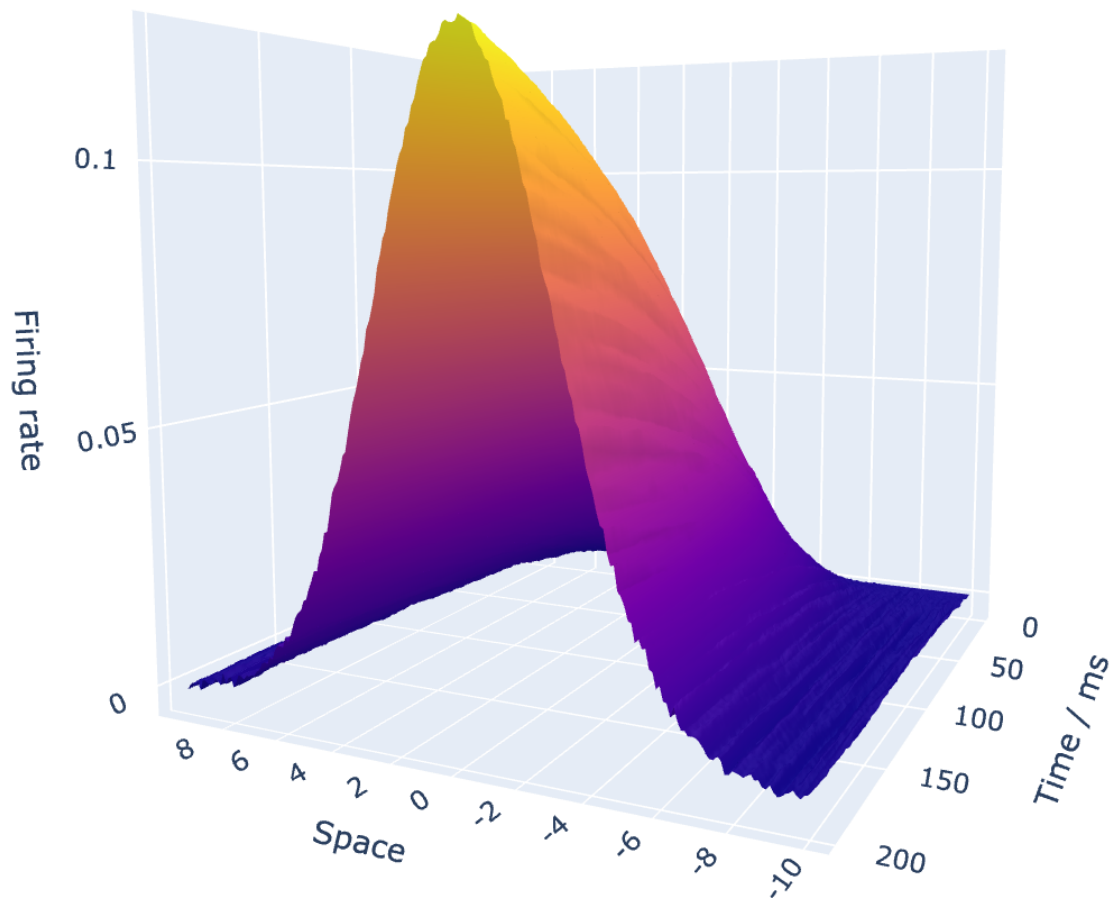
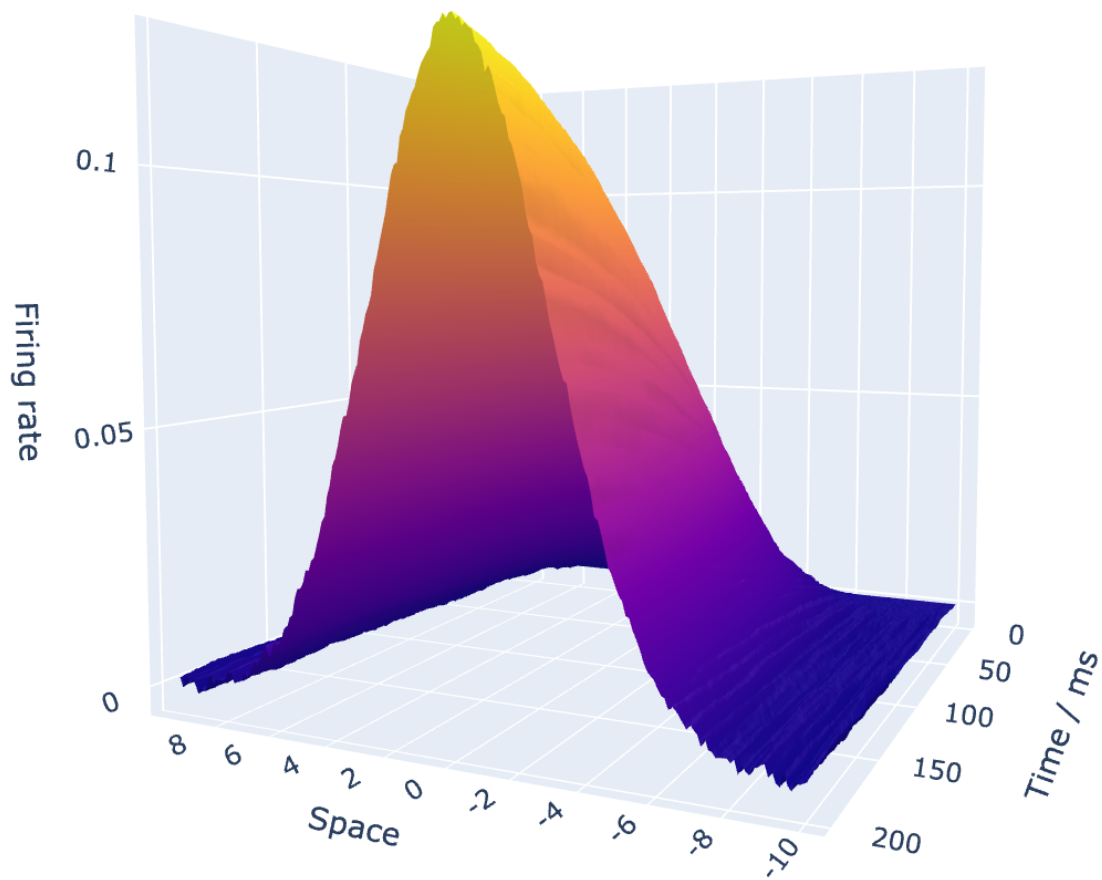Figure 3: Same as in Fig. 1 but with added Gaussian noise and $a = 0.7$.

Figure 4: Same as in Fig. 1 but with added Gaussian noise and $a = 1.5$.

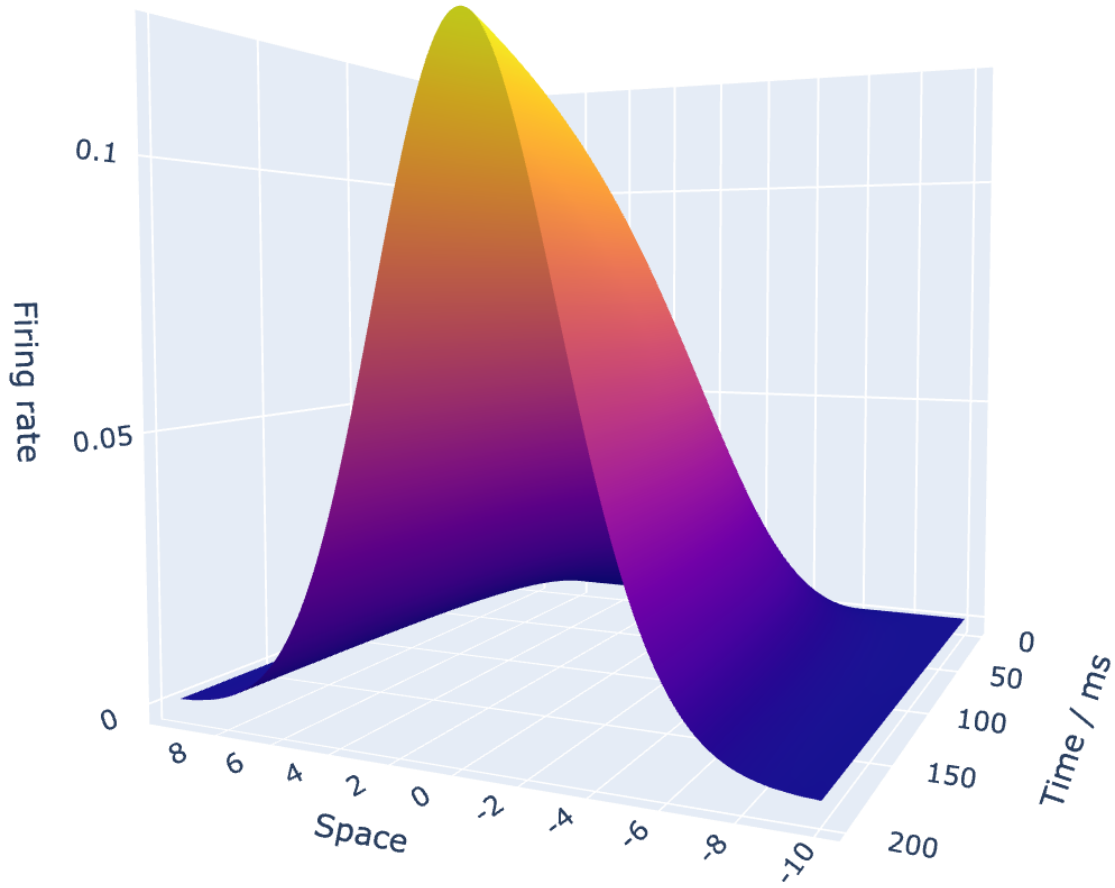Finally, let us discard the noise and keep $a = 1$ but now we set $k = 8$. This result is shown in figure 5.

Figure 5: Same as in Fig. 1 but with $k = 8$.

1.5 We can define a Green's function $g(x, t)$ for the neural field that describes its response to a delta input signal of the form $s(x, t) = \delta(x)\delta(t)$. If the function is know, then the response of the filed to $s(x, t)$ is characterised by

$$u(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x - x', t - t')s(x', t')dx'dt'$$

Note that we can apply a 2D Fourier Transform to the above equation (i.e. transform both space and time) to obtain $\mathcal{F}[g](k, \omega)$. First, we transorm the spatial domain (using the convolution theorem)

$$\widetilde{u}(k, t) = \int_{-\infty}^{\infty} \widetilde{g}(k, t - t')\widetilde{s}(k, t')dt'$$

Now, we transform the temporal domain, again applying the convolution theorem

$$\mathcal{F}[\tilde{u}](k, \omega) = \mathcal{F}[\tilde{g}](k, \omega)\mathcal{F}[\tilde{s}](k, \omega)$$

Hence, the 2D Fourier Transform of the Green's function of our neural field is

$$\mathcal{F}[\tilde{g}](k, \omega) = \frac{\mathcal{F}[\tilde{u}](k, \omega)}{\mathcal{F}[\tilde{s}](k, \omega)}$$

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1.6 We now assume a different interaction kernel $w(x)$ given by

$$w(x) = e^{-c|x|}\text{sign}(x)$$

where $c > 0$ and a time-dependent stimulus of the form

$$s(x,t) = c \exp\left(-\frac{(x - vt)^2}{4d_1^2}\right) / (2d_1\sqrt{\pi})$$

where v is the stimulus peak travelling speed. We run the simulation with the following parameters

- $c = 1$
- $d_1 = 0.5$
- $v = 0.1$

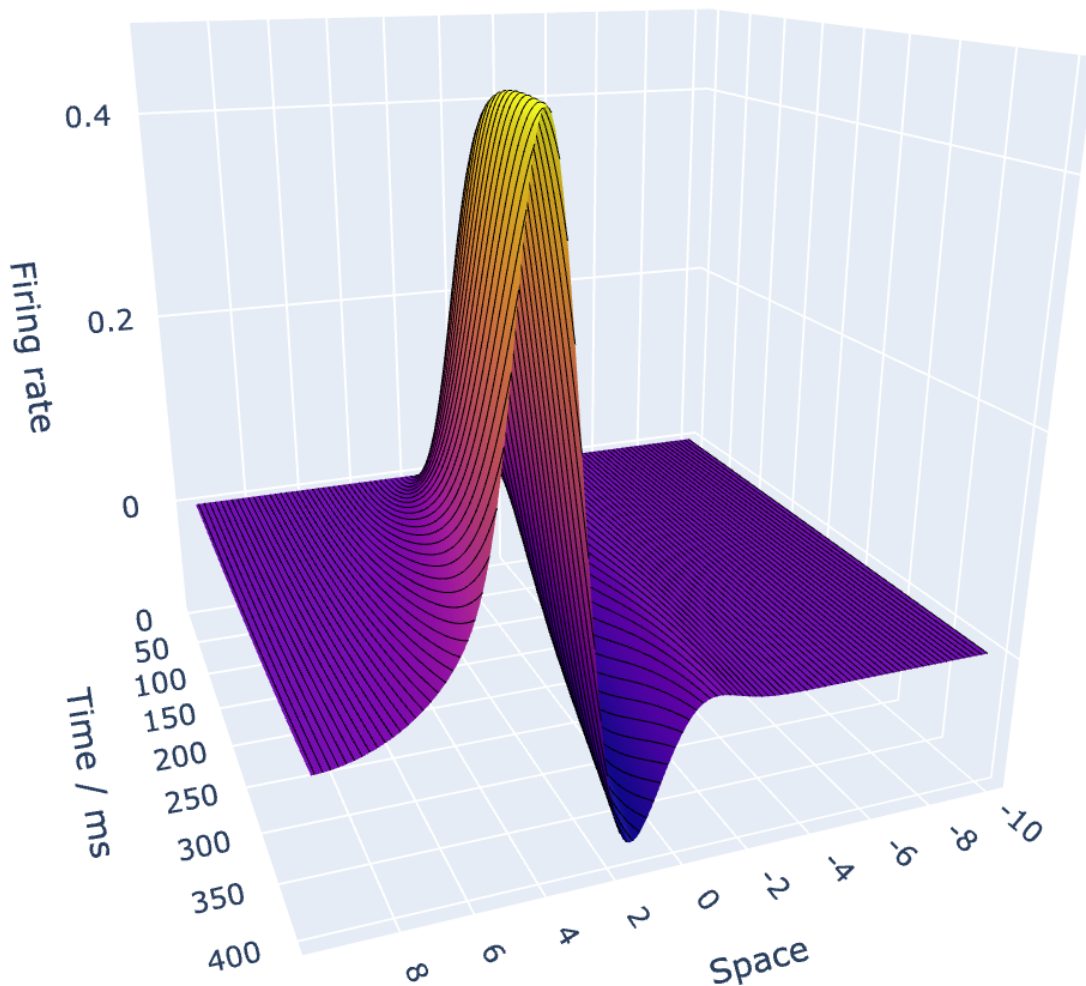The result is reported in Figure 6.

Figure 6: Linear neural field dynamics with different interaction kernel and input.

Now, if we change the stimulus travelling speed to $v = -0.1$, we yeild what is shown in Figure 7.
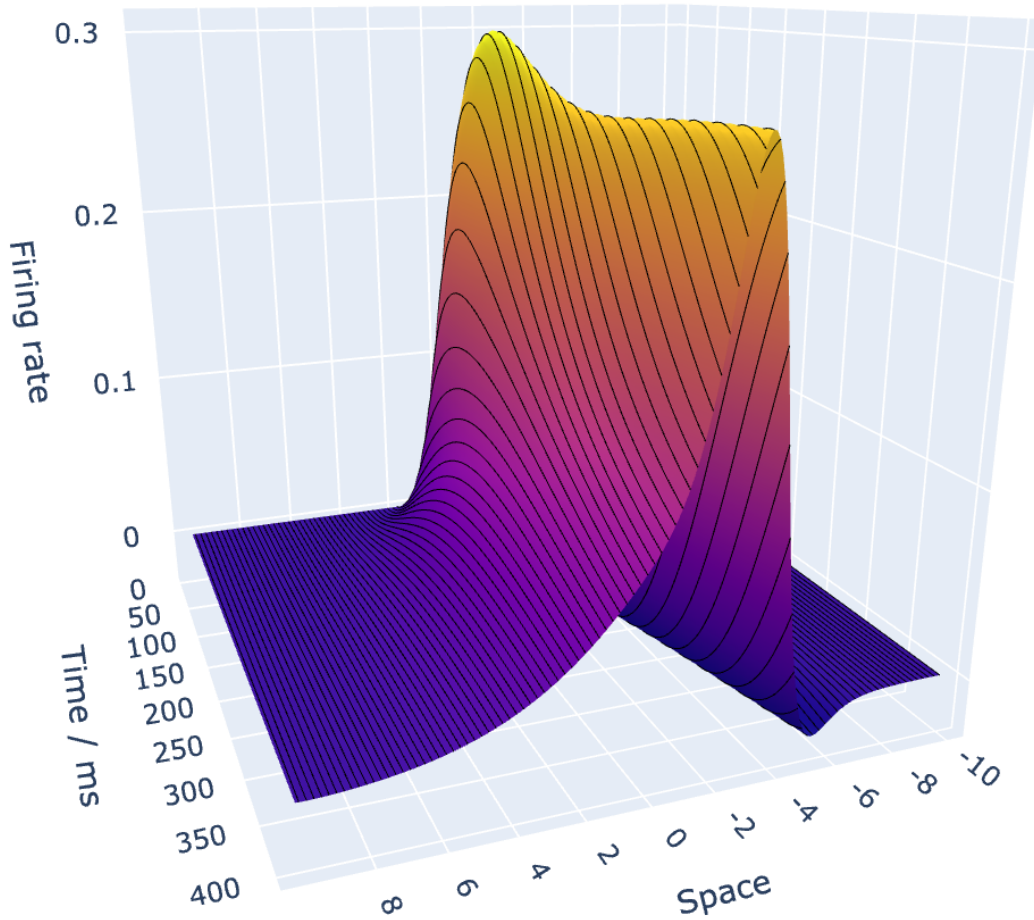

Figure 7: Same as in Fig. 6 but a negative stimulus speed.

1.7 We are required to compute the optimal input speed that leads to the highest output amplitude in the field.

Question 2. Noninear Neural Field.

2.1 We are given a neural field of Amari type which is described by the following equation

$$\tau\dot{u}(x,t) = -u(x,t) + \int_{-\infty}^{\infty} w(x-x')\mathbb{1}[u(x',t)]dx' + s(x,t) - h \tag{4}$$

We assume the resting level parameter, $h = 1$, and that both the input and recurrent interaction kernel are given by the following piecewise linear functions

$$s(x) = \begin{cases} C(1 - |x|/d), & |x| \leq d \\ 0, & \text{otherwise} \end{cases}$$

$$w(x) = \begin{cases} A, & |x| \leq a \\ -B, & a \leq |x| \leq b \\ 0, & \text{otherwise} \end{cases}$$

To simulate this neural field, similarly to Question 1, we use the Forward Euler approximation and we also express the convolution integral as a Riemann sum

$$\dot{u}(x, t + \Delta t) = u(x,t) + \frac{\Delta t}{\tau}\left(-u(x,t) + \sum_{i=0}^{N}\left[w(x-x_i)\mathbb{1}[u(x_i,t)]\Delta x\right] + s(x,t) - h\right)$$

Initially, we use the following parameters

- $A = 3$
- $B = 2$
- $C = 0.6$
- $a = 1$
- $b = 3$
- $d = 4$
- $h = 1$

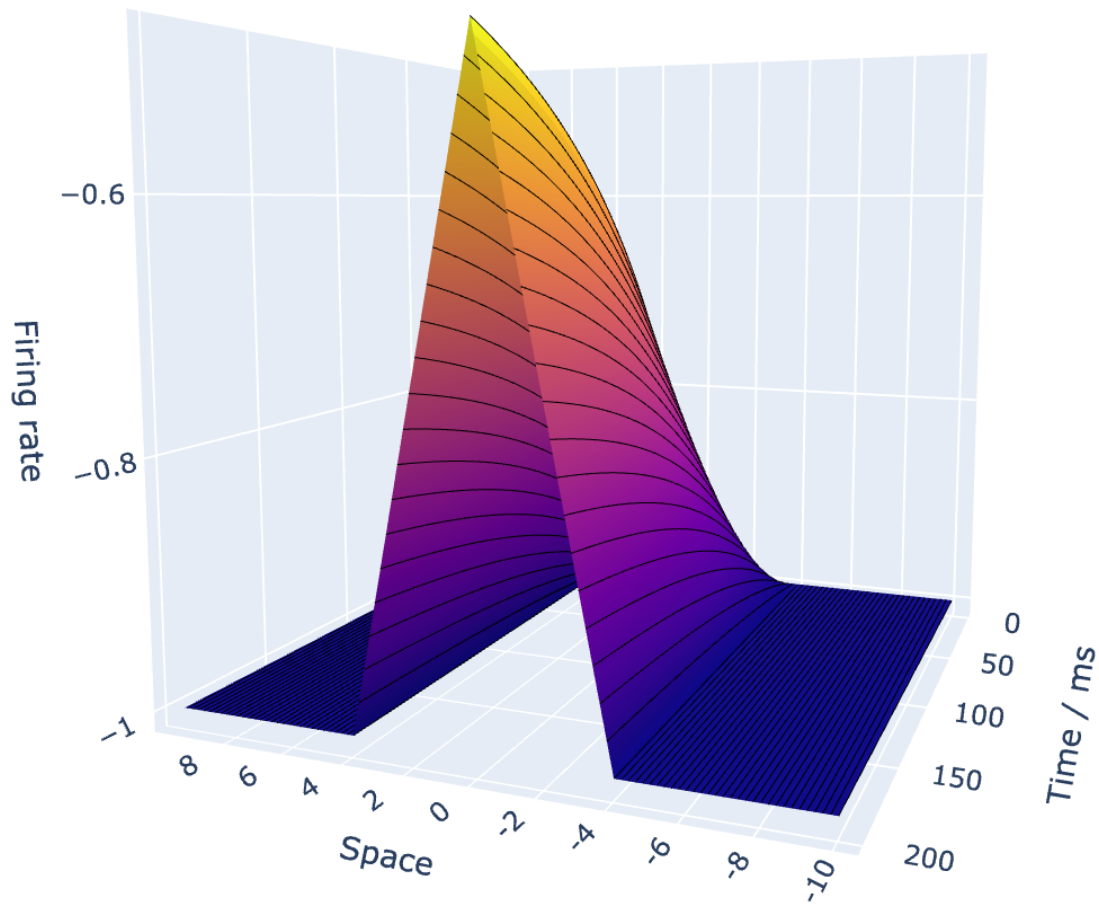The result of the simulation with the initial condition $u(x,0) = -h$ is shown in Figure 8.

Figure 8: Dynamics of the nonlinear neural field defined by equation 4.

Now, we run another simulation but this time with a different initial condition, $u(x, 0) = 3.3s(x) - h$. This simulation appears in Figure 9.
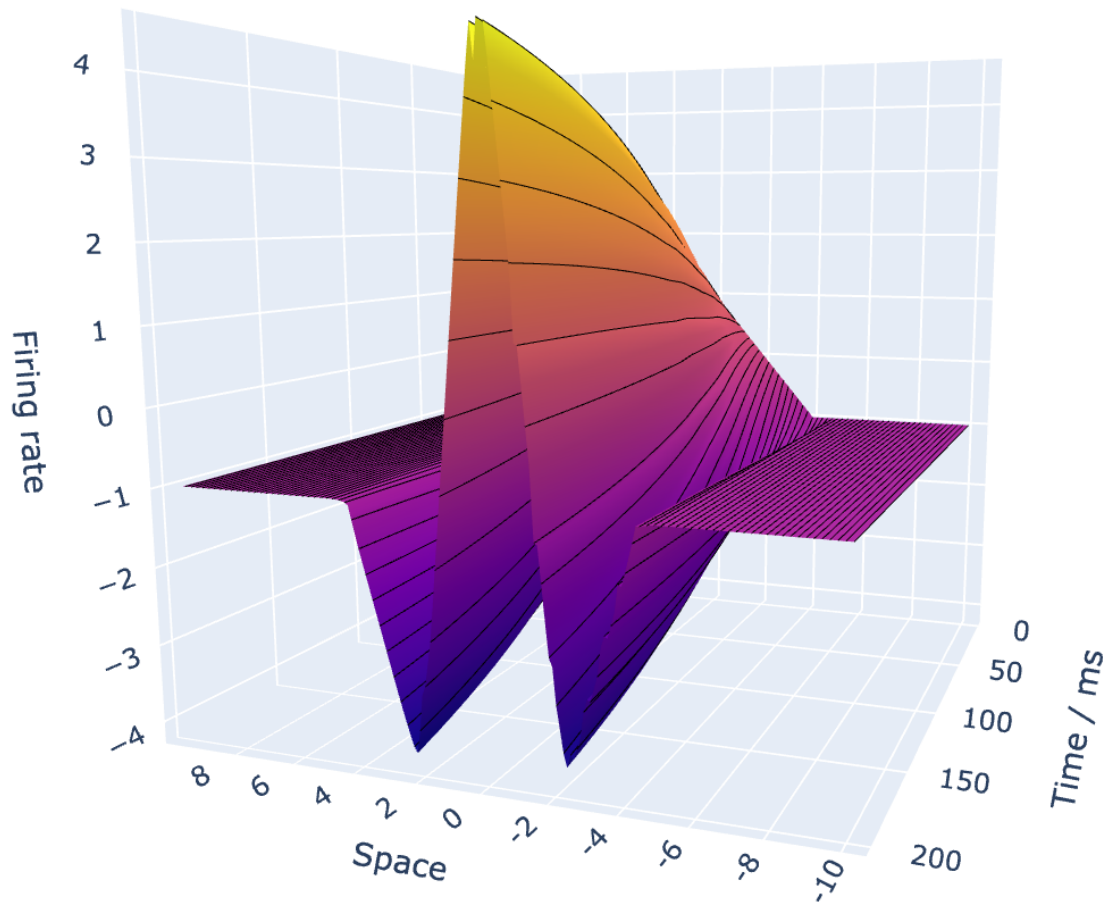
Figure 9: Same as in Figure 8 but with the initial condition $u(x, 0) = 3.3s(x) - h$.

Finally, we reduce the strength of the interaction kernel by one order of magnitude and run the simulation again with the remaining parameters as before. This simulation result is shown in Figure 10.
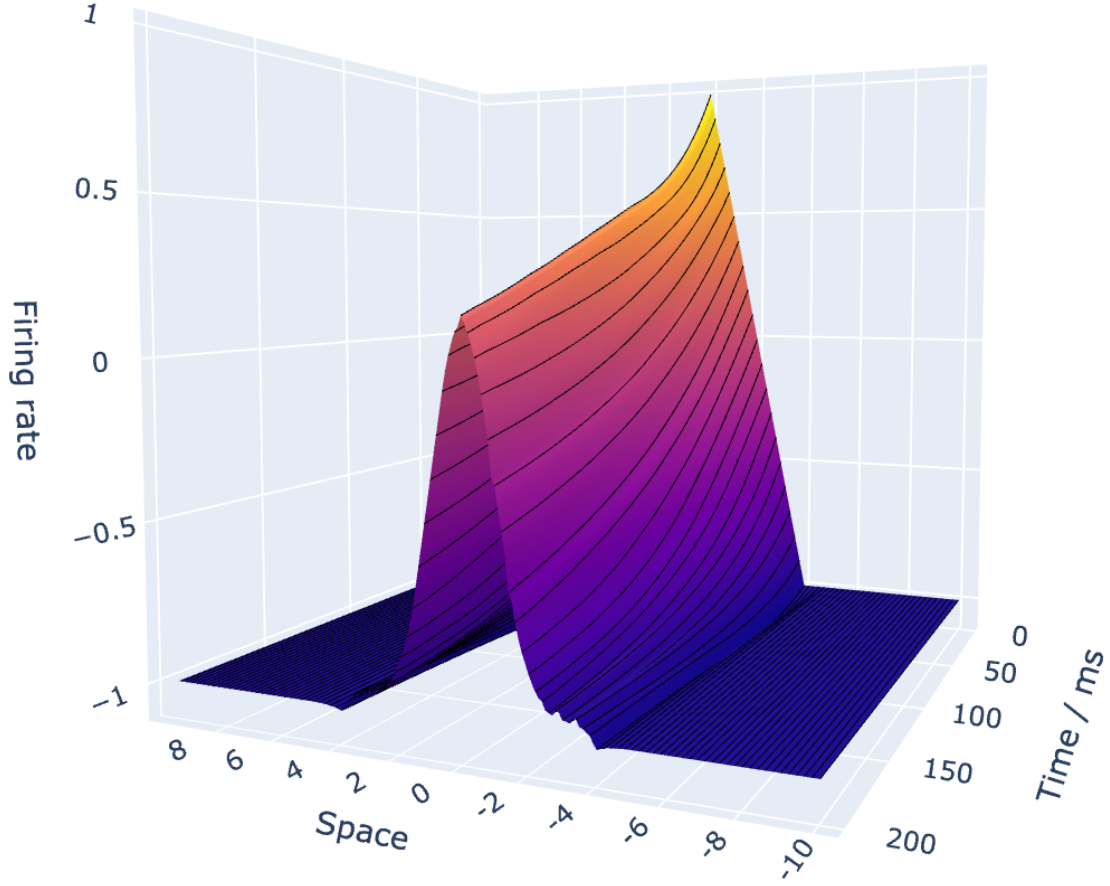
Figure 10: Same as in Figure 9 but with the interaction kernel $w(x)$ weakend by one order of magnitude.

2.2 As can be clearly seen from Figure 8, we have a localised activation peak

$$u^\star(x) = \begin{cases} \geq -h, & x_1 \leq x \leq x_2 \\ < -h, & \text{otherwise} \end{cases}$$

Hence, we can express the field dynamics within the peak by

$$\tau \frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{x_1(t)}^{x_2(t)} w(x - x')dx' + s(x) - h$$

2.4 Now we simulate the same neural field with the initial condition $u(x,0) = 3.3s(x) - h$ and the initial interaction kernel; however, we set the input current to zero (but not for the initial condition). The result is shown in Figure 11.
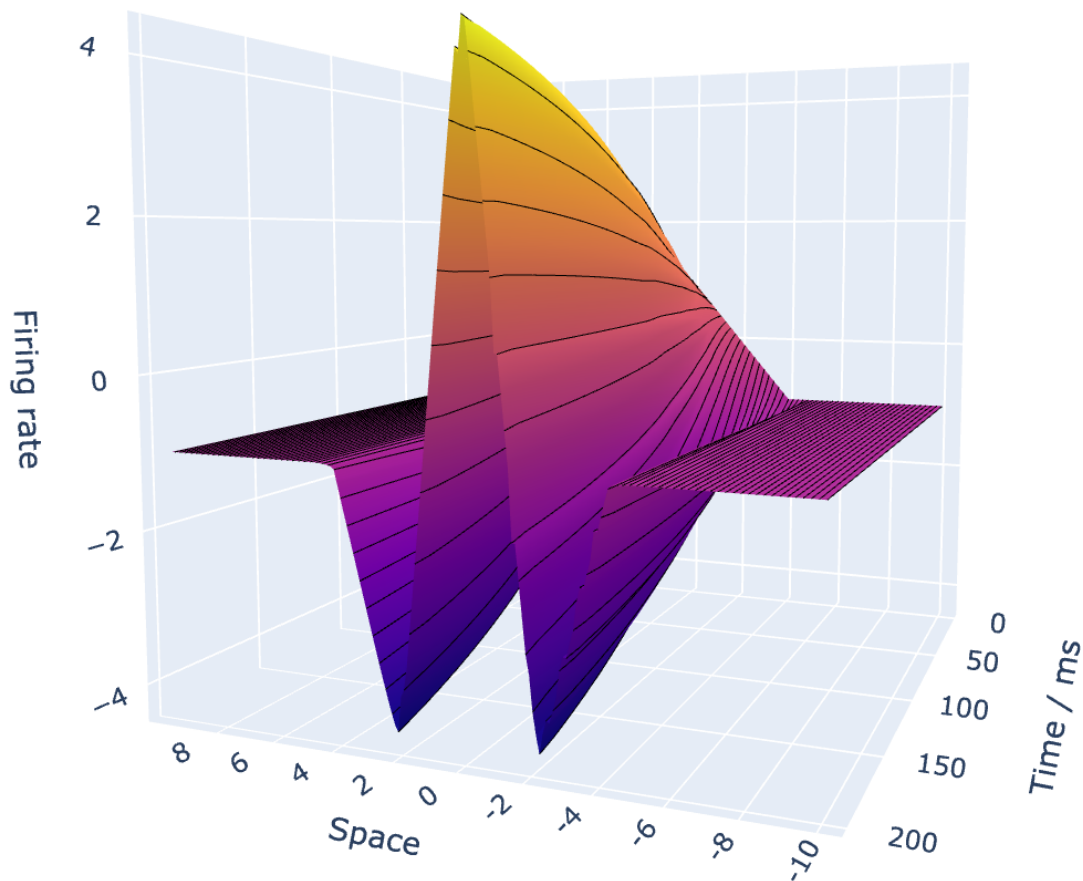
Figure 11: Nonlinear neural field dynamics with the initial condition $u(x, 0) = 3.3s(x) - h$ and the initial interaction kernel, but no input current.

2.5 Finally, we run the simulatiom with the same conditions as above, except that now we add a non-symmetric interaction kernel to our previous kernel, i.e. effectively we have $w(x) + w_u(x)$, where

$$w_u(x) = \begin{cases} 0.8x, & |x| \leq b \\ 0, & \text{otherwise} \end{cases}$$

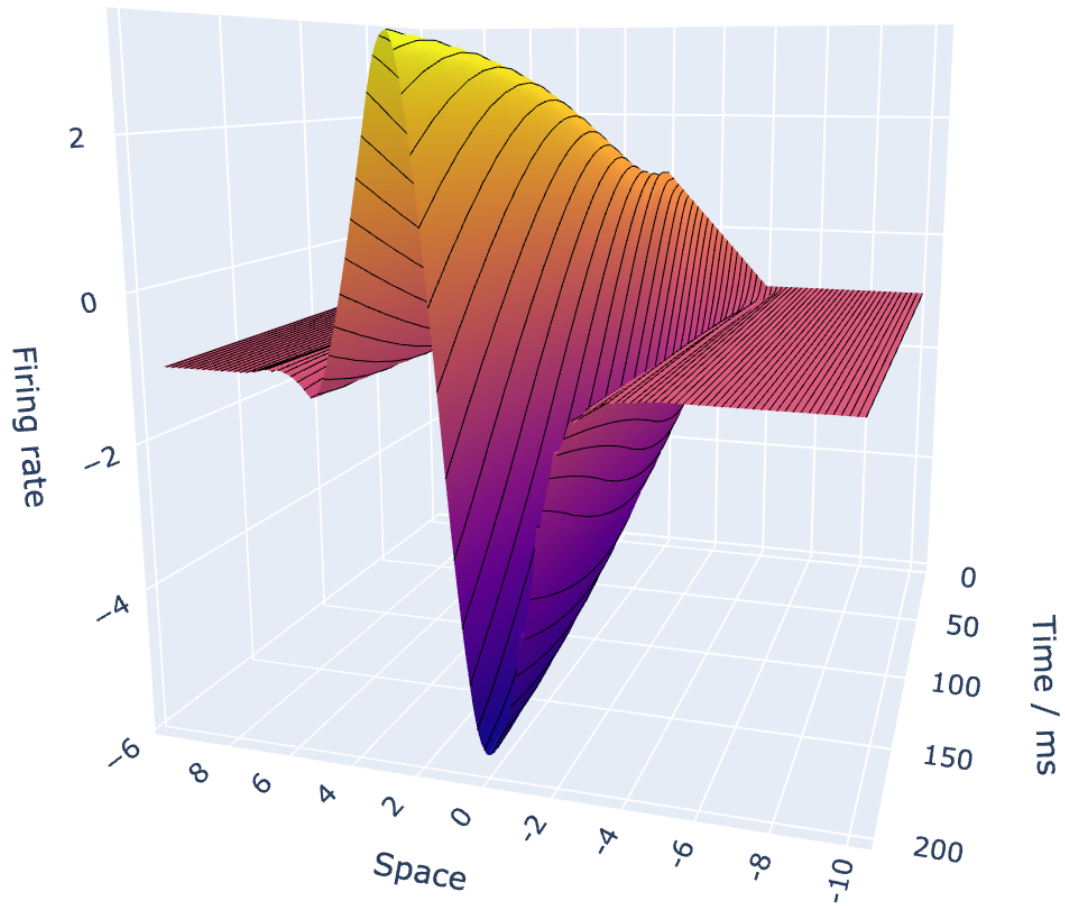The emerging solution appears in Figure 12.

Figure 12: Nonlinear neural field dynamics with the initial condition $u(x, 0) = 3.3s(x) - h$ and the interaction kernel $w(x) + w_u(x)$.