Question 1. Nonlinear network with two divisive inhibitory neurons.

1.1 We have a network of two nonlinear neurons with divisive feedback given by the following system of differential equations

$$\tau \dot{u}_1(t) = -u_1(t) + \frac{s_1}{1 + u_2(t)}$$
$$\tau \dot{u}_2(t) = -u_2(t) + \frac{s_2}{1 + u_1(t)}$$

where s_1 and s_2 are two nonnegative inputs, respectively. We are required to prove that when the initial condition $\mathbf{u(0)}$ lies in the first quadrant of the state space with u_1, $u_2 \geq 0$, the system stays in the first quadrant forever.

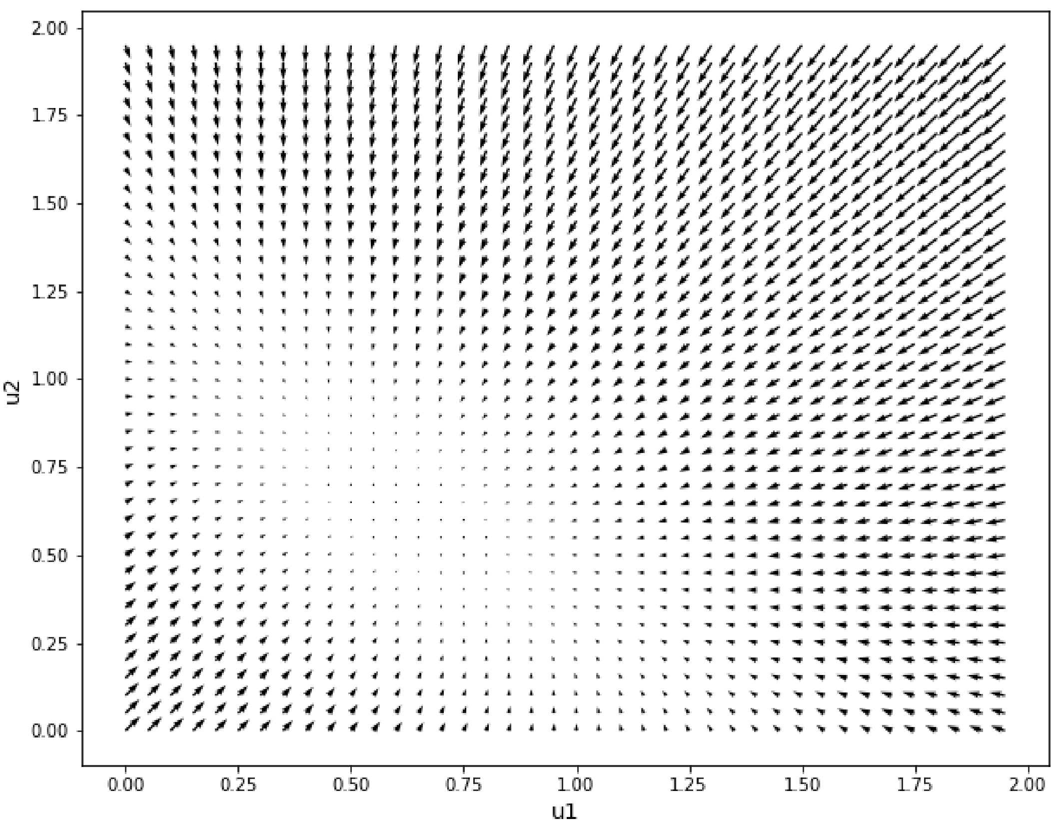

Figure 1: Vector field dynamics for the given nonlinear dynamical system for $u_1, u_2 \geq 0$.

The dynamics of the vector field for $s_1 = 1$, $s_2 = 1$, and $\tau = 1\,\mathrm{ms}$ appears in Figure 1. Note that in order for the system to stay in the first quadrant, whenever u_1 or u_2 is at the boundary, its derivative has to be nonnegative. Thus, we have

$$u_1 = 0 \implies \frac{du_1}{dt} \geq 0 \Leftrightarrow u_2 > -1$$

$$u_2 = 0 \implies \frac{du_2}{dt} \geq 0 \Leftrightarrow u_1 > -1$$

1.2 To compute the fixed points for an input of the form $s_1 = s_2 \geq 0$, we can plot the isoclines

$$\frac{du_1}{dt} = 0 \implies u_1 = \frac{s_1}{1+u_2}$$
$$\frac{du_2}{dt} = 0 \implies u_2 = \frac{s_2}{1+u_1}$$

A stationary point is then given by the intersection of these two isoclines

$$u_1 = \frac{s_1(1+u_1)}{1+u_1+s_2} \Leftrightarrow u_1^2 + u_1(s_2 - s_1 + 1) - s_1 = 0$$

Since we are given that $s_1 = s_2$, this simplifies to

$$u_1^2 + u_1 - s_1 = 0$$

And we therefore find that

$$u_1 = \frac{-1+\sqrt{1+4s_1}}{2}$$
$$u_2 = \frac{2s_2}{1+\sqrt{1+4s_1}}$$

The two isoclines are plotted in Figure 2.
For $s_1 = s_2 = 0$ the fixed point appears at $u_1 = u_2 = 0$ and for $s_1 = s_2 = 3/4$ it is at $u_1 = u_2 = 0.5$.

1.3 Recall that for the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$$

points with $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ are called fixed points, for the dynamics at these points does not change. Also recall that the linear dynamical system

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t)$$

where

$$\mathbf{A} = \frac{\partial \mathbf{f}(\mathbf{u}_0)}{\partial \mathbf{u}}$$

is called the linearised dynamics at the point $\mathbf{u}_0$. For the system given, the linearised dynamics at $\mathbf{u}_0$ is of the form

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{\mathbf{u}_0} = \frac{1}{\tau}\begin{pmatrix} -1 & -\frac{s_1}{(1+u_{0,2})^2} \\ -\frac{s_2}{(1+u_{0,1})^2} & -1 \end{pmatrix}$$

After eigendecomposition, the eigenvectors of the linearised dynamics matrix are

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

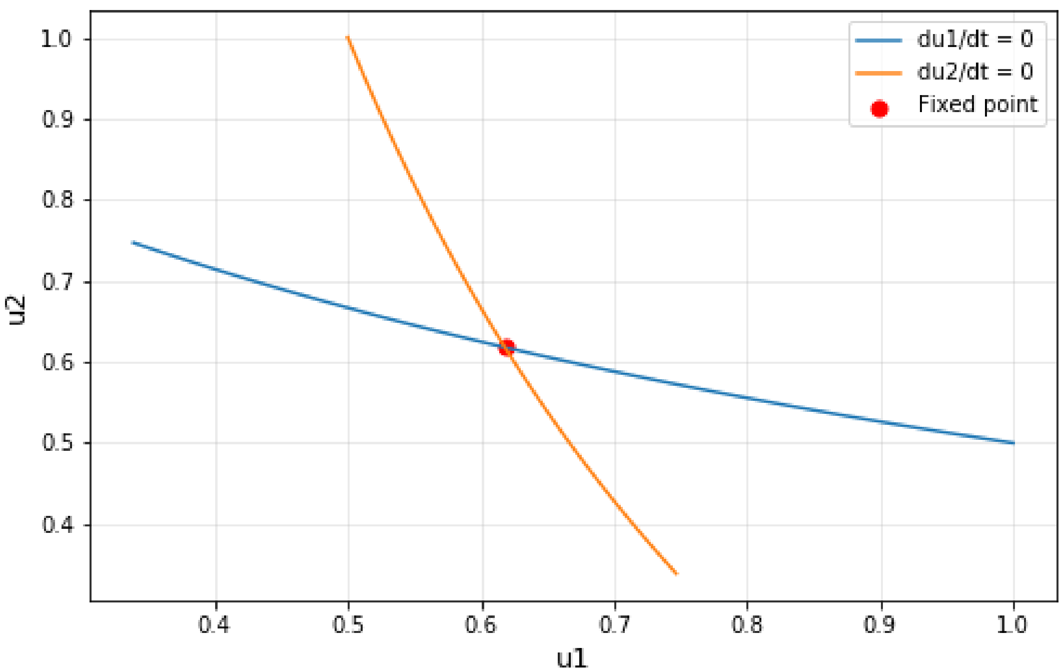

Figure 2: Computed fixed point given by the intersection of the two isoclines for $s_1 = s_2 = 1$.

and the corresponding eigenvalues are

$$\lambda_1 = \frac{-s\tau - u_0^2 - 2u_0 - 1}{\tau^2(u_0+1)^2}$$
$$\lambda_2 = \frac{s\tau - u_0^2 - 2u_0 - 1}{\tau^2(u_0+1)^2}$$

where $s = s_1 = s_2$ and $u_0 = u_{0,1} = u_{0,2}$. To analyse the stability of the system we need to fix τ and consider inputs of varying strength. For example, consider $\tau = 1\,\text{ms}$ and $s = 1$. Then the fixed point coordinates are $u_0 = 0.618$ and the linearised dynamics matrix is

$$\mathbf{A} = \begin{pmatrix} -1 & -0.381966 \\ -0.381966 & -1 \end{pmatrix}$$

The eigenvlues for this dynamics are

$$\lambda_1 = \frac{-2 - \sqrt{0.58359}}{2} = -1.38196,\ \lambda_2 = \frac{-2 + \sqrt{0.58359}}{2} = -0.6180$$

Both of the eigenvalues are negative, and hence the fixed point is stable.

1.4 Phase portraits for two different inputs appear in Figures 3 & 4.

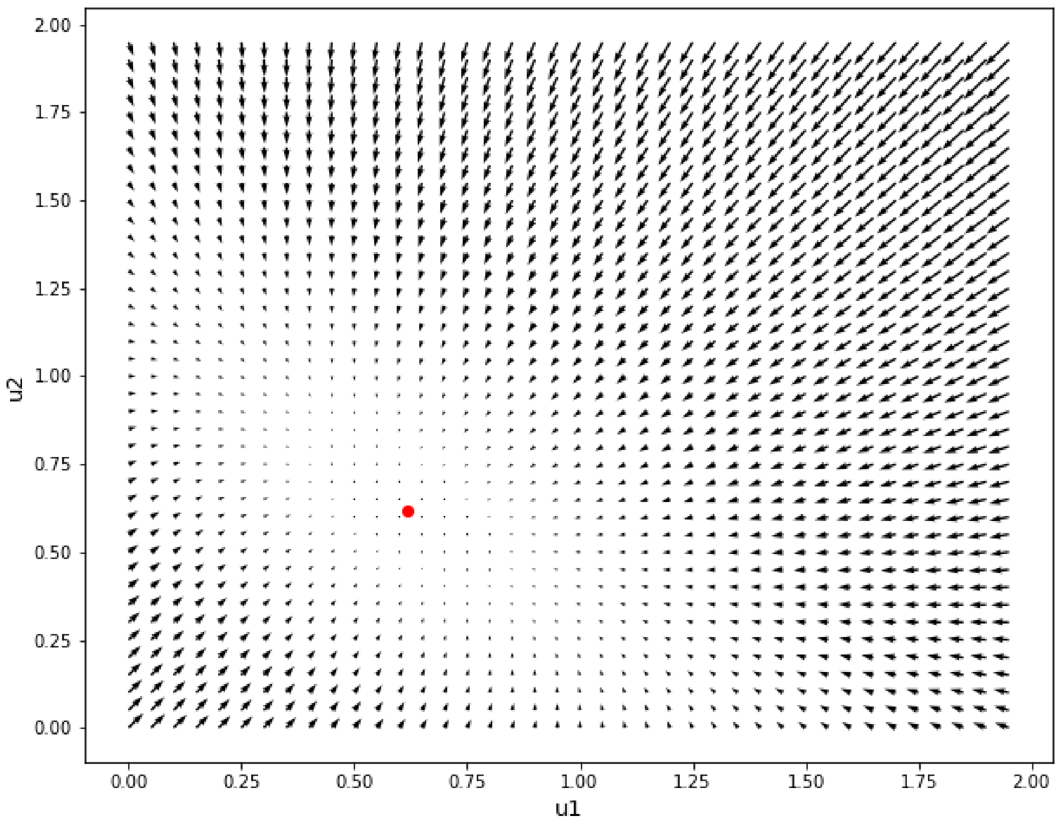

Figure 3: Phase portrait for $s = 1$. The fixed point is shown in red.

1.5 We are required to derive a Lyapunov function for the given system. Let us denote

$$F(u_1, u_2) = \frac{1}{\tau}\left(-u_1 + \frac{s_1}{1 + u_2}\right)$$

$$G(u_1, u_2) = \frac{1}{\tau}\left(-u_2 + \frac{s_2}{1 + u_1}\right)$$

Then, since both F and G are C^1 and have a finite number of joint zeroes, then by the Lyapunov theorem the following function with $|\epsilon| < 1$

$$E(\mathbf{u}) = \frac{1}{2}F^2(\mathbf{u}) + \epsilon F(\mathbf{u})G(\mathbf{u}) + \frac{1}{2}G^2(\mathbf{u})$$

is positive definite in regions around each zero. The resulting Lyapunov function for

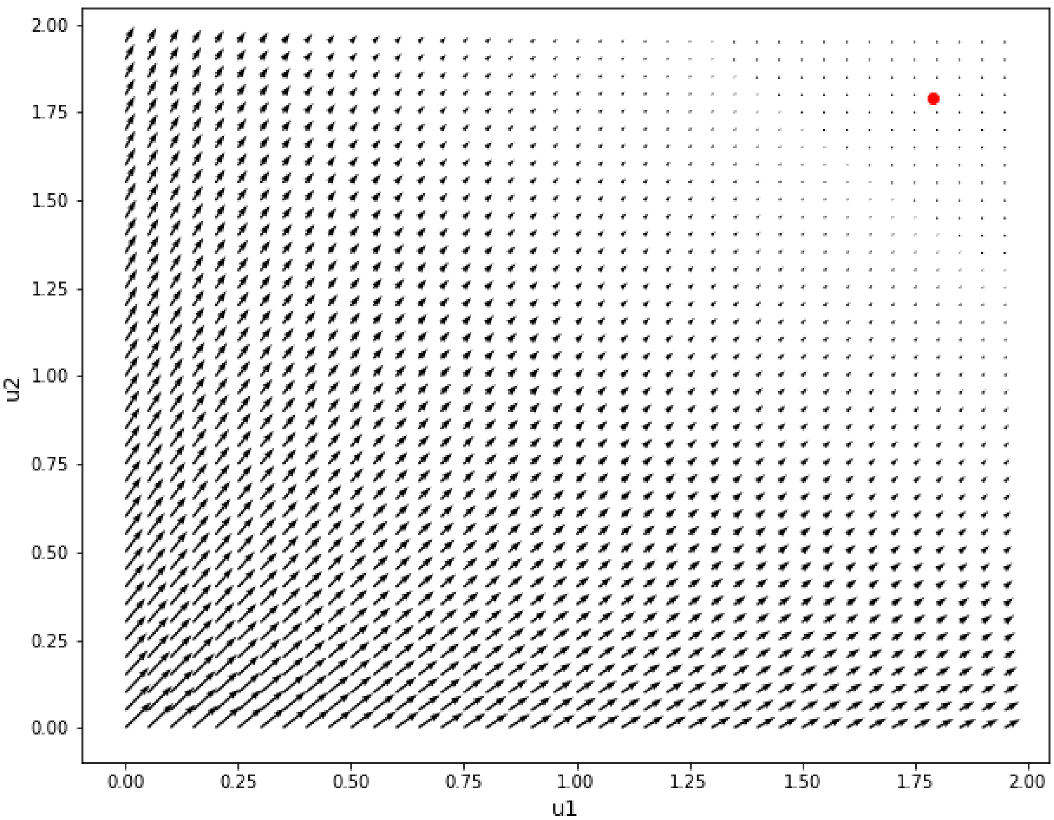

Figure 4: Phase portrait for $s = 5$. The fixed point is shown in red.

$|\epsilon| = 0$ is

$$\begin{aligned} E(u_1, u_2) &= \frac{1}{2}F^2(u_1, u_2) + \frac{1}{2}G^2(u_1, u_2) \\ &= \frac{1}{2\tau}\left(\left(-u_1 + \frac{s_1}{1+u_2}\right)^2 + \left(-u_2 + \frac{s_2}{1+u_1}\right)^2\right) \end{aligned}$$

The derivative of Lyapunov function is of the form

$$\dot{E}(u_1, u_2) = F^2\frac{\partial F}{\partial u_1} + FG\left(\frac{\partial F}{\partial u_2} + \frac{\partial G}{\partial u_1}\right) + G^2\frac{\partial G}{\partial u_2}$$

Hence, if we let $a = c = \frac{\partial F}{\partial u_1} = \frac{\partial G}{\partial u_2} = -\frac{1}{\tau}$ then, if the derivative $\dot{E}(u_0, u_1)$ is negative definite in the regions surrounding the fixed points, then the function $E(\mathbf{u})$ is truly Lyapunov. Thus, we check the corollary condition

$$|b| = \left|\frac{\partial F}{\partial u_2} + \frac{\partial G}{\partial u_1}\right| = \frac{1}{\tau}\left| - \frac{s}{(1+u_2)^2} - \frac{s}{(1+u_1)^2}\right| < \frac{2}{\tau}$$

which simplifies to

$$\frac{s}{(1+u_2)^2} + \frac{s}{(1+u_1)^2} < 2$$

The condition is fulfilled when

$$\begin{aligned} (1+u_0)^2 &> s \\ \implies 1 + 2u_0 + u_0^2 &> s \\ \implies u_0^2 + 2u_0 + (1-s) &> 0 \end{aligned}$$

where $u_0 = u_1 = u_2$. Therefore, we have

$$u_0 = \frac{-2 \pm \sqrt{3-s}}{2}$$

For $s = 3/4$, we find

$$u_0 = -1.75;\ -0.25$$

Thus, the function is Lyapunov for $s = 3/4$ when $u_0 \in (-0.25, +\infty)$.

Also observe that the left hand side of the condition is at maximum when $u_1 = u_2 = 0$. Hence, for the condition to be fulfilled, we require $s < 1$.