

Deep Generative Models

Lecture 5

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AI Masters

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Recap of previous lecture

LVM

$$p(\mathbf{x}|\theta) = \int p(\mathbf{x}, \mathbf{z}|\theta) d\mathbf{z} = \int p(\mathbf{x}|\mathbf{z}, \theta) p(\mathbf{z}) d\mathbf{z}$$

- ▶ More powerful $p(\mathbf{x}|\mathbf{z}, \theta)$ leads to more powerful generative model $p(\mathbf{x}|\theta)$.
- ▶ Too powerful $p(\mathbf{x}|\mathbf{z}, \theta)$ could lead to posterior collapse: $q(\mathbf{z}|\mathbf{x})$ will not carry any information about \mathbf{x} and close to prior $p(\mathbf{z})$.

Autoregressive decoder

$$p(\mathbf{x}|\mathbf{z}, \theta) = \prod_{j=1}^m p(x_j | \mathbf{x}_{1:j-1}, \mathbf{z}, \theta)$$

- ▶ Global structure is captured by latent variables.
- ▶ Local statistics are captured by limited receptive field autoregressive model.

Recap of previous lecture

Decoder weakening

- ▶ Powerful decoder $p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta})$ makes the model expressive, but posterior collapse is possible.
- ▶ PixelVAE model uses the autoregressive PixelCNN model with small number of layers to limit receptive field.

KL annealing

$$\mathcal{L}(q, \boldsymbol{\theta}, \beta) = \mathbb{E}_{q(\mathbf{z}|\mathbf{x}, \phi)} \log p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}) - \beta \cdot KL(q(\mathbf{z}|\mathbf{x}, \phi) || p(\mathbf{z}))$$

Start training with $\beta = 0$, increase it until $\beta = 1$ during training.

Free bits

Ensure the use of less than λ bits of information:

$$\mathcal{L}(q, \boldsymbol{\theta}, \lambda) = \mathbb{E}_{q(\mathbf{z}|\mathbf{x}, \phi)} \log p(\mathbf{x}|\mathbf{z}, \boldsymbol{\theta}) - \max(\lambda, KL(q(\mathbf{z}|\mathbf{x}, \phi) || p(\mathbf{z}))).$$

This results in $KL(q(\mathbf{z}|\mathbf{x}, \phi) || p(\mathbf{z})) \geq \lambda$.

Recap of previous lecture

VAE objective

$$\log p(\mathbf{x}|\theta) \geq \mathcal{L}(q, \theta) = \mathbb{E}_{q(\mathbf{z}|\mathbf{x}, \phi)} \log \frac{p(\mathbf{x}, \mathbf{z}|\theta)}{q(\mathbf{z}|\mathbf{x}, \phi)} \rightarrow \max_{q, \theta}$$

IWAE objective

$$\mathcal{L}_K(q, \theta) = \mathbb{E}_{\mathbf{z}_1, \dots, \mathbf{z}_K \sim q(\mathbf{z}|\mathbf{x}, \phi)} \log \left(\frac{1}{K} \sum_{k=1}^K \frac{p(\mathbf{x}, \mathbf{z}_k|\theta)}{q(\mathbf{z}_k|\mathbf{x}, \phi)} \right) \rightarrow \max_{\phi, \theta}.$$

Theorem

1. $\log p(\mathbf{x}|\theta) \geq \mathcal{L}_K(q, \theta) \geq \mathcal{L}_M(q, \theta) \geq \mathcal{L}(q, \theta)$, for $K \geq M$;
 2. $\log p(\mathbf{x}|\theta) = \lim_{K \rightarrow \infty} \mathcal{L}_K(q, \theta)$ if $\frac{p(\mathbf{x}, \mathbf{z}|\theta)}{q(\mathbf{z}|\mathbf{x}, \phi)}$ is bounded.
- ▶ IWAE makes the variational bound tighter and extends the class of variational distributions.
 - ▶ Gradient signal becomes really small, training is complicated.
 - ▶ IWAE is a standard quality measure for VAE models.

Recap of previous lecture

Change of variable theorem (CoV)

Let \mathbf{x} be a random variable with density function $p(\mathbf{x})$ and $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a differentiable, invertible function (diffeomorphism). If $\mathbf{z} = f(\mathbf{x})$, $\mathbf{x} = f^{-1}(\mathbf{z}) = g(\mathbf{z})$, then

$$p(\mathbf{x}) = p(\mathbf{z}) |\det(\mathbf{J}_f)| = p(\mathbf{z}) \left| \det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) \right| = p(f(\mathbf{x})) \left| \det \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right) \right|$$
$$p(\mathbf{z}) = p(\mathbf{x}) |\det(\mathbf{J}_g)| = p(\mathbf{x}) \left| \det \left(\frac{\partial \mathbf{x}}{\partial \mathbf{z}} \right) \right| = p(g(\mathbf{z})) \left| \det \left(\frac{\partial g(\mathbf{z})}{\partial \mathbf{z}} \right) \right|.$$

Inverse function theorem

If function f is invertible and Jacobian is continuous and non-singular, then

$$\mathbf{J}_f = \mathbf{J}_{g^{-1}} = \mathbf{J}_g^{-1}, \quad |\det(\mathbf{J}_f)| = \frac{1}{|\det(\mathbf{J}_g)|}$$

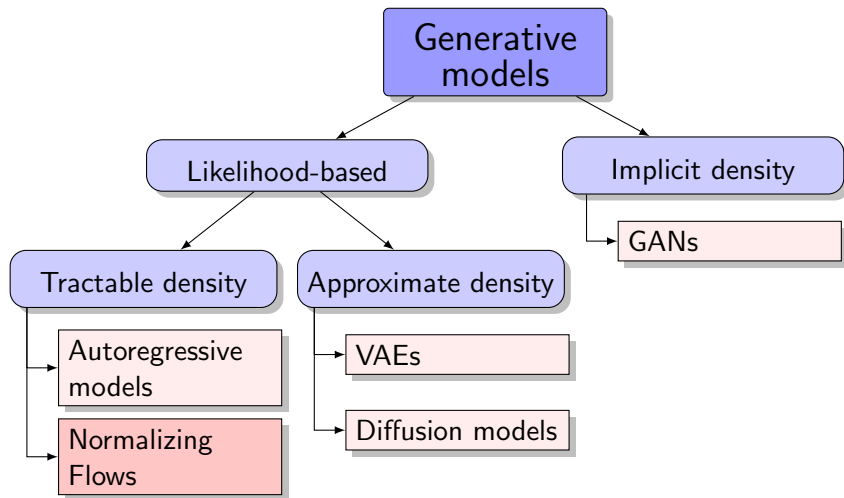
Outline

1. Normalizing flows
2. Forward and Reverse KL for Normalizing flows
3. Residual and Linear flows

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Generative models zoo



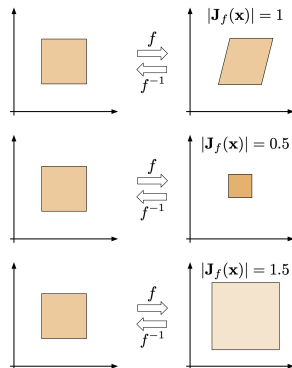
Jacobian determinant

Inverse function theorem

If function f is invertible and Jacobian matrix is continuous and non-singular, then

$$\mathbf{J}_f = \mathbf{J}_{g^{-1}} = \mathbf{J}_g^{-1}, \quad |\det(\mathbf{J}_f)| = \frac{1}{|\det(\mathbf{J}_g)|}$$

- ▶ \mathbf{x} and \mathbf{z} have the same dimensionality (\mathbb{R}^m).
- ▶ $f(\mathbf{x}, \boldsymbol{\theta})$ could be parametric function.
- ▶ Determinant of Jacobian matrix $\mathbf{J} = \frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}}$ shows how the volume changes under the transformation.

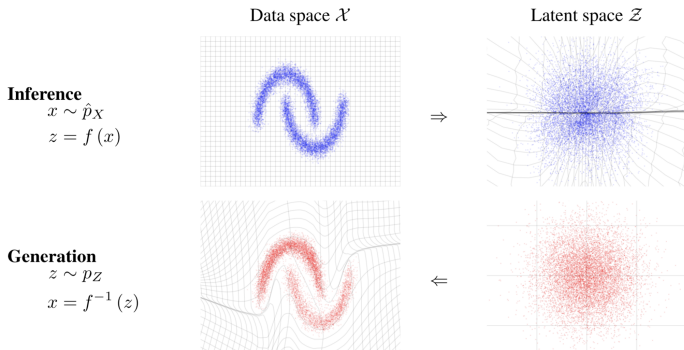


Fitting flows

MLE problem

$$p(\mathbf{x}|\boldsymbol{\theta}) = p(\mathbf{z}) \left| \det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) \right| = p(f(\mathbf{x}, \boldsymbol{\theta})) \left| \det \left(\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}} \right) \right|$$

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(f(\mathbf{x}, \boldsymbol{\theta})) + \log |\det(\mathbf{J}_f)| \rightarrow \max_{\boldsymbol{\theta}}$$

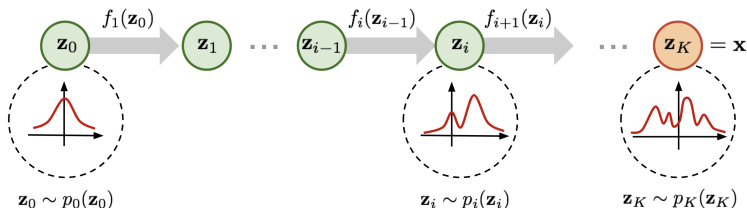


Composition of flows

Theorem

Diffeomorphisms are **composable** (If $\{f_k\}_{k=1}^K$ satisfy conditions of the change of variable theorem, then $\mathbf{z} = f(\mathbf{x}) = f_K \circ \dots \circ f_1(\mathbf{x})$ also satisfies it).

$$\begin{aligned} p(\mathbf{x}) &= p(f(\mathbf{x})) \left| \det \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \right) \right| = p(f(\mathbf{x})) \left| \det \left(\frac{\partial \mathbf{f}_K}{\partial \mathbf{f}_{K-1}} \dots \frac{\partial \mathbf{f}_1}{\partial \mathbf{x}} \right) \right| = \\ &= p(f(\mathbf{x})) \prod_{k=1}^K \left| \det \left(\frac{\partial \mathbf{f}_k}{\partial \mathbf{f}_{k-1}} \right) \right| = p(f(\mathbf{x})) \prod_{k=1}^K |\det(\mathbf{J}_{f_k})| \end{aligned}$$



Flows

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(f(\mathbf{x}, \boldsymbol{\theta})) + \log |\det(\mathbf{J}_f)|$$

Definition

Normalizing flow is a *differentiable, invertible* mapping from data \mathbf{x} to the noise \mathbf{z} .

- ▶ **Normalizing** means that the inverse flow takes samples from $\pi(\mathbf{x})$ and normalizes them into samples from the density $p(\mathbf{z})$.
- ▶ **Flow** refers to the trajectory followed by samples from $p(\mathbf{z})$ as they are transformed by the sequence of transformations

$$\mathbf{z} = f_K \circ \dots \circ f_1(\mathbf{x}); \quad \mathbf{x} = f_1^{-1} \circ \dots \circ f_K^{-1}(\mathbf{z}) = g_1 \circ \dots \circ g_K(\mathbf{z})$$

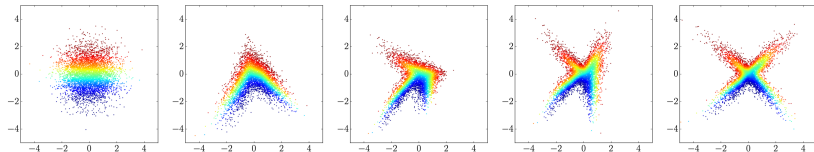
Log likelihood

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(f_K \circ \dots \circ f_1(\mathbf{x})) + \sum_{k=1}^K \log |\det(\mathbf{J}_{f_k})|,$$

where $\mathbf{J}_{f_k} = \frac{\partial \mathbf{f}_k}{\partial \mathbf{f}_{k-1}}$.

Flows

Example of a 4-step flow



Flow log likelihood

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(f(\mathbf{x}, \boldsymbol{\theta})) + \log |\det(\mathbf{J}_f)|$$

What is the complexity of the determinant computation?

What we want

- ▶ Efficient computation of the Jacobian matrix $\mathbf{J}_f = \frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}}$;
- ▶ Efficient sampling from the base distribution $p(\mathbf{z})$;
- ▶ Efficient inversion of $f(\mathbf{x}, \boldsymbol{\theta})$.

Outline

1. Normalizing flows
2. Forward and Reverse KL for Normalizing flows
3. Residual and Linear flows

Forward KL vs Reverse KL

Forward KL

$$\begin{aligned} KL(\pi||p) &= \int \pi(\mathbf{x}) \log \frac{\pi(\mathbf{x})}{p(\mathbf{x}|\boldsymbol{\theta})} d\mathbf{x} \\ &= -\mathbb{E}_{\pi(\mathbf{x})} \log p(\mathbf{x}|\boldsymbol{\theta}) + \text{const} \rightarrow \min_{\boldsymbol{\theta}} \end{aligned}$$

Maximum likelihood estimation is equivalent to minimization of the Monte-Carlo estimation of forward KL.

Forward KL for flow model

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(f(\mathbf{x}, \boldsymbol{\theta})) + \log |\det(\mathbf{J}_f)|$$

- ▶ We need to be able to compute $f(\mathbf{x}, \boldsymbol{\theta})$ and its Jacobian.
- ▶ We need to be able to compute the density $p(\mathbf{z})$.
- ▶ We don't need to think about computing the function $g(\mathbf{z}, \boldsymbol{\theta}) = f^{-1}(\mathbf{z}, \boldsymbol{\theta})$ until we want to sample from the flow.

Forward KL vs Reverse KL

Reverse KL

$$\begin{aligned} KL(p||\pi) &= \int p(\mathbf{x}|\boldsymbol{\theta}) \log \frac{p(\mathbf{x}|\boldsymbol{\theta})}{\pi(\mathbf{x})} d\mathbf{x} \\ &= \mathbb{E}_{p(\mathbf{x}|\boldsymbol{\theta})} [\log p(\mathbf{x}|\boldsymbol{\theta}) - \log \pi(\mathbf{x})] \rightarrow \min_{\boldsymbol{\theta}} \end{aligned}$$

Reverse KL for flow model

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(\mathbf{z}) + \log |\det(\mathbf{J}_f)| = \log p(\mathbf{z}) - \log |\det(\mathbf{J}_g)|$$

$$KL(p||\pi) = \mathbb{E}_{p(\mathbf{z})} [\log p(\mathbf{z}) - \log |\det(\mathbf{J}_g)| - \log \pi(g(\mathbf{z}, \boldsymbol{\theta}))]$$

- ▶ We need to be able to compute $g(\mathbf{z}, \boldsymbol{\theta})$ and its Jacobian.
- ▶ We need to be able to sample from the density $p(\mathbf{z})$ (do not need to evaluate it) and to evaluate(!) $\pi(\mathbf{x})$.
- ▶ We don't need to think about computing the function $f(\mathbf{x}, \boldsymbol{\theta})$.

Flow KL duality

Theorem

Fitting flow model $p(\mathbf{x}|\boldsymbol{\theta})$ to the target distribution $\pi(\mathbf{x})$ using forward KL (MLE) is equivalent to fitting the induced distribution $p(\mathbf{z}|\boldsymbol{\theta})$ to the base $p(\mathbf{z})$ using reverse KL:

$$\arg \min_{\boldsymbol{\theta}} KL(\pi(\mathbf{x})||p(\mathbf{x}|\boldsymbol{\theta})) = \arg \min_{\boldsymbol{\theta}} KL(p(\mathbf{z}|\boldsymbol{\theta})||p(\mathbf{z})).$$

- ▶ $p(\mathbf{z})$ is a base distribution; $\pi(\mathbf{x})$ is a data distribution;
- ▶ $\mathbf{z} \sim p(\mathbf{z})$, $\mathbf{x} = g(\mathbf{z}, \boldsymbol{\theta})$, $\mathbf{x} \sim p(\mathbf{x}|\boldsymbol{\theta})$;
- ▶ $\mathbf{x} \sim \pi(\mathbf{x})$, $\mathbf{z} = f(\mathbf{x}, \boldsymbol{\theta})$, $\mathbf{z} \sim p(\mathbf{z}|\boldsymbol{\theta})$;

$$\log p(\mathbf{z}|\boldsymbol{\theta}) = \log \pi(g(\mathbf{z}, \boldsymbol{\theta})) + \log |\det(\mathbf{J}_g)|;$$

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(f(\mathbf{x}, \boldsymbol{\theta})) + \log |\det(\mathbf{J}_f)|.$$

Flow KL duality

Theorem

Fitting flow model $p(\mathbf{x}|\boldsymbol{\theta})$ to the target distribution $\pi(\mathbf{x})$ using forward KL (MLE) is equivalent to fitting the induced distribution $p(\mathbf{z}|\boldsymbol{\theta})$ to the base $p(\mathbf{z})$ using reverse KL:

$$\arg \min_{\boldsymbol{\theta}} KL(\pi(\mathbf{x})||p(\mathbf{x}|\boldsymbol{\theta})) = \arg \min_{\boldsymbol{\theta}} KL(p(\mathbf{z}|\boldsymbol{\theta})||p(\mathbf{z})).$$

Proof

$$\begin{aligned} KL(p(\mathbf{z}|\boldsymbol{\theta})||p(\mathbf{z})) &= \mathbb{E}_{p(\mathbf{z}|\boldsymbol{\theta})} [\log p(\mathbf{z}|\boldsymbol{\theta}) - \log p(\mathbf{z})] = \\ &= \mathbb{E}_{p(\mathbf{z}|\boldsymbol{\theta})} [\log \pi(g(\mathbf{z}, \boldsymbol{\theta})) + \log |\det(\mathbf{J}_g)| - \log p(\mathbf{z})] = \\ &= \mathbb{E}_{\pi(\mathbf{x})} [\log \pi(\mathbf{x}) - \log |\det(\mathbf{J}_f)| - \log p(f(\mathbf{x}, \boldsymbol{\theta}))] = \\ &= \mathbb{E}_{\pi(\mathbf{x})} [\log \pi(\mathbf{x}) - \log p(\mathbf{x}|\boldsymbol{\theta})] = KL(\pi(\mathbf{x})||p(\mathbf{x}|\boldsymbol{\theta})). \end{aligned}$$

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Jacobian structure

Flow log-likelihood

$$\log p(\mathbf{x}|\boldsymbol{\theta}) = \log p(f(\mathbf{x}, \boldsymbol{\theta})) + \log \left| \det \left(\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}} \right) \right|$$

The main challenge is a determinant of the Jacobian matrix.

What is the $\det(\mathbf{J})$ in the following cases?

1. Consider a linear layer $\mathbf{z} = \mathbf{W}\mathbf{x}$.
2. Let \mathbf{z} be a permutation of \mathbf{x} .
3. Let z_j depend only on \mathbf{x}_j .

$$\log \left| \det \left(\frac{\partial f(\mathbf{x}, \boldsymbol{\theta})}{\partial \mathbf{x}} \right) \right| = \log \left| \prod_{j=1}^m f'_j(x_j, \boldsymbol{\theta}) \right| = \sum_{j=1}^m \log |f'_j(x_j, \boldsymbol{\theta})|.$$

4. Let z_j depend only on $\mathbf{x}_{1:j}$ (autoregressive dependency).

Linear flows

$$\mathbf{z} = f(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{W}\mathbf{x}, \quad \mathbf{W} \in \mathbb{R}^{m \times m}, \quad \boldsymbol{\theta} = \mathbf{W}, \quad \mathbf{J}_f = \mathbf{W}$$

In general, we need $O(m^3)$ to invert matrix.

Invertibility

- ▶ Diagonal matrix $O(m)$.
- ▶ Triangular matrix $O(m^2)$.
- ▶ It is impossible to parametrize all invertible matrices.

Invertible 1x1 conv

$\mathbf{W} \in \mathbb{R}^{c \times c}$ - kernel of 1x1 convolution with c input and c output channels. The computational complexity of computing or differentiating $\det(\mathbf{W})$ is $O(c^3)$. Cost to compute $\det(\mathbf{W})$ is $O(c^3)$. It should be invertible.

Linear flows

$$\mathbf{z} = f(\mathbf{x}, \boldsymbol{\theta}) = \mathbf{W}\mathbf{x}, \quad \mathbf{W} \in \mathbb{R}^{m \times m}, \quad \boldsymbol{\theta} = \mathbf{W}, \quad \mathbf{J}_f = \mathbf{W}$$

Matrix decompositions

- ▶ LU-decomposition

$$\mathbf{W} = \mathbf{P}\mathbf{L}\mathbf{U},$$

where \mathbf{P} is a permutation matrix, \mathbf{L} is lower triangular with positive diagonal, \mathbf{U} is upper triangular with positive diagonal.

- ▶ QR-decomposition

$$\mathbf{W} = \mathbf{Q}\mathbf{R},$$

where \mathbf{Q} is an orthogonal matrix, \mathbf{R} is an upper triangular matrix with positive diagonal.

Kingma D. P., Dhariwal P. *Glow: Generative Flow with Invertible 1x1 Convolutions*, 2018

Hoogeboom E., Van Den Berg R., and Welling M. *Emerging convolutions for generative normalizing flows*, 2019

Residual Flows

Matrix determinant lemma

$$\det(\mathbf{I}_m + \mathbf{V}\mathbf{W}^T) = \det(\mathbf{I}_d + \mathbf{W}^T\mathbf{V}), \quad \text{where } \mathbf{V}, \mathbf{W} \in \mathbb{R}^{m \times d}.$$

Planar flow

$$\mathbf{x} = g(\mathbf{z}, \boldsymbol{\theta}) = \mathbf{z} + \mathbf{v} \sigma(\mathbf{w}^T \mathbf{z} + b).$$

Here $\boldsymbol{\theta} = \{\mathbf{v}, \mathbf{w}, b\}$, $\sigma(\cdot)$ is a smooth element-wise non-linearity.

$$\left| \det \left(\frac{\partial g(\mathbf{z}, \boldsymbol{\theta})}{\partial \mathbf{z}} \right) \right| = \left| \det(\mathbf{I} + \sigma'(\mathbf{w}^T \mathbf{z} + b) \mathbf{v} \mathbf{w}^T) \right| = \left| 1 + \sigma'(\mathbf{w}^T \mathbf{z} + b) \mathbf{w}^T \mathbf{v} \right|$$

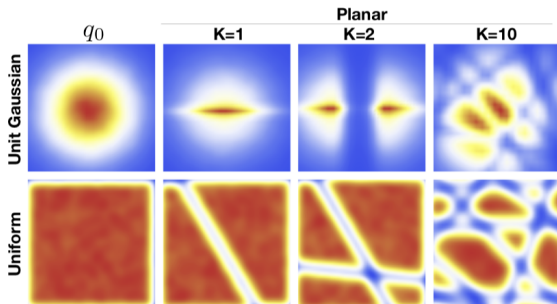
The transformation is invertible, for example, if

$$\sigma = \tanh; \quad \sigma'(\mathbf{w}^T \mathbf{z} + b) \mathbf{w}^T \mathbf{v} \geq -1.$$

Residual Flows

Expressiveness of planar flows

$$\mathbf{z}_K = g_1 \circ \dots \circ g_K(\mathbf{z}); \quad g_k = g(\mathbf{z}_k, \boldsymbol{\theta}_k) = \mathbf{z}_k + \mathbf{v}_k \sigma(\mathbf{w}_k^T \mathbf{z}_k + b_k).$$



Sylvester flow: planar flow extension

$$g(\mathbf{z}, \boldsymbol{\theta}) = \mathbf{z} + \mathbf{V} \sigma(\mathbf{W}^T \mathbf{z} + \mathbf{b}).$$

Rezende D. J., Mohamed S. *Variational Inference with Normalizing Flows*, 2015
Berg R. et al. *Sylvester normalizing flows for variational inference*, 2018

Summary

- ▶ Flow models transform a simple base distribution to a complex one via a sequence of invertible transformations with tractable Jacobian.
- ▶ Flow models have a tractable likelihood that is given by the change of variable theorem.
- ▶ Flows could be fitted using forward and reverse KL minimization. We will consider each of the scenarios later in the course.
- ▶ Linear flows try to parametrize set of invertible matrices via matrix decompositions.
- ▶ Planar and Sylvester flows are residual flows which use matrix determinant lemma.