Introduction to the Theory of Graphs

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To those who gave us permission
to write this book
(our wives):

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Preface

We have always found the theory of graphs to be an exciting branch of mathematics. There are problems and questions within this area to fascinate youthful minds and to challenge great minds. It is truly a mathematics subject for all ages. Although there are numerous applications of graph theory both within mathematics and outside mathematics, we feel that a strong background in the subject matter must be considered essential, no matter what may be the individual's reasons for pursuing the field. Therefore, this is a mathematics book with emphasis on theory and techniques of proof. We firmly believe such material to be necessary not only for those mathematicians wishing to study graph theory but for others as well who wish to apply graph theory to other areas.

Work on this book actually began in the summer of 1964 when the co-authors first considered the possibility of such a project. Indeed, notes written then were organized and typed by Carolyn Burk Bardsley whom we thank for actually starting the book. The notes were bound under the title "An Introduction to the Theory of Graphs." It is this title which, of course, inspired the title of the present volume. Since then, the contents have been altered many times and various versions have been used for courses taught at Western Michigan University, Pahlavi University (Shiraz, Iran), and the National University of Iran in Tehran.

The book is intended for a one year introductory course in the theory of graphs at the beginning graduate level. To be sure, there are many proofs (some of which are quite lengthy), and the instructor may find himself in the necessary position of being selective as to which theorems are proved in class. In those instances we would hope that the student take it upon himself to investigate the technique of proof employed. Much of the material (including exercises) is sequential in nature; thus, we would suggest if class time does not permit a proof of a theorem or the assignment of certain exercises that at least an understanding of the meaning of the theorem or exercise be gained before proceeding. We have tried to select problems which range from routine to challenging (but not impossible). There is a bibliography at the end of each chapter and keyed number references within the text for the student who may wish to pursue a topic in greater detail.

The first chapter is quite elementary, on purpose. We hope to give the student a feeling for the subject he is about to encounter. The theory actually begins with Chapter 2. Graph theory is often an intuitive and sometimes even a picturesque area. We have made every attempt to be rigorous, but not at the expense of taking the "fun" out of the subject.

The theory of graphs is still young enough that there is certainly a wide variation of opinion as to what should be included in a book such as this. Naturally, the authors have taken it upon themselves to select the material most appropriate for the purpose of this book. We can only hope that we have chosen wisely.

We would like to think that the student is about to embark on the reading of an error free book; however, we are not that naïve. We apologize for any mistakes which exist and hope such errors cause a minimum of inconvenience.

M.B.
G.C.

Acknowledgment

Seldom is a textbook the result of the efforts of only the author or, in this case, the co-authors. There are several individuals whom we would like to thank.

Above all, we are grateful to our "academic father" E. A. Nordhaus, who taught us graph theory. We are indebted to Frank Harary who has been an "academic foster father" to us; furthermore, we have been influenced by his books and papers. Thank you Professors Nordhaus and Harary for all that you have done for us.

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M.B.
G.C.

I

Graphs and Subgraphs

Throughout the many branches of mathematics one frequently encounters the fundamental concepts of "set" and "relation". The theory of graphs offers no exception to this. In fact, a graph may be defined as a finite, nonempty set A together with a symmetric, irreflexive relation R on A; i.e., a, $b \in A$ and $(a, b) \in R$ imply $(b, a) \in R$, and for all $a \in A$, $(a, a) \notin R$.

In this first chapter we introduce many of the basic terms of graph theory and present examples and some results to reinforce the meaning of these concepts. We delay until Chapter 2 a discussion of the structure of graphs and fundamental theorems related to this.

1.1 Graphs

A *graph* G (sometimes called an *ordinary graph*) is a finite, nonempty set V together with a (possibly empty) set E (disjoint from V) of two-element subsets of (distinct) elements of V. Each element of V is referred to as a *vertex* and V itself as the *vertex set* of G; the members of the *edge set* E are called *edges*. By an *element of a graph* we shall mean a vertex or an edge. (It should be noted at the outset that the terms introduced here are not used universally. For example, point and node are often synonyms for vertex, and line is sometimes used instead of edge. Indeed, the reader of an article on graph theory might do well to check the author's interpretation of the word "graph" itself.) In general, we represent the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. If it is clear from the context, we denote the vertex and edge sets of G by V and E, respectively.

The edge $e = \{u, v\}$ is said to *join* the vertices u and v. If $e = \{u, v\}$ is an edge of a graph G, then u and v are *adjacent vertices* while u and e are *incident*, as are v and e. Furthermore, if e_1 and e_2 are distinct edges of G incident with a common vertex, then e_1 and e_2 are *adjacent edges*. It is convenient to henceforth denote an edge by uv or vu rather than $\{u, v\}$.

One of the most appealing features of graph theory lies in the geometric or pictorial aspect of the subject. Given a graph G, it is often useful to express it diagrammatically, whereby each element of V is represented by a point in the plane and each edge by a line segment or Jordan arc joining appropriate distinguished points. So that confusion is eliminated, no point of such a line segment or Jordan arc (except the two endpoints) should correspond to a vertex of G. It is convenient to refer to such a diagram of G as G itself, since the sets V and E are easily discernible. In Fig. 1.1, a graph G is shown with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{v_1v_2, v_1v_5, v_2v_3, v_2v_4, v_3v_4, v_3v_5, v_4v_5\}$.

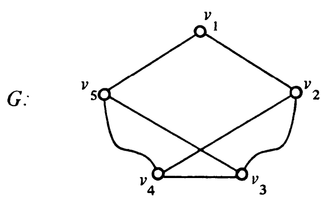


Figure 1.1 A graph

It is extremely important to understand what we shall mean for two graphs to be the same or different. Often apparently different geometric diagrams of graphs represent the same graph. For this purpose, we introduce the notion of isomorphism. A graph G_1 is *isomorphic* to a graph G_2 if there exists a one-to-one mapping ϕ, called an isomorphism, from $V(G_1)$ onto $V(G_2)$ such that ϕ preserves adjacency and nonadjacency; i.e., $uv \in E(G_1)$ if and only if $\phi u \, \phi v \in E(G_2)$. It is easy to see that the relation "isomorphic to" is an equivalence relation on graphs; hence this relation divides the collection of all graphs into equivalence classes, two graphs being *non-isomorphic* and considered different if they belong to different equivalence classes. If G_1 is isomorphic to G_2 then we say G_1 and G_2 are *isomorphic graphs* and write $G_1 = G_2$. The graphs G_1 and G_2 of Fig. 1.2 are isomorphic; for example the mapping ϕ: $V(G_1) \to V(G_2)$ defined by

$$\phi v_1 = v_1, \ \phi v_2 = v_3, \ \phi v_3 = v_5, \ \phi v_4 = v_2, \ \phi v_5 = v_4, \ \phi v_6 = v_6$$

is an isomorphism. On the other hand, $G_1 \neq G_3$ since G_3 contains three mutually adjacent vertices but G_1 does not. Of course, $G_2 \neq G_3$.

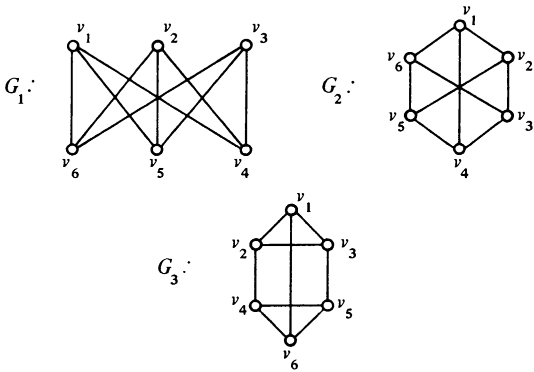


Figure 1.2 Isomorphic and non-isomorphic graphs

As usual, $|S|$ denotes the number of elements in a set S. For a graph G, if $|V| = p$ and $|E| = q$, then G is called a (p, q) *graph*; the number p is also referred to as the *order* of G. Thus, all (both) graphs of Fig. 1.2 are $(6, 9)$ graphs. The symbols $p(G)$ and $q(G)$ will be reserved exclusively for the number of vertices and number of edges, respectively, of a graph G. If it is clear from the context, then we write simply p and q rather than $p(G)$ and $q(G)$. Indeed, for any parameter $f(G)$ of a graph G, we write only f if the graph under consideration is understood. By definition $p \geq 1$ and $0 \leq q \leq \binom{p}{2} = p(p-1)/2$. There is only one $(1, 0)$ graph, and this is referred to as the *trivial graph*. A nontrivial graph then has $p \geq 2$.

A graph G is *labeled* (or *vertex-labeled*) if its p vertices are associated with p distinct labels in a one-to-one manner. We may then refer to the set of labels as the vertex set of G. Common examples of labels are $v_1, v_2, \ldots, v_p$ and $u_1, u_2, \ldots, u_p$. An *edge-labeled* graph is defined analogously. Two labeled graphs G_1 and G_2 having the same labels are *identical* if $G_1 = G_2$ and there exists a (label-preserving) isomorphism ϕ between G_1 and G_2 with $\phi v = v$ for all $v \in V(G_1)$. If G_1 and G_2 are identical, then we write $G_1 \equiv G_2$. Two labeled graphs may be isomorphic yet not identical; in fact, this is the case for graphs G_1 and G_2 of Fig. 1.2.

All 20 non-identical $(4,3)$ graphs having vertex set $\{1,2,3,4\}$ are shown in Fig. 1.3. Among those graphs only three are non-isomorphic, however. The total number of non-identical graphs having vertex set $\{1,2,3,4\}$ is 64, which is a consequence of our first result.

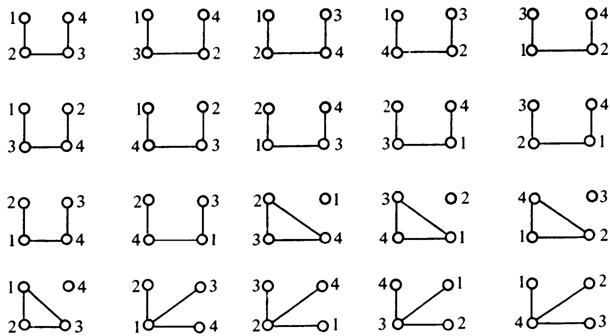


Figure 1.3 The non-identical $(4,3)$ graphs having vertex set $\{1,2,3,4\}$

Theorem 1.1 The number of non-identical graphs of order p with the same vertex set is $2^{p(p-1)/2}$.

Proof For $p = 1$ the theorem is obvious, so we assume $p \geq 2$. Let G be a graph with vertex set V. For each pair u, v of elements of V there are two possibilities depending on whether uv

is or is not an edge of G. Since there are $p(p-1)/2$ distinct pairs of elements of V, there are $2^{p(p-1)/2}$ such non-identical graphs G. ■

Although Theorem 1.1 provides the number of non-identical graphs with p given vertices, this theorem does not imply a corresponding result for non-isomorphic graphs; indeed we must wait until Chapter 18 for a discussion of this latter problem.

PROBLEM SET 1.1

1.1 Figure 1.2 shows two non-isomorphic (6, 9) graphs. Give an example of two other non-isomorphic graphs H_1 and H_2 such that $p(H_1) = p(H_2)$ and $q(H_1) = q(H_2)$.

1.2 Determine all non-isomorphic graphs of order five.

1.3 Determine all non-identical (4, 4) graphs having vertex set $\{1, 2, 3, 4\}$.

1.4 With the aid of Exercise 1.3 and Fig. 1.3, determine all 64 non-identical graphs having vertex set $\{1, 2, 3, 4\}$. How many of these are non-isomorphic?

1.5 Prove or disprove: Let $V = \{v_1, v_2, \ldots, v_p\}$. The number of non-identical (p, q) graphs having vertex set V equals the number of non-identical $\left(p, \binom{p}{2} - q\right)$ graphs having vertex set V.

1.2 Variations of Graphs

This section could very well be titled "Graphs Which Are Not Graphs". Although our only concern here is graphs, there are occasions when this concept is not entirely suitable for a particular problem or topic under investigation. It is advisable therefore to consider the following variations of graphs.

A *loop-graph* is a finite nonempty set V together with a set E (disjoint from V) consisting of one- or two-element subsets of V, each one-element subset being referred to as a *loop*. Each element of V is referred to as a vertex here also, and each element of E is an edge; hence every loop is an edge. Every graph is therefore a loop-graph, but not conversely. In terms of diagrams, the loop-graph L with $V(L) = \{v_1, v_2, \ldots, v_7\}$ and $E(L) = \{\{v_1\}, \{v_4\}, \{v_6\}, \{v_7\}, \{v_1, v_2\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_5, v_6\}\}$ may be illustrated as in Fig. 1.4.

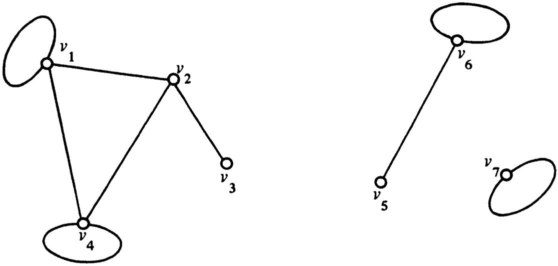


Figure 1.4 A loop-graph

A *directed graph* or *digraph* is a finite nonempty set V (of vertices) together with a set E (disjoint from V) of ordered pairs of distinct elements of V. In this case, we refer to the elements of E as *arcs*. The digraph D with $V(D) = \{u, v, w\}$ and $E(D) = \{(u, v), (u, w), (w, u)\}$ is shown in Fig. 1.5; in this example note that (u, v) is an arc of D, while (v, u) is not.

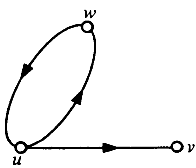


Figure 1.5 A digraph

If for each arc (u, v) of a digraph D, $(v, u) \in E(D)$, then the two arcs (u, v) and (v, u) of D may be denoted together as $\{u, v\}$. Thus, in a sense, a graph is a "symmetric" digraph. There is a theory of digraphs which is essentially distinct from that of graphs. The books [**2, 5**] deal with this theory.

According to the definition of "graph", two distinct vertices are joined by one edge or no edges. If one allows more than one edge (but a finite number) to join pairs of vertices, the result is then called a *multiple graph* or *multigraph*. If two or more edges join the same two vertices in a multigraph, then these edges are referred to as *multiple edges*. In describing a multigraph, it is necessary therefore to specify not only the vertex set and edge set, but the number of edges joining a given pair of vertices as well. A multigraph M is shown pictorially in Fig. 1.6.

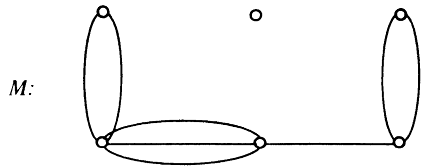


Figure 1.6 A multigraph

It is possible to arrive at other variations of graphs by using combinations of the aforementioned concepts. For instance, a *pseudograph* is a loop-graph which admits multiple edges (including multiple loops). A pseudograph is shown in Fig. 1.7.

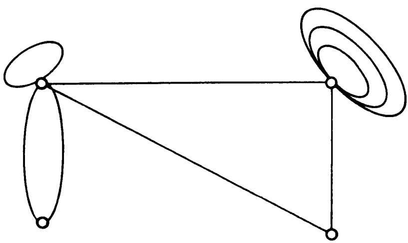


Figure 1.7 A pseudograph

If in the definition of "graph", the set V is taken to be infinite, then we are dealing with *infinite graphs*. Although we shall not be concerned with these "graphs", there are numerous interesting problems in this area [6].

PROBLEM SET 1.2

1.6 Let G be defined in terms of sets V and E where $|V|=p$. Determine the maximum value of $|E|$ if G is a

(a) loop-graph
(b) digraph
(c) "loop-digraph".

1.7 Give a definition of "G_1 is isomorphic to G_2" if G_1 and G_2 are
(a) loop-graphs
(b) digraphs
(c) multigraphs.

1.8 Replace "two-element subsets" in the definition of a graph by "r-element subsets", where r is an integer exceeding 1, to obtain the notion of an "r-graph". For the concepts defined thus far for graphs, define logical analogues for r-graphs.

1.3 Subgraphs

It is often the case that a graph which is under study is contained within some larger graph also being investigated. We consider several instances of this now. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; in such a case, we also say that G is a *supergraph* of H. Any graph isomorphic to a subgraph of G is also referred to as a *subgraph* of G.

The simplest type of subgraph of a graph G is that obtained by the deletion of a vertex or edge. If $v \in V(G)$, $|V(G)| \geq 2$, then $G - v$ denotes the subgraph with vertex set $V(G) - \{v\}$ and whose edges are all those of G not incident with v; if $e \in E$, then $G - e$ is the subgraph having vertex set $V(G)$ and edge set $E(G) - \{e\}$. The deletion of a set of vertices or set of edges is defined analogously. These concepts are illustrated in Fig. 1.8.

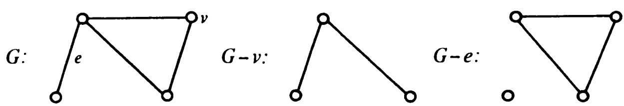


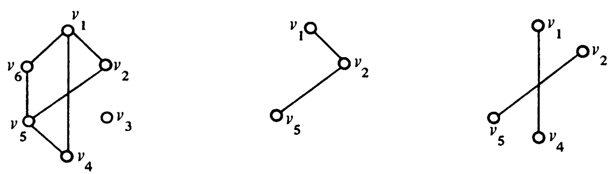
Figure 1.8 The deletion of an element of a graph

If u and v are nonadjacent vertices of a graph G, then $G+f$, where $f = uv$, denotes the graph with vertex set $V(G)$ and edge set $E(G) \cup \{f\}$. The graph $G+f$ is therefore a supergraph of G.

We have seen that $G - e$ has the same vertex set as G and that G has the same vertex set as $G+f$. Whenever a subgraph H of a graph G has the same order as that of G, then H is called a *spanning subgraph* of G.

The most important subgraphs which we shall encounter are the "induced subgraphs." If U is a nonempty subset of the vertex set $V(G)$ of a graph G, then the subgraph $\langle U \rangle$ of G *induced by* U is the graph having vertex set U and whose edge set consists of those edges of G incident with two elements of U. A subgraph H of G is called *vertex-induced* or *induced* if $H = \langle U \rangle$ for some subset U of V. Similarly, if F is a nonempty subset of $E(G)$, then the subgraph $\langle F \rangle$ *induced by* F is the graph whose vertex set consists of those vertices of G incident with at least one edge of F and whose edge set is F. A subgraph H of G is *edge-induced* if $H = \langle F \rangle$ for some subset F of $E(G)$. It is a simple consequence of the definitions that every induced subgraph of a graph G can be obtained by the removal of vertices from G while every subgraph of G can be obtained by the deletion of vertices and edges. These concepts are illustrated in Fig. 1.9 for the graph G, where

$$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\},\ U = \{v_1, v_2, v_5\}, \text{ and } F = \{v_1v_4, v_2v_5\}.$$



G $\langle U \rangle$ $\langle F \rangle$

Figure 1.9 Vertex-induced and edge-induced subgraphs

Before concluding this section, the reader should be aware of possible confusion between non-isomorphic and non-identical subgraphs. For example, in the graph G of Fig. 1.1, how many subgraphs of G have three vertices and three edges? The answer is obviously two (and this is independent of the fact that G is labeled) since what is certainly desired here is the number of non-identical such subgraphs. The reader could incorrectly give an answer of "one" here, interpreting the question as the number of non-isomorphic such subgraphs. Hence the reader must consider carefully the context in which the question is posed.

PROBLEM SET 1.3

1.9 Determine all non-isomorphic subgraphs of the graph G of Fig. 1.8. How many of these are induced? How many are edge-induced?

1.10 For a graph G, let $V_1, V_2 \subseteq V(G)$, where $V_1, V_2, V_1 \cap V_2 \neq \emptyset$. Prove that

(a) $E(\langle V_1 \rangle) \cup E(\langle V_2 \rangle) \subseteq E(\langle V_1 \cup V_2 \rangle)$,

(b) $E(\langle V_1 \rangle) \cap E(\langle V_2 \rangle) = E(\langle V_1 \cap V_2 \rangle)$.

1.11 Show that it is not always true that every edge-induced subgraph of G can be obtained by removing edges from G.

1.12 How many subgraphs of the graph G of Fig. 1.1 contain four vertices and four edges?

1.4 Degree Sequences

With the exception of the number of vertices and number of edges, the collection of numbers which one encounters most frequently in the study of graphs are the degrees of its vertices. The *degree* or *valency of a vertex* v in a graph G is the number of edges of G incident with v; a vertex of degree n is called *n–valent*. The degree of a vertex v in G is denoted $\deg_G v$ or simply $\deg v$ if G is clear from the context. A vertex is called *odd* or *even* depending on whether its degree is odd or even. A 0-valent vertex of G is called an *isolated vertex* and a 1-valent vertex is an *end-vertex* of G. In Fig. 1.10 a graph is shown together with the degree of each vertex.

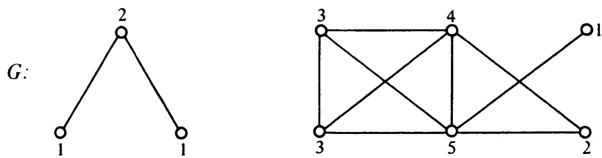


Figure 1.10 The degree of the vertices of a graph

Observe that for the graph G of Fig. 1.10, $p = 9$ and $q = 11$, while the sum of the degrees of its nine vertices is 22. The fact that this last number equals $2q$ for the graph G is not merely a coincidence. Every edge is incident with two vertices; hence, when the degrees of the vertices are summed each edge is counted twice. We state this as our next theorem.

Theorem 1.2 Let G be a (p,q) graph where $V(G)=\{v_1, v_2, \ldots, v_p\}$. Then

$$\sum_{i=1}^{p} \deg v_i = 2q.$$

This result has an interesting consequence.

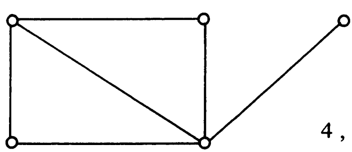
Corollary 1.2a In any graph, there is an even number of odd vertices.

Proof Let G be a (p,q) graph. If G has no odd vertices, the corollary, of course, is obvious. Otherwise, let $v_1, v_2, \ldots, v_k$ be the odd vertices of G so that $v_{k+1}, v_{k+2}, \ldots, v_p$ are the even vertices. By Theorem 1.2,

$$\sum_{i=1}^{p} \deg v_i = \sum_{i=1}^{k} \deg v_i + \sum_{i=k+1}^{p} \deg v_i = 2q.$$

Certainly, $\sum_{i=k+1}^{p} \deg v_i$ is even; hence $\sum_{i=1}^{k} \deg v_i$ is even, implying that k is even and thereby proving the corollary. ∎

A sequence $d_1, d_2, \ldots, d_p$ of nonnegative integers is called a *degree sequence* of a graph G if the vertices of G can be labeled $v_1, v_2, \ldots, v_p$ so that $\deg v_i = d_i$ for all i. Often we express the sequence so that $d_1 \geq d_2 \geq \cdots \geq d_p$. For example, a degree sequence of the graph of Fig. 1.11 is $4, 3, 2, 2, 1$.



$4, 3, 2, 2, 1$

Figure 1.11 A degree sequence of a graph

Given a graph G, a degree sequence of G can be easily determined, of course. On the other hand, if a sequence s: $d_1, d_2, \ldots, d_p$

of nonnegative integers is given, then under what conditions is s a degree sequence of some graph? If such a graph exists, then s is called a *graphical sequence*. Certainly the conditions $d_i \leq p - 1$ for all i and $\sum_{i=1}^{p} d_i$ is even are necessary for a sequence to be graphical, but these conditions are not sufficient. The sequence $3, 3, 3, 1$ is not graphical, for example. A necessary and sufficient condition for a sequence to be graphical was found by Havel [3] and later rediscovered by Hakimi [1].

Theorem 1.3 A sequence s: $d_1, d_2, \ldots, d_p$ of integers with $d_1 \geq d_2 \geq \cdots \geq d_p$, $p \geq 2$, $d_1 \geq 1$, is graphical if and only if the sequence s_1: $d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_p$ is graphical.

Proof Suppose s_1 is a graphical sequence. Then there exists a graph G_1 with $V(G_1) = \{v'_2, v'_3, \ldots, v'_p\}$ such that

$$\deg v'_i = \begin{cases} d_i - 1 & 2 \leq i \leq d_1 + 1 \\ d_i & d_1 + 2 \leq i \leq p. \end{cases}$$

A new graph G can now be constructed by adding a new vertex v_1 and the d_1 edges $v_1 v'_i$, $2 \leq i \leq d_1 + 1$. Relabeling the vertices v'_i, $2 \leq i \leq p$, as v_i in G, we now have $\deg v_i = d_i$ for $1 \leq i \leq p$. Hence the sequence s: $d_1, d_2, \ldots, d_p$ is graphical.

Conversely, let s be a graphical sequence. Thus a graph G exists with $V(G) = \{v_1, v_2, \ldots, v_p\}$ such that $\deg v_i = d_i$ for $1 \leq i \leq p$. At this point we distinguish two cases.

CASE 1. *Suppose* G *contains a vertex* u *of degree* d_1 *such that* u *is adjacent with vertices having degrees* $d_2, d_3, \ldots, d_{d_1+1}$. In this case the graph $G - u$ has degree sequence s_1 so that s_1 is graphical.

CASE 2. *Suppose no vertex* u *exists as in Case 1*. Since v_1 is not adjacent to all vertices v_i, $2 \leq i \leq d_1 + 1$, there exist vertices v_j and v_k with $d_j > d_k$ such that v_1 is adjacent to v_k but not to v_j. Since the degree of v_j exceeds that of v_k, there exists a vertex v_n such that v_n is adjacent to v_j but not to v_k. (See Fig. 1.12(a).) The removal of the edges $v_1 v_k$ and $v_j v_n$ and addition of the edges $v_1 v_j$ and $v_k v_n$ (see Fig. 1.12(b)) results in a graph G' having the same degree sequence as G. However, in G' the sum of the degrees of the vertices adjacent with v_1 is larger than that in G.

If we continue this procedure we necessarily arrive at a graph in which v_1 satisfies the hypothesis of Case 1 from which it follows that s_1 is graphical. ■

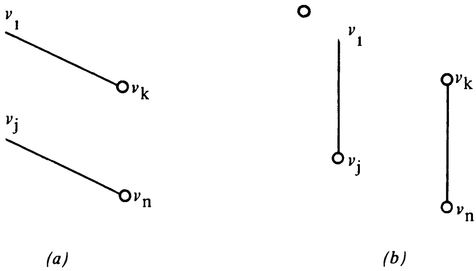


Figure 1.12 A step in the proof of the Havel–Hakimi Theorem

Theorem 1.3 is now illustrated, and we determine whether the sequence

$$5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1, 0, 0$$

is graphical. Since 0 can correspond only to isolated vertices, the answer depends on whether the sequence

$$s\colon\ 5, 3, 3, 3, 3, 2, 2, 2, 1, 1, 1$$

is graphical. No term exceeds $p-1$ and the sum of the terms is even; hence we apply Theorem 1.3, getting

$$s'_1\colon\ 2, 2, 2, 2, 1, 2, 2, 1, 1, 1.$$

We rearrange this sequence obtaining

$$s_1\colon\ 2, 2, 2, 2, 2, 2, 1, 1, 1, 1.$$

Not recognizing whether s_1 is graphical, we continue:

$$\begin{aligned} s'_2\colon&\ \ 1, 1, 2, 2, 2, 1, 1, 1, 1 \\ s_2\colon&\ \ 2, 2, 2, 1, 1, 1, 1, 1, 1 \\ s'_3 = s_3\colon&\ \ 1, 1, 1, 1, 1, 1, 1, 1. \end{aligned}$$

The sequence s_3 is now easily seen to be graphical since it is the degree sequence of the graph G_3 of Fig. 1.13. By Theorem 1.3, each of the sequences s_2, s_1 and s are in turn graphical. To construct a graph with degree sequence s_2, we proceed in reverse from s'_3

to s_2 observing that a vertex should be added to G_3 so that it is adjacent to two vertices of degree 1. We thus obtain a graph G_2 with degree sequence s_2 (or s'_2). Proceeding from s'_2 to s_1, we again add a new vertex joining it to two vertices of degree 1 in G_2. This gives a graph G_1 with degree sequence s_1 (or s'_1). Finally, we obtain a graph G with degree sequence s by considering s'_1, i.e., a new vertex is added to G_1, joining it to vertices of degrees $2, 2, 2, 2, 1$. The graph G is then completed by inserting two isolated vertices.

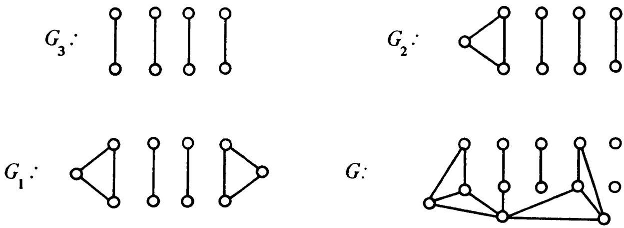


Figure 1.13 Construction of a graph G with given degree sequence

It should be pointed out that the graph G of Fig. 1.13 is not the only graph with degree sequence s. Indeed, in general, it is not known how many non-isomorphic graphs have a given sequence.

PROBLEM SET 1.4

1.13 Determine whether the following sequences are graphical. If so, construct a graph with the appropriate degree sequence.

(a) $4, 4, 3, 2, 1, 0$
(b) $3, 3, 2, 2, 2, 2, 1, 1, 0$
(c) $7, 4, 3, 3, 2, 2, 2, 1, 1, 1, 0$.

1.14 Show that no nontrivial sequence with distinct terms is graphical.

1.15 Prove that the sequence $d_1, d_2, \ldots, d_p$ is graphical if and only if the sequence $p - d_1 - 1, p - d_2 - 1, \ldots, p - d_p - 1$ is graphical.

1.16 (a) Show that s: $7, 6, 5, 4, 4, 3, 2, 1$ is graphical.
(b) Prove that there exists exactly one graph with degree sequence s.

1.17 Define the concept of "degree" for multigraphs. State and prove analogues to Theorem 1.2 and Corollary 1.2a for multigraphs.

1.5 Special Graphs

There are certain classes of graphs which occur so often that they deserve special mention and in some cases, special notation. We describe the most prominent of these in this section.

A graph G is *regular of degree r* if for each vertex v of G, $\deg v = r$; such graphs are also called *r-regular*. The 3-regular graphs are referred to as *cubic* graphs. A graph is *complete* if every two of its vertices are adjacent. A complete (p, q) graph is therefore a regular graph of degree $p-1$ having $q = p(p-1)/2$; we denote this graph by K_p. In Fig. 1.14 are shown all regular graphs with $p = 4$, including the complete graph K_4.

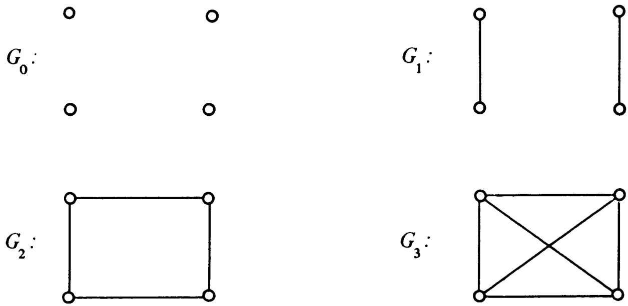


Figure 1.14 The regular graphs of order 4

Every graph is a subgraph of a regular graph; indeed, if G is a graph of order p, then G is a spanning subgraph of K_p. Of course, every vertex of K_p has degree $p-1$ while no vertex of G need have degree $p-1$. This suggests the problem of determining whether every given graph G is a spanning subgraph of a regular graph H, whose degree of regularity is equal to the largest degree among the vertices of G. The answer to this question is in the negative, however, for no graph H can exist if G has odd order and the largest degree among its vertices is also odd. If we replace "spanning" by "induced," we do obtain an affirmative answer.

By $\Delta(G) = \Delta$, we shall mean the *maximum degree* among the vertices *of G;* the symbol $\delta(G) = \delta$ will represent the *minimum degree of G*.

Theorem 1.4 For any graph G, there exists a supergraph H of G such that

(i) H is regular of degree $\Delta(G)$, and
(ii) G is an induced subgraph of H.

Proof Let $\Delta = d_1, d_2, \ldots, d_p$ be a degree sequence of G. If G is regular, then we let $H = G$. Otherwise, we assume G is not regular so that $\Delta \geq 1$ and $p \geq 3$. Suppose

$$\sum_{i=1}^{p} (\Delta - d_i) = n.$$

We consider two cases, according to the parity of n.

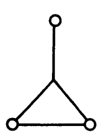
CASE 1. *Suppose* n *is even.* Let e be any edge of the graph $K_{1+\Delta}$, and consider $n/2$ disjoint copies of $K_{1+\Delta} - e$. Join the n vertices of degree $\Delta - 1$ of these copies to the vertices of G whose degrees are less than Δ to obtain a regular graph H of degree Δ. (There may very well be several graphs H obtainable in this manner.) We now observe that G is an induced subgraph of H.

CASE 2. *Suppose* n *is odd.* Again let e be any edge of $K_{1+\Delta}$, and consider $(n-1)/2$ disjoint copies of $K_{1+\Delta} - e$. We join the $n-1$ vertices of degree $\Delta - 1$ suitably to the vertices of G having degree less than Δ to obtain a graph H_1 with the property that all but one vertex v_1 of H_1 have degree Δ and deg $v_1 = \Delta - 1$. Let H_2 be another copy of H_1, where v_2 denotes the vertex of H_2 having degree $\Delta - 1$. The graph H with the desired properties can now be obtained from H_1 and H_2 by adding the edge v_1v_2. ■

We illustrate the two cases of Theorem 1.4 with two graphs. First, let G' be as shown in Fig. 1.15. Here $\Delta = 3$ and $n = 4$. Hence we take two copies of $K_4 - e$ and form the Δ-regular graph H' (the added edges indicated by dashed lines).

In Fig. 1.16 we have $\Delta = 3$ and $n = 5$ for the graph G. This graph

G':



H':

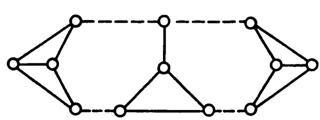


Figure 1.15 Construction of a regular supergraph (*Case 1*)

therefore comes under Case 2 of Theorem 1.4. Then H_1, H_2, and the Δ-regular graph H are illustrated (again the added edges shown with dashed lines).

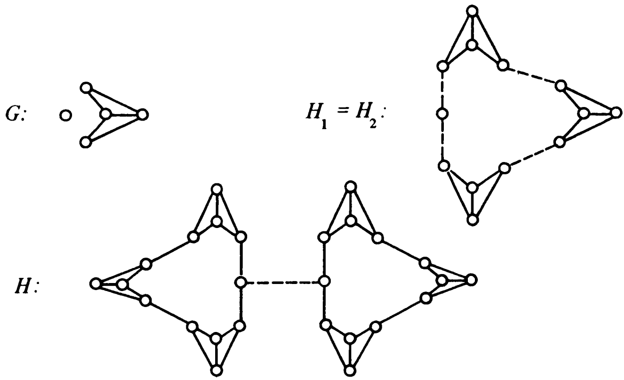


Figure 1.16 Construction of a regular supergraph (*Case 2*)

The *complement* $\overline{G}$ of a graph G is the graph with the same vertex set as that of G and such that two vertices are adjacent in $\overline{G}$ if and only if these vertices are not adjacent in G. Hence if G is a (p,q) graph, then $\overline{G}$ is a $(p,\bar{q})$ graph, where $q+\bar{q}=p(p-1)/2$. In Fig. 1.14 the graphs G_0 and G_3 are complementary, as are G_1 and G_2. The complement $\overline{K}_p$ of the complete graph K_p has p vertices and no edges and is referred to as the *empty graph* of order p. A graph G is *self-complementary* if $G=\overline{G}$.

A graph G is *n-partite*, $n \geq 2$, if it is possible to partition V into n subsets $V_1, V_2, \ldots, V_n$ such that every element of E joins a vertex of V_i to a vertex of V_j, $i \neq j$. For $n=2$, such graphs are called *bigraphs or bipartite graphs;* this class of graphs is particularly important and will be encountered many times. In Fig. 1.17 a bipartite graph G_1 is shown; a second graph G_2, isomorphic to G_1, is also given to emphasize the bipartite character of G_1.

A *complete n-partite* graph G is an n-partite graph with vertex set partition $V_1, V_2, \ldots, V_n$ having the added property that if $u \in V_i, v \in V_j$, $i \neq j$, then $uv \in E(G)$. If $|V_i| = p_i$, then this graph is denoted by $K(p_1, p_2, \ldots, p_n)$. Note that a complete n-partite graph is complete if and only if $p_i = 1$ for all i, in which case it is K_n.

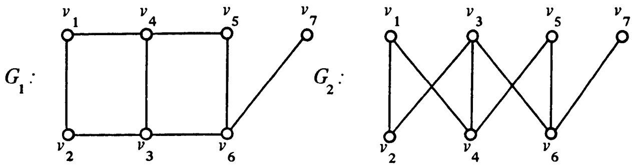


Figure 1.17 A bipartite graph

A graph with vertex partition V_1, V_2, where $|V_1| = m$ and $|V_2| = n$, is usually denoted by $K(m, n)$; the graph $K(1, n)$ is called a *star graph*. We note that $K(m, n) = K(n, m)$.

PROBLEM SET 1.5

1.18 Let G be a cubic (p, q) graph, where $q = 2p - 3$. What can be said about G?

1.19 (a) Let G be a self-complementary graph of order p. Show that $p \equiv 0$ or $p \equiv 1$ (modulo 4).

(b) Show that if there exists a self-complementary graph of order p, where $p \equiv 0$ (modulo 4), then there exists a self-complementary graph of order $p + 1$.

1.20 Determine the number of edges in a complete n-partite graph.

1.21 Let G be a nonempty graph with the property that whenever $uv \notin E$ and $vw \notin E$, then $uw \notin E$. Prove that G has this property if and only if it is a complete n-partite graph for some $n \geq 2$.

1.22 Let $G = K(1, 3)$. Use the proof of Theorem 1.4 to determine a graph H which satisfies (i) and (ii) of Theorem 1.4. Further, determine a graph H of smallest order which satisfies (i) and (ii).

1.23 Let $G = K(1, n)$, $n \geq 2$. Determine (with proof) the minimum order of a graph H which satisfies (i) and (ii) of Theorem 1.4.

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2

Connected and Disconnected Graphs

In this chapter we begin our investigation of the structure of graphs. The most basic property that a graph may possess is that of being connected. It is this topic that we consider here. We precede a discussion of this concept, however, with a few elementary definitions.

2.1 Paths and Cycles

Let u and v be (not necessarily distinct) vertices of a graph G. By a u–v *walk* of G is meant a finite, alternating sequence of vertices and edges of G, beginning with u and ending with v, such that every edge is immediately preceded and succeeded by the two vertices with which it is incident. A *trivial walk* is one containing no edges. We note that in a walk there may be repetition of vertices and edges. Often only the vertices of a walk are indicated since the edges present are then evident. Two u–v walks, $u = u_0, u_1, \ldots, u_n = v$ and $u = v_0, v_1, \ldots, v_m = v$ are considered to be *equal* if and only if $n = m$ and $u_i = v_i$ for $0 \leq i \leq n$; otherwise, they are *different.* Observe that the edges of two different u–v walks of G may very well induce the same subgraph of G.

A u–v walk is *closed* or *open* depending on whether $u = v$ or

$u \neq v$. A u–v *trail* is a u–v walk in which no edge is repeated, while a u–v *path* is a u–v walk in which no vertex is repeated; a vertex u forms the *trivial u–u path*. Every path is therefore a trail. A nontrivial closed trail of G is referred to as a *circuit* of G. It is evident that if a vertex v of degree 2 lies on a circuit C of G, then C also contains the two edges incident with v. In the graph G of Fig. 2.1, W_1: $v_1, v_2, v_3, v_2, v_5, v_3, v_4$ is a v_1–v_4 walk which is not a trail, W_2: $v_1, v_2, v_5, v_1, v_3, v_4$ is a v_1–v_4 trail which is not a path, while W_3: v_1, v_3, v_4 is a v_1–v_4 path.

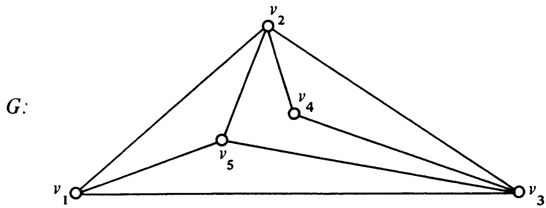

Figure 2.1 Walks, trails, and paths

By definition, every path is a walk. Although the converse of this statement is not true, we do have the following.

Theorem 2.1 Every u–v walk W contains a u–v path.

Proof If W is closed, the result is trivial. Let W: $u = u_0, u_1, u_2, \ldots, u_n = v$ be an open u–v walk of a graph G. (It may be the case here that a vertex has received more than one label.) If no vertex of G occurs in W more than once, then W is a u–v path. Otherwise, there are vertices of G which occur in W twice or more. Let j be the smallest positive integer such that there exists $i < j$ with $u_i = u_j$. If the terms $u_i, u_{i+1}, \ldots, u_{j-1}$ are deleted from W, a u–v walk W_1 is obtained having fewer terms than that of W. If there is no repetition of vertices in W_1, then W_1 is a u–v path. If this is not the case, we continue the above procedure until finally arriving at a u–v walk which is a u–v path. ■

A circuit of a graph G in which no vertices are repeated (except the first and last) is called a *cycle* of G. An *acyclic graph* has no cycles. The subgraph of G induced by the edges of a trail, path, circuit, or cycle is also referred to as a *trail, path, circuit,* or *cycle* of G. The number of occurrences of edges in a walk is called its *length.* A cycle is *even* if its length is even; otherwise it is *odd.* A cycle of length n is an *n-cycle;* a 3-cycle is also called a *triangle.* A graph of order n which is a path or cycle is denoted by P_n or C_n, respectively.

We are now in a position to consider the chief concept of this section. Two vertices u and v of a graph G are said to be *connected* if there exists a u–v path in G; the graph G itself is *connected* if every two of its vertices are connected. A graph which is not connected is *disconnected.*

The relation "is connected to" is an equivalence relation on the vertex set of any graph G. Each subgraph induced by the vertices in an equivalence class is called a *component* of G. Equivalently, a component of a graph G is a connected subgraph of G not properly contained in any other connected subgraph of G; i.e., a component of G is a subgraph which is maximal with respect to the property of being connected. The number of components of G is denoted by $c(G)$; of course, $c(G) = 1$ if and only if G is connected. For the graph G of Fig. 2.2, $c(G) = 6$.

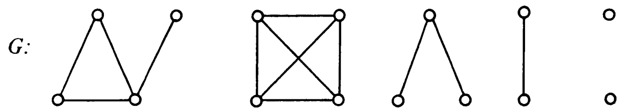

Figure 2.2 The components of a graph

With the concepts at hand, it is now possible to present an interesting characterization of bipartite graphs.

Theorem 2.2 A nontrivial graph is bipartite if and only if it contains no odd cycles.

Proof Let G be a bipartite graph with $V(G) = V_1 \cup V_2$ and V_1, $V_2 \neq \emptyset$, where each edge of G is incident with a vertex of V_1 and a vertex of V_2. Suppose C: $v_1, v_2, \ldots, v_k$, v_1 is a cycle of G. Without loss of generality, we may assume $v_1 \in V_1$. However, then, $v_2 \in V_2, v_3 \in V_1, v_4 \in V_2$, etc. This implies $k = 2s$ for some positive integer s; hence C has even length.

For the converse, it suffices to prove that every nontrivial connected graph G without odd cycles is bipartite, since a nontrivial graph is bipartite if and only if each of its nontrivial components is bipartite. Let $v \in V(G)$ and denote by V_1 the subset of $V(G)$ consisting of v and all vertices u of G with the property that any shortest u–v path of G has even length. Let $V_2 = V(G) - V_1$. We now prove that the partition $V_1 \cup V_2$ of $V(G)$ has the appropriate properties to show that G is bipartite.

Let u and w be elements of V_1, and suppose $uw \in E(G)$. Necessarily, then, neither u nor w is the vertex v. Let $v = u_1, u_2, \ldots, u_{2n+1} = u$, $n \geq 1$, and $v = w_1, w_2, \ldots, w_{2m+1} = w$, $m \geq 1$, be a shortest v–u path and a shortest v–w path of G, respectively, and suppose w' is the last vertex which the two paths have in common (w' may be v). The two v–w' subpaths so determined are then shortest v–w' paths. Thus, there exists an i such that $w' = u_i = w_i$. However, $u_i, u_{i+1}, \ldots, u_{2n+1}, w_{2m+1}, w_{2m}, \ldots, w_i = u_i$ is an odd cycle of G, which is a contradiction to our hypothesis. Similarly, no two vertices of V_2 are adjacent. ■

PROBLEM SET 2.1

2.1 Let u and v be arbitrary vertices of a connected graph G. Show that there exists a u–v walk containing all vertices of G.

2.2 Prove that every circuit of G contains a cycle of G.

2.3 Prove that if a vertex is repeated in a trail of a graph G, then the trail contains a cycle of G.

2.4 Prove that "is connected to" is an equivalence relation on the vertex set of a graph.

2.5 Let G be a graph of order p such that $\delta(G) \geq (p-1)/2$. Prove that G is connected.

2.6 Show that if G is a (p, q) graph for which $q < p - 1$, then G is disconnected.

2.7 Prove that there exists no graph G such that G and $\overline{G}$ are disconnected.

2.8 If $G = K(p_1, p_2, \ldots, p_n)$, determine $c(\overline{G})$.

2.2 Cut-vertices and Bridges

For the most part, our attention will be directed toward connected graphs. There are some connected graphs, however, which are connected so slightly that they may be disconnected by the removal of a single vertex or single edge. Such vertices and edges play a special role in graph theory, and we discuss these in this section.

A vertex v of a graph G is called a *cut-vertex* of G if $c(G - v) > c(G)$. Thus, a vertex of a connected graph is a cut-vertex if its removal produces a disconnected graph. In general, a vertex v of a graph G is a cut-vertex of G if its removal disconnects a component of G.

According to this definition, no isolated vertex is a cut-vertex. The complete graphs have no cut-vertices while, at the other extreme, each nontrivial path contains only two vertices which are not cut-vertices. In order to see this is the other extreme, we prove the following theorem.

Theorem 2.3 Every nontrivial graph contains at least two vertices which are not cut-vertices.

Proof We establish the theorem for nontrivial connected graphs, as the result then follows for arbitrary graphs. We proceed by induction on p. The only connected graph with $p = 2$ is K_2, and neither of its vertices is a cut-vertex. Assume that all nontrivial connected graphs of order less than p, where $p > 2$, contain at least two vertices which are not cut-vertices, and let G be a connected graph of order p. If G has no cut-vertices, then the theorem follows. Suppose v is a cut-vertex of G, and let $G_1, G_2, \ldots, G_k$, $k \geq 2$, be the components of $G - v$. If any G_i is trivial, then, necessarily, the vertex of G_i is not a cut-vertex of G. Assume some G_i, say G_1, is nontrivial. By the induction hypothesis, G_1 has two vertices v_1 and v_2 which are not cut-vertices of G_1. If either v_1 or v_2 is not adjacent to v in G, then this vertex is not a cut-vertex of G; on the other hand, if both vv_1 and vv_2 are edges of G, then neither v_1 nor v_2 is a cut-vertex of G. Hence each G_i contains at least one vertex which is not a cut-vertex of G; this gives the desired result. ■

A characterization of cut-vertices is presented in the next theorem.

Theorem 2.4 A vertex v of a connected graph G is a cut-vertex of G if and only if there exist vertices u and w ($u, w \neq v$) such that v is on every u–w path of G.

Proof Let v be a cut-vertex of G so that the graph $G - v$ is disconnected. If u and w are vertices in different components of $G - v$, then there are no u–w paths in $G - v$; however, since G is connected, there are u–w paths in G. Therefore, every u–w path of G contains v.

Conversely, assume that there exist vertices $u, w \in V(G)$ such that the vertex v lies on every u–w path of G. Then there are no u–w paths in $G - v$, implying that $G - v$ is disconnected and that v is a cut-vertex of G. ■

Analogous to the cut-vertex is the concept of a bridge. A *bridge* (or *cut-edge*) of a graph G is an edge e such that $c(G-e) > c(G)$. If e is a bridge of G, then it is immediately evident that $c(G-e) = c(G) + 1$. Furthermore, if $e = uv$, then u and v are cut-vertices of G if and only if $\deg u > 1$ and $\deg v > 1$. Indeed, the complete graph K_2 is the only connected graph containing a bridge but no cut-vertices. Bridges are characterized in a manner similar to that of cut-vertices; the proof too is similar to that of Theorem 2.4 and is omitted.

Theorem 2.5 An edge e of a connected graph G is a bridge of G if and only if there exist vertices u and w such that e is on every u–w path of G.

| For bridges, there is another useful characterization. |

Theorem 2.6 An edge e of a graph G is a bridge of G if and only if e is on no cycle of G.

Proof We assume G to be connected. Let $e = uv$ be an edge of G, and suppose e lies on a cycle C of G. Further, let w_1 and w_2 be arbitrary distinct vertices of G. If e does not lie on a w_1–w_2 path P of G, then P is also a w_1–w_2 path of $G - e$. If, however, e lies on a w_1–w_2 path Q of G, then replacing e by the u–v path (or v–u path) on C not containing e produces a w_1–w_2 walk in $G - e$. By Theorem 2.1, there is a w_1–w_2 path in $G - e$. Hence e is not a bridge.

Conversely, suppose $e = uv$ is an edge of G which is on no cycle of G, and assume e is not a bridge. Thus $G - e$ is connected. Hence there exists a u–v path P in $G - e$; however, P together with e produce a cycle in G containing e, which is a contradiction. ■

From Theorem 2.6 it is natural to define a *cycle edge* of a graph G as an edge which is not a bridge of G.

PROBLEM SET 2.2

2.9 Prove Theorem 2.5.

2.10 Determine the maximum number of bridges possible in a graph of order $p \geq 2$.

2.11 Show that every connected $(p,p-1)$ graph, $p \geq 3$, contains a cut-vertex.

2.12 Prove that every connected (p,q) graph, $3 \leq p \leq q$, contains a cycle edge.

2.3 Blocks

Many of the graphs which we encounter fail to contain cut-vertices. It is this class of graphs which we discuss next. A nontrivial connected graph with no cut-vertices is called a *block*. Nontrivial connected graphs which are not blocks contain special subgraphs in which we are also interested. A *block of a graph G* is a subgraph of G, which is itself a block and which is maximal with respect to that property. A block is necessarily an induced subgraph, and, moreover, the blocks of a graph partition its edge set. Every two blocks have at most one vertex in common, namely a cut-vertex. The graph of Fig. 2.3 has five blocks B_i, $1 \leq i \leq 5$, as indicated. The vertices v_3, v_5, and v_8 are cut-vertices while v_3v_5 and v_4v_5 are bridges.

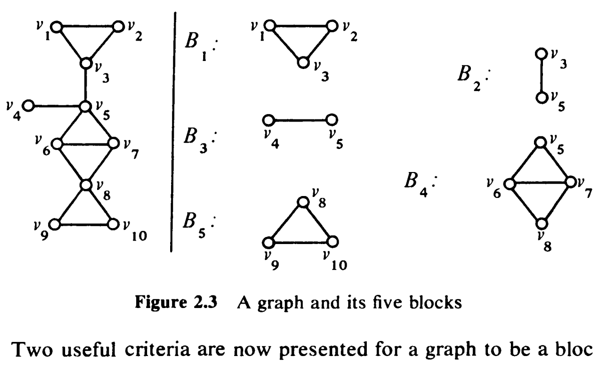

Figure 2.3 A graph and its five blocks

Two useful criteria are now presented for a graph to be a block.

Theorem 2.7 A graph G of order $p \geq 3$ is a block if and only if every two vertices of G lie on a common cycle of G.

Proof Let G be a graph such that each two of its vertices lie on a cycle. Thus G is connected. Suppose G is not a block; hence G contains a cut-vertex v. By Theorem 2.4, there exist vertices u and w such that v is on every u–w path. Let C be a cycle of G containing u and w. The cycle C determines two distinct u–w paths, one of which cannot contain v, contradicting the fact that every u–w path contains v. Therefore, G is a block.

Conversely, let G be a block with $p \geq 3$ vertices. We show that every two vertices of G lie on a common cycle of G. Let u be an arbitrary vertex of G, and denote by U the set of all vertices which lie on a cycle containing u. We now prove $U = V$. Assume $U \neq V$ so that there exists a vertex $v \in V - U$. Since G is a block, it contains no cut-vertices, and furthermore, since $p \geq 3$, the graph G contains no bridge. By Theorem 2.6, every edge of G lies on a cycle of G; hence, every vertex adjacent with u is an element of U. Since G is connected, there exists a u–v path W: $u = u_0, u_1, u_2, \ldots, u_n = v$ in G. Let i be the smallest integer, $2 \leq i \leq n$, such that $u_i \notin U$; thus $u_{i-1} \in U$. Let C be a cycle containing u and u_{i-1}. Because u_{i-1} is not a cut-vertex of G, there exists a u_i–u path P: $u_i = v_0, v_1, v_2, \ldots, v_m = u$ not containing u_{i-1}. If the only vertex common to P and C is u, then a cycle containing u and u_i exists, which produces a contradiction. Hence P and C have a vertex in common which is different from u. Let j be the smallest integer, $1 \leq j \leq m$, such that v_j belongs to both P and C. A cycle containing u and u_i can now be constructed by beginning with the u_i–v_j subpath of P, proceeding along C from v_j to u and then to u_{i-1}, and finally taking the edge $u_{i-1}u_i$ back to u_i. Thus, a contradiction arises again, implying that the vertex v does not exist and that every two vertices lie on a cycle. ■

Two u–v paths in a graph G are called *vertex-disjoint* (or simply *disjoint*) if they have no vertices in common, other than u and v. *Edge-disjoint paths* are defined analogously. A second characterization of blocks is now immediate.

Corollary 2.7a A graph G of order $p \geq 3$ is a block if and only if there exist two disjoint u–v paths for every two distinct vertices u and v of G.

Theorem 2.7 suggests the following definitions: A block of order $p \geq 3$ is called a *cyclic block* while the block K_2 is called the *acyclic block*.

We now state a theorem of which Theorem 2.3 is a corollary. (Note Exercises 2.16 and 2.17.)

Theorem 2.8 Let G be a connected graph with one or more cut-vertices. Then G contains at least two blocks, each of which contains exactly one cut-vertex of G.

In view of Theorem 2.8, we define an *end-block* of a graph G to be a block containing exactly one cut-vertex of G. Hence every connected graph with at least one cut-vertex contains at least two end-blocks.

We define the *block index* $b(v)$ of a vertex v of a graph G to be the number of blocks of G to which v belongs. It therefore follows that $b(v) = 0$ if and only if v is an isolated vertex and that $b(v) > 1$ if and only if v is a cut-vertex of G. The following formula [2] gives the number $b(G)$ of blocks of G in terms of the number $c(G)$ of components of G and the block indices of its vertices.

Theorem 2.9 For any graph G,

$$b(G) = c(G) + \sum_{v \in V} [b(v) - 1].$$

Proof We first consider the case where G is connected and prove

$$b(G) = 1 + \sum_{v \in V} [b(v) - 1].$$

If G is trivial, then $b(G) = 0$ and the formula is obvious. Hence assume G is nontrivial. We employ induction on the number of blocks of G. If $b(G) = 1$, the formula is immediate. Then suppose the formula to be true for all connected graphs with $n \geq 1$ blocks, and assume G has $n + 1$ blocks. By Theorem 2.8, there exists a block B of G containing exactly one cut-vertex u of G. Let G' be the graph obtained by deleting from G all vertices of B different from u. Each vertex so removed has block index 1 in G

while the block index of u in G' is one less than the block index of u in G. By hypothesis,

$$b(G') = 1 + \sum_{v \in V(G')} [b(v) - 1],$$

where $b(v)$ denotes the block index of v in G'. Since the numbers $b(G)$ and $1 + \sum_{v \in V} [b(v) - 1]$ are each one greater than the respective numbers $b(G')$ and $1 + \sum_{v \in V(G')} [b(v) - 1]$, the formula follows.

If G is disconnected, then consider the components G_i, $1 \le i \le c(G)$. For each i, we have

$$b(G_i) = 1 + \sum_{v \in V(G_i)} [b(v) - 1].$$

Summing over all i, we obtain

$$b(G) = c(G) + \sum_{v \in V} [b(v) - 1],$$

which is the desired formula. ■

PROBLEM SET 2.3

2.13 Determine the maximum number of cut-vertices which can lie in a single block of a connected graph of order $p \ge 3$.

2.14 Write out the details of a proof of Corollary 2.7a.

2.15 Let G be a block of order $p \ge 3$, and let u and v be distinct vertices of G. If P is a given u–v path of G, does there always exist a u–v path Q such that P and Q are disjoint u–v paths?

2.16 Prove Theorem 2.8.

2.17 Assuming Theorem 2.8, prove Theorem 2.3.

2.18 Let G be a graph with at least one cut-vertex. Prove that G contains a cut-vertex v with the property that, with at most one exception, all blocks of G containing v are end-blocks.

2.19 Prove that a graph G of order $p \ge 3$ is a block if and only if each pair of elements of G lie on a common cycle of G.

2.20 Let G be a graph having four blocks with $V(G) = \{v_1, v_2, \ldots, v_8\}$. Suppose each v_i, $1 \le i \le 6$, lies in exactly one block while each of v_7 and v_8 belongs to exactly two blocks. Prove that G is disconnected.

2.4 Critical and Minimal Blocks

Although no block G of order $p \geq 3$ contains a cut-vertex or bridge, this need not be the case in $G - v$ or $G - e$ for $v \in V(G)$ and $e \in E(G)$. This observation is considered in more detail in the present section.

A graph G is a *critical block* if G is a block and for every vertex v, $G - v$ is not a block. Hence a block G is non-critical if and only if there exists some vertex v of G such that $G - v$ is also a block. There is an analogous concept concerning edges. A graph G is a *minimal block* if G is a block and for every edge e, $G - e$ is not a block.

The block G_1 of Fig. 2.4 is minimal and non-critical while the block G_2 is critical but non-minimal.

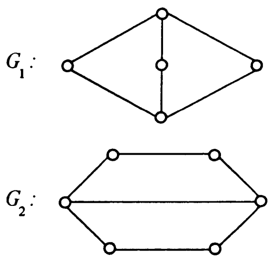

Figure 2.4 Minimal and critical blocks

In each of the graphs of Fig. 2.4 there are vertices of degree two. This is an important observation as is pointed out in the next theorem, credited to Kaugars [3].

Theorem 2.10 If G is a critical block of order $p \geq 3$, then G contains a vertex of degree two.

Proof The result is obvious if G has order 3, since then $G = K_3$. Thus we assume G is a critical block with $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geq 4$. Since G is critical, for each vertex v of G there exists a vertex w such that $G - v - w$ is disconnected. Any two vertices v_1 and v_2 such that $G - v_1 - v_2$ is disconnected will be referred to as a *cut-pair* of G. Hence every vertex of G belongs to some cut-pair of G.

Let S denote the set of all two-element subsets $\{i, j\}$ of the set $\{1, 2, \ldots, p\}$ such that $\{v_i, v_j\}$ is a cut-pair of G. For $\{i, j\} \in S$, denote by $m_{ij} = m_{ji}$ the fewest number of vertices in any component of $G - v_i - v_j$, and, furthermore, let m be the minimum value of m_{ij} over all $\{i, j\} \in S$.

Select $\{a, b\} \in S$ such that $m_{ab} = m$, and denote by C the vertex set of a component of $G - v_a - v_b$ of order m. Also, let $A = V - C - \{v_a, v_b\}$. Since G is a cyclic block, if $u \in C$ and $v \in A$, there exist disjoint u–v paths P_1 and P_2. Because u and v lie in different components of $G - v_a - v_b$, one of P_1 and P_2 contains v_a while the other contains v_b. This implies that, for each vertex $w \in V - \{v_a, v_b\}$, there exists a w–v_a path and a w–v_b path such that only w is on both paths. From this it follows that each of the induced subgraphs $\langle A \cup \{v_a\}\rangle$ and $\langle A \cup \{v_b\}\rangle$ is connected.

Let $v_c \in C$, and let $\{v_c, v_d\}$ be a cut-pair of G. Necessarily, $v_d \in A$, for otherwise $v_d \in C$ or $v_d \in \{v_a, v_b\}$ which implies $\langle A \cup \{v_a\}\rangle$ or $\langle A \cup \{v_b\}\rangle$ is a subgraph of $G - v_c - v_d$. However, these two subgraphs have order $p - m - 1$, implying that some component of $G - v_c - v_d$ has order at most $m - 1$, contradicting the minimality of m.

We now claim that the graph $G - v_c - v_d$ has exactly two components, one containing v_a and one containing v_b. Suppose this were not the case so that there is some vertex v in $G - v_c - v_d$ connected to neither v_a nor v_b. This implies that given a v–v_a path and a v–v_b path in G, which have only v in common, one of these paths contains v_c while the other contains v_d. However, then, a v_c–v_d path exists containing neither v_a nor v_b, contradicting the fact that v_c and v_d belong to different components of $G - v_a - v_b$. Because v_c is a cut-vertex of $G - v_d$, it also follows that v_c lies on every v_a–v_b path of $G - v_d$. In fact, since $v_d \in A$ and $\langle V - A\rangle$ is connected, v_c lies on every v_a–v_b path of $\langle V - A\rangle$ as well. By Theorem 2.4, v_c is a cut-vertex of $\langle V - A\rangle$, implying that v_a and v_b lie in different components of $\langle V - A\rangle - v_c$. Let G_a denote the component of $\langle V - A\rangle - v_c$ containing v_a, and let G_b denote that subgraph consisting of the remaining components.

Suppose $w \neq v_a$ is a vertex of G_a; then $w \in C$. Any w–v path of G, $v \in A$, contains v_a or v_b; hence, any w–v_b path of G contains v_a or v_c. Thus $\{v_a, v_c\}$ is a cut-pair of G. Since $\langle A \cup \{v_b\}\rangle$ is a connected subgraph of order $p - m - 1$, the graph $G - v_a - v_c$ contains a component of order less than m, which is contradictory.

Therefore, no such vertex w exists and G_a consists only of v_a. This implies, however, that C consists only of v_c and that $\deg v_c = 2$. ■

An analogous result can now be presented for minimal blocks.

Corollary 2.10a If G is a minimal block of order $p \geq 3$, then G contains a vertex of degree two.

Proof Suppose that G is a minimal block of order at least 3, but that G contains no vertices of degree two. By Theorem 2.10, G is not a critical block; thus, G contains a vertex w such that $G - w$ is a block. Let e be an edge of G incident with w. Since G is a minimal block, $G - e$ is not a block and therefore $G - e$ contains a cut-vertex $u \neq w$. Hence $G - e - u$ is disconnected while $G - u - w$ is connected, implying that w has degree one in $G - u$. However, then w has degree at most two in G, and this is a contradiction. ■

Further studies have been made on minimal blocks, and the interested reader is referred to [1,4] for additional information.

PROBLEM SET 2.4

2.21 Does there exist a non-critical block G containing an edge $e = uv$ such that $G - e$ is a block, but neither $G - u$ nor $G - v$ is a block?

2.22 Does there exist a graph other than K_2 and the n-cycles, $n \geq 4$, which is a critical block as well as a minimal block?

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3

Eulerian Graphs

3.1 The Königsberg Bridge Problem

It is difficult to say just when and where graphs originated, but there is some justification to the belief that graphs and graph theory may have begun in Switzerland in the early 18th century. In any case, it is evident that the great Swiss mathematician Leonhard Euler [2] was thinking in graphical terms when he considered the problem of the seven Königsberg bridges.

We quote from Newman's [4] account of the Königsberg Bridge Problem: "In the town of Königsberg there were in the 18th century seven bridges which crossed the river Pregel. They connected two islands in the river with each other and with the opposite banks. The townsfolk had long amused themselves with this problem: Is it possible to cross the seven bridges in a continuous walk without recrossing any of them?" In Fig. 3.1, a map of Königsberg is shown with the land areas denoted by the letters A, B, C, and D (as Euler himself did).

Euler proved that such a continuous walk over the bridges of Königsberg is impossible—a fact of which many of the people of Königsberg had already convinced themselves. However, it is probable that Euler's approach to the problem was a bit more sophisticated. He observed that if such a walk were possible it

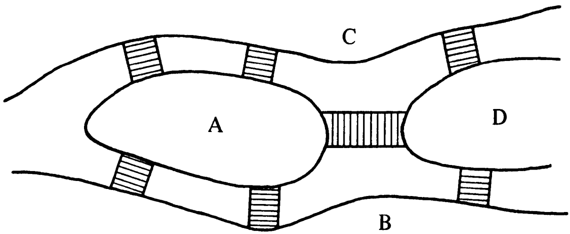

Figure 3.1 The bridges of Königsberg

could be represented by a sequence of eight letters, each chosen from A, B, C, and D. A term of the sequence would indicate the particular land area to which the walk had progressed while two consecutive terms would denote a bridge traversed while proceeding from one land area to another. Since each bridge was to be crossed only once, the letters A and B would necessarily appear in the sequence as consecutive terms twice, as would A and C. Also, since five bridges lead to region A, Euler saw that the letter A must appear in the sequence a total of three times—twice to indicate an entrance to and exit from land area A, and once to denote either an entrance to A or exit from A. Similarly, each of the letters B, C, and D must appear in the sequence twice. However, this implies nine terms are needed in the sequence, an impossibility; hence the desired walk through Königsberg is also impossible.

The Königsberg Bridge Problem has graphical overtones in many ways; indeed, even Euler's representation of a walk through Königsberg is essentially that of a walk in a graph. Suppose each land region of Königsberg is taken as a vertex and two vertices are joined by a number of edges equal to the number of bridges joining corresponding land areas; then the multigraph of Fig. 3.2 results. The Königsberg Bridge Problem is then equivalent to the problem of determining whether the multigraph of Fig. 3.2 has a trail containing all its edges.

PROBLEM SET 3.1

3.1 In present-day Königsberg (Kaliningrad), there are two additional bridges, one between regions B and C, and one between regions B and D. Is it now possible to make a "continuous walk" over the bridges of Königsberg without recrossing any of them?

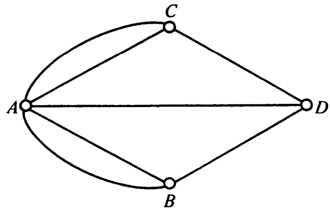

Figure 3.2 The multigraph of Königsberg

3.2 A Characterization of Eulerian Graphs

The Königsberg Bridge Problem suggests the following two concepts. An *eulerian trail* of a connected graph G is an open trail of G containing all the edges of G, while an *eulerian circuit* of G is a circuit containing all the edges of G. A graph possessing an eulerian circuit is called an *eulerian graph*. The graph G_1 of Fig. 3.3 contains an eulerian trail while G_2 is an eulerian graph.

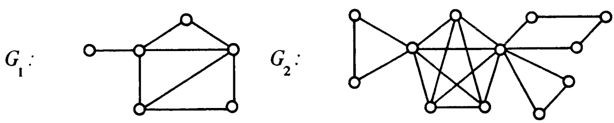

Figure 3.3 Graphs with eulerian trails and eulerian circuits

Simple but useful characterizations of both eulerian graphs and graphs with eulerian trails exist; in fact, in each case the characterization was known to Euler [2].

Theorem 3.1 Let G be a nontrivial connected graph. Then G is eulerian if and only if every vertex of G is even.

Proof Let G be an eulerian graph with eulerian circuit C, and let v be an arbitrary vertex of G. If v is not the initial vertex of C (and therefore not the final vertex either), then each time v is encountered on C, it is entered and left via distinct edges. Thus each occurrence of v in C represents a contribution of two to the degree of v so that v has even degree. If v is the initial vertex of C, then C begins and ends with v, each term representing a contribution of one to its degree while every other occurrence of v indicates an addition of two to its degree. This gives an even degree to v. In either case, v is even.

Conversely, let G be a nontrivial connected graph in which every vertex is even. We employ induction on the number q of edges of G. For $q = 3$, the smallest possible value, there is only one such graph; namely K_3, and this graph is eulerian. Assume then that all nontrivial connected graphs having only even vertices and with fewer than q edges, $q \geq 4$, are eulerian; and let G be such a graph with q edges.

Select some vertex u in G, and let W be a u–u circuit of G. Such a circuit exists in G since if W' is any u–v trail of G, $u \neq v$, then necessarily an odd number of edges of G incident with v are present in W', implying that W' can be extended to a trail W'' containing more edges than that of W'. Hence W' can be extended to a u–u circuit W of G.

If the circuit W contains every edge of G, then W is an eulerian circuit of G and G is eulerian. Otherwise, there are edges of G not in W. Remove from G all those edges which are in W together with any resulting isolated vertices, obtaining the graph G'. Since each vertex of W is incident with an even number of edges of W, every vertex of G' is even. Every component of G' is a nontrivial graph with fewer than q edges and is eulerian by hypothesis. Since G is connected, every component of G' has a vertex which also belongs to W. Hence an eulerian circuit of G can be constructed by inserting an eulerian circuit of each component H' of G' at a vertex of H' also belonging to W. ■

A characterization of graphs containing eulerian trails can now be presented.

Theorem 3.2 Let G be a nontrivial connected graph. Then G contains an eulerian trail if and only if G has exactly two odd vertices.

Proof If G contains an eulerian u–v trail, then, as in the proof of Theorem 3.1, every vertex of G different from u and v is even. It is likewise immediate that each of u and v is odd.

Conversely, let G be a connected graph having exactly two odd vertices u and v. If G does not contain the edge $e = uv$, then the graph $G + e$ is eulerian. If the edge e is deleted from an eulerian circuit of $G + e$, then an eulerian trail of G results. In any case, however, a new vertex w can be added to G together with the edges uw and vw, obtaining a graph H in which every vertex is even. Therefore H is eulerian and contains an eulerian circuit C. The circuit C necessarily contains uw and vw as consecutive edges so that their deletion from C yields an eulerian trail of G. ■

|The results of Theorem 3.2 can be extended.|

Theorem 3.3 Let G be a connected graph with $2n$ odd vertices, $n \geq 1$. Then $E(G)$ can be partitioned into subsets $E_1, E_2, \ldots, E_n$ so that $\langle E_i \rangle$ is an open trail for each i.

Proof Let u_i and v_i, $1 \leq i \leq n$, denote the odd vertices of G. Furthermore, let H denote the graph obtained by adding to G the n new vertices w_i together with the edges u_iw_i and v_iw_i, $1 \leq i \leq n$. The graph H is eulerian and therefore contains an eulerian circuit C. For each $i = 1, 2, \ldots, n$, u_iw_i and v_iw_i occur as consecutive edges in C so that if all such edges are deleted, n open trails of G result having the property that each edge of G lies on precisely one such trail. ■

We note that analogues to Theorems 3.1, 3.2, and 3.3 exist for multigraphs where the concepts of degree, trail, eulerian trail, and eulerian circuit are defined in a natural manner. It therefore follows that the multigraph of Fig. 3.2 contains neither an eulerian trail nor an eulerian circuit.

PROBLEM SET 3.2

3.2 Prove that a nontrivial connected graph G is eulerian if and only if every block of G is eulerian.

3.3 Prove that a nontrivial connected graph G is eulerian if and only if $E(G)$ can be partitioned into subsets E_i, $1 \leq i \leq n$, where $\langle E_i \rangle$ is a cycle of G for each i.

3.4 Regarding the graph G of Theorem 3.3, prove that $E(G)$ cannot be partitioned into subsets $E_1, E_2, \ldots, E_m$ so that $\langle E_i \rangle$ is an open trail for each i for $m < n$.

3.3 Randomly Eulerian Graphs

In this section we consider those eulerian graphs which have the added property that for some vertex v, any trail W beginning with v, proceeding to any incident edge and then thereafter to any adjacent edge not yet encountered in W automatically results in an eulerian circuit. More formally, we define a graph G to be *randomly eulerian from a vertex* v if every trail of G having initial vertex v can be extended to an eulerian v–v circuit of G. Of course, if G is randomly eulerian from a vertex, then it is eulerian. The eulerian graph G of Fig. 3.4 is randomly eulerian from both the vertices u and v, but from no others.

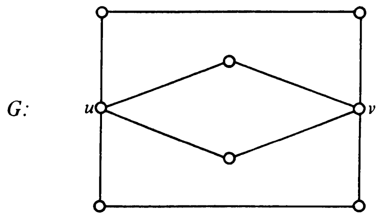

Figure 3.4 A graph which is randomly eulerian from two vertices

Graphs which are randomly eulerian from a vertex have been characterized by Ore [5].

Theorem 3.4 An eulerian graph G is randomly eulerian from a vertex v if and only if every cycle of G contains v.

Proof Let G be an eulerian graph having a vertex v which lies on every cycle of G, and assume G is not randomly eulerian from v. Hence there exists some v–w trail W of G which cannot be extended to an eulerian circuit of G. Hence W can be extended to a v–v circuit C which contains all edges of G incident with v but does not contain all edges of G. Thus there exists a vertex $u \neq v$ such that u is incident with edges not in C. Remove all edges of C from G. This results in a graph H, each of whose vertices is even. In H the vertex v is isolated while u belongs to a component which itself is an eulerian graph. Thus there is a circuit and therefore a cycle containing u which does not contain

v. This contradicts our hypothesis so that G is randomly eulerian from v.

Conversely, assume G is a graph which is randomly eulerian from a vertex v, and suppose there is some cycle C not containing v. Let G' be the graph obtained by removing the edges of C from G. Again every vertex of G' is even and the component of G' containing v is itself eulerian and therefore has an eulerian v–v circuit C'. The circuit C' necessarily contains all edges of G incident with v and therefore cannot be extended to include additional edges of G. This, however, contradicts the fact that G is randomly eulerian from v. ∎

A graph is said to be *randomly eulerian* if it is randomly eulerian from each of its vertices. In view of the preceding theorem, it follows that a graph G is randomly eulerian if and only if it is C_p for some $p \geq 3$. With the aid of Theorem 3.4, another observation [1] can now be made.

Corollary 3.4a If a graph G is randomly eulerian from v, then $\Delta(G) = \deg v$.

Proof Let C: $v = v_0, v_1, v_2, \ldots, v_{n-1}, v_n = v$ be an eulerian v–v circuit of G. For any vertex $u \neq v$, every two occurrences of u in C must have an occurrence of v between them. Otherwise, a cycle of G exists not containing v and this would contradict Theorem 3.4. Hence the number of occurrences of u cannot exceed the number of occurrences of v. Since each occurrence of a vertex indicates a degree of two for that vertex, except for the first and last terms (which represent a degree of one), the degree of u cannot exceed the degree of v so that $\Delta(G) = \deg v$. ∎

PROBLEM SET 3.3

3.5 Prove that if a graph G is randomly eulerian from v, then v belongs to every block of G (see [3]).

3.6 Prove that if a graph G is randomly eulerian from two or more of its vertices, then G is a block.

3.7 Let G be an eulerian graph of order $p \geq 3$. Prove that G is randomly eulerian from exactly none, one, two, or all of its vertices.

3.8 Let G be a graph which is randomly eulerian from a vertex v. If $\deg u = \Delta(G)$, where $u \neq v$, then prove G is randomly eulerian from u.

3.9 Using Exercises 3.7 and 3.8, determine a necessary condition for an eulerian graph to be randomly eulerian from one or more vertices.

3.10 Define a concept analogous to "randomly eulerian" for connected graphs having exactly two odd vertices, and then characterize those graphs possessing this property.

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4

Matrices and Graphs

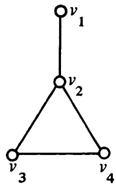
The theory of graphs might properly be classified as a subfield of matrix theory, for each (labeled) graph can be represented (in several ways) by a matrix. As expected, however, most graphical properties one usually considers are easier to distinguish in a given graph G by means of a diagram of G rather than by a matrix associated with G. Indeed it is often the case that a particular labeling of G is selected so that a prescribed property of G can be identified. There are several instances when certain matrices are extremely useful in obtaining results about graphs. The most important of these matrices are the adjacency matrix and incidence matrix, which we consider first.

4.1 The Adjacency and Incidence Matrices

The *adjacency matrix* A of a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$ is the p–by–p matrix $[a_{ij}]$, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and $a_{ij} = 0$, otherwise. It is evident that A is a (0, 1) symmetric matrix with zero diagonal. (A (0, 1) *matrix* is a matrix, each of whose entries is 0 or 1.) Likewise, it is clear that these conditions are sufficient for a matrix to be the adjacency matrix of some graph; thus, the set of all such matrices for all positive integers p represents the class of all graphs.

Figure 4.1 shows a labeled graph and its adjacency matrix.

G:



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Figure 4.1 The adjacency matrix of a graph

The entries of the nth power A^n of A have a particularly nice interpretation.

Theorem 4.1 If A is the adjacency matrix of a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$, then the (i,j) entry of A^n, $n \geq 1$, is the number of different v_i–v_j walks of length n in G.

Proof The proof is by induction on n. The result is obvious for $n = 1$ since there exists a v_i–v_j walk of length 1 if and only if $v_iv_j \in E(G)$. Let $A^{n-1} = [a_{ij}{}^{(n-1)}]$ and assume $a_{ij}{}^{(n-1)}$ is the number of different v_i–v_j walks of length $n-1$ in G; furthermore, let $A^n = [a_{ij}{}^{(n)}]$. Since $A^n = A^{n-1} \cdot A$, we have

$$a_{ij}{}^{(n)} = \sum_{k=1}^{p} a_{ik}{}^{(n-1)} a_{kj}. \tag{4.1}$$

Every v_i–v_j walk of length n in G consists of a v_i–v_k walk of length $n-1$, where v_k is adjacent to v_j, followed by the edge v_kv_j and the vertex v_j. Thus by the inductive hypothesis and equation (4.1), we have the desired result. ■

The preceding theorem has some immediate consequences. The *trace* $\operatorname{tr}(M)$ of a square matrix M is the sum of the diagonal entries of M.

Corollary 4.1a If $A^n = [a_{ij}{}^{(n)}]$ is the nth power of the adjacency matrix A of a graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$, then

(i) $a_{ij}^{(2)}$, $i \neq j$, is the number of v_i–v_j paths of length two,
(ii) $a_{ii}^{(2)} = \deg v_i$, and
(iii) $\frac{1}{6}\,\mathrm{tr}(A^3)$ is the number of triangles of G.

A second matrix of interest which can be associated with a nonempty graph G whose vertices and edges are labeled, say $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{e_1, e_2, \ldots, e_q\}$, is the *incidence matrix* B, defined as that p–by–q matrix $[b_{ij}]$ for which $b_{ij} = 1$ if vertex v_i is incident with edge e_j and $b_{ij} = 0$ otherwise. The matrix B has the properties that (i) it is a (0, 1) matrix, (ii) no two columns are identical, and (iii) the sum of the entries in any column is 2. These conditions prove to be sufficient as well as necessary for a p–by–q matrix to be the incidence matrix of some graph. It might also be noted that the sum of the entries in any row is the degree of the corresponding vertex, as is the case for the adjacency matrix.

If G is a disconnected graph with no isolated vertices having components $G_1, G_2, \ldots, G_k$ and the vertices and edges of G are appropriately labeled, then the incidence matrix B of G can be expressed as:

$$B = \left[\begin{array}{c|c|c|c} B_1 & 0 & & 0 \\ \hline 0 & B_2 & & 0 \\ \hline & & & \\ \hline 0 & 0 & & B_k \end{array}\right],$$

where B_i represents the incidence matrix of the component G_i.

We shall have occasion to encounter both the adjacency and incidence matrices again.

PROBLEM SET 4.1

4.1 Show that a graph G is a bipartite if and only if G can be labeled in such a way that its adjacency matrix A can be represented in the form:

$$A = \left[\begin{array}{c|c} 0 & A_{12} \\ \hline A_{21} & 0 \end{array}\right],$$

where A_{12} and A_{21} are two submatrices of A, and A_{12} is the transpose of A_{21}.

4.2 Prove that a graph G is disconnected if and only if G can be labeled so that its adjacency matrix A can be represented in the form:

$$A = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right].$$

4.3 Prove that the inner product of every pair of distinct row-vectors of the adjacency matrix of a graph G is at most 1 if and only if G does not contain a 4-cycle.

4.4 Let G be a connected graph with adjacency matrix A and incidence matrix B. Characterize G so that $A = B$.

4.2 Distance in Graphs

For a connected graph G, we define the *distance* $d(u,v)$ between two vertices u and v as the length of any shortest u–v path. Under this distance function, the set $V(G)$ is a metric space. There are several references which we shall make to this distance function; however, in this section we describe some concepts which are intimately related to distance and which are of interest in their own right.

The *eccentricity* $e(v)$ of a vertex v of a connected graph G is the number $\max\limits_{u \in V(G)} d(u,v)$. The *radius* rad G is defined as $\min\limits_{v \in V(G)} e(v)$ while the *diameter* diam G is $\max\limits_{v \in V(G)} e(v)$. It therefore follows that diam $G = \max\limits_{u,v \in V(G)} d(u,v)$. A vertex v is a *central vertex* if $e(v) =$ rad G and the *center* of G consists of its central vertices. The radius and diameter are related by the following inequalities.

Theorem 4.2 For every connected graph G,

$$\text{rad } G \leq \text{diam } G \leq 2 \text{ rad } G.$$

Proof The inequality rad $G \leq$ diam G is a direct consequence of the definitions. In order to verify the second inequality, select vertices u and v in G such that $d(u,v) =$ diam G. Furthermore, let w be a central vertex of G. Since d is a metric on $V(G)$,

$$d(u,v) \le d(u,w) + d(w,v) \le 2 \operatorname{rad} G. \quad \blacksquare$$

The center of a graph G may consist of a single vertex or $V(G)$ itself, but there are restrictions as were pointed out by Harary and Norman [1].

Theorem 4.3 The center of every connected graph G lies in a single block of G.

Proof Suppose G is a connected graph whose center S does not lie within a single block of G. Then G has a cut-vertex v such that $G-v$ contains components G_1 and G_2, each of which contains elements of S. Let u be a vertex such that $d(u,v) = e(v)$, and let P_1 be a v–u path of G having length $e(v)$. At least one of G_1 and G_2, say G_2, contains no vertices of P_1. Let w be an element of S belonging to G_2, and let P_2 be a shortest w–v path. The paths P_1 and P_2 together form a u–w path P_3, which is necessarily a shortest u–w path. However, then, $e(w) > e(v)$, which contradicts the fact that $w \in S$. Thus S lies in a single block of G. $\blacksquare$

In a very natural way we may now define a matrix in terms of the aforementioned distance concept. The *distance matrix* $D = [d_{ij}]$ of a connected graph G of order p with $V(G) = \{v_1, v_2, \ldots, v_p\}$ is that p–by–p matrix for which d_{ij} is the distance between v_i and v_j. The distance matrix is therefore a symmetric matrix with nonnegative integer entries having zero diagonal. The following theorem characterizes those matrices which are the distance matrix of some graph.

Theorem 4.4 A p–by–p matrix $D = [d_{ij}]$ is the distance matrix of a graph of order p if and only if D has the following properties:

(i) d_{ij} is a nonnegative integer for all i, j,
(ii) $d_{ij} = 0$ if and only if $i = j$,
(iii) D is symmetric,
(iv) $d_{ij} \le d_{ik} + d_{kj}$ for all i, j, k, and
(v) For $d_{ij} > 1$, there exists $k \ne i, j$, such that $d_{ij} = d_{ik} + d_{kj}$.

Proof If $D = [d_{ij}]$ is the distance matrix of a graph G, then properties (i)—(iv) follow immediately from the fact that the distance function is a metric on $V(G)$. To verify (v), let v_i, $v_j \in$

$V(G)$ such that $d_{ij} = d(v_i, v_j) > 1$ and let P be a shortest v_i–v_j path. Select a vertex v_k on P different from v_i and v_j; then both the v_i–v_k subpath and the v_k–v_j subpath of P are shortest such paths so that $d_{ij} = d_{ik} + d_{kj}$.

For the converse, let $D = [d_{ij}]$ be a p-by-p matrix satisfying (i)—(v). We show D is the distance matrix of a graph. Define a graph G by letting $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $E(G) = \{v_iv_j \mid d_{ij} = 1\}$. Let v_i and v_j be arbitrary elements of $V(G)$; we show $d(v_i, v_j) = d_{ij}$. If $i = j$, $d(v_i, v_j) = 0$; however, $d_{ii} = 0$ and the implication follows. If $v_iv_j \in E(G)$, then $d(v_i, v_j) = 1$ and, necessarily, $d_{ij} = 1$ from the manner in which G was defined.

Suppose then, $i \neq j$ and $v_iv_j \notin E(G)$ so that $d_{ij} \geq 2$. By repeated application of (v), there exist integers $i_1, i_2, \ldots, i_k$ such that

$$d_{ij} = d_{ii_1} + d_{i_1i_2} + \cdots + d_{i_kj}, \tag{4.2}$$

where each term on the right-hand side of (4.2) has value 1. This implies that $v_iv_{i_1}, v_{i_1}v_{i_2}, \ldots, v_{i_k}v_j \in E(G)$ so that G has a v_i–v_j walk of length d_{ij}. Hence, G has a v_i–v_j path of length at most d_{ij} by Theorem 2.1; this implies that G is connected and $d(v_i, v_j) \leq d_{ij}$. If $d(v_i, v_j) < d_{ij}$, then there exists a path $v_i, v_{j_1}, v_{j_2}, \ldots, v_{j_m}, v_j$ of length less than d_{ij}; however, then $d_{ij_1} = d_{j_1j_2} = \cdots = d_{j_mj} = 1$. By (iv), we have $d_{ij} \leq d_{ij_1} + \cdots + d_{j_mj} < d_{ij}$, which is a contradiction; thus, $d(v_i, v_j) = d_{ij}$. ■

There is an interesting class of graphs which one can associate with a given graph G of order p based on the distance concept. These are the powers of G, which we now describe. The nth *power* G^n of G is that graph with $V(G^n) = V(G)$ for which $uv \in E(G^n)$ if and only if $1 \leq d(u,v) \leq n$ in G. The graphs G^2 and G^3 are also referred to as the *square* and *cube*, respectively, of G. The graph G^n receives its name, however, from an alternative but equivalent definition. Let A be the adjacency matrix of G, and let I be the p-by-p identity matrix. If we compute $(A + I)^n - I$ using boolean arithmetic $(1 + 1 = 1)$, then we arrive at a matrix which is the adjacency matrix of some graph. This graph is G^n. This observation is a direct consequence of the fact that employing boolean arithmetic, $(A + I)^n - I = A^n + A^{n-1} + \cdots + A$, and this matrix has (i,j) entry 1 if there is a path of length k, $1 \leq k \leq n$, between v_i and v_j and has (i,j) entry 0 otherwise.

A graph with its square and cube are shown in Fig. 4.2.

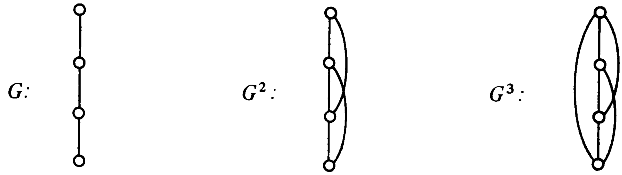


Figure 4.2 The square and cube of a graph

A graph H is an nth *root* of G if $H^n = G$. The square roots of K_4 are shown in Fig. 4.3.

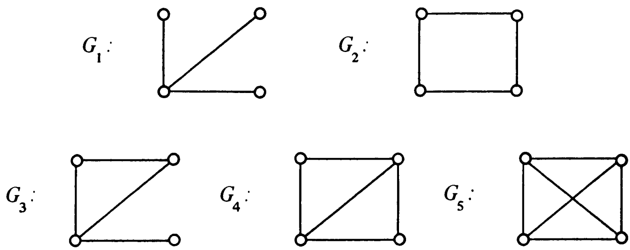


Figure 4.3 The square roots of K_4

Ordinarily, it is a difficult task to determine whether a given graph has an nth root. For square roots, however, a criterion [**3**] has been obtained.

Theorem 4.5 A connected graph G of order p with $V(G) = \{v_1, v_2, \ldots, v_p\}$ has a square root if and only if G contains a collection of complete subgraphs $G_1, G_2, \ldots, G_p$ such that

(i) $\cup\, E(G_i) = E(G)$,
(ii) G_i contains v_i, and
(iii) G_i contains v_j if and only if G_j contains v_i.

Proof Let H be a square root of a connected graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$; hence $V(H) = V(G)$. For each i, let $V_i = \{v_i\} \cup \{v_j | v_i v_j \in E(H)\}$. Thus in G, $\langle V_i \rangle = G_i$ is a complete subgraph of G. Let $e = v_j v_k \in E(G)$. If $e \in E(H)$, then $e \in E(G_j)$

and $e \in E(G_k)$; otherwise, $e \in E(G_i)$, where $v_iv_j, v_iv_k \in E(H)$, so that (i) is satisfied. The property (ii) follows from the definition of G_i. If G_i contains v_j, then G_j contains v_i; hence (iii) is satisfied.

In order to verify the sufficiency, let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$ containing complete subgraphs G_1, $G_2, \ldots, G_p$; thus (i)—(iii) are satisfied. We define a graph H by letting $V(H) = V(G)$ and $E(H) = \{v_iv_j \in E(G) \mid G_i \text{ contains } v_j\}$. We now show $H^2 = G$. Let $e = v_iv_j$ be an arbitrary edge of G. By (i), $e \in E(G_k)$ for some k so that G_k contains v_i and v_j. By (iii), G_i and G_j contain v_k; hence $v_iv_k, v_jv_k \in E(H)$ so that in H, $d(v_i,v_j) \leq 2$. Conversely, suppose v_i, $v_j \in V(H)$, $v_i \neq v_j$, such that in H, $d(v_i,v_j) \leq 2$. If $d(v_i,v_j) = 1$, then $v_iv_j \in E(H)$ implying that $v_iv_j \in E(G)$. If $d(v_i,v_j) = 2$, then there exists a path v_i, v_k, v_j in H. Therefore, G_k contains v_i and v_j; thus, $v_iv_j \in E(G)$. This completes the proof. ■

PROBLEM SET 4.2

4.5 Prove that a finite metric space (M,d) is the metric space of a graph if and only if (i) the distance between every two points of M is an integer and (ii) if a, $b \in M$ and $d(a,b) \geq 2$, then there exists a point $x \in M$ such that $d(a,x) > 0$, $d(x,b) > 0$ and $d(a,b) = d(a,x) + d(x,b)$.

4.6 Give another proof of Theorem 2.2 using the distance concept introduced in Section 4.2.

4.7 Show that the square of a nontrivial connected graph is a block.

4.8 Let A be the adjacency matrix of a connected graph and let v_i and v_j be two vertices of G. Prove that $d(v_i,v_j)$ is the smallest positive integer n for which $a_{ij}^{(n)}$, the (i,j) entry of A^n, is not zero.

4.9 Determine a class of digraphs for which concepts of distance, radius, diameter, and center can be defined in a manner analogous to that of graphs.

4.10 Prove that the bounds given in Theorem 4.2 for the diameter of G are best possible; i.e., find an infinite class of graphs for which the radius and diameter are equal and an infinite class of graphs for which the diameter is twice the radius.

4.11 Let n and m be positive integers such that $n \leq m \leq 2n$. Prove that there exists a graph G such that rad $G = n$ and diam $G = m$. Find the minimum order of such a graph G.

4.3 An Appendix of Matrix Theorems

We have already remarked that properties of matrices associated with a given graph are often beneficial in determining results concerning the graph. There are certain matrix theorems which we shall have occasion to employ during the course of our study; it is our intention to state these in the present section. Since, for the most part, they are well-known or consequences of well-known results and since we wish only to emphasize proofs of a graph-theoretic nature, these theorems are presented without proof.

The determinant of a square matrix M is denoted by det M. A square matrix $M=[m_{ij}]$ is *lower* (*upper*) *triangular* if $m_{ij}=0$ for $j>i$ $(i>j)$.

Theorem 4A If $M=[m_{ij}]$ is a lower (upper) triangular matrix, then $\det M=\prod_i m_{ii}$.

The *(i,j) cofactor of a square matrix* $M=[m_{ij}]$ is $(-1)^{i+j}\cdot\det M_{ij}$, where M_{ij} is the submatrix obtained from M by deleting the ith row and jth column.

Theorem 4B If the sum of the entries of every row and of every column of a square matrix M is zero, then every two cofactors of M are equal.

If M and M' are m-by-n and n-by-m matrices, respectively, $m\leq n$, then the m-by-m submatrix M_i of M is said to *correspond* to the m-by-m submatrix M'_i of M' if the column numbers of M determining M_i are the same as the row numbers of M' determining M'_i. (By a *submatrix* of a matrix A is meant the matrix obtained by deleting a set of rows and a set of columns from A.)

Theorem 4C Let M and M' be m-by-n and n-by-m matrices, respectively, with $m\leq n$. Then $\det(M\cdot M')=\Sigma(\det M_i)(\det M'_i)$, where the sum is taken over all m-by-m submatrices M_i of M, and where M'_i is the m-by-m submatrix of M' corresponding to M_i.

Probably the results which are most useful in a matrix connection are those concerning the eigenvalues of a square matrix M, namely those numbers λ which satisfy the equation $\det(M-\lambda I)=0$, where

I is the identity matrix of the same order as M. We define the *eigenvalues* of a graph G to be the eigenvalues of the adjacency matrix A of G. The next statement follows since A is always a symmetric matrix with real entries.

Theorem 4D The eigenvalues of a graph are real numbers.

In general, we denote the eigenvalues of a graph G of order p by λ_i, $1 \leq i \leq p$.

Since the sum of the eigenvalues of a square matrix with real entries equals its trace, we have the following result.

Theorem 4E For any graph G with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$,

$$\sum_{i=1}^{p} \lambda_i = 0.$$

This theorem implies that the maximum eigenvalue of G is a nonnegative real number.

Theorem 4F Let G' be a spanning subgraph of G. Then $\lambda \geq \lambda'$, where λ is the maximum eigenvalue of G and λ' is the maximum eigenvalue of G'.

Theorem 4G If λ_1 is an eigenvalue of a graph G with adjacency matrix $A = [a_{ij}]$, then an integer i exists such that

$$|\lambda_1| \leq \sum_j a_{ij} \leq \Delta(G).$$

The next statement actually gives a formula for the maximum eigenvalue of a graph.

Theorem 4H Let A be the adjacency matrix of a graph G of order p whose maximum eigenvalue is λ. Then

$$\lambda = \max \frac{(Ax, x)}{(x, x)},$$

where the maximum is taken over all real nonzero row-vectors x with p entries. (The symbol (y, z) denotes the inner product of y and z.)

Theorem 4I If φ is a polynomial with real coefficients in a single indeterminate and A is an n-by-n matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of the matrix $\varphi(A)$ are $\varphi(\lambda_i)$, $1 \leq i \leq n$.

Many of the theorems given in this section, as well as several related results, may be found in [**2**,**4**].

PROBLEM SET 4.3

4.12 The *determinant* det G of a graph G is defined as the determinant of the adjacency matrix of G. Show the following.

(a) $\det K_p = (-1)^{p-1}(p-1)$ for $p \geq 2$.

(b) If two nonadjacent vertices of a graph G are adjacent to the same vertices of G, then $\det G = 0$.

(c) If $G = K(p_1, p_2, \ldots, p_n)$, $n \geq 2$, is not complete, then $\det G = 0$.

4.13 Determine the maximum eigenvalue of

(a) K_p,

(b) C_{2n+1}.

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5

Trees and Treelike Graphs

For the most part, our main interest is with graphs which are connected. Among the connected graphs, the simplest in structure and perhaps most important for applications are the trees, so named for their appearance. This class of graphs has been the subject of study by several outstanding scientists and mathematicians; Kirchhoff [6] discovered trees while investigating electric networks and Cayley [2] encountered them through organic chemistry.

5.1 Trees

A *tree* is an acyclic connected graph and a *forest* is an acyclic graph. Thus each component of a forest is a tree.

There are several elementary observations which can be made regarding trees. First, by Theorem 2.6, it follows that every edge of a tree G is a bridge, i.e., every block of G is acyclic. Conversely, if every edge of a connected graph G is a bridge, then G is a tree.

If u and v are any two nonadjacent vertices of a tree G, then $G + uv$ contains precisely one cycle C. If, in turn, e is any edge of C in $G + uv$, then the graph $G + uv - e$ is once again a tree.

In a nontrivial tree G, it is immediate that $b(v) = \deg v$ for every vertex v of G. Thus, every vertex of G which is not an end-vertex is necessarily a cut-vertex, and by Theorem 2.3, we can conclude the following.

Theorem 5.1 Every nontrivial tree G has at least two end-vertices.

> There are a number of alternative ways to define trees (e.g., see [**1**], p. 152); three of these are particularly useful.

Theorem 5.2 A (p, q) graph is a tree if and only if it is acyclic and $p = q + 1$.

Proof If G is a tree, then it is acyclic by definition. To verify the equality $p = q + 1$, we employ induction on p. For $p = 1$, the result (and graph) is trivial. Assume, then, that the equality $p = q + 1$ holds for all (p, q) trees with $p \geq 1$ vertices, and let G_1 be a tree with $p + 1$ vertices. Let v be an end-vertex of G_1. The graph $G_2 = G_1 - v$ is a tree of order p, and so $p = |E(G_2)| + 1$. Since G_1 has one more vertex and one more edge than does G_2, $|V(G_1)| = p + 1 = (|E(G_2)| + 1) + 1 = |E(G_1)| + 1$.

Conversely, let G be an acyclic (p, q) graph with $p = q + 1$. To show G is a tree, we need only verify that G is connected. Denote by $G_1, G_2, \ldots, G_k$ the components of G, where $k \geq 1$. Furthermore, let G_i be a (p_i, q_i) graph. Since each G_i is a tree, $p_i = q_i + 1$. Hence

$$p - 1 = q = \sum_{i=1}^{k} q_i = \sum_{i=1}^{k} (p_i - 1) = p - k$$

so that $k = 1$ and G is connected. ■

The proof of Theorem 5.2 provides us with the following observation.

Corollary 5.2a A forest G of order p has $p - c(G)$ edges.

Theorem 5.3 A (p, q) graph G is a tree if and only if G is connected and $p = q + 1$.

Proof Let G be a (p, q) tree. By definition, G is connected and by Theorem 5.2, $p = q + 1$. For the converse, we assume G is a connected (p, q) graph with $p = q + 1$. It suffices to show that G is acyclic. If G contains a cycle C and e is an edge of C, then $G - e$ is a connected graph of order p having $p - 2$ edges. This is impossible by Exercise 2.6; therefore, G is acyclic and is a tree. ∎

Hence, any two of the properties (1) connected, (2) acyclic, (3) $p = q + 1$ characterize a tree. There is yet another interesting property of trees which deserves mention.

Theorem 5.4 A graph G is a tree if and only if every two distinct vertices of G are joined by a unique path of G.

Proof If G is a tree, then certainly every two vertices u and v are joined by at least one path. If u and v are joined by two different paths, then a cycle of G is determined, producing a contradiction.

On the other hand, suppose G is a graph for which every two distinct vertices are joined by a unique path. This implies that G is connected. If G has a cycle C containing vertices u and v, then u and v are joined by at least two paths. This contradicts our hypothesis. Thus, G is acyclic so that G is a tree. ∎

In Fig. 5.1 all trees of order six are shown.

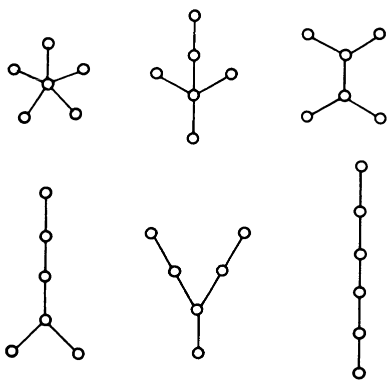

Figure 5.1 The trees of order six

Although no convenient formula is known for the number of non-isomorphic trees of order p, such a formula does exist when one considers non-identical trees. This result is due to Cayley [2], but since the original proof, the result has been established by a variety of mathematicians using a variety of methods [7]. The proof given here is due to Clarke [4].

Theorem 5.5 (Cayley's Tree Formula) The number of non-identical trees of order p is p^{p-2}.

Proof Let N be the number of trees on the p vertices labeled $v_1, v_2, \ldots, v_p$, no two of which are identical. For $d = 1, 2, \ldots, p-1$, denote by N_d the number of such trees with $\deg v_p = d$. We refer to these N_d graphs as *trees of type d*. We note further that

$$N = \sum_{d=1}^{p-1} N_d .$$

For $d \geq 2$, let G and G' be trees of type d and $d-1$, respectively. Also, let $v_i \neq v_p$ be one of the $p-d$ vertices of G' not adjacent with v_p. Suppose that v_j is the vertex on the unique v_i–v_p path which is adjacent with v_i. Define a *linkage* as an ordered pair (G, G') of trees for which $G \equiv G' + v_iv_p - v_iv_j$. Since each tree of type $d-1$ is linked to $p-d$ trees of type d and no two of these trees of type d are identical, the total number of linkages is $(p-d)N_{d-1}$.

We now derive another expression for the number of linkages. Let G be a tree of type d, and assume that the vertices adjacent with v_p are $v_1, v_2, \ldots, v_d$. Denote by G_i the component of $G - v_p$ containing v_i, $i = 1, 2, \ldots, d$, and let G_i have order p_i. Any tree of type $d-1$ linked to G may be obtained by adding to $G - v_iv_p$ (for some $i = 1, 2, \ldots, d$) an edge joining v_i and a vertex v_j not in the same component of $G - v_p$ containing v_i. The number of different such edges is $p - 1 - p_i$; hence the number of trees linked to G is $\sum_{i=1}^{d} (p - 1 - p_i)$, and no two of these trees are identical. Since

$$\sum_{i=1}^{d} (p - 1 - p_i) = (d-1)(p-1),$$

it follows that the total number of linkages is $(d-1)(p-1)N_d$.

We therefore arrive at the recursive relation $(p-d)N_{d-1} = (d-1)(p-1)N_d$.

Using the fact that $N_{p-1} = 1$, we see that $N_{p-2} = (p-2)(p-1)$, and calculating N_{d-1} from N_d, we arrive at

$$N_d = \binom{p-2}{d-1}(p-1)^{p-d-1}.$$

Thus,

$$N = \sum_{d=1}^{p-1} \binom{p-2}{d-1}(p-1)^{p-d-1} = \sum_{d=0}^{p-2} \binom{p-2}{d}(p-1)^{p-d-2},$$

the latter expression being equal to the binomial expansion

$$\sum_{d=0}^{p-2} \binom{p-2}{d}(p-1)^{p-2-d}1^d = [(p-1)+1]^{p-2} = p^{p-2},$$

which completes the proof. ■

This proof of Cayley's Theorem further provides the following information.

Corollary 5.5a The number of non-identical trees of order p such that $\deg v_i = d$, $1 \le d \le p-1$, for a fixed i, $1 \le i \le p$, is $\binom{p-2}{d-1}(p-1)^{p-d-1}$.

The preceding theorem might be considered as a formula for determining the number of non-identical spanning trees in the labeled graph K_p. We now consider the same question for graphs in general. The next theorem, due to Kirchhoff [6], is often referred to as the Matrix-Tree Theorem; the proof given here is based on that given in [5]. The *degree matrix* Q of a graph G of order p is that p-by-p matrix whose ith diagonal entry is $\deg v_i$ and all other entries of which are zero.

Theorem 5.6 (The Matrix-Tree Theorem) If G is a nontrivial labeled graph with adjacency matrix A and degree matrix Q, then the number of non-identical spanning trees of G is the value of any cofactor of the matrix $Q-A$.

Proof We note first that the sum of the entries of row i (column i) of A is $\deg v_i$ so that every row (column) sum of $Q-A$

is zero; hence by Theorem 4B all cofactors of $Q-A$ have the same value.

Assume first that G is disconnected and that G_1 is a component of G with $V(G_1)=\{v_1, v_2, \ldots, v_n\}$. Let Q' be the $(p-1)$-by-$(p-1)$ matrix obtained by deleting from $Q-A$ the last row and last column. Since the sum of the first n rows of Q' is the zero vector with $p-1$ entries, the rows of Q' are linearly dependent, implying that $\det Q'=0$. Hence one cofactor of $Q-A$ has value zero. This is, of course, the number of spanning trees of G.

We henceforth assume G to be connected. Let B denote the incidence matrix of G and in each column of B, replace one of the two nonzero entries by -1. Denote the resulting matrix by $M=[m_{ij}]$. We now show that the product of M and its transpose M^t is $Q-A$. The (i,j) entry of MM^t is

$$\sum_{k=1}^{q} m_{ik}\, m_{jk},$$

which has the value $\deg v_i$ if $i=j$, -1 if $v_iv_j \in E(G)$, and 0 otherwise. Therefore, $MM^t=Q-A$.

Consider a spanning subgraph H of G containing $p-1$ edges. Let M' be the $(p-1)$-by-$(p-1)$ submatrix of M determined by the columns associated with the edges of H and by all rows of M with one exception, say row k.

We now determine $|\det M'|$. If H is not connected, then H has a component H_1 not containing v_k. The sum of the row-vectors of M' corresponding to the vertices of H_1 is the zero vector with $p-1$ entries; hence $\det M'=0$.

Assume now that H is connected so that H is (by Theorem 5.3) a spanning tree of G. Let $u_1 \neq v_k$ be an end-vertex of H, and e_1 the edge incident with it. Next, let $u_2 \neq v_k$ be an end-vertex of the tree $H-u_1$ and e_2 the edge of $H-u_1$ incident with u_2. We continue this procedure until finally only v_k remains. A matrix $M''=[m''_{ij}]$ can now be obtained by a permutation of the rows and columns of M' such that $|m''_{ij}|=1$ if and only if u_i and e_j are incident. From the manner in which M'' was defined, any vertex u_i is incident only with edges e_j, where $j \leq i$. This, however, implies that M'' is lower triangular, and since $|m''_{ij}|=1$ for all i, we conclude by Theorem 4A that $|\det M''|=1$. However, the permutation of rows and columns of a matrix affects only the sign of its determinant, implying that $|\det M'|=|\det M''|=1$.

Since every cofactor of $Q-A$ has the same value, we evaluate only the ith principal cofactor, i.e., the determinant of the matrix obtained by deleting from $Q-A$ both row i and column i. Denote by M_i the matrix obtained from M by removing row i, so that the aforementioned cofactor equals det $(M_iM_i^t)$, which by Theorem 4C implies that this number is the sum of the products of the corresponding major determinants of M_i and M_i^t. However, corresponding major determinants have the same value and their product is 1 if the defining columns correspond to a spanning tree of G and is 0 otherwise. This completes the proof. ■

PROBLEM SET 5.1

5.1 Determine the number of non-identical trees having vertices v_1, v_2, v_3, v_4 with deg $v_4 = 2$. Draw all of them.

5.2 Prove that every tree of order $p \geq 3$ contains a cut-vertex v such that every vertex adjacent with v, with at most one exception, is an end-vertex.

5.3 Prove Theorem 5.5 as a corollary to Theorem 5.6.

5.4 Prove that a graph G is acyclic if and only if every induced subgraph of G contains a vertex of degree one or less.

5.5 Show that the center of every tree contains at most two vertices.

5.2 Decomposition of Graphs into Acyclic Subgraphs

One of the most common problems in graph theory deals with the decomposition of a graph into various subgraphs possessing some prescribed property. There are ordinarily two problems of this type, one dealing with a decomposition of the vertex set and the other with a decomposition of the edge set. One such property which has been the subject of investigation is that of being acyclic, which we now consider.

For any graph G, it is possible to partition $V(G)$ into subsets V_i, $1 \leq i \leq n$, such that each induced subgraph $\langle V_i \rangle$ is acyclic, i.e., is a forest. This can always be done by selecting each V_i so that $|V_i| \leq 2$; however, the major problem is to so partition $V(G)$ that as few subsets as possible are involved. This suggests our next concept. The *vertex-arboricity* $a(G)$ of a graph G is the fewest number of subsets into which $V(G)$ can be partitioned so that each subset induces an acyclic subgraph. It is obvious that $a(G) = 1$ if and only if G is acyclic. For a few classes of graphs, the vertex-arboricity is easily determined. For example, $a(C_p) = 2$. If p is even, $a(K_p) = p/2$; while if p is odd, $a(K_p) = (p + 1)/2$. Also, $a(K(m,n)) = 1$ if $m = 1$ or $n = 1$, and $a(K(m,n)) = 2$ otherwise. No formula is known in general, however, for the vertex-arboricity of a graph although some bounds for this number exist. First, it is clear that for any graph G of order p,

$$a(G) \leq \{p/2\}. \tag{5.1}$$

(For any real number x, we denote by $\{x\}$ the smallest integer not less than x.) The bound (5.1) is not particularly good. In order to present a sharper bound, a new concept is introduced at this point.

A graph G is called *critical with respect to vertex-arboricity* if $a(G - v) < a(G)$ for all vertices v of G. This is the first of several occasions when a graph will be defined as critical with respect to a certain parameter. In order to avoid cumbersome phrases, we will simply use the term "critical" when the parameter involved is clear by context. In particular, a graph G which is critical with respect to vertex-arboricity will be referred to in this section as a critical graph and, further, as an n-critical graph if $a(G) = n$. The complete graph K_{2n+1} is n-critical while each cycle is 2-critical. It is not difficult to locate critical graphs; indeed, every graph G with $a(G) = n \geq 2$ contains an induced n-critical subgraph. In fact, any induced subgraph G' of G with $a(G') = n$ and containing the fewest number of vertices is n-critical.

Before presenting the aforementioned bound for $a(G)$, another result is given.

Theorem 5.7 If G is a graph having $a(G) = n \geq 2$ which is critical with respect to vertex-arboricity, then $\delta(G) \geq 2(n - 1)$.

Proof Let G be an n-critical graph, $n \geq 2$, and suppose G contains a vertex v of degree $2n - 3$ or less. Since G is n-critical, $a(G - v) = n - 1$ and there is a partition $V_1, V_2, \ldots, V_{n-1}$ of the vertex set of $G - v$ such that each subgraph $\langle V_i \rangle$ is acyclic. Because $\deg v \leq 2n - 3$, at least one of these subsets, say V_j, contains at most one vertex adjacent with v in G. The subgraph $\langle V_j \cup \{v\} \rangle$ is necessarily acyclic. Hence $V_1, V_2, \ldots, V_j \cup \{v\}, \ldots, V_{n-1}$ is a partition of the vertex set of G into $n - 1$ subsets, each of which induces an acyclic subgraph. This contradicts the fact that $a(G) = n$. ■

We are now in position to present the desired upper bound [3]. The notation $H \prec G$ indicates that H is an induced subgraph of G. The symbol $[x]$, for a real number x, represents the largest integer not exceeding x.

Theorem 5.8 For any graph G,

$$a(G) \leq 1 + \left[\frac{\max \delta(G')}{2}\right],$$

where the maximum is taken over all induced subgraphs G' of G.

Proof The result is obvious for acyclic graphs; thus, let G be a graph with $a(G) = n \geq 2$. Furthermore, let H be an induced n-critical subgraph of G. Since H itself is an induced subgraph of H, we have

$$\delta(H) \leq \max_{H' \prec H} \delta(H'). \tag{5.2}$$

Moreover, every induced subgraph of H is an induced subgraph of G so that

$$\max_{H' \prec H} \delta(H') \leq \max_{G' \prec G} \delta(G'). \tag{5.3}$$

By Theorem 5.7, $\delta(H) \geq 2n - 2$ so by (5.2) and (5.3),

$$\max_{G' \prec G} \delta(G') \geq 2n - 2 = 2a(G) - 2.$$

This inequality now produces the desired result. ■

Since $\delta(G') \leq \Delta(G)$ for $G' \prec G$, we note the following consequence of the preceding result.

Corollary 5.8a For any graph G,

$$a(G) \leq 1 + \left[\frac{\Delta(G)}{2}\right].$$

We now turn to the second decomposition problem, namely, the decomposition of the edge set of a graph G so that each subset of edges induces an acyclic subgraph. Before we deal with this problem directly, however, a few remarks are in order.

Every connected graph G contains a spanning tree. If G itself is a tree, then this is a trivial observation; if G is not a tree, then a spanning tree of G may be obtained by removing cycle edges, one at a time, until finally only bridges remain. If G has q edges, then, of course, it is necessary to delete $q - p + 1$ edges in order to obtain a spanning tree of G. A much stronger statement than this can be made however. A spanning subgraph H of a connected graph G is said to be *distance-preserving from a vertex* v in G if $d(u, v)$ in H equals $d(u, v)$ in G for every vertex u. (The following result can be found in [10], p. 102.)

Theorem 5.9 For every vertex v of a connected graph G, there exists a tree H which is distance-preserving from v.

Proof For $i = 0, 1, 2, \ldots,$ let

$$d_i(v) = \{u \in V(G) \mid d(u, v) = i\}.$$

Since G is connected, for $u \neq v$ it follows that $u \in d_i(v)$ for some $i \neq 0$. Furthermore, such a vertex u is adjacent with at least one vertex of $d_{i-1}(v)$ and possibly with vertices in $d_i(v)$ and $d_{i+1}(v)$ as well. Delete all but one edge of the type uw, $w \in d_{i-1}(v)$. Also, remove every edge of the type uw, $w \in d_i(v)$. Repeat this process for each $u \neq v$; the resulting graph is denoted by H.

From the manner in which H was constructed, it is clear that H is connected since a u–v path exists in H for each $u \neq v$. It is likewise obvious that H is distance-preserving from v. To verify that H is a tree, it remains to show that H is acyclic. Suppose H contains a cycle C. Let w be a vertex of C whose distance from v is maximum, and let w_1 and w_2 be the vertices adjacent with w on C. Suppose $w \in d_k(v)$; hence $w_i \in d_k(v)$ or $w_i \in d_{k-1}(v)$ for $i = 1, 2$. If either $w_1 \in d_k(v)$ or $w_2 \in d_k(v)$, then we have reached a contradiction, due to the manner in which H was constructed. Thus, $w_1 \in d_{k-1}(v)$ and $w_2 \in d_{k-1}(v)$, which again gives a contradiction. Therefore H is acyclic and hence is a tree. ■

A connected graph may very well contain several spanning trees; indeed, Nash-Williams [8] and Tutte [11] have proved that a nontrivial graph or multigraph G has n mutually edge-disjoint spanning trees if and only if for every partition π of $V(G)$,

$$q_\pi \geq n(|\pi| - 1), \tag{5.4}$$

where q_π is the number of edges joining distinct elements of π.

The *edge-arboricity* or simply the *arboricity* $a_1(G)$ of a nonempty graph G is the minimum number of subsets into which $E(G)$ can be partitioned so that each subset induces a forest. As with vertex-arboricity, a nonempty graph has arboricity one if and only if it is a forest. Unlike vertex-arboricity, however, a formula exists for the arboricity of any graph [9].

Theorem 5.10 For any nonempty graph G,

$$a_1(G) = \max_{H \prec G} \left\{ \frac{|E(H)|}{|V(H)| - 1} \right\},$$

where the maximum is taken over all nontrivial induced subgraphs H of G.

Proof Let G be a nonempty graph, and let $\max_{H \prec G} \left\{ \frac{|E(H)|}{|V(H)| - 1} \right\} = n$. Certainly, for any nontrivial induced subgraph H of G, we have $a_1(G) \geq |E(H)|/(|V(H)| - 1)$, since any forest of G contains at most $|V(H)| - 1$ edges of H. Hence,

$$a_1(G) \geq \max_{H \prec G} \left\{ \frac{|E(H)|}{|V(H)| - 1} \right\} = n.$$

Now we show that $E(G)$ can indeed be partitioned into n subsets, each of which induces a forest.

In the remainder of the proof, we find it convenient to employ multigraphs and to introduce some relevant notation and definitions. For any induced sub-multigraph M_1 of a multigraph M, we define the number

$$\psi(M_1) = n[|V(M_1)| - 1] - |E(M_1)|.$$

A multigraph M is said to be *dense* if, for every induced sub-multigraph M_1, $\psi(M_1) \geq 0$. Further, if $\psi(M_1) = 0$, then M_1 is called a *root* of ψ.

We have already observed that $\psi(H) \geq 0$ for all induced subgraphs H of G; hence G is dense.

We now show the existence of a dense multigraph J which is a root of ψ such that G is a spanning subgraph of J. If G is a root of ψ, then we can take $J = G$; otherwise, we construct J in the following manner.

Let $u \in V(G)$. The trivial subgraph containing only u is clearly a root of ψ. If the subgraphs G_1 and G_2 of G are both roots of ψ containing u, then so too are $G_3 = \langle V(G_1) \cup V(G_2) \rangle$ and

$G_4 = \langle V(G_1) \cap V(G_2) \rangle$ roots of ψ, for by Exercise 1.10, we have

$$|E(G_3)| + |E(G_4)| \geq |E(G_1)| + |E(G_2)| . \tag{5.5}$$

From the definition of ψ and by (5.5), it follows that $\psi(G_3) + \psi(G_4) \leq \psi(G_1) + \psi(G_2) = 0$, and so $\psi(G_3) = \psi(G_4) = 0$. This further implies that the subgraph G' induced by the union of all vertex sets of roots of ψ containing u is also a root of ψ containing u.

Since G is not a root of ψ, $V(G) \neq V(G')$ so that there exists a vertex $v \in V(G) - V(G')$. Moreover, there is no root of ψ containing both u and v. Therefore the addition to G of an edge joining u and v produces a multigraph (or graph if $uv \notin E(G)$) J_1 with the property that $\psi(J_1) = \psi(G) - 1 \geq 0$. Moreover, J_1 is dense; for if H_1 is any induced sub-multigraph not containing both u and v, then it is also an induced subgraph of G and, as such, $\psi(H_1) \geq 0$; however, if H_1 contains both u and v, then H_1 minus an edge uv is an induced subgraph of G which is not a root of ψ and again, $\psi(H_1) \geq 0$.

If J_1 is a root of ψ, we take $J = J_1$; if not, we repeat the process until finally arriving at a dense multigraph J_m which is a root of ψ, and we let $J = J_m$.

Let $\pi = \{V_1, V_2, \ldots, V_k\}$, $k \geq 2$, be any partition of $V(J) = V(G)$, and let q_π denote the number of edges joining distinct elements of π. Then

$$q_\pi = |E(J)| - \sum_{i=1}^{k} |E(\langle V_i \rangle)| . \tag{5.6}$$

Since J is a root of ψ,

$$|E(J)| = n(|V(J)| - 1) . \tag{5.7}$$

Since J is dense,

$$n(|V_i| - 1) \geq |E(\langle V_i \rangle)|, \quad i = 1, 2, \ldots, k. \tag{5.8}$$

Combining (5.7) and (5.8) with (5.6), we obtain

$$\begin{aligned} q_\pi &\geq n(|V(J)| - 1) - n \sum_{i=1}^{k} (|V_i| - 1) \\ &= nk - n = n(|\pi| - 1) . \end{aligned}$$

By (5.4), J contains n mutually edge-disjoint spanning trees. By deleting the elements of $E(J) - E(G)$ from these n trees, we arrive at n mutually edge-disjoint forests of G, which produce a a partition of $E(G)$. Hence $a_1(G) = n$. ∎

PROBLEM SET 5.2

5.6 Prove that the maximum number of mutually edge-disjoint spanning trees of a connected graph with n blocks $B_1, B_2, \ldots, B_n$ equals the minimum of the corresponding numbers for B_i, $i = 1, 2, \ldots, n$.

5.7 Give an example of two non-isomorphic spanning trees which are both distance-preserving from the same vertex v of a graph G.

5.8 What upper bounds for $a(K(1, n))$ are given by Theorem 5.8 and Corollary 5.8a?

5.3 Treelike Graphs

Naturally, not all connected graphs are trees; however, it often occurs that a graph which is not a tree possesses a structure which, in some sense, resembles that of a tree. This resemblance is reflected in the following way. With every nonempty graph G there is associated another graph $BC(G)$, called the *block–cut–vertex graph of* G, whose vertex set can be put in one-to-one correspondence with the set of blocks and cut-vertices of G such that two vertices of $BC(G)$ are adjacent if and only if one vertex corresponds to a cut-vertex of G and the other corresponds to a block of G containing that cut-vertex. A graph and its block–cut–vertex graph are shown in Fig. 5.2.

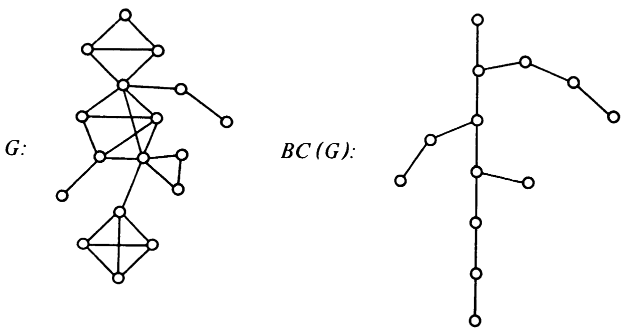

Figure 5.2 A graph and its block–cut–vertex graph

A graph G is called *a block–cut–vertex graph* if there exists a graph H such that $G = BC(H)$. Block–cut–vertex graphs are characterized in the following theorem.

Theorem 5.11 A connected graph G is a block–cut–vertex graph if and only if it is a tree with the property that the distance between every two end-vertices is even.

Proof Suppose G is a block–cut–vertex graph, and let $G = BC(H)$. If H is a block, then G is the trivial tree and the theorem is satisfied vacuously. Assume the assertion to be valid if H has $n-1$ blocks, and suppose H has n blocks, $n \geq 2$. Let B be an end-block of H containing the cut-vertex v such that, with at most one exception, every block of H containing v is an end-block of H (see Exercise 2.18). Delete from H all vertices of B different from v, obtaining the graph G', which has $n-1$ blocks. By the induction hypothesis, $BC(H')$ is a tree in which the distance between every two end-vertices is even. If v is also a cut-vertex of H', then G consists of $BC(H')$ together with a new vertex adjacent only to the vertex of $BC(H')$ which corresponds to v; otherwise, G consists of $BC(H')$ together with a vertex u corresponding to B, a vertex w corresponding to v, the edge uw, and an edge joining w with the vertex corresponding to the block of H' containing v. Thus in any case, $G = BC(H)$ has the desired properties.

Conversely, let G be a tree of order p with the property that the distance between every two of its end-vertices is even. We employ induction on p. If $G = K_1$, then $G = BC(K_2)$, for example. Furthermore, if $G = K(1, n)$ for some $n \geq 2$, then $BC(G) = G$. If G is neither trivial nor isomorphic to $K(1, n)$ for some $n \geq 2$, then G has at least three cut-vertices. Assume that every tree of order less than p with the property that the distance is even between every two end-vertices is a block–cut–vertex graph. Let v be a cut-vertex of G such that every vertex which is adjacent to v, except one, say u, is an end-vertex (see Exercise 5.2). Remove from G the vertex v and all end-vertices of G adjacent with v. The resulting graph G' satisfies the inductive hypothesis so that there exists a graph H' such that $G' = BC(H')$. Corresponding to u is necessarily a block B of H'. We now construct a graph H with the property that $BC(H) = G$. If B contains a vertex w which is not a cut-vertex of G', then H is formed by adding $\deg v - 1$ new vertices to H' and joining each to w. If on the other hand every vertex of B is a cut-vertex of H', then we add a new vertex w adjacent with two or more vertices of B and then construct H as before. It is now easily observed that $BC(H) = G$. ■

In view of the preceding result, we refer to $BC(G)$ as the *block-cut-vertex tree* of G. Just as there are some trees which are similar to graphs which are not trees, there are some graphs which are not trees whose properties are reminiscent of those of trees. We now consider a few such graphs.

We have already noted that in a tree every two vertices are joined by a unique path. There are other graphs which possess this property. A graph G is *geodetic* if every two vertices u and v of G have a unique $u-v$ path of length $d(u, v)$. There are several classes of geodetic graphs; for example, odd cycles, complete graphs, and, of course, trees. Geodetic graphs have not been characterized.

A *Husimi tree* is a connected graph G every block of which is a complete subgraph of G. Since each block of a tree is the complete graph K_2, every tree is a Husimi tree. Furthermore, because each vertex of every shortest $u-v$ path of a Husimi tree G different from u and v is a cut-vertex of G, it follows that G is geodetic as well.

Another "treelike" graph defined by its block properties is the cactus. A *cactus* is a connected graph every cyclic block of which is a cycle. Once again, every tree is a cactus. A sufficient condition for a graph to be a cactus is now presented.

Theorem 5.12 If every cycle of a connected graph G is odd, then G is a cactus.

Proof Assume G is not a cactus so that G has a cyclic block B which is not simply a cycle. Let C: $v_1, v_2, \ldots, v_n = v_1$ be a shortest cycle of B. There exists a vertex u_1 of B not on C which is adjacent with a vertex, say v_1, of C. By Exercise 2.19, there exists a cycle in G containing the edges u_1v_1 and v_1v_2. Hence, there exists a u_1-v_2 path $u_1, u_2, \ldots, u_m = v_2$ not containing v_1. Let k be the smallest integer such that u_k belongs to C, say $u_k = v_j$. The cycle $v_1, u_1, u_2, \ldots, u_k, v_{j+1}, v_{j+2}, \ldots, v_{n-1}, v_1$ or the cycle $v_1, u_1, u_2, \ldots, u_k, v_{j-1}, v_{j-2}, \ldots, v_2, v_1$ is even, and this is a contradiction to the hypothesis of the theorem. Thus G is a cactus. ■

A special type of cactus is that in which only one block is not K_2. A *unicyclic graph* is a connected graph which contains exactly one cycle. The trees and unicyclic graphs are remarkably similar in many ways. We present one of these similarities now.

Theorem 5.13 A (p,q) graph G is unicyclic if and only if G is connected and $p=q$.

Proof Let G be a (p,q) unicyclic graph, and let e be an edge of the cycle of G. The $(p,q-1)$ graph $G-e$ is a tree and, as such, is connected and $p-1=q-1$. Thus G is connected and $p=q$.

Conversely, let G be a connected (p,q) graph with $p=q$. Not every edge of G is a bridge; for otherwise, G is a tree and $q=p-1$. Thus G contains a cycle edge $e=uv$ belonging to some cycle C. The $(p,q-1)$ graph $G-e$ is connected, and since $p=q$, $G-e$ has $p-1$ edges and hence is a tree. The addition of e to $G-e$ can produce only one cycle in G, and G is therefore unicyclic. ■

PROBLEM SET 5.3

5.9 Determine a necessary and sufficient condition such that $G=BC(G)$.

5.10 If G is a geodetic graph, show that any cycle of G of smallest length is odd.

5.11 If every cycle of a graph G is odd, prove that G is geodetic.

5.12 Determine a necessary and sufficient condition for a cactus to be geodetic.

5.13 Give an example of a geodetic graph which is neither a cactus nor a Husimi tree.

5.14 Prove that a (p,q) graph G is unicyclic if and only if G has exactly one cycle and $p=q$.

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6

The Reconstruction Problem

Like any other mathematical subject, graph theory has its collection of unsolved problems. In Chapter 16 we shall consider the most famous of these; however, in the present chapter we discuss another problem which has been the object of many investigations.

6.1 The Kelly-Ulam Conjecture

The problem with which we are chiefly concerned in this chapter is centered around a conjecture due to Kelly [**5**] and Ulam ([**6**], p. 29).

THE KELLY-ULAM CONJECTURE. Let G and H be graphs with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $V(H) = \{u_1, u_2, \ldots, u_p\}$, $p \geq 3$. If $G - v_i = H - u_i$ for $i = 1, 2, \ldots, p$, then $G = H$.

The condition $p \geq 3$ is obviously necessary in this conjecture since $G = K_2$ and $H = \overline{K}_2$ satisfy all other conditions but $G \neq H$. Furthermore, it is understood that although G and H are labeled, the subgraphs $G - v_i$ and $H - u_i$ are unlabeled; for otherwise there is no problem.

The Kelly-Ulam Conjecture has been reformulated by Harary [3] in another manner, which we now describe. A graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geq 3$, is said to be *reconstructable* if it is determined (uniquely) by the p subgraphs $G - v_i$, $1 \leq i \leq p$. If G is reconstructable, then we say that G can be *reconstructed* from the subgraphs $G - v$, $v \in V(G)$. Harary's version of the Kelly-Ulam Conjecture may now be stated.

THE RECONSTRUCTION CONJECTURE. Every graph of order at least three is reconstructable.

It should be emphasized that if a graph G is reconstructable, then it does not necessarily imply the existence of a special technique to construct or display G from the subgraphs $G - v$, $v \in V(G)$. Of course, from a single subgraph $G - v$, one can determine the order, say p, of G. In a finite number of steps, one can then write down all graphs of order p. Again, after a finite number of steps, the graph G can be located from among the graphs of order p. Certainly, this procedure is likely to be highly inefficient; thus, even when a graph G or class of graphs G has been proven reconstructable the problem always remains to determine the most effective algorithm to display G from its subgraphs $G - v$, $v \in V(G)$. This is quite a different problem. There is another problem in this context which we shall not consider, namely: Given graphs $G_1, G_2, \ldots, G_p$, does there exist any graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$ such that $G_i = G - v_i$ for $i = 1, 2, \ldots, p$?

From what we have said, it should be clear that the Reconstruction Conjecture and the Kelly-Ulam Conjecture are equivalent. The unsolved *Reconstruction Problem* is thus to settle the Reconstruction Conjecture (or, of course, the Kelly-Ulam Conjecture). There are instances when it is preferable to consider one form of the conjecture rather than the other.

There are several properties of a graph G which can be found by considering the subgraphs $G - v$, $v \in V(G)$. We begin with the most elementary properties.

Theorem 6.1 If G is a (p, q) graph with $p \geq 3$, then p and q as well as the degrees of the vertices of G are determined from the p subgraphs $G - v$, $v \in V(G)$.

Proof As we have already mentioned, it is obvious that the number p of vertices of G is one greater than the order of any subgraph $G-v$. Also, p is equal to the number of subgraphs $G-v$. Label these subgraphs by G_i, $i=1, 2, \ldots, p$, and let $G_i = G - v_i$, $v_i \in V(G)$. Further, denote the number of edges in G_i by q_i. Let e be an arbitrary edge of G, say $e = v_jv_k$. Then e is an edge of $p-2$ of the subgraphs G_i, namely, all but G_j and G_k. Hence $\sum_{i=1}^{p} q_i$ counts each edge $p-2$ times, i.e., $\sum_{i=1}^{p} q_i = (p-2)q$. Therefore,

$$q = \frac{\left(\sum_{i=1}^{p} q_i\right)}{p-2}.$$

The degrees can be determined by simply noting that $\deg v_i = q - q_i$, $i = 1, 2, \ldots, p$. ■

We illustrate this theorem with the ten subgraphs $G-v$ shown in Fig. 6.1 of some undetermined graph G. From these subgraphs we establish p, q, and $\deg v_i$, for $i = 1, 2, \ldots, 10$. Clearly, $p = 10$, and by calculating q_i, $1 \le i \le 10$, we find that $q = 13$. Thus $\deg v_1 = \deg v_2 = \deg v_3 = \deg v_4 = 2$, $\deg v_5 = 3$, $\deg v_6 = \deg v_7 = 2$, $\deg v_8 = 3$, and $\deg v_9 = \deg v_{10} = 4$.

From Theorem 6.1, it is immediately evident whether a given graph is regular as well as what degree of regularity it possesses. Thus, when considering the reconstructability of a graph which is regular, there is no loss in generality if the regularity of the graph is assumed.

Corollary 6.1a Every regular graph of order $p \ge 3$ is reconstructable.

Proof Let G be a regular graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$, where $p \ge 3$. By Theorem 6.1, the p subgraphs $G - v_i$ determine the degree of regularity, say r, of G. Thus, by selecting one of these subgraphs, say $G - v_1$, adding the vertex v_1 and those edges v_1v_j for which $\deg v_j = r - 1$ in $G - v_1$, the graph G is reconstructed. ■

From the p subgraphs $G - v_i$, it is immediately discernible as to whether G is connected.

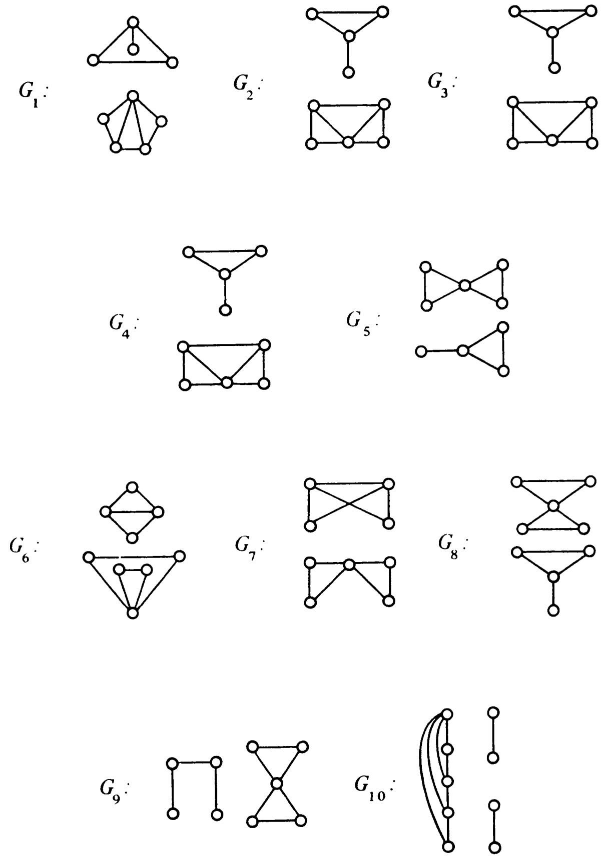

Figure 6.1 The subgraphs $G - v$ of a graph G

Theorem 6.2 If G is a graph with $V(G)=\{v_1, v_2, \ldots, v_p\}$, for $p \geq 3$, then G is connected if and only if at least two of the subgraphs $G-v_i$ are connected.

Proof Let G be a connected graph. By Theorem 2.3, G contains at least two vertices which are not cut-vertices, implying the result.

Conversely, assume there exist vertices v_1, $v_2 \in V(G)$ such that both $G-v_1$ and $G-v_2$ are connected. Thus, in $G-v_1$ and also in G, v_2 is connected to each v_i, $i \geq 3$. Moreover, in $G-v_2$ and thus in G, v_1 is connected to each v_i, $i \geq 3$. Hence every pair of vertices of G are connected and so G is connected. ■

Of course, the preceding result also gives a criterion for determining whether a graph G is disconnected. We now show that the disconnected graphs are reconstructable. The proof here is given in [1].

Theorem 6.3 If G is a disconnected graph of order $p \geq 3$, then G is reconstructable.

Proof Let G be a disconnected (p,q) graph, $p \geq 3$, with $V(G)= \{v_1, v_2, \ldots, v_p\}$, and suppose $G-v_i$ has q_i edges for $1 \leq i \leq p$. By Theorem 6.1, it is possible to calculate p, q, and the degrees of the vertices v_i from the subgraphs $G-v_i$. If $q_t = q$ for some t, $1 \leq t \leq p$, then v_t is an isolated vertex so that G consists of $G-v_t$ together with one additional isolated vertex.

Suppose next that G has no isolated vertices. Since G contains vertices which are not cut-vertices, it follows that

$$c(G) = \min\{c(G-v_i) \mid i=1,2,\ldots,p\} = k.$$

Each subgraph $G-v_i$ having k components is necessarily obtained by removing a vertex v_i from G which is not a cut-vertex of G. We henceforth consider only those subgraphs $G-v_i$ possessing k components.

Among all subgraphs $G-v_i$ with k components, choose one containing a component of minimum order m. Suppose $G-u$ is a subgraph containing the component F of order m. We note that F is the only component of $G-u$ having order m and that, necessarily, F is obtained by the removal of u from a component of G; i.e., $F=F_1-u$ for some component F_1 of G. Denote the remaining components of $G-u$ by $F_2, F_3, \ldots, F_k$; hence each F_i, $2 \le i \le k$, has order at least $m+1$.

It thus follows that $k-1$ of the k components of G are $F_2, F_3, \ldots, F_k$, which are immediately discernible from $G-u$. Hence to reconstruct G, it remains only to identify F_1. We now consider three cases, depending on the orders of the components F_i, $2 \le i \le k$.

CASE 1 *Some component F_i, $2 \le i \le k$, has order at least $m+3$.* Among these $k-1$ components, assume there are r components of order $m+1$. (It may occur that $r=0$.) Select a subgraph $G-v_j$ with k components such that $r+1$ components have order $m+1$. Thus v_j belongs to a component of G of order exceeding $m+2$. Therefore, all components of $G-v_j$ having order $m+1$ are components of G. (One of these components is F_1.) Hence G consists of all components of order $m+1$ in $G-v_j$ together with all components among $F_2, F_3, \ldots, F_k$ having order greater than $m+1$.

CASE 2 *All components F_i, $2 \le i \le k$, have order $m+2$.* We consider all subgraphs $G-v_i$ with k components having two components of order $m+1$. Necessarily, in each such subgraph $G-v_i$, one of the two components of order $m+1$ is F_1. If there is only one graph which occurs in every pair of components of order $m+1$, then this graph is F_1 and G is determined. Otherwise, suppose that every pair of components of order $m+1$ consists of the same two (non-isomorphic) components, say F' and F''. One of F' and F'' is F_1, while the other is obtained by deleting a vertex which is not a cut-vertex from a component F_i, $2 \le i \le k$. Hence we need only remove a non-cut-vertex from F_2, say, to produce the graph among F' and F'' which is not F_1. The other graph is then F_1.

CASE 3 *At least one component among the F_i, $2 \le i \le k$, has order $m+1$ and all others have order $m+2$.* Consider all subgraphs $G-v_i$ with k components having a component of order m. In each such subgraph $G-v_i$, every component having order greater than m is a component of G. Thus, in this case, a graph H is a component of G if and only if H has order exceeding m and is a component of a subgraph $G-v_i$ with k components one of which has order m. If each such subgraph $G - v_i$ has $k - 1$ components isomorphic to H, then it follows immediately that all k components of G are isomorphic to H. If not all components of G are isomorphic to H, then one only needs to observe that the number of components in G isomorphic to H is the maximum number of components isomorphic to H among the subgraphs $G - v_i$ with k components, one of which has order m, except if G has components of order $m + 2$, H is a component of order $m + 1$ and every component of order $m + 1$ in each $G - v_i$ is isomorphic to H; in this latter situation the number of components of G isomorphic to H is one greater than the aforementioned maximum. ■

Corollary 6.3a If G is a disconnected graph of order $p \geq 3$, then the components of G are determined from the subgraphs $G - v$, $v \in V(G)$.

PROBLEM SET 6.1

6.1 Show that every graph of order 3 or 4 is reconstructable.

6.2 Reconstruct the graph G for the subgraphs $G - v$ given in Fig. 6.1.

6.3 Reconstruct the graph G with $V(G) = \{v_1, v_2, \ldots, v_7\}$ such that $G - v_i = K(2, 4)$ for $i = 1, 2, 3$ and $G - v_i = K(3, 3)$ for $i = 4, 5, 6, 7$.

6.4 (a) Prove that if G is reconstructable, then $\overline{G}$ is reconstructable.

(b) Show that every graph of order $p (\geq 3)$ whose complement is disconnected is reconstructable.

6.5 (a) Show that it is possible to determine whether a graph G is eulerian from the subgraphs $G - v$, $v \in V(G)$.

(b) Show that if G is an eulerian graph, then G is reconstructable.

6.6 Let G be a connected graph of order $p \geq 3$. Prove that the number of cut-vertices of G can be determined by the p subgraphs $G - v$, $v \in V(G)$.

6.7 (a) Show that it is possible to determine whether a graph G is a tree from the subgraphs $G - v$, $v \in V(G)$.

(b) Show that it is possible to determine whether a graph G is unicyclic from the subgraphs $G - v$, $v \in V(G)$.

6.8 Give examples of disconnected graphs which fall into each of the cases considered in Theorem 6.3.

6.2 The Reconstruction of Blocks of Graphs

In the preceding section we explained that the components of a graph G can be determined from the subgraphs $G - v$, $v \in V(G)$, provided G has two or more components. In the present section we show that the blocks of G can be determined from the subgraphs $G - v$, provided G has at least two blocks. Toward this end we consider the following theorem.

Theorem 6.4 Let G and H be graphs with $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $V(H) = \{u_1, u_2, \ldots, u_p\}$, $p \geq 3$, such that $G - v_i = H - u_i$ for $1 \leq i \leq p$. If G contains α subgraphs isomorphic with a graph F, where $2 \leq |V(F)| < p$, then H contains α subgraphs isomorphic with F. Furthermore, for each $i = 1, 2, \ldots, p$, u_i and v_i belong to the same number of subgraphs isomorphic with F.

Proof Suppose H contains β subgraphs isomorphic with F. Let α_i (respectively, β_i) denote the number of subgraphs of G (respectively, H) containing v_i (respectively, u_i) isomorphic with F. Certainly,

$$\alpha = \frac{\sum_1^p \alpha_i}{|V(F)|} \text{ and } \beta = \frac{\sum_1^p \beta_i}{|V(F)|}.$$

Since $G - v_i = H - u_i$, the number of subgraphs of G isomorphic with F and not containing v_i is equal to the number of subgraphs of H isomorphic with F and not containing u_i. Thus $\alpha - \alpha_i = \beta - \beta_i$ for $i = 1, 2, \ldots, p$, or, equivalently, $\alpha - \beta = \alpha_i - \beta_i$. Summing over all i, we have

$$\sum_1^p (\alpha_i - \beta_i) = p(\alpha - \beta);$$

however,

$$\sum_1^p (\alpha_i - \beta_i) = \sum_1^p \alpha_i - \sum_1^p \beta_i = |V(F)|(\alpha - \beta).$$

Therefore, $\alpha = \beta$ so that $\alpha_i = \beta_i$ for $i = 1, 2, \ldots, p$. ■

At this point, it is convenient to introduce some additional concepts and notation. The *union* $G_1 \cup G_2$ of two subgraphs G_1 and G_2 of a graph G is that subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The union of any finite number of subgraphs is defined similarly.

We further denote the ordered pair (p, q) of a (p, q) graph G by $o(G)$. If $o(G') = (p', q')$, then we write $o(G) < o(G')$ if either $p < p'$ or $p = p'$ and $q < q'$. (Note that $o(G) = o(G')$ if $p = p'$ and $q = q'$.) The graph G is *smaller than* G', denoted $G < G'$, if $o(G) < o(G')$.

Theorem 6.5 If G is a graph with cut-vertices having $V(G) = \{v_1, v_2, \ldots, v_p\}$, then the subgraphs $G - v_i$, $i = 1, 2, \ldots, p$, determine the blocks of G.

Proof If G is disconnected, then the subgraphs $G - v_i$, $i = 1, 2, \ldots, p$, determine the two or more components of G and, thus, determine the blocks of G. Hence we may assume that G is connected. We now prove the result in the following equivalent formulation: If G is a connected graph having $V(G) = \{v_1, v_2, \ldots, v_p\}$ and blocks $B_1, B_2, \ldots, B_m$, $m \geq 2$, and H is a graph having $V(H) = \{u_1, u_2, \ldots, u_p\}$ with blocks $B'_1, B'_2, \ldots, B'_n$ such that $G - v_i = H - u_i$ for $i = 1, 2, \ldots, p$, then $m = n$ and $B_i = B'_i$ for $1 \leq i \leq m$ (after a possible relabeling).

We first note that the connectedness of G and Theorem 6.2 imply that H is connected. Also, we observe that H has cut-vertices since G has cut-vertices, so that $n \geq 2$. Without loss of generality, we may order the blocks of G and the blocks of H so that

$$o(B_1) \geq o(B_2) \geq \cdots \geq o(B_m)$$

and

$$o(B_1) \geq o(B'_1) \geq o(B'_2) \geq \cdots \geq o(B'_n).$$

Next we apply Theorem 6.4, letting $F = B_1$. The graph H therefore has a subgraph $H_1 = B_1$. Since H_1 has no cut-vertices, H_1 is a subgraph of B'_j for some j. However, $o(H_1) = o(B_1) \geq o(B'_1) \geq o(B'_j)$, and this implies $H_1 = B'_j$. By relabeling (if necessary), we may assume that $B_1 = B'_1$. Employing a mathematical induction argument, we now assume $B_i = B'_i$ for $1 \leq i \leq k$, where $1 \leq k < m$. Consider now the block B_{k+1}. Again, by Theorem 6.4, G and H have the same number of subgraphs isomorphic with B_{k+1}. Moreover, $\bigcup_{i=1}^{k} B_i$ and $\bigcup_{i=1}^{k} B'_i$ also have the same number of subgraphs isomorphic with B_{k+1}. Hence H has a subgraph $H_{k+1} = B_{k+1}$ which is necessarily a subgraph of B'_j for some $j > k$. This implies the existence of the block B'_{k+1}.

We now consider two cases.

CASE 1. *Assume* $o(B_{k+1}) \geq o(B'_{k+1})$. Then

$$o(H_{k+1}) = o(B_{k+1}) \geq o(B'_{k+1}) \geq o(B'_j)$$

for $j > k$. This implies that $H_{k+1} = B'_j$ and, again, by relabeling if needed, we conclude that $B_{k+1} = B'_{k+1}$. Thus, $m = n$ and $B_i = B'_i$ for all i, $1 \leq i \leq m$.

CASE 2. *Assume* $o(B'_{k+1}) \geq o(B_{k+1})$. Then by Theorem 6.4, G and H have the same number of subgraphs isomorphic with B'_{k+1}. Moreover, $\bigcup_{i=1}^{k} B_i$ and $\bigcup_{i=1}^{k} B'_i$ contain the same number of subgraphs isomorphic with B'_{k+1}. Hence G has a subgraph $G_{k+1} = B'_{k+1}$ which is necessarily a subgraph of B_j for some $j > k$. Thus

$$o(G_{k+1}) = o(B'_{k+1}) \geq o(B_{k+1}) \geq o(B_j)$$

for $j > k$. This implies that $G_{k+1} = B_j$. By relabeling (if necessary), we may conclude that $B_{k+1} = B'_{k+1}$. Hence $m = n$ and $B_i = B'_i$, $1 \leq i \leq m$, in this case also. ∎

A note of warning should be made regarding the preceding theorem. Although the blocks of every connected graph G having cut-vertices can be determined from the subgraphs $G - v$, $v \in V(G)$, Theorem 6.5 does not imply the reconstructability of G, for there are numerous non-isomorphic graphs, in general, which have the same collection of blocks. A reconstructable class of connected graphs having cut-vertices is presented next. This result is due to Bondy [2].

Theorem 6.6 Let G and H be connected graphs having cut-vertices but no end-vertices. If

$$V(G) = \{v_1, v_2, \ldots, v_p\},$$
$$V(H) = \{u_1, u_2, \ldots, u_p\},$$

and

$$G - v_i = H - u_i$$

for $i = 1, 2, \ldots, p$, then $G = H$.

Proof Let B_1 be an end-block of G, and let v be the cut-vertex of G contained in B_1. Furthermore, let G_1 be the subgraph of G obtained by deleting all vertices of B_1 except v.

For a positive integer s, define $G_{1,s}$ to be the graph obtained by adding to G_1 a total of s new vertices and an edge joining v and each new vertex. Since G has no end-vertices, B_1 is a cyclic block, implying that $G_{1,1}$ is a proper subgraph of G. By Theorem 6.4, H contains a subgraph $H_{1,1}$ isomorphic with $G_{1,1}$. Let ψ be such an isomorphism.

Let w be the end-vertex of $H_{1,1}$, and denote $H_{1,1} - w$ by H_1. It is immediate that $G_1 = H_1$. The restriction ψ_1 of ψ to $V(G_1)$ is an isomorphism from G_1 to H_1. We now show that ψ_1 can be extended to an isomorphism from G to H.

By Theorem 6.5, $b(H_1) = b(H) - 1$ since $b(H_1) = b(G_1) = b(G) - 1 = b(H) - 1$. Furthermore, by Theorem 6.5, the blocks of G are the same as the blocks of H. Since $G_1 = H_1$, the block B_1 is necessarily isomorphic with that block of H which is not in H_1. Let this block be B'_1 so that $B_1 = B'_1$. We observe that B'_1 and H_1 have only one vertex in common, and this vertex must be the cut-vertex of B'_1 in H_1. Since the edge $w\psi v$ is in H but not in H_1, it must be an edge of B'_1. Also, since ψv is in B'_1 as well as in H_1, it is the vertex contained in both. This implies that $\psi_1 v = \psi v$ is the cut-vertex of B'_1 in H_1. Thus it remains to show that there is an isomorphism from B_1 to B'_1 which maps v to ψv.

Let $B_{1,1}$ be the graph obtained from B_1 by adding a new vertex and an edge joining it to v. Let $B'_{1,1}$ be the graph obtained from B'_1 by adding a new vertex and an edge joining it to ψv.

Since for each $i = 1, 2, \ldots, p$, $G - v_i = H - u_i$, it follows that v_i, $i = 1, 2, \ldots, p$, is a cut-vertex of G if and only if u_i is a cut-vertex of H. Moreover, for each i $(1 \leq i \leq p)$, v_i and u_i have the same degrees. Since $G_1 = H_1$,

$$\deg_G v = \deg_H \psi v = r + s,$$

where r edges of G_1 and s edges of B_1 are incident with v. Thus ψv is incident with r edges of H_1 and s edges of B'_1. If there are α subgraphs of $G_{1,s}$ isomorphic with $B_{1,1}$, then G has $\alpha + r$ subgraphs isomorphic with $B_{1,1}$. However, then, since $H_{1,s} = G_{1,s}$, the graph $H_{1,s}$ contains α subgraphs isomorphic with $B_{1,1}$. By Theorem 6.4, there are $\alpha + r$ subgraphs of H isomorphic with $B_{1,1}$. This implies that $B'_{1,1}$ contains at least one subgraph isomorphic with $B_{1,1}$. However, $B_1 = B'_1$ implies that $B_{1,1} = B'_{1,1}$.

Therefore, there is an isomorphism from B_1 to B'_1 which maps v to ψv, and completes the proof. ■

Although Theorem 6.6 gives a large collection of graphs which are reconstructable, there are, of course, many graphs which do not satisfy the hypothesis of this theorem. In particular, it is not known whether every block is reconstructable. An important type of graph not covered by Theorem 6.6 is the tree. However, Kelly [5] has verified that the Kelly-Ulam Conjecture holds for trees. Indeed Harary and Palmer [4] have shown that a tree G can be reconstructed from the subgraphs $G - v$ which are also trees, i.e., those subgraphs $G - v$ for which v is an end-vertex of G.

PROBLEM SET 6.2

6.9 Let G be a tree of order $p \geq 3$, and let M denote the set of all subgraphs $G - v$ which are trees.

(a) Prove that G is reconstructable if $|M| \leq 4$.

(b) Let $|M| \geq 5$. Prove that the center of G consists of a single vertex if and only if at most two elements of M have a center consisting of two vertices.

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7

Planar Graphs and the Euler Polyhedral Formula

In this chapter we introduce an important class of graphs which are defined from a geometric standpoint. We then present a formula which plays a major role with respect to these graphs.

7.1 Planar Graphs and Euler's Formula

A (p, q) graph G is said to be *realizable* or *embeddable* on a surface S if it is possible to associate a collection of p distinct points on S which correspond to the vertices of G and a collection of q Jordan arcs, mutually disjoint except possibly for endpoints, on S which correspond to the edges of G such that if an arc a corresponds to the edge $e = uv$, then only the endpoints of a correspond to vertices of G, namely u and v. Intuitively, G is embeddable on S if G can be drawn on S so that edges (more precisely, the arcs corresponding to edges) intersect only at a vertex (that is, a point corresponding to a vertex) mutually incident with them. In this chapter we are concerned chiefly with the case in which S is a plane or sphere (in the topological sense).

A graph is *planar* if it can be embedded in a plane (or, equivalently, on a sphere). If a planar graph is embedded in the plane, then it is called a *plane graph*. The graph $G_1 = K(2, 3)$ of Fig. 7.1 is planar though, as drawn, it is not plane; however, $G_2 = K(2, 3)$ is both planar and plane. The graph $G_3 = K(3, 3)$ is nonplanar. This last statement will be proved presently.

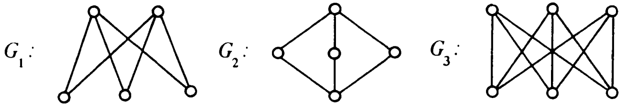

Figure 7.1 Planar, plane, and nonplanar graphs

Given a plane graph G, a *region* of G is a maximal portion of the plane for which any two points may be joined by a Jordan arc a such that any point of a neither corresponds to a vertex of G nor lies on any Jordan arc corresponding to an edge of G. For a plane graph G, the *boundary* of a region R consists of all those points x corresponding to vertices and edges of G having the property that x can be joined to a point of R by an arc all of whose points different from x belong to R. Every plane graph G contains an unbounded region called the *exterior region* of G. If G is embedded on a sphere, then no region of G can be regarded as being exterior. On the other hand, it is equally clear that a plane graph G can always be embedded in the plane so that a given region of G becomes the exterior region. Hence a plane graph G can always be realized in the plane so that any vertex or edge lies on the boundary of its exterior region. The plane graph G_2 of Fig. 7.1 has three regions, and the boundary of each is a 4-cycle.

The order, number of edges, and number of regions for any connected plane graph are related by a well-known formula, attributed to Euler.

Theorem 7.1 (Euler's Formula) If G is a connected plane graph with p vertices, q edges, and r regions, then

$$p - q + r = 2.$$

Proof We employ induction on q, the result being obvious for $q = 0$ since in this case $p = 1$ and $r = 1$. Assume the result is true for all connected plane graphs with fewer than q edges, where $q \geq 1$, and suppose G has q edges. If G is a tree, then $p = q + 1$ and $r = 1$ so that the desired formula follows. On the other hand, if G is not a tree, let e be a cycle edge of G and consider $G - e$. The connected plane graph $G - e$ has p vertices, $q - 1$ edges, and $r - 1$ regions so that by the inductive hypothesis, $p - (q - 1) + (r - 1) = 2$, which implies that $p - q + r = 2$. ■

From the preceding theorem, it follows that any two embeddings of a planar graph in the plane result in plane graphs having the same number of regions; thus one can speak of the number of regions of a planar graph.

A planar graph G is called *maximal planar* if for every pair of non-adjacent vertices u and v of G, the graph $G + uv$ is non-planar. Thus in any embedding of a maximal planar graph G having order $p \geq 3$, the boundary of every region of G is a triangle. For this reason, maximal planar graphs are also referred to as *triangulated planar graphs.*

On a given number p of vertices, a planar graph is quite limited as to how large its number q of edges can be. This is spelled out next.

Theorem 7.2 If G is a maximal planar (p, q) graph with $p \geq 3$, then

$$q = 3p - 6.$$

Proof Denote by r the number of regions of G. In G the boundary of every region is a triangle, and each edge is on the boundary of two regions. Therefore, if the number of edges on the boundary of a region is summed over all regions, the result is $3r$. On the other hand, such a sum counts each edge twice so that $3r = 2q$. Applying Theorem 7.1, we obtain $q = 3p - 6$. ■

Corollary 7.2a If G is a planar (p, q) graph with $p \geq 3$, then

$$q \leq 3p - 6.$$

Proof Add to G sufficiently many edges so that the resulting (p', q') graph G' is maximal planar. Clearly, $p = p'$ and $q \leq q'$. By Theorem 7.2, $q' = 3p - 6$ and so $q \leq 3p - 6$. ■

An immediate but important consequence of Corollary 7.2a is the following.

Corollary 7.2b Every planar graph contains a vertex of degree at most five.

Proof Let G be a planar (p, q) graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. If $p \leq 5$, then the result is obvious. Otherwise, $q \leq 3p - 6$ implies that

$$\sum_{i=1}^{p} \deg v_i = 2q \leq 6p - 12.$$

Not all p vertices of G have degree six or more, for then $2q \geq 6p$. Thus G contains a vertex of degree five or less. ■

An interesting feature of any planar graph is that there always exists a realization of it in the plane in which each edge is a straight line segment. This result was proved independently by Fáry [**1**] and Wagner [**2**].

Theorem 7.3 Each planar graph can be embedded in the plane so that every edge is a straight line segment.

Proof It suffices to prove the theorem for maximal planar graphs. In order to see that this is sufficient, assume every maximal planar graph can be embedded in the plane so that each of its edges is a straight line segment, and let H be an arbitrary planar graph. We have observed that H is a spanning subgraph of a maximal planar graph G, which, we have assumed, can be embedded in the plane so that each of its edges is a straight line segment. By deleting the appropriate edges from G, an embedding of H is produced in which each edge is a straight line segment.

Thus we let G_0 be a maximal planar graph and further assume it to be a plane graph. Without loss of generality, we suppose G_0 to have order $p \geq 4$. As we have already noted, the boundary of every region of G_0 (including the exterior region) is a triangle.

Our procedure will be to disassemble G_0 by removing one vertex at a time until only the exterior triangle remains. We then reassemble G_0 using only straight line segments.

Select a vertex of G_0 not belonging to the exterior triangle and denote it by v_1. Delete v_1 and its incident edges from G_0, and leave all other vertices and edges of G_0 exactly as originally placed. The boundary of all regions of $G_1 = G_0 - v_1$ are triangles except possibly one, namely the region in which v_1 would be inserted to reproduce G_0. Denote this region by R_1 and the cycle which forms its boundary by C_1. Among the vertices of C_1 not belonging to the exterior triangle of G_1, let v_2 be any one vertex having degree two. If there is no such vertex of degree two, select as v_2 any vertex of C_1 not on the exterior triangle. Let $G_2 = G_1 - v_2$, and denote by R_2 the region of G_2 into which v_2 would be inserted to obtain G_1. We continue this procedure until only the exterior triangle remains; thus we obtain vertices $v_1, v_2, \ldots, v_{p-3}$, the plane graphs $G_1, G_2, \ldots, G_{p-3}$, and the regions $R_1, R_2, \ldots, R_{p-3}$. From the manner in which the vertices v_i, $1 \leq i \leq p-3$, were selected, it follows that for each $i = 1, 2, \ldots, p-3$, the vertex v_i together with at least two edges are placed in region R_i to produce the graph G_{i-1}.

Before we proceed further with the proof, an observation is useful here. For the purpose of making this observation, we define an additional term. A region of a plane graph is said to be *starlike* if there exists an open set $\mathcal{O}$ in the region such that for any point $x \in \mathcal{O}$, a straight line segment can be drawn from x to each vertex on the boundary of the region such that the entire line segment (except the vertex on the boundary) lies wholly in the region. It then follows that if a vertex v is placed in the distinguished open set of a starlike region R and straight line segments are drawn from v to two or more vertices on the boundary of R, the resulting regions on whose boundary v lies are all starlike.

We may now complete the proof. Denote by G_{p-3} the graph K_3. Surely G_{p-3} may be embedded in the plane so that each of its edges is a straight line segment. It is obvious that the interior region of G_{p-3} is starlike; indeed in this case, the open set of interest may be taken as the entire region. Place a vertex in the interior region R_{p-3} of G_{p-3}, label it v_{p-3}, and join it to the two or three vertices of G_{p-3}, depending on the number of vertices of G_{p-3} to which v_{p-3} is adjacent in the final step of disassembling G_0. Denote the resulting graph by G_{p-4}. As we have just observed, the two or three new regions formed in constructing G_{p-4} are starlike. Label the appropriate region R_{p-4}, insert the vertex v_{p-4} in the suitable open set of R_{p-4}, and join v_{p-4} to the appropriate vertices of G_{p-4} by straight line segments. We thus obtain an embedding of G_{p-5} in the plane in which every edge is a straight line segment. We may now continue this procedure until arriving at an embedding of G_0 in which every edge is a straight line segment. ■

PROBLEM SET 7.1

7.1 Give an example of a planar graph which contains no vertex of degree less than five.

7.2 Apply the proof of Theorem 7.3 to the graph G of Fig. 7.2, and thereby obtain an embedding of G in the plane in which every edge is a straight line segment.

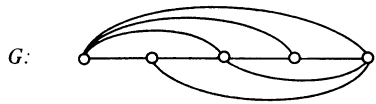

Figure 7.2 A maximal planar graph

7.3 Show that every planar graph of order $p \geq 4$ has at least four vertices with degree less than or equal to 5.

7.2 Planar Graphs and Polyhedra

The theory of planar graphs is very closely allied with the study of polyhedra; in fact, it is the case that with every polyhedron P, there is associated a connected planar graph $G(P)$ whose vertices and edges are the vertices and edges of P. Necessarily, then, every vertex of $G(P)$ has degree at least three. Moreover, if $G(P)$ is a plane graph, then the faces of P are the regions of $G(P)$ and every edge of $G(P)$ is on the boundary of two regions. A polyhedron and its associated plane graph are shown in Fig. 7.3.

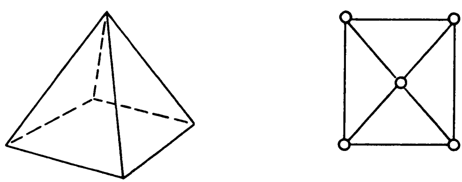

Figure 7.3 A polyhedron and its associated graph

It is customary to denote the number of vertices, edges, and faces of a polyhedron P by V, E, and F, respectively. However, these are the number of vertices, number of edges, and number of regions of a connected planar graph, namely $G(P)$. According to Theorem 7.1, V, E, and F are related. In this form, the statement of this result is known as the Euler Polyhedron Formula.

Theorem 7.4 (Euler Polyhedron Formula) If V, E, and F are the number of vertices, edges, and faces of a polyhedron, then

$$V - E + F = 2.$$

For a given polyhedron P (as well as for the graph $G(P)$) we represent the number of vertices of degree n by V_n and the number of faces (regions) bounded by an n-cycle by F_n. It follows then that

$$2E = \sum_{n\geq 3} nV_n = \sum_{n\geq 3} nF_n. \tag{7.1}$$

By Corollary 7.2b, every polyhedron has at least one vertex of degree 3, 4, or 5. As an analogue to this result, we have the following.

Theorem 7.5 At least one face of every polyhedron is bounded by an n-cycle for some $n = 3, 4, 5$.

Proof Assume that $F_3 = F_4 = F_5 = 0$ so that by (7.1),

$$2E = \sum_{n\geq 6} nF_n \geq \sum_{n\geq 6} 6F_n = 6\sum_{n\geq 6} F_n = 6F. \tag{7.2}$$

Hence $E \geq 3F$. Also,

$$2E = \sum_{n\geq 3} nV_n \geq \sum_{n\geq 3} 3V_n = 3\sum_{n\geq 3} V_n = 3V. \tag{7.3}$$

By Theorem 7.4, $V - E + F = 2$; therefore, $E \leq \frac{2}{3}E + \frac{1}{3}E - 2 = E - 2$. This is a contradiction. ■

A *regular polyhedron* is a polyhedron whose faces are bounded by congruent regular polygons and whose polyhedral angles are congruent. In particular, for a regular polyhedron, $V = V_k$ for some k and $F = F_h$ for some h. For example, a cube is a regular polyhedron with $V = V_3$ and $F = F_4$. There are only four other regular polyhedra.

Theorem 7.6 There are exactly five regular polyhedra.

Proof Let P be a regular polyhedron and $G(P)$ its associated planar graph. Then $V-E+F=2$, where V, E, and F denote the number of vertices, edges, and faces of P and $G(P)$. Therefore,

$$\begin{aligned} -8 &= 4E-4V-4F \\ &= 2E+2E-4V-4F \\ &= \sum_{n\geq 3} nF_n + \sum_{n\geq 3} nV_n - 4\sum_{n\geq 3} V_n - 4\sum_{n\geq 3} F_n \\ &= \sum_{n\geq 3} (n-4)(F_n+V_n). \end{aligned}$$

Since P is regular, there exist integers $h(\geq 3)$ and $k(\geq 3)$ such that $F=F_h$ and $V=V_k$. Hence $-8=(h-4)F_h+(k-4)V_k$. Moreover, we note that $3\leq h\leq 5$, $3\leq k\leq 5$, and $hF_h=2E=kV_k$. This gives us nine cases to consider.

CASE 1. $(h=3, k=3)$ Here we have

$$-8=-F_3-V_3 \quad\text{and}\quad 3F_3=3V_3,$$

so that $F_3=V_3=4$. Thus P is the *tetrahedron*. (That the tetrahedron is the only regular polyhedron with $V_3=F_3=4$ follows from geometric considerations.)

CASE 2. $(h=3, k=4)$ Therefore,

$$-8=-F_3 \quad\text{and}\quad 3F_3=4V_3.$$

Hence $F_3=8$ and $V_4=6$, implying that P is the *octahedron*.

CASE 3. $(h=3, k=5)$ In this case,

$$-8=-F_3+V_5 \quad\text{and}\quad 3F_3=5V_5,$$

so that $F_3=20$, $V_5=12$, and P is the *icosahedron*.

CASE 4. $(h=4, k=3)$ We find here that

$$-8=-V_3 \quad\text{and}\quad 4F_4=3V_3.$$

Thus $V_3=8$, $F_4=6$, and P is the *cube*.

CASE 5. $(h=4, k=4)$ This is impossible since $-8\neq 0$.

CASE 6. $(h=4, k=5)$ This case too cannot occur; for otherwise $-8=V_5$.

CASE 7. $(h=5, k=3)$ For these values,

$$-8=F_5-V_3 \quad\text{and}\quad 5F_5=3V_3.$$

Solving for F_5 and V_3, we find that $F_5 = 12$ and $V_3 = 20$ so that P is the *dodecahedron.*

CASE 8. ($h = 5$, $k = 4$) Here $-8 = F_5$, which is impossible.

CASE 9. ($h = 5$, $k = 5$) This, too, is impossible since $-8 \neq F_5 + V_5$. This completes the proof. ∎

The graphs of the five regular polyhedra are shown in Fig. 7.4.

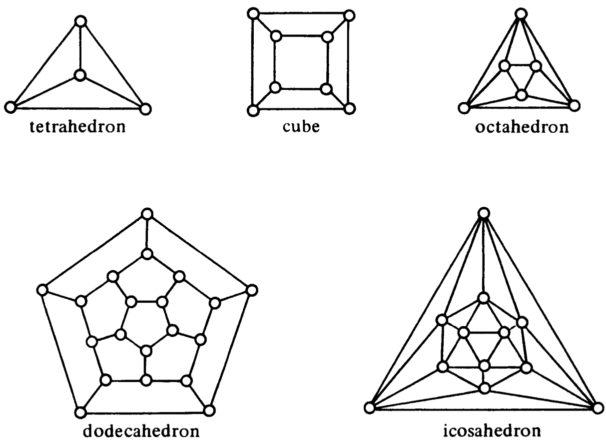

Figure 7.4 The graphs of the regular polyhedra

PROBLEM SET 7.2

7.4 Show that the graph of the octahedron is a complete n-partite graph for some n.

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2. K. WAGNER, Bemerkungen zum Vierfarbenproblem. *Jber. Deutsch. Math. Verein.* 46 (1936), pp. 21–22.

8

Characterizations of Planar Graphs

In this chapter we present two useful, interesting criteria for planar graphs. We begin by describing two relations on graphs.

8.1 Homeomorphism and Contraction

An *elementary subdivision* of a nonempty graph G is a graph obtained from G by the removal of some edge $e = uv$ and the addition of a new vertex w and the edges uw and vw. A *subdivision* of G is a graph obtained from G by a succession of elementary subdivisions. A graph H is defined to be *homeomorphic from* G if either $H = G$ or H is a subdivision of G. A graph G_1 is *homeomorphic with* a graph G_2 if there exists a graph G_3 such that each of G_1 and G_2 is homeomorphic from G_3. Furthermore, if H is homeomorphic from G, then H is a *homeomorphic image* of G while if H is homeomorphic with G, then H is a *homeomorph* of G.

In Fig. 8.1, the graphs G_1 and G_2 are homeomorphic with each other since each is homeomorphic from G_3. However, neither G_1 nor G_2 is homeomorphic from the other.

It is a straightforward exercise to verify that "homeomorphic with" is an equivalence relation on graphs. We thus refer to two

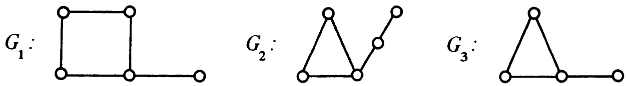

Figure 8.1 Homeomorphism

graphs as being *homeomorphic* if either is homeomorphic with the other. Hence the set of graphs may be partitioned into equivalence classes, two graphs belonging to the same class if and only if they are homeomorphic.

Theorem 8.1 In every class $\mathcal{C}$ of homeomorphic graphs, there exists a unique graph H such that if $G \in \mathcal{C}$, then G is homeomorphic from H.

Proof Let $\mathcal{C}$ be a class of homeomorphic graphs, and let H be an element of $\mathcal{C}$ with the minimum number of vertices. If $G \in \mathcal{C}$, then H and G are homeomorphic graphs. Thus there exists a graph H_1 such that each of H and G is homeomorphic from H_1. Since H is homeomorphic from H_1, either $H = H_1$ or H is a subdivision of H_1, the latter implying that H_1 has fewer vertices than H, which is a contradiction because $H_1 \in \mathcal{C}$. Hence $H = H_1$ and G is homeomorphic from H. It is easily seen that H is unique. ■

The preceding result suggests the following definition. A graph H is *homeomorphically irreducible* if whenever a graph G is homeomorphic with H, then G is homeomorphic from H. By Theorem 8.1 then, every class of homeomorphic graphs contains exactly one homeomorphically irreducible graph. Homeomorphically irreducible graphs are characterized below.

Theorem 8.2 A graph H is homeomorphically irreducible if and only if every vertex of degree two of H lies on a triangle of H.

Proof If a graph H contains a vertex v of degree two adjacent with vertices u and w, which are themselves nonadjacent, then $H - v + uw$ is homeomorphic with H but not homeomorphic from H. This implies that H is not homeomorphically irreducible.

Conversely, suppose H is a graph in which every vertex of degree two lies on a triangle and suppose, to the contrary, that H is not homeomorphically irreducible. Accordingly, there exists a graph G which is homeomorphic with H but not from H. Hence, a graph H_1 exists such that both H and G are homeomorphic from H_1. Since $H_1 \neq H$, H is a subdivision of H_1, and H therefore can be obtained from H_1 by a sequence of elementary subdivisions. However, any new vertex introduced in an elementary subdivision has degree two and is adjacent to two nonadjacent vertices. This is contradictory to the hypothesis so that H is is homeomorphically irreducible. ■

As a direct consequence of both the definition of "homeomorphically irreducible" and of the preceding theorem, we note that if H is a graph containing no vertices of degree two, then H is homeomorphically irreducible.

For graphs G_1 and G_2, a mapping φ from $V(G_1)$ onto $V(G_2)$ is called an *elementary contraction* if there exist exactly two adjacent vertices u and v of G_1 such that (1) $\varphi u = \varphi v$, (2) $\{u_1, v_1\} \cap \{u, v\} = \emptyset$ implies $u_1v_1 \in E(G_1)$ if and only if $\varphi u_1 \varphi v_1 \in E(G_2)$, and (3) for $w \in V(G_1)$, $w \neq u, v$, then $uw \in E(G_1)$ or $vw \in E(G_1)$ if and only if $\varphi u \varphi w \in E(G_2)$. We say here G_2 is obtained from G_1 by the *identification of the adjacent vertices* u and v. A *contraction* is then a mapping from $V(G_1)$ onto $V(G_2)$ which is either an isomorphism or a composition of finitely many elementary contractions.

If there exists a contraction from $V(G_1)$ onto $V(G_2)$, then G_2 is a *contraction of* G_1 and G_1 *contracts to* or is *contractible to* G_2. A *subcontraction* of a graph G is an edge-induced subgraph of a contraction of G.

There is an alternative and more intuitive manner in which to define "contraction." A graph G may be defined as a contraction of a graph H if there exists a one-to-one correspondence between $V(G)$ and the elements of a partition of $V(H)$ such that each element of the partition induces a connected subgraph of H and two vertices of G are adjacent if and only if the subgraph induced by the union of the corresponding subsets is connected.

In Fig. 8.2, the graph G is a contraction of H, obtained by the identification of v_2 and v_5. It might also be considered as the contraction resulting from the partition $V(H) = \{v_1\} \cup \{v_2, v_5\} \cup \{v_3\} \cup \{v_4\}$.

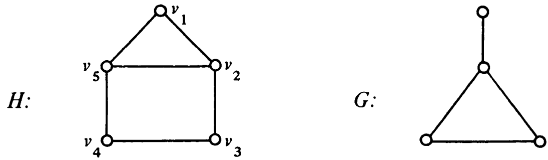

Figure 8.2 Contraction

A relationship between contraction and homeomorphism is given in the following theorem.

Theorem 8.3 If a graph H is homeomorphic from a graph G, then G is a contraction of H.

Proof If $G = H$, then clearly G is a contraction of H. Hence we may assume H is obtained from G by a sequence of elementary subdivisions. Suppose G' is an elementary subdivision of G, say $G' = G - uv + w + uw + vw$. However, then, G' is contractible to G by an elementary contraction φ, which fixes every element of $V(G)$ and $\varphi w = \varphi u$. Hence G can be obtained from H by a mapping which is a composition of finitely many elementary contractions so that G is a contraction of H. ■

The following corollary will actually prove to be of more use than the theorem itself.

Corollary 8.3a If a graph H contains a subgraph homeomorphic from a connected nontrivial graph G, then G is a subcontraction of H.

PROBLEM SET 8.1

8.1 Determine the homeomorphically irreducible graph for the class of homeomorphic graphs containing the graphs of Fig. 8.1.

8.2 Determine the homeomorphically irreducible trees of order six.

8.3 Prove that if a graph G_2 is homeomorphic with an eulerian graph G_1, then G_2 is eulerian.

8.4 Let H be a homeomorphically irreducible graph of order p. What are the maximum number of vertices of degree two which H can possess if

(a) H is disconnected?
(b) H is connected?
(c) H is a block?

8.5 Prove that a graph G is a contraction of a graph H if and only if there exists a one-to-one correspondence between $V(G)$ and the elements of a partition of $V(H)$ such that each element of the partition induces a connected subgraph of H and where two vertices of G are adjacent if and only if the subgraph induced by the union of the corresponding subsets is connected.

8.6 Show that the converse of Theorem 8.3 is not true in general.

8.7 Show that a graph G is a forest if and only if G does not have K_3 as a subcontraction.

8.2 Characterizations of Planar Graphs

There are two graphs, namely K_5 and $K(3, 3)$ (shown in Fig. 8.3), which play an important role in the study of planar graphs. We verify the nonplanarity of these two graphs.

K_5: $K(3, 3)$:

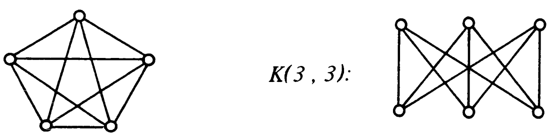

Figure 8.3 The nonplanar graphs K_5 and $K(3, 3)$

Theorem 8.4 The graphs K_5 and $K(3, 3)$ are nonplanar.

Proof Suppose, to the contrary, that K_5 is a planar graph. Since K_5 has $p = 5$ vertices and $q = 10$ edges,

$$10 = q > 3p - 6 = 9,$$

which contradicts Corollary 7.2a. Thus K_5 is nonplanar.

Suppose next that $K(3, 3)$ is a planar graph, and consider any plane graph of it. Since $K(3, 3)$ is bipartite, it has no triangles; thus each of its regions is bounded by at least four edges. Let the number of edges bounding a region be summed over all r regions of $K(3, 3)$, denoting the result by N. Thus, $N \geq 4r$. Since the sum N counts each edge twice and $K(3, 3)$ contains $q = 9$ edges, $N = 18$ so that $r \leq 9/2$. However, by Theorem 7.1, $r = 5$ and this is a contradiction. Hence $K(3, 3)$ is nonplanar. ■

It should be clear that any homeomorph of a graph G is planar or nonplanar according to whether G is planar or nonplanar. Also it is an elementary observation that if a graph G contains a nonplanar subgraph H, then G too is nonplanar. Combining these facts with our preceding results, we state the following.

Theorem 8.5 If a graph G contains a subgraph homeomorphic with either K_5 or $K(3, 3)$, then G is nonplanar.

The remarkable property of Theorem 8.5 is that its converse is also true. These two results provide a characterization of planar graphs that may very well be one of the most significant theorems in the theory of graphs. Before presenting a proof of this result, due to Kuratowski [4], we need one additional fact about planar graphs.

Theorem 8.6 A graph is planar if and only if each of its blocks is planar.

Proof Certainly, a graph G is planar if and only if each of its components is planar, so we may assume G to be connected. It is equally clear that if G is planar, then each block of G is planar. For the converse, we employ induction on the number of blocks of G. If G has only one block and this block is planar, then, of course, G is planar. Assume every graph with fewer than $n \geq 2$ blocks, each of which is planar, is a planar graph, and suppose G has n blocks, all of which are planar. Let B be an endblock of G, and denote by v the cut-vertex of G common to B. Delete from G all vertices of B different from v, calling the resulting graph G'. By the inductive hypothesis, G' is a planar graph. Since the block B is planar, it may be embedded in the plane so that v lies on the exterior region. In any region of a plane graph of G' containing v, the aforementioned plane block B may now be suitably placed so that the two vertices of G' and B labeled v are identified. The result is a plane graph of G; hence G is planar. ■

| A characterization of planar graphs may now be given. |

Theorem 8.7 (Kuratowski's Theorem) A graph is planar if and only if it contains no subgraph homeomorphic with K_5 or $K(3, 3)$.

Proof The necessity is precisely the statement of Theorem 8.5; thus we need only consider the sufficiency. In view of Theorem 8.6, it is sufficient to show that if a block contains no subgraph homeomorphic with K_5 or $K(3, 3)$, then it is planar. Assume, to the contrary, that such is not the case. Hence among all nonplanar blocks not containing subgraphs homeomorphic with either K_5 or $K(3, 3)$, let G be one with a minimum number of edges.

First we verify that $\delta(G) \geq 3$. Since G is a block, it contains no end-vertices. Suppose that G contains a vertex v with $\deg v = 2$, such that v is adjacent with u and w. If $uw \in E(G)$, then $G - v$ is also a block. Since $G - v$ is a subgraph of G, it follows that $G - v$ too contains no subgraph homeomorphic with K_5 or $K(3, 3)$; however, G is a nonplanar block with the fewest number of edges having this property so that $G - v$ is planar. It is now clear that in any plane graph of $G - v$, the vertex v and edges uv and vw may be inserted so that the resulting graph G is plane, which is contradictory. If $uw \notin E(G)$, then the graph $G' = G - v + uw$ has fewer edges than does G. Furthermore, the graph G' contains no subgraph homeomorphic with either K_5 or $K(3, 3)$; for suppose it contained such a subgraph F. If F failed to contain the edge uw, then G would also contain F which is impossible; thus F contains uw. If to $F - uw$ we add the vertex v and edges uv and vw, the resulting graph F' is homeomorphic from F. However, F' is a subgraph of G and, once again, a contradiction arises. This implies that $\delta(G) \geq 3$.

By Corollary 2.10a, G is not a minimal block so that there exists an edge $e = uv$, such that $H = G - e$ is also a block. Since H has no subgraph homeomorphic with K_5 or $K(3, 3)$ and H has fewer edges than does G, the graph H is planar. Since H is a cyclic block, it follows by Theorem 2.7 that H possesses cycles containing both u and v. We henceforth assume H to be a plane graph having a cycle, say C, containing u and v such that the number of regions interior to C is maximum. Assume C to be given by

$$u = v_0, v_1, \ldots, v_i = v, \ldots, v_n = u,$$

where $1 < i < n - 1$.

Several observations regarding the plane graph H can now be made. In order to do this, it is convenient to define two special subgraphs of H. By the *exterior subgraph (interior subgraph)* of H, we mean the subgraph of G induced by those edges lying exterior (interior) to the cycle C. First, since the graph G is nonplanar, both the exterior and interior subgraphs exist; for otherwise, the edge e could be added to H (either exterior to C or interior to C) so that the resulting graph, namely G, is planar.

We note further that no two distinct vertices of the set $\{v_0, v_1, \ldots, v_i\}$ are joined by a path in the exterior subgraph of H, for this would contradict the choice of C as being that cycle containing u and v having the maximum number of regions interior to it. A similar statement can be made regarding the set $\{v_i, v_{i+1}, \ldots, v_n\}$. These remarks in connection with the fact that $H+e$ is nonplanar imply the existence of a v_j–v_k path P, $0 < j < i < k < n$, in the exterior subgraph of H such that no vertex of P different from v_j and v_k belongs to C. This structure is illustrated in Fig. 8.4. We further note that no vertex of P different from v_j and v_k is adjacent to a vertex of C other than v_j or v_k, and, moreover, any path joining a vertex of P with a vertex of C must contain at least one of v_j and v_k.

H:

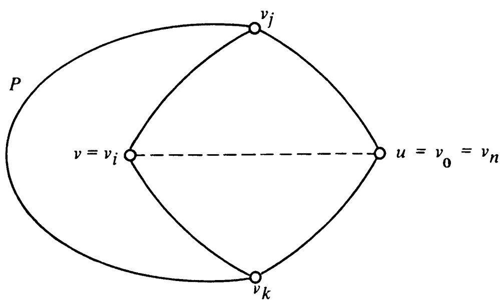

Figure 8.4 Structure of the graph H of Theorem 8.7

Let H_1 be the component of $H - \{v_m \mid 0 \le m < n, m \ne j, k\}$ containing P. By the choice of C, the subgraph H_1 cannot be inserted in the interior of C in a plane manner. This, together with the assumption that G is nonplanar, implies that the interior subgraph of H must contain one of the following:

a. A v_r–v_s path Q, $0 < r < j$, $i < s < k$, none of whose vertices different from v_r and v_s belongs to C (or, equivalently, $j < r < i$ and $k < s < n$).
b. A vertex w not on C which is connected to C by three mutually disjoint paths such that the end-vertex of one such path is one of v_0, v_j, v_i, v_k, say v_0, and the end-vertices of the other two paths are v_r and v_s, where $j \leq r < i$ and $i < s \leq k$, but not both $r = j$ and $s = k$ hold.
c. A vertex w not on C which is connected to C by three mutually disjoint paths P_1, P_2, P_3 such that the end-vertices of the paths (different from w) are three of the four vertices v_0, v_j, v_i, v_k, say v_0, v_i, v_j respectively, together with a v_t–v_k path $P_4(v_t \neq v_0, v_i, w)$, where v_t is on P_1 or P_2, and P_4 is disjoint from P_1, P_2, and C except for v_t and v_k.
d. A vertex w not on C which is connected to the vertices v_0, v_j, v_i, v_k by four mutually disjoint paths.

These four cases exhaust the possibilities. (This is a fact of which one must convince himself individually.) In each of the first three cases, the graph G has a subgraph homeomorphic with $K(3,3)$ while in the fourth case, G has a subgraph homeomorphic with K_5. However, in any case, this is contrary to assumption. Thus no such graph G exists, and the proof is complete. ■

As an illustration of the preceding theorem, we consider the graph of Fig. 8.5. This graph P is referred to as *the Petersen graph* and will be encountered on several occasions. The Petersen graph is nonplanar since it has a subgraph homeomorphic with $K(3,3)$. Despite its resemblance to the complete graph K_5, P does not contain a subgraph homeomorphic with K_5. However, the Petersen graph does contain K_5 as a subcontraction, indeed as a contraction. This can be seen by considering the partition $V(P) = \cup V_i$, $1 \leq i \leq 5$, where $V_i = \{v_i, v_{i+5}\}$. This observation also implies that P is nonplanar, as we now see [2, 3, 5].

Theorem 8.8 A graph is planar if and only if it contains neither K_5 nor $K(3, 3)$ as a subcontraction.

Proof Let G be a nonplanar graph, which contains (by Theorem 8.7) a subgraph homeomorphic with (or equivalently here,

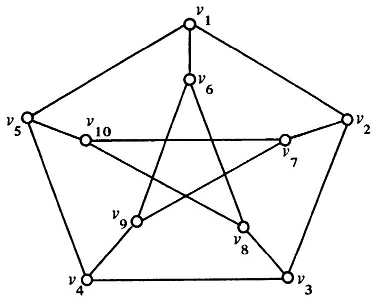

Figure 8.5 The Petersen graph

homeomorphic from) K_5 or $K(3, 3)$. Thus by Corollary 8.3a, K_5 or $K(3, 3)$ is a subcontraction of G.

In order to verify the converse, we first suppose that G is a graph containing $H = K(3, 3)$ as a subcontraction. We show, in this case, that G contains a subgraph homeomorphic with $K(3, 3)$, implying that G is nonplanar. Denote the vertices of H by u_i and u'_i, $1 \le i \le 3$, such that every edge of H is of the type $u_i u'_j$. Taking the alternate definition of contraction, we let G_i, $1 \le i \le 3$, be the connected subgraph of G corresponding to u_i and let G'_i correspond to u'_i. Since $u_i u'_j \in E(H)$ for $1 \le i \le 3$, $1 \le j \le 3$, in the graph G there exists a vertex v_{ij} of G_i adjacent with a vertex v'_{ij} of G'_j. Among the vertices v_{i1}, v_{i2}, v_{i3} of G_i, two or possibly all three may actually represent the same vertex. If $v_{i1} = v_{i2} = v_{i3}$, we set each $v_{ij} = v_i$; otherwise, we define v_i to be a vertex of G_i joined to the distinct elements of $\{v_{i1}, v_{i2}, v_{i3}\}$ with disjoint paths in G_i. (It is possible that $v_i = v_{ij}$ for some j.) We now proceed as above with the subgraphs G'_i, thereby obtaining vertices v'_i. The subgraph of G induced by the nine edges $v_{ij}v'_{ij}$ together with the edge sets of any necessary aforementioned paths from a vertex v_i or v'_i is homeomorphic with $K(3, 3)$.

Assume now that G contains $H = K_5$ as a subcontraction. Let $V(H) = \{u_i \mid 1 \le i \le 5\}$, and suppose G_i is the connected subgraph of G which corresponds to u_i. As before, there exists a vertex v_{ij} of G_i adjacent with a vertex v_{ji} of G_j, $i \ne j$, $1 \le i, j \le 5$. For a fixed i, $1 \le i \le 5$, we consider the vertices v_{ij}, $j \ne i$. If the vertices v_{ij} represent the same vertex, we denote this vertex by

v_i. If the vertices v_{ij} are distinct and there exists a vertex (possibly some v_{ij}) from which there are disjoint paths (one of which may be trivial) to the v_{ij}, then denote this vertex by v_i. If three of the vertices v_{ij} are the same vertex, call this vertex v_i. If two vertices v_{ij} are the same while the other two are distinct, then denote the two coinciding vertices by v_i if there exist disjoint paths to the other two vertices. Hence in several instances we have defined a vertex v_i, for $1 \le i \le 5$. Should v_i exist for each $i = 1, 2, \ldots, 5$, then G contains a subgraph homeomorphic with K_5. Otherwise, for some i, there exist distinct vertices w_i and w'_i of G_i, each of which is joined to two of the v_{ij} by disjoint (possibly trivial) paths of G_i while w_i and w'_i are joined by a path of G_i, none of whose vertices different from w_i and w'_i are the vertices v_{ij}. Without loss of generality, we assume $i = 1$ and that w_1 is joined to v_{12} and v_{13} while w'_1 is joined to v_{14} and v_{15} by these paths. Denote the edge set of these five paths of G_1 by E_1. We now turn to G_2. If $v_{21} = v_{24} = v_{25}$, we set $E_2 = \varnothing$; otherwise, there is a vertex w_2 of G_2 (which may coincide with v_{21}, v_{24}, or v_{25}) joined by mutually disjoint (possibly trivial) paths in G_2 to the distinct elements of $\{v_{21}, v_{24}, v_{25}\}$. We then let E_2 denote the edge sets of these paths. In an analogous manner, we define accordingly the sets E_3, E_4, and E_5 with the aid of the sets $\{v_{31}, v_{34}, v_{35}\}$, $\{v_{41}, v_{42}, v_{43}\}$, and $\{v_{51}, v_{52}, v_{53}\}$, respectively. The subgraph induced by the union of the sets E_i and the edges $v_{ij}v_{ji}$ contains a subgraph homeomorphic with $K(3, 3)$. In either case, G is nonplanar. ■

PROBLEM SET 8.2

8.8 Show that the Petersen graph of Fig. 8.5 is nonplanar by showing that it has

(a) a subgraph homeomorphic with $K(3, 3)$.

(b) $K(3, 3)$ as a subcontraction.

8.9 Show that Kuratowski's Theorem holds for pseudographs as well as graphs.

8.10 Which complete n-partite graphs are planar?

8.3 Outerplanar Graphs

In this section we describe briefly a special class of planar graphs which possess properties which are remarkably similar to those of planar graphs themselves. A graph G is *outerplanar* if it can be embedded in the plane so that every vertex of G lies on the boundary of some region. From our earlier remarks, we may take this region to be the exterior—which we do. According to the definition, then, every tree, indeed every forest, is an outerplanar graph. Figure 8.6 shows an outerplanar graph.

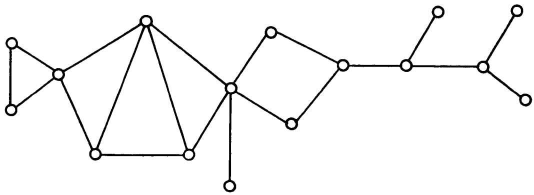

Figure 8.6 An outerplanar graph

Two planar graphs which are not outerplanar and which will prove to be important in the present context are K_4 and $K(2,3)$.

Kuratowski's Theorem 8.7 may be reformulated so that it reads as follows: A graph is planar if and only if it contains no subgraph homeomorphic from K_5 or $K(3, 3)$. We note the similarity of this statement and the following characterization [1] of outerplanar graphs.

Theorem 8.9 A graph is outerplanar if and only if it contains no subgraph homeomorphic from K_4 or $K(2, 3)$.

Proof We have already observed that neither K_4 nor $K(2, 3)$ is outerplanar; hence, no graph having a subgraph homeomorphic from K_4 or $K(2, 3)$ is outerplanar.

To prove the sufficiency, let G be a graph containing no subgraph homeomorphic from K_4 or $K(2, 3)$. Certainly, by Theorem 8.7, G is planar. If G is not outerplanar, then it must have a block which is not outerplanar. Hence without loss of generality, we assume G is a cyclic block which is not outerplanar. We further assume that G is so embedded in the plane that the exterior region contains the maximum number of vertices. The exterior region is bounded by a cycle C. Since not all vertices of G lie on C, there are one or more vertices lying interior to C. If there exists a vertex u interior to C and three mutually disjoint paths between u and three distinct vertices of C, then G contains a subgraph homeomorphic from K_4. Otherwise, since G is a cyclic block, there must exist a vertex u and two disjoint paths between u and two distinct vertices v and w of C. Moreover, from the choice of C, the edge vw does not belong to C. This implies that G contains a subgraph homeomorphic from $K(2, 3)$. ■

Outerplanar graphs may also be characterized in a manner similar to that for planar graphs in Theorem 8.8.

Theorem 8.10 A graph is outerplanar if and only if it has neither K_4 nor $K(2, 3)$ as a subcontraction.

PROBLEM SET 8.3

8.11 Show that if G is a (p, q) outerplanar graph with $p \geq 2$, then $q \leq 2p - 3$.

8.12 Prove that if G is an outerplanar graph with $p \geq 3$ vertices, then G has at least n vertices of degree less than or equal to n, $n = 2, 3$.

8.13 Prove that a graph is outerplanar if and only if each of its blocks is outerplanar.

8.14 Prove Theorem 8.10.

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9

Topological Parameters

There are numerous parameters or integer-valued functions defined on graphs, a few of which we have already mentioned and several others we have yet to discuss. In this chapter we introduce the concept of the genus of a graph together with some additional parameters. In most instances the development of a formula which gives the value of these parameters for a given graph appears to be an impossible task; however, there are special classes of graphs for which formulas have been found. In order to define one such class and to help in describing many types of graphs, we present two operations on graphs in the first section.

9.1 The Sum and Product of Graphs

Let G_1 and G_2 be two nonempty graphs for which $V(G_1) = V(G_2)$ and $E(G_1) \cap E(G_2) = \emptyset$; then the graph G is the *sum* of G_1 and G_2, written $G = G_1 + G_2$, if $V(G) = V(G_1)$ and $E(G) = E(G_1) \cup E(G_2)$. Thus, for any nonempty, noncomplete graph G of order p, we have $G + \overline{G} = K_p$. Furthermore, any graph with $q(>1)$ edges can be expressed as the sum of q of its subgraphs, each possessing a single edge. It is clear that the operation of sum is both associative and commutative.

The *product* $G_1 \times G_2$ of two graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$ is the graph with vertex set $V(G_1) \times V(G_2)$ such that two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \times G_2$ are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. Here also the operation of product is both associative and commutative. In addition, K_1 serves as an identity for product, so that under this operation the set of graphs with distinct vertex sets forms a commutative semigroup with identity. The operation of product is illustrated in Fig. 9.1 for $G_1 = K_2$ and $G_2 = K_3$.

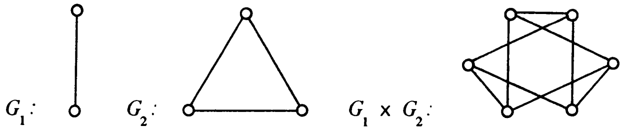

Figure 9.1 The product of two graphs

A nontrivial graph G is called *prime* if $G = G_1 \times G_2$ implies either $G_1 = K_1$ or $G_2 = K_1$. In general, it is difficult to determine whether a given graph is prime, although the following theorem presents a sufficient condition for a graph to be prime.

Theorem 9.1 If G is a nontrivial connected graph containing a vertex which lies on no 4-cycle of G, then G is prime.

Proof Assume G satisfies the hypotheses of the theorem and $G = G_1 \times G_2$, where neither G_1 nor G_2 is trivial. Let (u_1, u_2) be an arbitrary vertex of G. Since each of G_1 and G_2 is necessarily connected, there exist vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. However, then, (u_1, u_2), (u_1, v_2), (v_1, v_2), and (v_1, u_2) determine a 4-cycle of G, which is contrary to hypothesis. Hence G is prime. ■

It should be clear that every nontrivial connected graph can be expressed as a product of primes; however, it is further true that such a decomposition is unique. A proof of this latter result was supplied by Sabidussi [**10**].

An important class of graphs is defined in terms of products; these are the "cubes", which we now describe. The *n-cube* Q_n is the graph K_2 if $n = 1$, while for $n > 1$, Q_n is defined inductively as $Q_{n-1} \times K_2$. The cube Q_n can also be considered as that graph whose vertices are labeled by the binary sequences $a_1, a_2, \ldots, a_n$ (i.e., a_i is 0 or 1 for $1 \leq i \leq n$) and such that two vertices are adjacent if and only if their corresponding sequences differ in precisely one position. It is easily observed that Q_n, $n \geq 2$, is an n-regular cyclic block of order 2^n. The 2-cube and 3-cube are shown in Fig. 9.2.

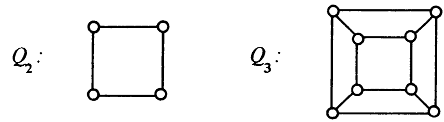

Figure 9.2 Cubes

PROBLEM SET 9.1

9.1 Show that arboricity can be defined in terms of sums of subgraphs.

9.2 If G_1 is regular of degree r_1 and G_2 is regular of degree r_2, prove that $G_1 \times G_2$ is regular of degree $r_1 + r_2$.

9.3 Prove that the converse of Theorem 9.1 is not true, in general.

9.4 Let G_i, $i = 1, 2$, be a connected graph different from K_1 and K_2. Show $G_1 \times G_2$ is planar if and only if both G_1 and G_2 are paths or one is a cycle and the other a path.

9.5 Prove that $G \times K_2$ is planar if and only if G is outerplanar.

9.6 Let G_1 and G_2 be graphs such that between every two distinct vertices of G_i, $i = 1, 2$, there exist at least m_i (>0) mutually disjoint paths. Prove that between every two distinct vertices of $G_1 \times G_2$, there exist at least $m_1 + m_2$ mutually disjoint paths.

9.7 Show that the two definitions of "n-cube" given in the text are equivalent.

9.2 The Genus of a Graph

While only certain graphs can be embedded in a plane, it is evident that any graph is embeddable in 3-space. In this section we consider the problem of embedding graphs on special surfaces in 3-space called compact orientable 2-manifolds. Topologically speaking, a compact orientable 2-manifold may be considered as a sphere on which has been placed a number of "handles" (or, equivalently, a sphere in which has been inserted a number of holes). This number of handles (or holes) is referred to as the genus of the compact orientable 2-manifold. The *genus* $\gamma(G)$ *of a graph* G is the smallest genus among all compact orientable 2-manifolds on which G can be embedded. Every graph has a genus; in fact, it is a relatively simple observation that a graph with q edges can be embedded on a sphere with q handles.

Since the embedding of graphs on spheres and planes is equivalent, the graphs of genus 0 are precisely the planar graphs. The graphs with genus 1 are therefore the nonplanar graphs which can be realized on a torus. The graphs K_5 and $K(3,3)$ have genus 1; embeddings of $K(3,3)$ on a torus and on a sphere with two handles are shown in Fig. 9.3 (a), (b).

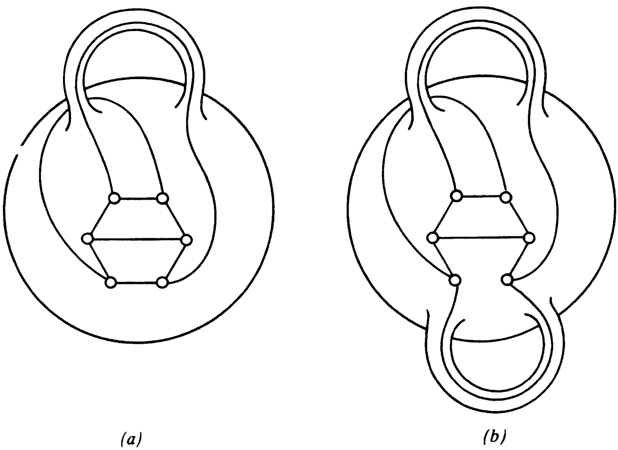

Figure 9.3 Embeddings of $K(3,3)$ on compact orientable 2-manifolds of genus 1 and 2

For graphs embedded on compact orientable 2-manifolds of positive genus, regions and their boundaries are defined in entirely the same manner as for embeddings in the plane. In Fig. 9.3(a) there are three regions and in Fig. 9.3(b) there are two regions. Figure 9.3(b) also serves as an illustration of a property of some embeddings on surfaces of positive genus, which we now mention. For an embedding of a connected graph on a sphere, each region is necessarily *simply connected*, i.e., any simple closed curve in a region can be continuously deformed in that region to a single point. This need not be the case for graphs embedded on surfaces of positive genus. For example, in Fig. 9.3(b) there are two regions, only one of which is simply connected. The boundary of the simply connected region is a 4-cycle while the boundary of the other region consists of all vertices and edges of $K(3,3)$.

We are now in a position to present an extension of Theorem 7.1 to compact orientable 2-manifolds of positive genus. Here, however, it is necessary to require each region to be simply connected. This result is also the work of Euler.

Theorem 9.2 Let G be a connected (p,q) graph embedded on a compact orientable 2-manifold of genus n such that each of the r regions so defined is simply connected. Then

$$p - q + r = 2 - 2n. \tag{9.1}$$

Proof We proceed by induction on n, the formula holding for $n = 0$ by Theorem 7.1. Assume (9.1) to be satisfied for all connected graphs embedded on compact orientable 2-manifolds of genus $n - 1$, every region of which is simply connected. Now let G be a connected (p,q) graph embedded on a compact orientable 2-manifold S of genus n, $n \geq 1$, such that each of its r regions is simply connected. Without loss of generality, we assume every vertex of G lies on the sphere of S (or, equivalently, on a single handle). Since every region is simply connected, each handle of S contains one or more edges of G. We now add new vertices and edges (and thereby new regions) to G in the following manner. Let H be a handle of S containing, say, edges $e_1, e_2, \ldots, e_k$, $k \geq 1$, labeled consecutively about H. On the handle H, draw disjoint simple closed curves C_1 and C_2 so that each curve intersects each of the edges e_i once. For $i = 1, 2, \ldots, k$, denote by u_i the point of intersection of C_1 with e_i and by v_i the point of intersection of C_2 with e_i; furthermore, distinguish each of the u_i and v_i as vertices. Also we now consider any arc containing a newly added vertex as an edge. Thus, each former edge e_i gives rise to three edges while each of C_1 and C_2 produces k edges. Should $k = 1$ or $k = 2$ (i.e., if we have introduced loops or multiple edges by this process), we add u_2 and u_3 (or simply u_3 if u_2 exists) to C_1 and v_2 and v_3 (possibly only v_3) to C_2 along with the edges u_2v_2 and u_3v_3 (or, accordingly, only u_3v_3) embedded on H. It is not difficult to see that the resulting graph G' is also embedded on S so that every region is simply connected. Moreover, G' has $p+2k$ vertices, $q+4k$ edges, and $r+2k$ regions if $k \geq 3$; and $p+6$ vertices, $q+k+9$ edges, and $r+k+3$ regions if $k=1$ or $k=2$. Denote by G'' the graph obtained from G' by deleting all the edges u_iv_i. Clearly, the graph G'' can be embedded on the compact orientable 2-manifold S' of genus $n-1$ obtained by removing H from S and "filling in" the two resulting holes. On S', the vertices u_i determine a cycle which is a boundary of a region of G'' that is not a region of G'; such is also the case with the v_i. Thus, in proceeding from an embedding of G' on S to an embedding of G'' on S' a loss of k edges and $k-2$ regions occurs if $k \geq 3$, and a loss of 3 edges and 1 region should $k=1$ or $k=2$. Again, we observe that each region of G'' on S' is simply connected. By the inductive hypothesis,

$$(p+2k)-(q+3k)+(r+k+2)=2-2(n-1) \qquad \text{if } k \geq 3,$$

and

$$(p+6)-(q+k+6)+(r+k+2)=2-2(n-1) \qquad \text{if } k=1,2.$$

However, in either case $p-q+r=2-2n$, thus proving the theorem. ■

In connection with Theorem 9.2, the following result is particularly useful.

Theorem 9.3 If G is a connected graph embedded on a compact orientable 2-manifold of genus $\gamma(G)$, then every region of G is simply connected.

Proof Again, we may assume that every vertex of G lies on the sphere. Hence on each handle are embedded only edges. Suppose, to the contrary, that there exists a region R which is not simply connected. Hence there is a simple closed curve C in R which cannot be continuously deformed in R to a single point. For $\gamma(G)=0$, the curve C divides (by the Jordan Curve Theorem) the sphere into two parts, each of which necessarily contains parts of G. However, this implies G is disconnected, which is impossible. Thus $\gamma(G) \geq 1$.

We now consider two cases.

CASE 1. *Suppose C lies entirely on the sphere.* Again, C divides the sphere into two parts. Since C cannot be deformed to a single point, there exists a handle H with a base in each part of the sphere. If the surface is now cut along C and the two resulting holes are incorporated into the parts of the sphere from which they come, a compact orientable 2-manifold of genus $\gamma(G)-1$ is obtained. However, G is embedded on this surface as well, and this is impossible.

CASE 2. *Suppose C lies (at least partially) on some handle* H. Certainly, C cannot be wholly on a single handle; thus the curve C can be divided into two arcs, one of which, say A, does not lie on H. However, the edges of G lying on H can now be redrawn along A in the region R so that the result is an embedding of G on the surface in which the handle H is not used. This, however, contradicts the genus of G. ■

The preceding two results now immediately imply the following.

Corollary 9.3a If G is a connected (p, q) graph embedded on a compact orientable 2-manifold of genus $\gamma(G)$ with r regions, then

$$p-q+r=2-2\gamma(G).$$

An important conclusion which can be reached with the aid of Corollary 9.3a is that any two embeddings of a connected graph G on an orientable 2-manifold of genus $\gamma(G)$ result in the same number of regions. The theorems thus far obtained have other useful implications.

Corollary 9.3b If G is a connected (p, q) graph, then

$$\gamma(G) \geq \frac{q}{6}-\frac{p}{2}+1.$$

Proof Let G be embedded on an orientable 2-manifold of genus $\gamma(G)$ so that, by Corollary 9.3a, $p-q+r=2-2\gamma(G)$, where r is the number of regions (each being simply connected). Since every region is bounded by at least three edges and every edge lies on the boundary of at most two regions, $3r \leq 2q$, which gives the desired result. ■

The lower bound for $\gamma(G)$ can be improved when G has no triangles. The proof of the next corollary is entirely analogous to that of the preceding.

Corollary 9.3c If G is a connected (p, q) graph containing no triangles, then

$$\gamma(G) \geq \frac{q}{4} - \frac{p}{2} + 1 .$$

A formula for the genus of a graph is not known in general. Indeed, it is unlikely that such a formula will ever be developed in terms of quantities that are easily calculable. As is often the case in such circumstances, formulas (or partial formulas) have been established for certain classes of graphs. Ordinarily, the first classes to be considered are the complete graphs, complete bipartite graphs, and cubes. Such is the case with the genus.

In 1968 Ringel and Youngs [9] completed a proof of a result which has a remarkable history, namely

$$\gamma(K_p) = \left\{ \frac{(p-3)(p-4)}{12} \right\}, \qquad p \geq 3 . \tag{9.2}$$

This formula is intimately tied in with another topic of graph theory and will be revisited in Chapter 16. Formulas exist for both the genus of the complete bipartite graph [7] and the genus of the n-cube [**2, 8**]; we prove the last result to illustrate some of the techniques involved. We omit the obvious equality $\gamma(Q_1) = 0$.

Theorem 9.4 For $n \geq 2$,

$$\gamma(Q_n) = (n-4) \cdot 2^{n-3} + 1 .$$

Proof The n-cube is a $(2^n, n \cdot 2^{n-1})$ graph with no triangles. Thus by Corollary 9.3c,

$$\gamma(Q_n) \geq (n-4) \cdot 2^{n-3} + 1 .$$

To verify the inequality in the other direction, we employ induction on n. For $n \geq 2$, define the statement $A(n)$ as follows: the graph Q_n can be embedded on a compact orientable 2-manifold of genus $(n-4) \cdot 2^{n-3} + 1$ such that the boundary of every region is a 4-cycle and such that there exist 2^{n-2} regions with mutually disjoint boundaries. That the statements $A(2)$ and $A(3)$

are true is trivial. Assume $A(k-1)$ to be true, $k \geq 4$, and, accordingly, let S be an orientable 2-manifold of genus $(k-5) \cdot 2^{k-4} + 1$ on which Q_{k-1} is embedded such that the boundary of each region is a 4-cycle and such that there exist 2^{k-3} regions with mutually disjoint boundaries. We note that since Q_{k-1} has order 2^{k-1}, each vertex of Q_{k-1} belongs to the boundary of precisely one of the aforementioned 2^{k-3} regions. Now let Q_{k-1} be embedded on an orientable 2-manifold S' of genus $(k-5) \cdot 2^{k-4} + 1$ such that the embedding of Q_{k-1} on S' is a "mirror image" of the embedding of Q_{k-1} on S (that is, if v_1, v_2, v_3, v_4 are the vertices of a region of Q_{k-1} on S, where the vertices are listed clockwise about the 4-cycle, then there is a region on S', with the vertices v_1, v_2, v_3, v_4 on its boundary listed counterclockwise). We now consider the 2^{k-3} distinguished regions of S together with the corresponding regions of S', and join each pair of associated regions by a handle. The addition of the first handle produces an orientable 2-manifold of genus $2[(k-5) \cdot 2^{k-4} + 1]$ while the addition of each of the other $2^{k-3} - 1$ handles results in an increase of 1 to the genus. Thus, the orientable 2-manifold just constructed has genus $(k-4) \cdot 2^{k-3} + 1$. Now each set of four vertices on a distinguished region can be joined to the corresponding four vertices on the associated region so that the four edges are embedded on the handle joining the regions. It is now immediate that the resulting graph is isomorphic to Q_k and that every region is bounded by a 4-cycle. Furthermore, each added handle gives rise to four regions, "opposite" ones of which have disjoint boundary. Altogether, then, there exist 2^{k-2} regions of Q_k that are mutually disjoint.

Thus, $A(n)$ is true for all $n \geq 2$, proving the result. ∎

It should be noted in closing that it is possible to speak of embedding graphs on nonorientable surfaces such as the Möbius strip, projective plane, and Klein bottle. As might be expected, every planar graph (as well as some nonplanar graphs) can be embedded on each of these surfaces. Figure 9.4 shows K_5 embedded on the Möbius strip. This topic shall not be the subject of further discussion here, however.

PROBLEM SET 9.2

9.8 Show that Theorem 9.2 holds if G is a pseudograph.

9.9 Determine $\gamma(K(4,4))$.

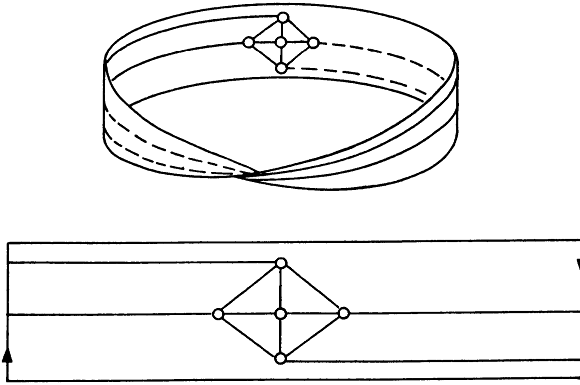

Figure 9.4 An embedding of K_5 on the Möbius strip

9.10 Prove Corollary 9.3c.

9.11 Show that

$$\gamma(K_p) \geq \left\{\frac{(p-3)(p-4)}{12}\right\}, \qquad p \geq 3.$$

9.12 Show that

$$\gamma(K(m,n)) \geq \left\{\frac{(m-2)(n-2)}{4}\right\}, \qquad 2 \leq m \leq n.$$

9.3 Packings and Coverings of Graphs

There is an extraordinarily large number of parameters which have been defined for graphs. Indeed, we have just been discussing one of these, namely the genus of a graph. Although there are several other parameters topological in nature, there is a great number of parameters (topological and nontopological) whose definitions bear a striking similarity.

Let P be any property possessed by the trivial graph K_1 (such as being acyclic or planar). By the *vertex covering number* of a graph G with respect to P is meant the minimum number of elements V_i in a partition of $V(G)$ such that each induced subgraph $\langle V_i \rangle$ has property P. The *vertex packing number* of G with respect to P is the maximum number of mutually disjoint nonempty subsets V_i of $V(G)$ such that no subgraph $\langle V_i \rangle$ has property P. In a completely analogous manner, one can define the *edge covering number* and *edge packing number* of a nonempty graph G *with respect to any property possessed by the graph* K_2.

We have already seen examples of these types of parameters. The arboricity $a_1(G)$ of a nonempty graph G (defined in Chapter 5) is the edge covering number of G with respect to the property of being acyclic while the vertex-arboricity $a(G)$ of G is the corresponding vertex covering number of G. The number which might be considered "dual" to $a_1(G)$ is the edge packing number of G with respect to the property of being acyclic. It is not difficult to see that this is the maximum number of edge-disjoint cycles contained in G. For this reason, this number is referred to as the *cycle multiplicity* of G; we denote it by $\bar{a}_1(G)$. As expected, no formula exists for the cycle multiplicity of an arbitrary graph G; however, formulas have been found [4] for $\bar{a}_1(G)$ when $G = K_p$ and $G = K(m,n)$, the latter of which we now prove.

Theorem 9.5 For $m \le n$,

$$\bar{a}_1(K(m,n)) = \begin{cases} \left[\dfrac{m}{2}\right]\left[\dfrac{n}{2}\right], & \text{if } mn \text{ is even} \\[2ex] \left[\dfrac{m}{2}\right]\left[\dfrac{n}{2}\right] + \left[\dfrac{m}{4}\right], & \text{if } mn \text{ is odd}. \end{cases}$$

Proof We begin with a few elementary observations. Because $K(m,n)$ is a bipartite graph, by Theorem 2.2 each of its cycles is even. Hence, every cycle of $K(m,n)$ contains at least four edges. Since $K(m,n)$ has mn edges, the maximum number of edge-disjoint cycles cannot exceed $[mn/4]$. On the other hand, if $V(K(m,n))$ is partitioned as $U \cup W$ with $|U| = m$ and $|W| = n$ such that every edge joins a vertex of U with a vertex of W, then U contains mutually disjoint subsets $U_1, U_2, \ldots, U_r$ and W contains mutually disjoint subsets $W_1, W_2, \ldots, W_s$, where $r = [m/2]$, $s = [n/2]$, and $|U_i| = |W_j| = 2$ for $1 \le i \le r$ and $1 \le j \le s$. Now, for each i and j such that $1 \le i \le r$ and $1 \le j \le s$, a 4-cycle is induced by $U_i \cup W_j$. The total number of these cycles is $rs = [m/2][n/2]$, and every two such cycles are edge-disjoint. Therefore, every complete bipartite graph $K(m,n)$ contains a set of $[m/2][n/2]$ edge-disjoint cycles, producing the inequalities

$$\left[\frac{m}{2}\right]\left[\frac{n}{2}\right] \leq \bar{a}_1(G) \leq \left[\frac{mn}{4}\right].$$

We now consider four cases.

CASE 1. If m and n are both even, then $[m/2][n/2] = [mn/4]$ so that $\bar{a}_1(K(m,n)) = [m/2][n/2]$.

CASE 2. Assume $m = 2r$ and $n = 2s + 1$, where $m < n$. In this case, every vertex in U has odd degree; however, in any collection of edge-disjoint cycles, every vertex is incident with an even number of edges belonging to the cycles. This implies that the number of edges in any set of edge-disjoint cycles is at most $mn - m = 4rs$ and that the number of edge-disjoint cycles is at most $rs = [m/2][n/2]$. However, a set with this number of cycles exists; thus, $\bar{a}_1(K(m,n)) = [m/2][n/2]$.

CASE 3. Assume $m = 2r + 1$ and $n = 2s$, where $m < n$. The argument here is exactly the one used in Case 2 so that here also $\bar{a}_1(K(m,n)) = [m/2][n/2]$.

CASE 4. Suppose $m = 2r + 1$ and $n = 2s + 1$, where $m \leq n$. As we noted in Case 2, since every vertex of W has odd degree, in any set of edge-disjoint cycles of $K(m,n)$ each vertex of W must be incident with at least one edge not in any of the cycles. Hence, the maximum number of edges in any collection of edge-disjoint cycles cannot exceed $mn - n = 4rs + 2r$. Thus, the maximum number of edge-disjoint cycles cannot be more than

$$\left[\frac{4rs + 2r}{4}\right] = rs + \left[\frac{r}{2}\right] = \left[\frac{m}{2}\right]\left[\frac{n}{2}\right] + \left[\frac{m}{4}\right].$$

We now show that such a collection of cycles always exists, which implies that $\bar{a}_1(K(m,n)) = [m/2][n/2] + [m/4]$. Of course, we have already seen that $K(m,n)$ always contains $[m/2][n/2]$ edge-disjoint cycles. If $m = 1$, then, $K(m,n)$ has no cycles while if $m = 3$, it is easily seen that the maximum number of edge-disjoint cycles is s. In either case, $\bar{a}_1(K(m,n))$ assumes the value $[m/2][n/2] + [m/4]$.

Assume now that $5 \leq m \leq n$. First we show that $\bar{a}_1(K(5,5)) = 5$. The graph $K(5,5)$ is shown in Fig. 9.5. The subgraphs induced by each of the following subsets of vertices are edge-disjoint 4-cycles: $\{u_1, u_2, w_1, w_2\}$, $\{u_2, u_3, w_4, w_5\}$, $\{u_1, u_4, w_3, w_5\}$, $\{u_4, u_5, w_2, w_4\}$, $\{u_3, u_5, w_1, w_3\}$. Since $[m/2][n/2] + [m/4] = 5$

when $m = n = 5$, we thus have $\bar{a}_1(K(5,5)) = 5$. We note also, for later reference, that none of the five cycles just described contains the edge u_5w_5.

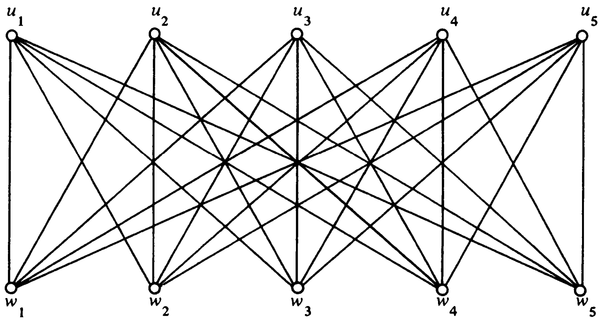

Figure 9.5 The graph $K(5,5)$

We construct $[m/2][n/2]$ edge-disjoint 4-cycles in $K(m,n)$, $5 \le m \le n$, as described at the beginning of this proof. Since m and n are both odd, there are vertices $u \in U$ and $w \in W$ such that $u \notin U_i$ for all $1 \le i \le r$ and $w \notin W_j$ for all $1 \le j \le s$. If we remove all those edges from the $[m/2][n/2]$ cycles which join $U_1 \cup U_2$ with $W_1 \cup W_2$, then we diminish the number of cycles by 4. However, we can construct five edge-disjoint 4-cycles on the two sets $U_1 \cup U_2 \cup \{u\}$ and $W_1 \cup W_2 \cup \{w\}$ so that the edge uw is not included, as we have seen. This procedure provides us with a net gain of one cycle. If U contains subsets U_3 and U_4, then we can proceed as before using $U_3 \cup U_4$ and $W_3 \cup W_4$, recalling that the edge uw is not needed in gaining an extra cycle. Since this procedure can be performed $[m/4]$ times, we see that we can accomplish a net gain of $[m/4]$ cycles over the original construction, obtaining $[m/2][n/2] + [m/4]$ cycles in all. This proves that $\bar{a}_1(K(m,n)) = [m/2][n/2] + [m/4]$. ∎

Formulas for the vertex analogue of cycle multiplicity are very easy to derive in the case of complete graphs and complete bipartite graphs; in fact, a formula exists [5] for any complete n-partite graph.

Another property which has given rise to parameters of the above type is planarity. The *edge-thickness* or simply the *thickness* $t_1(G)$ of a nonempty graph G is the edge-covering number of G with respect to planarity; that is, $t_1(G)$ is the minimum number of planar spanning subgraphs of G whose sum is G. Once again, it has been the complete graphs, complete bipartite graphs, and cubes that have received the most attention. The thickness of the n-cube has been found [**6**]; however, investigations with the complete bipartite graph [**3**] and the complete graph [**1**] have fallen short of solution. The vertex analogue of the edge-thickness is the vertex-thickness $t(G)$; very little is known about this parameter. There are other parameters of the type discussed in this section which will be encountered later in the book.

PROBLEM SET 9.3

9.13 Define the vertex analogue $\bar{a}(G)$ of the parameter cycle multiplicity. Derive formulas for $\bar{a}(K_p)$ and $\bar{a}(K(m,n))$.

9.14 Develop formulas for $\bar{a}(Q_n)$ and $\bar{a}_1(Q_n)$.

9.15 Prove that $t_1(K_p) \geq \left[\frac{p+7}{6}\right]$ for all positive integers p.

9.16 Verify that $t_1(K_p) = \left[\frac{p+7}{6}\right]$ for $p = 4, 5, 6, 7, 8$.

9.17 Give a definition for $t(G)$. Develop a formula for $t(K_p)$.

9.18 Define the parameters $\bar{t}(G)$ and $\bar{t}_1(G)$. Develop a formula for $\bar{t}(K_p)$ and an upper bound for $\bar{t}_1(K_p)$.

9.19 Define the vertex covering number of a graph G with respect to the property of being outerplanar. Find the value of this parameter for all complete graphs.

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10

Connectivity and Edge-Connectivity

In this chapter the concepts of cut-vertex, bridge, and block (introduced in Chapter 2) are generalized.

10.1 n-Connected and n-Edge Connected Graphs

The *vertex-connectivity* or simply *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G results in a disconnected graph or the trivial graph. The complete graph K_p cannot be disconnected by the removal of vertices, but the deletion of any $p-1$ vertices results in K_1; thus, $\kappa(K_p) = p - 1$. It is an immediate consequence of the definition that a nontrivial graph G has connectivity 0 if and only if G is disconnected. Furthermore, a graph G has connectivity 1 if and only if G is K_2 or is a connected graph with cut-vertices; $\kappa(G) \geq 2$ if and only if G is a cyclic block.

Connectivity has an edge analogue. The *edge-connectivity* $\kappa_1(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected or trivial graph. Connectivity and edge-connectivity are related as shown below [**9**].

Theorem 10.1 For any graph G,

$$\kappa(G) \leq \kappa_1(G) \leq \delta(G).$$

Proof Let $v \in V(G)$ such that $\deg v = \delta(G)$. The removal of the $\delta(G)$ edges of G incident with v results in a graph G' in which v is isolated so that G' is either disconnected or trivial; therefore, $\kappa_1(G) \leq \delta(G)$.

We now verify the second inequality. If $\kappa_1(G) = 0$, then G is disconnected or trivial so that $\kappa(G) = 0$. If $\kappa_1(G) = 1$, then G contains a bridge so that either G is K_2 or G is connected and contains cut-vertices; thus, $\kappa(G) = 1$. In each of these cases, $\kappa(G) = \kappa_1(G)$. We henceforth assume $\kappa_1(G) \geq 2$.

There exists a set of $\kappa_1(G)$ edges in G whose removal disconnects G. The removal of $\kappa_1(G) - 1$ of these edges results in a connected graph with a bridge $e = uv$. For each of the $\kappa_1(G) - 1$ edges, select an incident vertex different from u and v. If the removal of these vertices results in a graph H which is disconnected, then $\kappa(G) < \kappa_1(G)$. If, on the other hand, H is connected, then either H is K_2 or H has a cut-vertex. In either case, there exists a vertex of H whose removal results in a disconnected or trivial graph. Therefore, $\kappa(G) \leq \kappa_1(G)$. ■

A graph G is said to be *n-connected*, $n \geq 1$, if $\kappa(G) \geq n$. Thus, G is 1-connected if and only if G is nontrivial and connected, and G is 2-connected if and only if it is a cyclic block. It might be further noted that a graph G is n-connected if and only if the removal of fewer than n vertices results in neither a disconnected graph nor a trivial graph.

It is often the case that the knowledge that a graph is n-connected for some specified n is as valuable as knowing the connectivity itself. The following theorem gives a condition under which a graph is n-connected. The result is due to Bondy [1].

Theorem 10.2 Let G be a graph of order p which is not complete, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq \ldots \leq d_p$. Then G is n-connected if for each k, $1 \leq k \leq p - 1 - d_{p-n+1}$,

$$d_k \geq k + n - 1.$$

Proof Suppose $\kappa(G) < n$, and let S be a set of vertices of G such that $G - S$ is disconnected and $|S| = s = \kappa(G)$. Let H be a component of $G - S$ of minimum order k. Clearly, each vertex of H has degree at most $k - 1$ in $G - S$; therefore, the degree of each vertex of H is at most $k + s - 1$ in G. Thus,

$$d_k \leq k + s - 1 < k + n - 1 .$$

Hence, by the hypothesis of the theorem, $k > p - 1 - d_{p-n+1}$. From the manner in which H was chosen, $k + s \leq p - k$. Thus, for $v \in V(H)$,

$$\deg v \leq k + s - 1 \leq p - k - 1 .$$

For $u \in V(G) - V(H) - S$, $\deg u \leq p - k - 1$. Hence, there are at most s vertices (namely those of S) having degree exceeding $p - k - 1$. Thus $d_{p-s} \leq p - k - 1$. Now $k > p - 1 - d_{p-n+1}$ implies $d_{p-s} < d_{p-n+1}$. Hence $p - s < p - n + 1$ so that $s \geq n$, but this is a contradiction. The result now follows. ■

A graph G is *n-edge connected,* $n \geq 1$, if $\kappa_1(G) \geq n$. Equivalently, G is n-edge connected if the removal of fewer than n edges from G results in a connected graph. The class of n-edge connected graphs is characterized in the following theorem.

Theorem 10.3 A graph G is n-edge connected if and only if there exists no nonempty proper subset W of $V(G)$ such that the number of edges joining W and $V(G) - W$ is less than n.

Proof First, assume that there exists no nonempty proper subset W of $V(G)$ for which the number of edges joining W and $V(G) - W$ is less than n and that G is not n-edge connected. This implies that there exist k edges, $0 < k < n$, such that their deletion from G results in a disconnected graph H. Let H_1 be a component of H. Since the number of edges joining $V(H_1)$ and $V(G) - V(H_1)$ is at most k, where $k < n$, this is a contradiction.

Conversely, suppose G is an n-edge connected graph. If there should exist a subset W of $V(G)$ such that j edges, $j < n$, join W and $V(G) - W$, then the deletion of these j edges produces a disconnected graph—again a contradiction. The characterization now follows. ■

PROBLEM SET 10.1

10.1 (a) Generalize the concepts of connectivity and edge-connectivity for multigraphs.
(b) Generalize Theorem 10.1 for multigraphs.

10.2 Prove that if G is a graph of order p and $\delta(G) \geq p/2$, then $\kappa_1(G) = \delta(G)$.

10.3 (a) Prove that for a graph G of order $p \geq 2$ and for $1 \leq n < p$, the conditions (i) and (ii) are sufficient for G to be n-connected.
(i) for every k such that $n - 1 \leq k < (p + n - 3)/2$, the number of vertices of degree not exceeding k does not exceed $k + 1 - n$.
(ii) the number of vertices of degree not exceeding $(p + n - 3)/2$ does not exceed $p - n$.
(b) Prove that if $\delta(G) \geq (p + n - 2)/2$, then G is n-connected.

10.4 Let a, b, c be positive integers with $a \leq b \leq c$. Prove that there exists a graph G with $\kappa(G) = a$, $\kappa_1(G) = b$, and $\delta(G) = c$.

10.5 Let G be a graph with $\kappa(G) = n \geq 1$ and $\kappa_1(G) = m \geq 1$. What are the possible values for the following numbers:

$\kappa(G-v)$,
$\kappa(G-e)$,
$\kappa_1(G-v)$,
$\kappa_1(G-e)$?

10.2 Menger's Theorem

A graph G is connected (or, equivalently, 1-connected) if between every two distinct vertices of G there exists at least one path. This fact can be generalized in many ways, most of which involve, either directly or indirectly, a theorem due to Menger [**8**]. In this section, we discuss the major ones of these, beginning with Dirac's proof [3] of Menger's Theorem itself.

A set S of vertices (or edges) of a graph G is said to *separate* two vertices u and v of G if the removal of the elements of S from G produces a disconnected graph in which u and v lie in different components.

Theorem 10.4 (Menger's Theorem) If u and v are distinct nonadjacent vertices of a graph G, then the maximum number of disjoint $u-v$ paths in G equals the minimum number of vertices of G which separate u and v.

Proof If u and v are two vertices in different components of a graph G, then the result is obvious. Hence we may assume, without loss of generality, that the graphs under consideration are connected.

If the minimum number of vertices which separate two distinct nonadjacent vertices u and v is n, then clearly the maximum number of disjoint $u-v$ paths of G cannot exceed n. Thus, the theorem is true for $n = 1$.

If u and v are distinct nonadjacent vertices of a connected graph, then we denote by $S_n(u, v)$ the property that no fewer than n vertices separate u and v.

If the theorem is not true, then there is a smallest positive integer m for which there exist connected graphs G containing distinct nonadjacent vertices u and v such that $S_m(u, v)$ holds, but there is no set of m disjoint $u-v$ paths. Among such graphs G, let F be one of minimum order. Let H be a spanning subgraph of F which satisfies $S_m(u, v)$ but which fails to do so upon the removal of any edge. We now derive three properties of H.

(1) For any edge $e = v_1v_2$ of H, the graph $H - e$ does not satisfy $S_m(u, v)$. This implies that in $H - e$ there is a set $U(e)$ of fewer than m vertices which separates u and v. Since v_1 and v_2 are adjacent in H, at least one of v_1 and v_2, say v_1, is neither u nor v. Necessarily, $|U(e)| = m - 1$, for if $|U(e)| < m - 1$, then since $U(e) \cup \{v_1\}$ separates u and v, H would not satisfy $S_m(u, v)$. Hence, in any case, there exists a set of m vertices which separates u and v in H; namely $U(e) \cup \{v_1\}$ or $U(e) \cup \{v_2\}$.

(2) For any vertex w of H, $w \neq u, v$, not both of uw and vw are elements of $E(H)$; for otherwise, $H - w$ satisfies $S_{m-1}(u, v)$ so that $H - w$ contains a set of $m - 1$ disjoint $u-v$ paths. However, then, H contains m disjoint $u-v$ paths, which is impossible.

(3) If $W = \{w_1, w_2, \ldots, w_m\}$ is any set of m vertices which separates u and v in H, then either $uw_i \in E(H)$ for all $i = 1, 2, \ldots, m$ or $vw_i \in E(H)$, for all $i = 1, 2, \ldots, m$. In order to verify this, we first note that every $u-v$ path contains at least one vertex of W. (If this were not so, then the removal of W from H would still leave u and v connected.) Let H_u denote the subgraph determined by all $u-w_i$ paths of H, each of which contains exactly one element of W; the subgraph H_v is defined similarly. Certainly, $V(H_u) \cap V(H_v) = W$. Suppose neither

$uw_i \in E(H)$ for all $i = 1, 2, \ldots, m$ nor $vw_i \in E(H)$ for all $i = 1, 2, \ldots, m$. Denote by $H_u{}^*$ the graph obtained from H_u by adding a new vertex v^* and the edges v^*w_i, $i = 1, 2, \ldots, m$; denote by $H_v{}^*$ the graph obtained by adding a new vertex u^* to H_v together with the edges u^*w_i. Each of $H_u{}^*$ and $H_v{}^*$ has fewer vertices than H; and moreover, $H_u{}^*$ satisfies $S_m(u, v^*)$ and $H_v{}^*$ satisfies $S_m(u^*, v)$. This implies that $H_u{}^*$ contains a set of m disjoint u–v^* paths and, similarly, $H_v{}^*$ contains m disjoint u^*–v paths. However, then, these $2m$ paths determine a set of m disjoint u–v paths in H. This is a contradiction.

Now let P be a shortest u–v path in H. By (2), the length of P is at least 3. Thus, we may denote P by $u, u_1, u_2, \ldots, v$, where $u_1, u_2 \neq v$. If we let $e = u_1u_2$, then by (1), $U(e) \cup \{u_1\}$ separates u and v. Since $uu_1 \in E(H)$, by (2) and (3) it follows that u is adjacent to each vertex in $U(e)$. However, $U(e) \cup \{u_2\}$ also separates u and v so that $uu_2 \in E(H)$. This, however, contradicts the fact that P is a shortest u–v path. ■

With the aid of Menger's Theorem, it is now possible to present Whitney's characterization [9] of n-connected graphs.

Theorem 10.5 A graph G is n-connected if and only if for each pair u, v of distinct vertices there are at least n disjoint u–v paths in G.

Proof Assume G is an n-connected graph and that the maximum number of disjoint u–v paths in G is m, where $m < n$. If $uv \notin E(G)$, then by Theorem 10.4, $\kappa(G) \leq m < n$, which is contrary to hypothesis. If $uv \in E(G)$, then the maximum number of disjoint u–v paths in $G - uv$ is $m - 1 < n - 1$; hence, $\kappa(G - uv) < n - 1$. Therefore, there exists a set U of fewer than $n - 1$ vertices such that $G - uv - U$ is a disconnected graph. Therefore, at least one of $G - (U \cup \{u\})$ and $G - (U \cup \{v\})$ is disconnected implying that $\kappa(G) < n$, and this produces a contradiction also.

Conversely, suppose that G is a graph which is not n-connected but in which every pair of distinct vertices are connected by at least n–disjoint paths. Certainly, G is not complete.

Since G is not n-connected, $\kappa(G) < n$. Let W be a set of $\kappa(G)$ vertices of G such that $G - W$ is disconnected, and let u and v

be in different components of $G - W$. The vertices u and v are necessarily nonadjacent; however by hypothesis, there are at least n disjoint u–v paths. By Theorem 10.4, u and v cannot be separated by fewer than n vertices, so that a contradiction arises. ■

Both Theorems 10.4 and 10.5 have "edge" analogues; the analogue to Menger's Theorem was proved in [4,5]. It is not surprising that the edge analogue of Menger's Theorem can be proved in a manner which bears a striking similarity to the proof of Menger's Theorem.

Theorem 10.6 If u and v are distinct vertices of a graph G, then the maximum number of edge-disjoint u–v paths in G equals the minimum number of edges of G which separate u and v.

Proof We actually prove a stronger result here by allowing G to be a multigraph. If u and v lie in different components of G, then the theorem is, of course, true. We henceforth assume G to be connected.

If the minimum number of edges of a multigraph G which separate two vertices u and v is n, then the maximum number of edge-disjoint u–v paths of G cannot exceed n. Hence the theorem follows if $n = 1$.

For distinct vertices u and v of a connected multigraph G, we denote by $S_n(u, v)$ the property that no fewer than n edges separate u and v.

If the theorem is not true, then there exists a least positive integer m for which there are multigraphs G containing vertices u and v such that $S_m(u, v)$ holds, but there is no set of m edge-disjoint u–v paths. Among all such multigraphs G, let F denote one with the fewest number of edges. Certainly, then, for any edge e of F, the multigraph $F - e$ does not satisfy $S_m(u, v)$. This implies that in $F - e$ there is a set of $m - 1$ edges which separates u and v. Therefore, the minimum number of edges of F which separate u and v is m.

If every u–v path of F has length one or two, then every set of edges which separates u and v must contain at least one edge from each path; however, since a minimum such set contains m elements there are necessarily m edge-disjoint u–v paths of F. This is contrary to the properties possessed by F. Thus F must contain at least one u–v path P of length three or more. Let e_1 be an edge of P incident with neither u nor v. The edge e_1 belongs to a set of m edges of F which separates u and v, say $E_1 = \{e_1, e_2, \ldots, e_m\}$. We now subdivide each of the edges e_i, $1 \leq i \leq m$; i.e., let $e_i = u_i v_i$, replace each e_i by a new vertex w_i, and add the $2m$ edges $u_i w_i$ and $w_i v_i$. The vertices w_i are now identified, producing a new vertex w and a new multigraph H. The vertex w in H is a cut-vertex, and every u–v path of H contains w.

Denote by H_u the submultigraph of H determined by all u–w paths of H; the submultigraph H_v is defined similarly. Each of the multigraphs H_u and H_v has fewer edges than does F (since F contains a u–v path of length three or more). Also, the minimum number of edges separating u and w in H_u is m, and the minimum number of edges separating v and w in H_v is m. Thus, the multigraph H_u satisfies $S_m(u, w)$, and the multigraph H_v satisfies $S_m(w, v)$. This implies that H_u contains a set of m edge-disjoint u–w paths and H_v contains a set of m edge-disjoint w–v paths. For each $i = 1, 2, \ldots, m$, a u–w path and w–v path can be paired off to produce a u–v path in H containing the two edges $u_i w$ and $w v_i$. These m u–v paths of H are edge-disjoint. The process of subdividing the edges $e_i = u_i v_i$ of F and identifying the vertices w_i to obtain w can now be reversed to produce m edge-disjoint u–v paths in F. This, however, produces a contradiction.

Since the theorem has been proved for multigraphs G, its validity follows in the case where G is a graph. ■

With the aid of Theorem 10.6, it is now possible to present an edge analogue of Theorem 10.5.

Theorem 10.7 A graph G is n–edge connected if and only if for every two distinct vertices u and v of G, there exist at least n edge-disjoint, u–v paths in G.

PROBLEM SET 10.2

10.6 Prove Theorem 10.7.

10.7 Use Exercise 9.6 to show that $\kappa(G \times H) \geq \kappa(G) + \kappa(H)$, where G and H are connected graphs.

10.8 Assume that G is a graph in the proof of Theorem 10.6. Does the proof go through? If not, where does it fail?

10.3 Critically n-Connected Graphs

In Chapter 2 we discussed critical blocks and minimal blocks. These graphs could also be described, respectively, as critical with respect to the property of being 2-connected or 2-edge connected. In the present section, these concepts are generalized to n-connected graphs and n-edge connected graphs, $n \geq 1$.

A graph G is *critically n-connected* if $\kappa(G) = n$ and $\kappa(G - v) = n - 1$ for each $v \in V(G)$, while G is *minimally n-connected* if $\kappa(G) = n$ and $\kappa(G - e) = n - 1$ for every $e \in E(G)$.

Since the 1-connected graphs are the nontrivial connected graphs and since every nontrivial connected graph possesses at least two vertices which are not cut-vertices, it follows that the only critically 1-connected graph is K_2. It is also easily observed that a graph is minimally 1-connected if and only if it is a nontrivial tree. In Fig. 2.4, the graph G_1 is minimally 2-connected but not critically 2-connected while G_2 is critically 2-connected but not minimally 2-connected. The graph G_1 is, of course, the complete bipartite graph $K(2,3)$; in general, the graph $K(n, n+1)$ is minimally n-connected but not critically n-connected. For $n \geq 3$, the graph consisting of $K(n, n)$ together with an additional edge is critically n-connected but not minimally n-connected.

According to Corollary 2.10a, a minimally 2-connected graph has minimum degree 2. This result has been extended to minimally n-connected graphs by Halin [**6**] who proved that the minimum degree of every minimally n-connected graph is n. By Theorem 2.10, a critically 2-connected graph has minimum degree 2. This fact has also been generalized to critically n-connected graphs; however, the result in this case is not analogous, in general, to the aforementioned theorem of Halin. We illustrate the techniques involved by presenting a necessary condition for a graph to be critically n-connected [**2**].

> A set S of vertices (or edges) of a connected graph G is called a *cut set* if $G - S$ is disconnected; the cut set S is an *n-cut set* if $|S| = n$. We note that every connected graph which is not complete contains cut sets of vertices, and every nontrivial connected graph contains cut sets of edges.

Theorem 10.8 If G is a critically n-connected graph, $n \geq 2$, then $\delta(G) < (3n - 1)/2$.

Proof Suppose the theorem to be false so that there exists a graph G of order p having $\kappa(G)=n$ and $\delta(G) \geq (3n-1)/2$ such that for every $v \in V(G)$, $\kappa(G-v)=n-1$. We note that since $\delta(G) \geq (3n-1)/2$, G is not complete. This implies that every vertex of G belongs to some n-cut set of G.

Among all n-cut sets of G, let S be one such that $G-S$ contains a component G_1 of smallest order; denote the order of G_1 by m. Furthermore, let

$$G_2 = G - S - V(G_1).$$

Let $v \in V(G_1)$ and $u \in V(G_2)$. By Theorem 10.5, there exist n disjoint u–v paths in G; necessarily, each such path contains precisely one vertex of S. Hence there exist n disjoint paths joining u and S (and also v and S).

Let $w \in V(G_1)$, and let S^* be an n-cut set of G containing w. Define $G^* = G - S^*$ and, furthermore, let $V_1 = V(G_1) \cap S^*$; $V_2 = V(G_2) \cap S^*$; and $V_3 = S \cap S^*$, where $|V_i| = n_i$, $i = 1, 2, 3$. We note that $n_1 + n_2 + n_3 = n$ and $n_1 \geq 1$.

We now show that $n_2 \geq n_1$. If $S^* \supseteq V(G_2)$, then this is obvious. Assume therefore that $V(G_2) - V_2 \neq \emptyset$. We have already noted that for each $u \in V(G_2)$, there exists in G a set of n disjoint paths joining u and S. If $u \in V(G_2) - V_2$, then at least $n - n_2 - n_3 = n_1$ of these paths contain no vertices of $V_2 \cup V_3$. In this case, denote the set of end-vertices in S of these n_1 (or more) paths by $R(u)$. Thus for each $u \in V(G_2) - V_2$, there exists a set $R(u) \subset S - V_3$ such that there are disjoint paths containing no elements of $V_2 \cup V_3$ which join u and $R(u)$, where $|R(u)| \geq n_1$. If there exist vertices $u_1, u_2 \in V(G_2) - V_2$ such that $R(u_1) \cap R(u_2) = \emptyset$, then $|S - V_3| \geq 2n_1$ so that $n - n_3 \geq 2n_1$ and $n_2 \geq n_1$. Otherwise, let $R = \cup R(u)$, the union taken over all $u \in V(G_2) - V_2$, and let $G' = \langle R \cup (V(G_2) - V_2)\rangle$. It is now easy to verify that every two vertices of G' are connected, so that G' itself is connected. Hence G' is a subgraph of a component of G^*. Since the order of G' is at least $n_1 + (p - m - n) - n_2$, there must be a component of G^* of order at most $m + n_2 - n_1$. Therefore, $m \leq m + n_2 - n_1$ so that $n_2 \geq n_1$. Thus in any case, $n_2 \geq n_1$.

The inequality $n_2 \geq n_1$ implies that $n_1 \leq n/2$. We next verify that $V(G_1) - V_1 \neq \emptyset$ or, equivalently, that $n_1 < m$. Assume that $n_1 = m$ so that $V(G_1) = V_1$. Hence for each $v \in V(G_1)$,

$$\deg v \leq (n_1 - 1) + n \leq (3n-2)/2,$$

which contradicts the fact that $\delta(G) \geq (3n-1)/2$. We conclude therefore that $n_1 < m$ and $V(G_1) - V_1 \neq \emptyset$.

Let $F = \langle [V(G_1) - V_1] \cup [S - V_3] \rangle$. We show that F is disconnected. Suppose, to the contrary, that F is a connected subgraph of G^*. Since G^* is not connected, $V(G_2) - V_2 \neq \emptyset$. Because each $u \in V(G_2) - V_2$ is joined to $S - V_3$ by at least n_1 paths in G^*, it follows that G^* is connected, an impossibility. Thus F is disconnected.

Denote the components of F by F_t, $t = 1, 2, \ldots, k$, where $k \geq 2$. Furthermore, for each $t = 1, 2, \ldots, k$, denote by W_t the set of vertices of F_t in S, where $|W_t| = s_t$. We note that each $W_t \neq \emptyset$; for otherwise, there would exist a component of F of order less than m contained in $\langle V(G_1) - V_1 \rangle$, and this would also be a component of G^*.

We claim that precisely one of the subgraphs F_t contains elements of $V(G_1) - V_1$. Assume this is not the case so that there are two subgraphs F_i and F_j, $i \neq j$, containing elements of $V(G_1) - V_1$. Let

$$W'_i = \cup\, W_t,\ t \neq i, \quad \text{where} \quad |W'_i| = s'_i.$$

Each of the sets $V_1 \cup V_3 \cup W_i$ and $V_1 \cup V_3 \cup W'_i$ is a cut set of G, for in each case the removal of the set from G produces a graph having a component contained in $\langle V(G_1) - V_1 \rangle$. This implies that $n_1 + n_3 + s_i \geq n$ and $n_1 + n_3 + s'_i \geq n$ so that $s_i \geq n_2$ and $s'_i \geq n_2$. However, the equality $n_1 + n_2 + n_3 = s_i + s'_i + n_3 = n$ together with the inequality $n_2 \geq n_1$ yield $s_i = s'_i = n_1 = n_2$. Therefore, $V_1 \cup V_3 \cup W_i$ is an n-cut set of G, but the graph $G - (V_1 \cup V_3 \cup W_i)$ has a component of order less than m. This produces a contradiction; hence, exactly one of the subgraphs F_t contains elements of $V(G_1) - V_1$. Let F_1 be the subgraph with this property.

Now $V_1 \cup V_3 \cup W_1$ is a cut set of G so that $n_1 + n_3 + s_1 \geq n$ or $s_1 \geq n_2$. Let G^*_1 be a component of G^* which contains vertices of W'_1. If $V(G^*_1) \supseteq W'_1$, then $s'_1 \geq m$, but this implies that

$$n = s_1 + s'_1 + n_3 \geq n_2 + m + n_3 > n_2 + n_1 + n_3 = n,$$

which is impossible. Therefore, G^*_1 contains vertices of $V(G_2) - V_2$, which incidentally shows that $V(G_2) - V_2 \neq \emptyset$.

We show next that $V_2 \cup V_3 \cup W'_1$ is a cut set of G. Suppose this is not so. Then $G' = G - (V_2 \cup V_3 \cup W'_1)$ is connected. Since F_1 is connected, the graph $G'' = G' - V_1$ is also connected. However, $G^* = \langle V(G'') \cup W'_1 \rangle$ is disconnected; therefore, G^* has a component which is a subgraph of $\langle W'_1 \rangle$, but we have seen that every component of G^* which contains elements of W'_1 also contains elements of $V(G_2) - V_2$. Hence $G - (V_2 \cup V_3 \cup W'_1)$ is disconnected so that $V_2 \cup V_3 \cup W'_1$ is a cut set of G. This produces the inequality

$$n_2 + n_3 + s'_1 \geq n$$

or

$$s'_1 \geq n_1.$$

We now know that $s_1 + s'_1 = n_1 + n_2$, $s_1 \geq n_2$, and $s'_1 \geq n_1$. From this we conclude that $s_1 = n_2$ and $s'_1 = n_1$. Returning to the cut set $V_1 \cup V_3 \cup W_1$, we note that this is an n–cut set. However, $G - (V_1 \cup V_3 \cup W_1)$ contains a component of order less than m. This produces a contradiction, and the desired result follows. ■

A graph G is *minimally n-edge connected* if $\kappa_1(G) = n$ and $\kappa_1(G - e) = n - 1$ for all $e \in E(G)$. It is shown in [7] that a minimally n-edge connected graph G has $\delta(G) = n$. This leaves one analogous concept, namely critically n-edge connected graphs. No theorem for this class of graphs has been developed; indeed, it is not entirely clear as to what the proper definition of this concept should be (see Exercise 10.5).

PROBLEM SET 10.3

10.9 Prove that the bound given in Theorem 10.8 is "best possible"; i.e., for each $n \geq 2$, construct a graph G_n which is critically n-connected and such that $\delta(G_n) = [(3n - 2)/2]$.

10.10 Prove that an n-connected graph G is minimally n-connected if and only if for each pair u, v of adjacent vertices of G there exist at most n disjoint u–v paths in G.

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11

Hamiltonian Graphs

We saw in Chapter 3 how the concept of eulerian graphs developed through Euler's connection with the Königsberg Bridge Problem. In this chapter we introduce a class of graphs which possess a striking similarity to eulerian graphs and yet, at the same time, differ markedly from them.

11.1 Sufficient Conditions for Hamiltonian Graphs

A graph G is defined to be *hamiltonian* if it has a cycle containing all the vertices of G. The word "hamiltonian" is derived from Sir William Hamilton, the well-known mathematician. Surprisingly, though, Hamilton's relationship with the graphs bearing his name is not of a strictly mathematical nature [1]. In 1857, Hamilton reportedly introduced a game consisting of a solid regular dodecahedron made of wood, twenty nails (one inserted at each corner of the dodecahedron), and a supply of string. Each corner was marked with an important city of the time. The aim of the game was to find a route along the edges of the dodecahedron which passes through each city exactly once and which ends at the city where the route began. In order for the player to recall which cities in a route he had already visited, the string was used to connect the appropriate nails in the appropriate order. The dodecahedron proved to be rather awkward to manage so that Hamilton also produced a "planar graph" version of the game (see Fig. 11.1). There has never been any indication that either version of the game proved successful.

The object of Hamilton's game may be described in graphical terms, namely to determine whether the graph of the dodecahedron has a cycle containing each of its vertices—hence the term "hamiltonian".

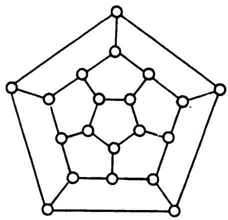


Figure 11.1 The graph of the dodecahedron

A cycle of a graph G containing every vertex of G is called a *hamiltonian cycle* of G; thus a hamiltonian graph is one which possesses a hamiltonian cycle. Because of the similarity in the definitions of eulerian graphs and hamiltonian graphs and because a particularly useful characterization of eulerian graphs exists, one might well expect an analogous criterion for hamiltonian graphs. However, such is not the case; indeed it must be considered one of the major unsolved problems of graph theory to develop an applicable characterization of hamiltonian graphs.

There have been several sufficient conditions established for a graph to be hamiltonian. We consider some of these in this section. The following result is due to Pósa [11], who proved this theorem in his early teens.

Theorem 11.1 Let G be a graph of order $p \geq 3$ such that for every integer j with $1 \leq j < p/2$, the number of vertices of degree not exceeding j is less than j. Then G is hamiltonian.

Proof Assume the theorem is false. Hence there exists a nonhamiltonian graph G which satisfies the hypothesis of the theorem. Without loss of generality, we assume that the addition of any edge to G results in a hamiltonian graph; for otherwise, edges could be added to G until a graph H is produced with the property that H is not hamiltonian, but the addition of any edge to H results in a hamiltonian graph. Furthermore, H satisfies the hypothesis of the theorem.

Among all pairs of nonadjacent vertices of G, let v_1 and v_p be two nonadjacent vertices such that $\deg v_1 + \deg v_p$ is a maximum. Suppose $\deg v_1 \leq \deg v_p$. As we have already noted, if the edge $v_1 v_p$ is added to G, the resulting graph G' is hamiltonian. Moreover, every hamiltonian cycle of G' must contain $v_1 v_p$. This implies that v_1 and v_p are the end-vertices of a path P: $v_1, v_2, \ldots, v_p$ in G containing every vertex of G. Now if a vertex v_i, $1 < i < p$, is adjacent to v_1, then v_{i-1} is not adjacent to v_p; for otherwise,

$$v_1, v_i, v_{i+1}, \ldots, v_p, v_{i-1}, v_{i-2}, \ldots, v_1$$

would be a hamiltonian cycle of G. Therefore, there are at least $\deg v_1$ vertices in G which are not adjacent to v_p, so that there are at most $p - 1 - \deg v_1$ vertices adjacent to v_p. Hence

$$\deg v_1 \leq \deg v_p \leq p - 1 - \deg v_1$$

or $\deg v_1 \leq (p-1)/2$. From the manner in which v_1 and v_p were chosen, it follows that $\deg v_{i-1} \leq \deg v_1$ for all vertices v_{i-1} not adjacent to v_p. Thus there are at least $\deg v_1$ vertices having degree not exceeding $\deg v_1$. However, $1 \leq \deg v_1 < p/2$, so, by hypothesis, there are fewer than $\deg v_1$ vertices having degree not exceeding $\deg v_1$. This presents a contradiction so that G is hamiltonian. ■

Intuitively, Pósa's theorem states that if a graph contains some vertices of small degree, but not a great number, then the graph is hamiltonian. In a certain sense, Pósa's result cannot be improved, for let $p \geq 4$ and suppose $1 \leq r < p/2$. If G is the connected graph having the two blocks K_{r+1} and K_{p-r}, then G satisfies the hypothesis of Pósa's theorem with one exception: It has at least r vertices of degree r. However, G is not hamiltonian.

Another sufficient condition for a graph to be hamiltonian is due to Ore [9]. It is a corollary of Theorem 11.1 although, chronologically, it preceded Pósa's theorem.

Corollary 11.1a If G is a graph of order $p \geq 3$ such that for all nonadjacent vertices u and v,

$$\deg u + \deg v \geq p,$$

then G is hamiltonian.

Proof Let k denote the number of vertices of G whose degree does not exceed n, where $1 \leq n < p/2$. These k vertices induce a subgraph H which is complete; for if any two vertices of H were not adjacent, then there would exist two nonadjacent vertices the sum of whose degrees is less than p. This implies that $k \leq n + 1$. However $k \neq n + 1$; for otherwise, each vertex of H is adjacent only to vertices of H, and if $u \in V(H)$ and $v \in V(G) - V(H)$, then

$$\deg u + deg\ v \leq n + (p - n - 2) = p - 2,$$

which is a contradiction. Furthermore, $k \neq n$; otherwise each vertex of H is adjacent to at most one vertex of G not in H. However, since $k = n < p/2$, there exists a vertex $w \in V(G) - V(H)$ adjacent to no vertex of H. Then if $u \in V(H)$,

$$\deg u + \deg w \leq n + (p - n - 1) = p - 1,$$

which again is a contradiction. Therefore, $k < n$, which implies that G satisfies the hypothesis of Theorem 11.1, so that G is hamiltonian. ■

Each of Theorem 11.1 and Corollary 11.1a implies that if a graph G of order $p \geq 3$ has no vertices of degree less than $p/2$, then G is hamiltonian. This fact was first discovered by Dirac [**5**]. We restate this below.

Corollary 11.1b If G is a graph of order $p \geq 3$ such that $\deg v \geq p/2$ for all $v \in V(G)$, then G is hamiltonian.

If the longest cycle of a graph G of order p has length p, then, of course, G has a hamiltonian cycle. If the longest path of a graph G of order p has length $p - 1$, then such a path is called a *hamiltonian path*. We now consider longest paths and longest cycles in nonhamiltonian graphs. By Corollary 11.1b, such graphs G satisfy $\delta(G) < p/2$.

Theorem 11.2 If G is a connected graph of order three or more which is not hamiltonian, then the length k of a longest path of G satisfies $k \geq 2\delta(G)$.

Proof Let P: $u_0, u_1, \ldots, u_k$ be a longest path in G. Since P is a longest path, each of u_0 and u_k is adjacent only to vertices of P.
If $u_0u_i \in E(G)$, $1 \leq i \leq k$, then $u_{i-1}u_k \notin E(G)$; for otherwise the cycle

$$C:\ u_0, u_1, \ldots, u_{i-1}, u_k, u_{k-1}, \ldots, u_i, u_0$$

of length $k+1$ is present in G. The cycle C cannot contain all vertices of G since G is not hamiltonian. Therefore there exists a vertex w not on C adjacent with a vertex of C; however, this implies G contains a path of length $k+1$, which is impossible. Hence for each vertex of $\{u_1, u_2, \ldots, u_k\}$ adjacent to u_0 there is a vertex of $\{u_0, u_1, \ldots, u_{k-1}\}$ not adjacent with u_k. Thus $\deg u_k \leq k - \deg u_0$ so that

$$k \geq \deg u_0 + \deg u_k \geq 2\delta(G)\,. \blacksquare$$

Theorem 11.2 implies the following result.

Corollary 11.2a If G is a graph of order p such that $\deg v \geq (p-1)/2$ for all $v \in V(G)$, then G contains a hamiltonian path.

PROBLEM SET 11.1

11.1 Show that the graphs of the five regular polyhedra are hamiltonian.

11.2 Let G be a graph with $\delta(G) \geq 2$. Prove that G contains a cycle of length at least $1 + \delta(G)$.

11.3 Prove that the n-cube Q_n, $n \geq 2$, is hamiltonian.

11.4 Let G be a (p, q) graph, where $p \geq 3$ and $q \geq (p^2 - 3p + 6)/2$. Prove that G is hamiltonian.

11.5 (a) Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$, $p \geq 3$, such that $\deg v_1 \leq \deg v_2 \leq \ldots \leq \deg v_p$. Suppose that $\deg v_k \leq k$ and $\deg v_n \leq n\ (k \neq n)$ imply $\deg v_k + \deg v_n \geq p$. Prove that G is hamiltonian. (See [2].)

(b) Prove that Theorem 11.1 follows as a corollary to the result in (a).

(c) Explain the statements in (a) in terms of the degree sequence of G.

11.2 Special Classes of Hamiltonian Graphs

As we have already indicated, it is an unsolved problem in graph theory to obtain an applicable characterization of hamiltonian graphs. In view of the lack of success in developing such a characterization, it is not surprising that special classes of hamiltonian graphs have been investigated, with the intended hope of characterizing these graphs. However, only in rare cases have criteria been found for these "highly" hamiltonian graphs. In fact, in many cases, the theorems present sufficient conditions, much like those for hamiltonian graphs themselves. In this section we consider three classes of hamiltonian graphs on which additional restrictions have been placed.

A graph G is called *hamiltonian-connected* if for every pair of distinct vertices u and v of G, there exists a hamiltonian u–v path. It is immediate that any hamiltonian-connected graph with at least three vertices is hamiltonian. As in the case of hamiltonian graphs, there is no characterization of hamiltonian-connected graphs but sufficient conditions for a graph to be hamiltonian-connected exist. The following theorem is due to Ore [**10**].

Theorem 11.3 If G is a graph of order $p(\geq 3)$ such that

$$\deg u + \deg v \geq p + 1$$

for every pair of nonadjacent vertices u and v, then G is hamiltonian-connected.

Proof Since $\deg u + \deg v \geq p + 1$ for every pair of nonadjacent vertices u and v, by Corollary 11.1a, G is hamiltonian. Let C: $v_1, v_2, \ldots, v_p, v_1$ be a hamiltonian cycle of G. Now suppose, to the contrary, that there exist two vertices v_i and v_j which are not joined by a hamiltonian path, where, say, $i < j$. The edge $v_{i+1}v_{j+1}$ (where $j + 1$ is expressed modulo p) cannot be present in G, for if it were, then

$$v_i, v_{i-1}, \ldots, v_{j+1}, v_{i+1}, v_{i+2}, \ldots, v_j$$

would be a hamiltonian v_i–v_j path of G.

Suppose there are n vertices v_k such that $v_{i+1}v_k \in E(G)$ and $i+1 < k < j+1$. For each such vertex v_k, the edge $v_{k-1}v_{j+1}$ is not present in G; for otherwise,

$$v_i, v_{i-1}, \ldots, v_{j+1}, v_{k-1}, v_{k-2}, \ldots, v_{i+1}, v_k, v_{k+1}, \ldots, v_j$$

would be a hamiltonian v_i–v_j path of G. Thus n vertices (including v_{i+1}) are ruled out to which v_{j+1} may be adjacent.

Assume there are m vertices v_k such that $v_{i+1}v_k \in E(G)$, where $k < i+1$ or $k > j+1$. For any such vertex v_k, the edge $v_{k+1}v_{j+1}$ is not in G; for, if this were the case, then

$$v_i, v_{i-1}, \ldots, v_{k+1}, v_{j+1}, v_{j+2}, \ldots, v_k, v_{i+1}, v_{i+2}, \ldots, v_j$$

would be a hamiltonian v_i–v_j path of G. Hence m vertices are eliminated to which v_{j+1} may be adjacent. However, one of these m vertices (namely v_{i+1}) is included among the previously described n vertices. Therefore, there are at least $n+m-1 = \deg v_{i+1} - 1$ vertices to which v_{j+1} is not adjacent, implying that

$$\deg v_{j+1} \leq (p-1) - (\deg v_{i+1} - 1)$$

or $\deg v_{i+1} + \deg v_{j+1} \leq p$, which contradicts our hypothesis. Thus G is hamiltonian-connected. ∎

A corollary now follows.

Corollary 11.3a If G is a graph such that $\deg v \geq (p+1)/2$ for all vertices v of G, then G is hamiltonian-connected.

Naturally every vertex (indeed every set of vertices) of a hamiltonian graph G lies on a hamiltonian cycle of G. In such a graph G, however, there may very well be edges which belong to no hamiltonian cycle of G. A hamiltonian graph G with the property that each of its edges belongs to a hamiltonian cycle of G is referred to as a *strongly hamiltonian graph*. The cycle C_p is strongly hamiltonian for all $p \geq 3$, while the graph G of Fig. 11.2 is hamiltonian but not strongly hamiltonian, since the edge e lies on no hamiltonian cycle of G.

Theorem 11.4 If G is a hamiltonian-connected graph of order $p \geq 3$, then G is strongly hamiltonian.

Proof Let $e = uv$ be an edge of G. Since G is hamiltonian-connected, there exists a hamiltonian u–v path P: $u, u_2, u_3, \ldots, u_{p-1}, v$ in G. However, $u, u_2, u_3, \ldots, u_{p-1}, v, u$ is a hamiltonian cycle of G containing e. Therefore, G is strongly hamiltonian. ∎

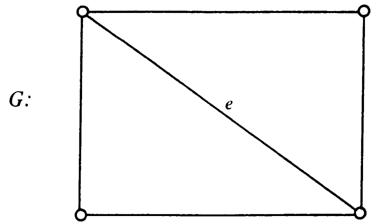


Figure 11.2 A hamiltonian graph which is not strongly hamiltonian

The converse of Theorem 11.4 is false, in general, since no cycle of order four (or more) is hamiltonian-connected. Combining Theorems 11.3 and 11.4, we arrive at a condition under which a graph is strongly hamiltonian.

Corollary 11.4a If G is a graph of order p such that $\deg u + \deg v \geq p + 1$ for every pair of nonadjacent vertices u and v, then G is strongly hamiltonian.

Finally, we consider a family of hamiltonian graphs whose definition is much like that of a class of eulerian graphs discussed in Section 3.3. A hamiltonian graph G is *randomly hamiltonian from a vertex* v of G if every path of G beginning at v can be extended to a hamiltonian v–v cycle of G. A graph is called simply *randomly hamiltonian* if it is randomly hamiltonian from each of its vertices.

Unlike the hamiltonian graphs and the subclasses of hamiltonian graphs described thus far, the randomly hamiltonian graphs have been characterized. In order to present this result (see [3]), a few preliminary definitions will be useful.

Let G be a hamiltonian graph and C: $v_1, v_2, \ldots, v_p, v_1$ a hamiltonian cycle of G. With respect to this cycle, every edge of G either lies on C (and will be called simply a *cycle edge* in this context) or joins two non-consecutive vertices of C and is referred to as a *diagonal*. Any cycle of G containing precisely one diagonal is an *outer cycle* of G (with respect to the fixed hamiltonian cycle C). Thus an outer n-cycle has $n - 1$ cycle edges and one diagonal.

Theorem 11.5 A graph G of order p is randomly hamiltonian if and only if G is isomorphic with one of the graphs C_p, K_p, or $K(p/2, p/2)$, the last being possible only if p is even.

Proof It is a routine matter to verify that each of the graphs C_p, K_p, and $K(p/2, p/2)$ is randomly hamiltonian.

Conversely, let G be a randomly hamiltonian graph of order $p, p \geq 3$. Since G is hamiltonian, it contains a hamiltonian cycle C: $v_1, v_2, \ldots, v_p, v_1$. If G consists only of the cycle C, then $G = C_p$. Suppose then that G contains diagonals and therefore outer cycles.

Suppose G contains the diagonal $v_j v_k$. We show now that G also contains the diagonal $v_{j+1} v_{k+1}$, where $j+1$ and $k+1$ are expressed modulo p. Consider the path which begins as follows: $v_{j+2}, v_{j+3}, \ldots, v_k$ (all subscripts expressed modulo p). We then proceed along the edge $v_j v_k$ to v_j and then take in turn v_{j-1}, $v_{j-2}, \ldots, v_{k+2}, v_{k+1}$. The path thus far contains every vertex of G with the exception of v_{j+1}; however, since G is randomly hamiltonian, v_{k+1} must be adjacent with v_{j+1}, thereby proving the existence of the diagonal $v_{j+1} v_{k+1}$ in G. By repeating this procedure, we conclude that if $v_j v_k$ is a diagonal of G, then $v_{j+i} v_{k+i}$ is a diagonal of G, for $i = 1, 2, \ldots, p-1$, where, as always, the numbers $j+i$ and $k+i$ are expressed modulo p.

Let n be the smallest value of m such that G contains an outer m-cycle. We next show that $n = 3$ or $n = 4$, for suppose $n \geq 5$. From what we have seen, $v_1, v_2, \ldots, v_n, v_1$ is an outer n-cycle so that $v_1 v_n$ is a diagonal of G, as is $v_2 v_{n+1}$.

We construct a path P by beginning with $v_4, v_5, \ldots, v_n$. We then proceed via the diagonal $v_1 v_n$ to v_1, next to v_2 and then to v_{n+1} along the diagonal $v_2 v_{n+1}$. We now follow along C by taking in order the vertices $v_{n+2}, v_{n+3}, \ldots, v_p$. Thus far, the path P contains every vertex of G except v_3. However, since G is randomly hamiltonian, v_p is adjacent with v_3, but then the vertices v_p, v_1, v_2, and v_3 form an outer 4-cycle. This produces a contradiction; hence the smallest outer cycle of G is an outer triangle or an outer 4-cycle.

Suppose G contains an outer triangle. We prove here that $G = K_p$. Since G has an outer triangle, G contains all the diagonals $v_i v_{i+2}$, $i = 1, 2, \ldots, p$. Let v_j and v_k be any two nonconsecutive vertices of G. We show that $v_j v_k \in E(G)$, thereby showing that G is complete. Now $v_{j-1} v_{j+1} \in E(G)$, as we have noted. We construct a path P by beginning with $v_{k+1}, v_{k+2}, \ldots, v_{j-1}$. We proceed via the diagonal $v_{j-1} v_{j+1}$ to v_{j+1}. Next we take, in order, the vertices $v_{j+2}, v_{j+3}, \ldots, v_k$. At this point, only the vertex v_j does not belong to P; but since G is randomly hamiltonian, v_k must be adjacent with v_j.

Assume next that G contains no outer triangle. Thus G contains as its smallest outer cycle a 4-cycle. Before proceeding, we first show that in this case, p is even. Suppose p is odd. Since G has an outer 4-cycle, G contains all the diagonals v_iv_{i+3}, $i = 1, 2, \ldots, p$. We construct a path P' which begins as follows: $v_5, v_4, v_7, v_6, v_9, v_8$. We continue this, proceeding in general to the vertices v_{2k+1}, v_{2k} for $k = 2, 3, \ldots, (p+1)/2$. Finally, we arrive at $v_{p-2}, v_{p-3}, v_p, v_{p-1}, v_2, v_1$. The path P' now contains all vertices of G except for v_3. Because G is randomly hamiltonian, however, the presence of the edge v_1v_3 in G is guaranteed. The vertices v_1, v_2, and v_3 now form an outer triangle, but this is contrary to the assumption that G contains no outer cycle of length less than four. Therefore, G contains an even number of vertices.

We now verify that G contains all edges of the type v_j and v_k, where j and k are of opposite parity. We may, of course, restrict our attention here to nonconsecutive vertices v_j and v_k. From what we have already shown, it is sufficient to prove that G contains all the diagonals v_1v_r, where r is even. Consider the path P_1 which begins at the vertex v_3, then proceeds in order to v_2, v_5, v_4, v_7, and v_6. We continue in this manner arriving at v_{r-1}, v_{r-2}, and then v_{r+1}. We follow C in the order $v_{r+2}, v_{r+3}, \ldots, v_p, v_1$. Since the path P_1 fails only to contain v_r and since G is randomly hamiltonian, v_1 is adjacent to v_r.

Finally, we show that G contains none of the diagonals v_jv_k, where j and k are of the same parity. Again, it is sufficient to show that G contains none of the diagonals v_1v_s, where s is odd. Suppose, to the contrary, that G has such an edge. We can then construct a path P_2 beginning at the vertex v_{s+2} and then proceeding, in order, to $v_{s+1}, v_{s+4}, v_{s+3}, v_{s+6}$. We continue in this manner until we arrive at v_{p-1}, v_{p-2}, and v_1 (since p is even). The path P_2 now proceeds along the edge v_1v_s to the vertex v_s and then about C as follows: $v_{s-1}, v_{s-2}, \ldots, v_3, v_2$. The path P_2 contains every vertex of G with the exception of v_p. Since G is randomly hamiltonian, G contains the diagonal v_2v_p. However, v_p, v_1, and v_2 form the vertices of an outer triangle, and this is a contradiction.

Thus if the smallest outer cycle of G is a 4-cycle, then p is even and v_jv_k is an edge of G if and only if j and k are of opposite parity; i.e., two vertices of G are adjacent if and only if one vertex belongs to V_1 and the other to V_2, where

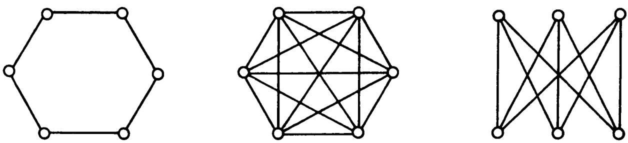
$$V_1 = \{v_{2m-1} \,|m = 1, 2, \ldots, p/2\}$$

and

$$V_2 = \{v_{2m} \,|m = 1, 2, \ldots, p/2\}.$$

Thus G is the complete bipartite graph $K(p/2, p/2)$. This completes the proof. ■

Hence Fig. 11.3 shows the only randomly hamiltonian graphs of order six.



C_6 K_6 $K(3, 3)$

Figure 11.3 The randomly hamiltonian graphs of order six

PROBLEM SET 11.2

11.6 (a) Give an example of a hamiltonian graph G_1 which is both hamiltonian-connected and strongly hamiltonian.

(b) Give an example of a hamiltonian graph G_2 which is neither hamiltonian-connected nor strongly hamiltonian.

11.7 Show that if G is a graph of order p such that $\deg v \geq (p+1)/2$ for every vertex v of G, then G is strongly hamiltonian.

11.8 Show that if G is a (p, q) graph, where $p \geq 4$ and $q \geq \binom{p-1}{2} + 3$, then G is hamiltonian-connected.

11.9 Prove that if G is a hamiltonian-connected graph of order $p \geq 4$, then $\delta(G) \geq 3$.

11.10 Let G be a graph of order $p \geq 3$ such that for every j with $2 \leq j \leq p/2$, the number of vertices of degree not exceeding j is less than $j - 1$. Prove that G is hamiltonian-connected.

11.11 Determine all graphs of order p which are randomly hamiltonian from at least one of their vertices. (See **[4]**.)

11.3 Hamiltonian Graphs and Powers of Graphs

In this section we revisit the concept of the power of a connected graph. Since the nth power G^n ($n \geq 2$) of a connected graph G contains G as a subgraph (a proper subgraph if G is not complete), it follows that G^n is hamiltonian if G is hamiltonian. Whether G is hamiltonian or not, it is clear that for sufficiently large n, the graph G^n is hamiltonian (provided, of course, G has order at least three) since G^d is complete if G has diameter d. It is therefore a natural question to inquire for the minimum n for which G^n is hamiltonian. Certainly, for connected graphs in general, $n = 2$ will not suffice since if H is the graph of Fig. 11.4, then H^2 is not hamiltonian.

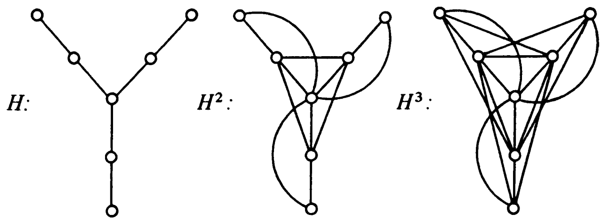


Figure 11.4 A graph whose square is not hamiltonian

Returning to Fig. 11.4, we see that the graph H^3 is hamiltonian. It is true, in general, that the cube of every connected graph of order at least three is hamiltonian. In fact, a stronger result exists, due to Karaganis [8] and Sekanina [12].

Theorem 11.6 Let G be a connected graph. Then G^3 is hamiltonian-connected.

Proof If H is a spanning tree of G and H^3 is hamiltonian-connected, then G^3 is hamiltonian-connected. Hence it is sufficient to prove that the cube of every tree is hamiltonian-connected. To show this we proceed by induction on p, the order of the tree. For small values of p, the result is obvious.

Assume for all trees H of order less than p that H^3 is hamiltonian-connected, and let T be a tree of order p. Let u and v be any two vertices of T. We consider two cases.

CASE 1. u *and* v *are adjacent in* T. Let $e = uv$, and consider the forest $T - e$. This forest has two components, one tree T_u of which contains u and the other tree T_v contains v. By hypothesis, $T_u{}^3$ and $T_v{}^3$ are hamiltonian-connected. Let u_1 be any vertex of T_u adjacent to u, and let v_1 be any vertex of T_v adjacent to v. If T_u or T_v is trivial, we define $u_1 = u$ or $v_1 = v$, respectively. Note that u_1 and v_1 are adjacent in T^3, since $d(u_1, v_1) \leq 3$ in T. Let P_u be a u–u_1 hamiltonian path (which may be trivial) of $T_u{}^3$, and let P_v be a v_1–v hamiltonian path of $T_v{}^3$. The path formed by beginning with P_u and then following with the edge u_1v_1 and the path P_v is a hamiltonian u–v path of T^3.

CASE 2. u *and* v *are not adjacent in* T. Since T is a tree, there exists a unique path between every two of its vertices. Let P be the unique u–v path of T, and let $f = uw$ be the edge of P incident with u. The graph $T - f$ consists of two trees, one tree T_u containing u and the other tree T_w containing w. By hypothesis, there exists a hamiltonian w–v path P_w in $T_w{}^3$. Let u_1 be a vertex of T_u adjacent to u, or let $u_1 = u$ if T_u is trivial, and let P_u be a hamiltonian $u - u_1$ path in $T_u{}^3$. Because the distance between u_1 and w does not exceed 2 in T, the edge u_1w is present in T^3. Hence the path formed by starting with P_u and then following with u_1w and P_w is a hamiltonian u–v path of T^3. ■

It is, of course, an immediate corollary that for any connected graph G of order at least three, G^3 is strongly hamiltonian and hamiltonian.

Although it is not true that the squares of all connected graphs of order at least three are hamiltonian, it was conjectured independently by Nash-Williams and Plummer that for 2-connected graphs this is the case. In 1971, Fleischner [6] proved the conjecture to be correct; indeed, Hobbs [7] then proceeded to verify that for any 2-connected graph G, its square G^2 is hamiltonian-connected.

PROBLEM SET 11.3

11.12 Show that the graph H^3 of Fig. 11.4 is hamiltonian, and prove that the graph H^2 is not hamiltonian.

11.13 Let H be any graph homeomorphic from the graph $K(2, 3)$. Show that H is not hamiltonian, but that H^2 is hamiltonian.

11.14 Prove that the square of every hamiltonian graph is hamiltonian-connected.

11.15 Prove that if v is any vertex of a connected graph G of order at least four, then the graph $G^3 - v$ is hamiltonian.

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12

Extremal Regular Subgraphs

In Chapter 11 we discussed hamiltonian cycles (and longest cycles in general). These might be interpreted as connected 2-regular subgraphs of maximum order. We consider here various other regular subgraphs of fixed degree having maximum or minimum order, beginning with the regular subgraphs of degrees 0 and 1.

12.1 Factorization and Matching

The maximum order among the induced regular subgraphs of degree 0 of a graph G is called the *independence number* of G and is denoted $\beta(G)$. The *edge independence number* $\beta_1(G)$ of G is the maximum number of edges among the regular subgraphs of degree 1 in G. For example, $\beta[K(m,n)] = \max(m,n)$ and $\beta_1[K(m,n)] = \min(m,n)$. In general, an *independent* set of vertices (or edges) is one whose elements are mutually nonadjacent. The independence number and edge independence number of a graph are related by means of an equation which involves two other parameters, referred to as covering numbers.

If a vertex and an edge are incident, then they are said to *cover* each other. The *vertex covering number* $\alpha(G)$ of a graph G having no isolated vertices is the minimum number of vertices which cover all edges of G; the *edge covering number* $\alpha_1(G)$ of G (having no isolated vertices) is the minimum number of edges of G which cover all vertices of G. As an example, we note that $\alpha[K(m,n)] = \min(m,n)$ and $\alpha_1[K(m,n)] = \max(m,n)$. The following is a theorem of Gallai [3].

Theorem 12.1 If G is a graph of order p having no isolated vertices, then

$$\alpha(G) + \beta(G) = p \tag{12.1}$$

and

$$\alpha_1(G) + \beta_1(G) = p. \tag{12.2}$$

Proof We begin with (12.1). Let U be an independent set of vertices of G with $|U| = \beta(G)$. Clearly, the set $V(G) - U$ of vertices of G covers all edges of G. Therefore, $\alpha(G) \leq p - \beta(G)$. If, however, W is a set of $\alpha(G)$ vertices which covers all edges of G, then $V(G) - W$ is independent; thus $\beta(G) \geq p - \alpha(G)$. This proves (12.1).

To verify (12.2), let E_1 be an independent set of edges of G with $|E_1| = \beta_1(G)$. Obviously, E_1 covers $2\beta_1(G)$ vertices of G. For each vertex of G not covered by E_1, select an incident edge and define E_2 to be the union of this set of edges and E_1. Necessarily, E_2 covers all vertices of G so that $|E_2| \geq \alpha_1(G)$. Also we note that $|E_1| + |E_2| = p$; hence $\alpha_1(G) + \beta_1(G) \leq p$. Now suppose E' is a set of edges of G which covers all vertices of G and such that $|E'| = \alpha_1(G)$. The minimality of E' implies that each component of $\langle E' \rangle$ is a star subgraph. Select from each component of $\langle E' \rangle$ one edge, denoting the resulting set of edges by E''. We observe that $|E''| \leq \beta_1(G)$ and that $|E'| + |E''| = p$. These two facts imply that $\alpha_1(G) + \beta_1(G) \geq p$, completing the proof of (12.2) and the theorem. ■

Every independent set E' of edges in a graph G produces, in a very natural way, a pairing of two disjoint subsets of vertices of G, each containing $|E'|$ elements. In this connection, we make the following definition. In a graph G, a nonempty subset U_1 of $V(G)$ is said to be *matched* to a subset U_2 of $V(G)$ disjoint from U_1 if the subgraph $\langle U_1 \cup U_2 \rangle$ of G contains a spanning 1-regular subgraph F such that each edge of F is incident with a vertex of U_1 and a vertex of U_2. A set of independent edges is thus often called a *matching* in G; a set of $\beta_1(G)$ such edges being referred to as a *maximum matching*. In general, no procedure or algorithm has been developed to determine a maximum matching in an arbitrary graph. (This has been referred to as "*The Matching Problem*".) Some results along this line have been obtained, however, in the case where G is a bipartite graph. (For some elementary applications of matchings see **[9]**.)

Let U_1 be a nonempty set of vertices of a graph G, and let U_2 be the set of all vertices disjoint from U_1 and adjacent with at least one element of U_1. Then the *deficiency* $\operatorname{def}(U_1)$ of U_1 in G is defined by

$$\operatorname{def}(U_1) = |U_1| - |U_2|.$$

The set U_1 is then said to be *nondeficient* (relative to G) if no subset of U_1 has positive deficiency.

Theorem 12.2 Let G be a bipartite graph with $V(G)$ partitioned as $V_1 \cup V_2$ such that every edge of G joins a vertex of V_1 with a vertex of V_2. The set V_1 can be matched to a subset of V_2 if and only if V_1 is nondeficient.

Proof If V_1 can be matched to a subset of V_2, then it is clear that V_1 is nondeficient. To verify the converse, we apply induction on $|V_1|$. If G is a bipartite graph for which V_1 is nondeficient and $|V_1| = 1$, then the result follows immediately. Thus, we assume for all bipartite graphs G having V_1 nondeficient and $|V_1| < n$, where $n \geq 2$, that V_1 can be matched to a subset of V_2. Let H be a bipartite graph with $V(H)$ partitioned as $W_1 \cup W_2$ such that each edge of H joins a vertex of W_1 with a vertex of W_2 and such that W_1 is nondeficient and $|W_1| = n$. We now consider two cases.

Case 1. *Assume that for each proper subset U of W_1*, $\operatorname{def}(U) < 0$. Since W_1 is nondeficient, each vertex of W_1 is adjacent to at least one vertex of W_2. Let $w_1 \in W_1$ and let $w_1w_2 \in E(H)$, where, of course, $w_2 \in W_2$. The graph $H' = H - w_1 - w_2$ is bipartite, and $V(H')$ may be partitioned as $W'_1 \cup W'_2$, where $W'_i = W_i - \{w_i\}$, $i = 1, 2$. Furthermore, each edge of H' is incident with a vertex of W'_1 and a vertex of W'_2. Since $|W'_1| < n$ and W'_1 is nondeficient, it follows by the inductive hypothesis that W'_1 can be matched to a subset of W'_2; i.e., $W_1 - \{w_1\}$ can be matched to a subset of $W_2 - \{w_2\}$. Since $w_1w_2 \in E(H)$, W_1 can be matched to a subset of W_2.

CASE 2. *Assume there exists a nonempty proper subset* U_1 *of* W_1 *such that* $\operatorname{def}(U_1) = 0$. Denote by U_2 the subset of W_2 consisting of those vertices adjacent with at least one element of U_1. Thus, $|U_1| = |U_2|$. Define the bipartite graphs

$$H_1 = \langle U_1 \cup U_2 \rangle$$

and

$$H_2 = \langle (W_1 - U_1) \cup (W_2 - U_2) \rangle.$$

The set U_1 of H_1 is nondeficient (since it is a subset of a nondeficient set W_1) and $|U_1| < n$; so by the inductive hypothesis, U_1 can be matched to a subset of U_2. We now show that the set $W_1 - U_1$ in the graph H_2 is nondeficient; for suppose there is a nonempty subset S_1 of $W_1 - U_1$ which has positive deficiency in H_2. Let S_2 be those vertices of $W_2 - U_2$ adjacent with at least one vertex of S_1; hence $|S_1| - |S_2| > 0$. However, then, in H,

$$\operatorname{def}(U_1 \cup S_1) = |U_1 \cup S_1| - |U_2 \cup S_2| > 0,$$

but this is contradictory. Therefore, $W_1 - U_1$ can be matched to a subset of $W_2 - U_2$. This implies that W_1 can be matched to a subset of W_2. ■

We are now in a position to present a well-known theorem due to Philip Hall [4]. A collection $S_1, S_2, \ldots, S_n$, $n \geq 1$, of finite nonempty sets is said to have a *system of distinct representatives* if there exists a set $\{s_1, s_2, \ldots, s_n\}$ of distinct elements such that $s_i \in S_i$ for $1 \leq i \leq n$.

Theorem 12.3 A collection $S_1, S_2, \ldots, S_n$, $n \geq 1$, of finite nonempty sets has a system of distinct representatives if and only if the union of any k of these sets contains at least k elements, for each $1 \leq k \leq n$.

Proof Let $S_1, S_2, \ldots, S_n$, $n \geq 1$, be a collection of finite nonempty sets and let V_1 be the set $\{v_1, v_2, \ldots, v_n\}$ of distinct vertices, where v_i corresponds to the set S_i. Furthermore, let V_2 be a set of vertices disjoint from V_1 such that $|V_2| = \left|\bigcup_{i=1}^{n} S_i\right|$, where there is a one-to-one correspondence between the elements of V_2 and those of $\bigcup_{i=1}^{n} S_i$.

We now construct a bipartite graph G such that $V(G) = V_1 \cup V_2$ and such that an edge of G joins a vertex v of V_1 with a vertex w of V_2 if and only if v corresponds to a set S_i and w corresponds to an element of S_i. From the manner in which G is defined, it follows that V_1 is nondeficient if and only if the union of any k of the sets S_i contains at least k elements. Now obviously, the sets S_i have a system of distinct representatives if and only if V_1 can be matched to a subset of V_2. Theorem 12.2 now produces the desired result. ■

It is obvious that no matching in a graph G of order p can possess more than $p/2$ edges; indeed, if G has a matching with $p/2$ edges then G contains a spanning 1-regular subgraph. This notion can be generalized.

A *factor* of a nonempty graph G is a nonempty spanning subgraph of G. If G is expressed as a sum of factors G_i, then this sum is called a *factorization* of G. We have actually considered various factorizations of a graph G already; namely, a nonempty graph G is the sum of $a_1(G)$ acyclic factors and $t_1(G)$ planar factors.

A factor of a graph G which is regular of degree $r \geq 1$ is called an *r-factor* of G. Thus G has a 1-factor if and only if $\beta_1(G) = p/2$. If there exists a factorization of G, each factor of which is an r-factor, then G is *r-factorable*. An r-factorable graph is necessarily regular having a degree which is a multiple of r. The problems involving these concepts which have received the most attention have been the determination of whether an arbitrary graph has a 1-factor and whether a regular graph is 1-factorable.

Graphs which contain 1-factors have been characterized **[11]**, but, as yet, no easily applicable criterion has been developed to determine whether a given graph has a 1-factor. Graphs which are 1-factorable have not been classified; indeed, only certain such regular graphs have been determined. We verify this for one class of regular graphs.

Theorem 12.4 Every regular bipartite graph of degree $r \geq 1$ is 1-factorable.

Proof We proceed by induction on r, the result being obvious for $r = 1$. Assume then that every regular bipartite graph of degree $r - 1$, $r \geq 2$, is 1-factorable, and let G be a regular bipartite graph of degree r, where $V(G) = V_1 \cup V_2$ and every edge of G is incident with a vertex of V_1 and one of V_2.

We now show V_1 is nondeficient, for suppose there exists a subset U_1 of V_1 having positive deficiency. If U_2 is the set of vertices of V_2 adjacent with at least one vertex of U_1, then $|U_1| - |U_2| > 0$. The number of edges of G incident with the vertices of U_1 is $r|U_1|$. These edges are, of course, also incident with the vertices of U_2. Since G is regular of degree r, the number of edges incident with the vertices of U_2 cannot exceed $r|U_2|$; hence, $r|U_2| \geq r|U_1|$. This, however, is contrary to our assumption that $|U_1| - |U_2| > 0$. Hence V_1 is nondeficient. By Theorem 12.2, V_1 can be matched to a subset of V_2. Since G is regular, we have $|V_1| = |V_2|$; thus, G has a 1-factor F. The removal of the edges of F from G results in a regular bipartite graph G' which is regular of degree $r-1$. By the inductive hypothesis, G' is 1-factorable, implying that G also is 1-factorable. ■

PROBLEM SET 12.1

12.1 Let G be an r-regular bipartite graph, and suppose that n is a positive integer which divides r. Prove that G is n-factorable.

12.2 Show that K_{2n}, $n \geq 1$, is 1-factorable.

12.3 Show that K_{2n+1}, $n \geq 1$, is not only 2-factorable but can be expressed as the sum of n hamiltonian cycles (i.e., connected 2-factors).

12.4 Prove that an r-regular complete n-partite graph, $r \geq 1$, $n \geq 2$, is 1-factorable if and only if rn is even. (See [**5**].)

12.5 Use Menger's Theorem to prove Theorem 12.2.

12.2 Petersen's Theorem

By definition every 1-regular graph contains a 1-factor and, trivially, is 1-factorable. A 2-regular graph G contains a 1-factor if and only if every component of G is an even cycle; such graphs are, of course, also 1-factorable. This brings us to the 3-regular or cubic graphs, which we may assume are connected. First, not all connected cubic graphs contain 1-factors, as is shown by the graph of Fig. 12.1.

Petersen [**10**], however, proved that every cubic graph which fails to contain a 1-factor possesses bridges. We follow the proof in [**7**] by beginning with a new definition.

Let $e = uv$ be an edge of a multigraph G such that e is the only edge joining u and v and $\deg u = \deg v = 3$. Suppose u is adjacent

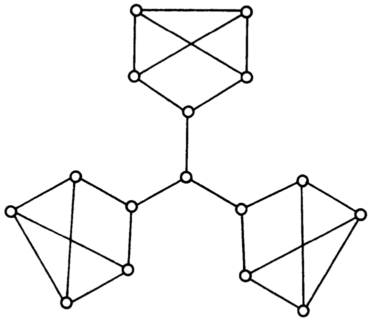

Figure 12.1 A cubic graph containing no 1-factors

to u_i, $i = 1, 2$, and v is adjacent to v_i, $i = 1, 2$. (The vertices u_1, u_2, v_1, v_2 need not be distinct.) Delete the vertices u and v from G and add the edges u_1v_1 and u_2v_2 to the resulting multigraph to obtain a pseudograph G_1. If we delete u and v and add u_1v_2 and u_2v_1, we obtain the pseudograph G_2. In the sequel, we shall refer to these added edges as the "new edges" of G_1 and G_2. We say that G_1 and G_2 are obtained from G by *splitting the edge* e of G. (Figure 12.2 shows the result of splitting an edge when the vertices u_1, u_2, v_1, v_2 are distinct.)

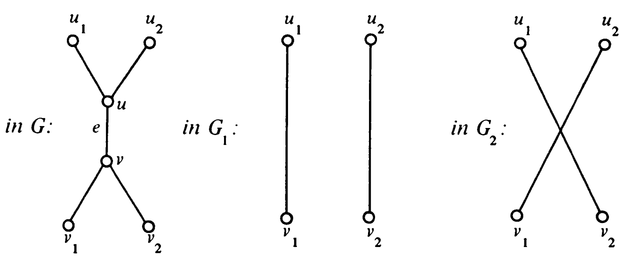

Figure 12.2 Splitting an edge

If G is a cubic multigraph, and G_1 and G_2 are obtained by splitting an edge of G, then it follows immediately that G_1 and G_2 are cubic pseudographs. The following result of Frink [2] will be useful in proving Petersen's theorem.

Theorem 12.5 Let G be a connected, bridgeless, cubic multigraph, and let $e = uv$ be the only edge of G joining u and v. If G_1 and G_2 are the two pseudographs obtained by splitting e, then at least one of G_1 and G_2 is a connected, bridgeless multigraph.

Proof The theorem is proved in five steps.

(1) *If* G_1 *is not connected, then it has exactly two components, one containing the new edge* u_1v_1 *and the other containing the new edge* u_2v_2. Suppose F is a component of G_1 which contains neither u_1v_1 nor u_2v_2. Then F is a submultigraph of G. Let u^* be a vertex of F. The multigraph G contains a u–u^* path P. This path contains at least one of u_1, v_1, u_2, and v_2 as well. If P contains either u_1 or v_1, then F contains the new edge u_1v_1, while if either u_2 or v_2 is in P, then F contains the new edge u_2v_2. This and the fact that G_1 is disconnected completes the proof of (1).

(2) *Neither the new edge* u_1v_1 *nor the new edge* u_2v_2 *is a bridge of* G_1. Suppose u_1v_1 is a bridge of G_1. Since u_1u is not a bridge of G, there is a u_1–u path in G which does not contain the edge u_1u. This path contains v_1, v_2, or u_2. Suppose that v_1 is the first among these three vertices to appear on this path. Then G_1 contains a u_1–v_1 path which does not contain the new edge u_1v_1, and (2) follows. In any other case, G_1 contains a u_1–u_2 path or u_1–v_2 path which excludes the new edges u_1v_1 and u_2v_2. Likewise, by noting that v_1v is not a bridge of G, we may assert that either (2) follows or G_1 contains a v_1–u_2 path or v_1–v_2 path which excludes the new edges u_1v_1 and u_2v_2. Let us assume that (2) does not hold, so that we have the existence of the aforementioned paths. The u_1–u_2 path or u_1–v_2 path (whichever exists), the v_1–u_2 path or v_1–v_2 path (whichever exists), and the edge u_2v_2 (if necessary) determine a u_1–v_1 path in G_1 which does not contain the new edge u_1v_1. Thus, the new edge u_1v_1 is not a bridge of G_1, which produces a contradiction. Similarly, the assumption that the new edge u_2v_2 is a bridge leads to a contradiction.

(3) *Suppose* G_1 *is connected and* $e^* = v_1^*v_2^*$ *is a bridge of* G_1. *Let* H_i *be the component of* $G_1 - e^*$ *which contains the vertex* v_i^*, $i = 1, 2$. *Then one of* H_1 *and* H_2 *contains the new edge* u_1v_1 *and the other contains the new edge*

u_1v_2. According to (2) the edge e^* belongs to G and is therefore contained in a cycle C of G. The cycle C must contain at least one of u_1, u_2, v_1, and v_2; for otherwise e^* would belong to a cycle of G_1. Consider the v_1^*-v_2^* path P on C not containing e^*. Assume that u_1 is the first of the vertices u_1, u_2, v_1, v_2 encountered on P. Then this $v_1^* - u_1$ path P^* is also a path in G_1 and does not contain the edge e^*. Thus the vertex u_1 and hence the new edge u_1v_1 belong to H_1. Next consider the $v_2^* - v_1^*$ path Q on C not containing e^*. The first vertex among u_1, u_2, v_1, v_2 on Q cannot be u_1 or v_1; for otherwise, G_1 would have a cycle containing e^*. Hence, the first vertex among u_1, u_2, v_1, v_2 on Q must be u_2 or v_2, which implies that u_1v_1 and u_2v_2 belong to different components of $G_1 - e^*$. The same conclusion is reached if u_2, v_1, or v_2 is the first vertex among u_1, u_2, v_1, v_2 encountered on the path P.

According to (2), there is a cycle C_i of G_1 which contains the new edge u_iv_i, $i = 1, 2$. We now proceed to the fourth step of the proof.

(4) *If* G_1 *is disconnected or contains a bridge, then the cycles* C_1 *and* C_2 *are disjoint.* If G_1 is disconnected, then the result follows from (1). Thus, we assume G_1 to be a connected graph possessing a bridge e'. By (3), C_1 and C_2 belong to different components of $G_1 - e'$; hence C_1 and C_2 are disjoint in G_1.

(5) *If* G_1 *is disconnected or contains a bridge, then* G_2 *is connected and bridgeless.* Delete the edges u_1v_1 and u_2v_2 from C_1 and C_2, respectively. By (4), then, G_1 contains a $u_1 - v_1$ path and a $u_2 - v_2$ path disjoint from it. These two paths also belong to G and G_2. The paths together with the new edges u_1v_2 and u_2v_1 produce a cycle C in G_2. Suppose G_2 is not both connected and bridgeless, and apply (4) with G_2 playing the role of G_1 and u_1v_2 and v_1u_2 replacing u_1v_1 and u_2v_2, respectively. Hence any cycle containing u_1v_2 must be disjoint from any cycle containing u_2v_1. This contradicts the existence of C and verifies (5).

It remains only to observe that if G_1 (or, equivalently, G_2) is connected and bridgeless, then it contains no loops. This is a consequence of the fact that G_1 is cubic and bridgeless. ■

Assume a cubic multigraph G is factored into a 1-factor F_1 and a 2-factor F_2. It is clear that any bridge of G must be an edge of F_1. If the edges of a cycle C in G belong alternately to F_1 and F_2, then C will be referred to as an *alternating cycle* of G (with respect to the given 1-factor and 2-factor).

We present one more result before proceeding to Petersen's theorem.

Theorem 12.6 Let G be a connected, bridgeless, cubic multigraph which is factored into a 1-factor F_1 and a 2-factor F_2. Then every edge of G belongs to an alternating cycle of G.

Proof Assume the theorem to be false. Among those multigraphs satisfying the hypotheses of the theorem but for which the conclusion fails, let G be one of minimum order. Certainly the order of G exceeds two since no pair of vertices can be joined by three edges. Suppose there exist vertices u and v in G joined by two edges. Then G contains a cycle of length two. Necessarily, G contains a vertex u_1 adjacent to u and a vertex v_1 adjacent to v. Furthermore, $u_1 \neq v_1$ since G is bridgeless. Delete the vertices u and v from G, and add a new edge u_1v_1, denoting the resulting multigraph by G'. Then G' too is connected, bridgeless, and cubic; in addition, $|V(G')| = |V(G)| - 2$.

We now show that G' can be factored into a 1-factor and a 2-factor. We consider the following two cases:

(i) u_1u and v_1v belong to F_1,
(ii) u_1u and v_1v belong to F_2.

In case (i), we construct a 1-factor F'_1 for G' by selecting all edges of F_1 except u_1u and v_1v and choosing the newly added edge u_1v_1. The remaining edges of G' determine a 2-factor F'_2 of G'. In case (ii), those edges of G' in F_1 determine a 1-factor F'_1 of G' and the remaining edges of G' (including the added edge u_1v_1) form a 2-factor F'_2. Hence G' can be factored into a 1-factor and a 2-factor.

Since G' has order less than that of G, every edge of G' belongs to an alternating cycle of G'. Let e be an edge of G' different from the added edge u_1v_1, and let C be an alternating cycle of G' containing e. If u_1v_1 is not in C, then C is an alternating cycle of G containing e. If u_1v_1 is in C, we may replace u_1v_1 by the path u_1, u, v, v_1, where the edge uv is arbitrary in case (i) and where uv is the edge in F_1 in case (ii), to arrive at an alternating cycle of G containing e. Hence the edges of G which belong to G' as well as the edges u_1u and v_1v belong to alternating cycles of G. It is easily seen that in both cases, (i) and (ii), the two edges joining u and v also belong to alternating cycles of G. This produces a contradiction; hence, G must not contain two vertices joined by two edges. Therefore, G must be a graph.

Let $e = u_0u_1$ be an arbitrary edge of the graph G which belongs to F_2. We show that e belongs to an alternating cycle of G. Let uu_1 be the other edge of F_2 which is incident with u_1, and let uv be the edge of F_1 incident with u. Further denote the third edge incident with u by u_2u and the other edges incident with v by v_1v and v_2v. (The vertices so introduced need not all be distinct.) Let G_1 and G_2 be the two pseudographs obtained by splitting uv; i.e., G_1 is produced by deleting u and v and adding the new edges u_1v_1 and u_2v_2, and G_2 is formed by deleting u and v and adding the new edges u_1v_2 and u_2v_1. By Theorem 12.5, at least one of G_1 and G_2 is a connected, bridgeless multigraph; hence, without loss of generality we assume G_1 to be a connected, bridgeless, cubic multigraph.

We note next that the edges of G_1 which belong to F_1 form a 1-factor F_1^* of G_1, while the edges of G_1 belonging to F_2 together with the new edges u_1v_1 and u_2v_2 determine a 2-factor F_2^* of G_1. Hence, G_1 can be expressed as the sum of a 1-factor F_1^* and a 2-factor F_2^*.

As we have already mentioned, the vertices $u, u_0, u_1, u_2, v, v_1, v_2$ need not all be distinct. (The situation where these vertices are distinct is illustrated in Fig. 12.3.) Thus we consider the possibility that e may be the same edge as uv, u_1u, u_2u, v_1v, or v_2v, none of which are edges of G_1. The edge e cannot be uv since, by definition, uv is an edge of F_1, and from the definition of u, the edges e and u_1u are distinct. The edge u_2u is distinct from e because only one edge joins u_1 and u_0. Suppose, then, that e is the same edge as v_1v. We cannot have $u_1 = v$, for otherwise there would exist two edges joining u and v; hence $u_1 = v_1$. However, then, u_1v_1 would be a loop in G_1, contradicting the fact that G_1 is a multigraph. Therefore, the only remaining possibility is that e is the same edge as v_2v, which we now assume. We have already noted that $u_1 \neq v$ so that $u_0 = v$ and $u_1 = v_2$.

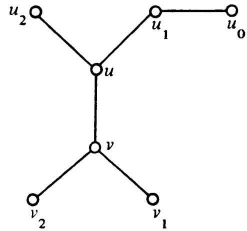

Figure 12.3 A step in the proof of Theorem 12.6

Since G_1 possesses two fewer vertices than does G, there is an alternating cycle C in G_1 which contains the new edge u_2v_2. Since two edges of F_2* cannot be adjacent in C, the edge u_1v_1, which is distinct from u_2v_2, is not in C. If the edge u_2v_2 is replaced by the path u_2, u, v, v_2 in G, then $e = u_0u_1 = vv_2$ is contained in an alternating cycle of G. (This consideration is valid in the case where not only $u_1 = v_2$, but also $u_2 = v_1$.)

We now assume that the edge $e = u_0u_1$ of F_2 is also an edge of G_1. From the manner in which G was selected, e is on an alternating cycle C_1 of G_1. The cycle C_1 cannot contain the new edge u_1v_1 of F_2* which is adjacent to e. Hence C_1 is either an alternating cycle of G or it contains u_2v_2. In the latter case, an alternating cycle of G containing e is obtained by replacing the edge u_2v_2 by the path u_2, u, v, v_2. Hence every edge of F_2 is contained in an alternating cycle of G.

Suppose next that $e = u_0u_1$ is an edge of F_1, and let $e' = u_1u_2$ be an edge of F_2 incident with u_1. We have shown that e' belongs to an alternating cycle of G, but this cycle also contains e since e is the only edge of F_1 incident with u_1. ■

We are now prepared to present a proof of Petersen's theorem.

Theorem 12.7 (Petersen's Theorem) Every bridgeless, cubic graph can be expressed as the sum of a 1-factor and a 2-factor.

Proof We actually give a stronger result by verifying the theorem for bridgeless, cubic multigraphs. Suppose the theorem is false, and let G be a bridgeless, cubic multigraph of minimum order which cannot be expressed as a sum of a 1-factor and a 2-factor. Clearly, G is connected and has order exceeding two. Necessarily, G contains two vertices u and v joined by exactly one edge, say $e = uv$. Denote the vertices adjacent to u by u_1 and u_2 and those adjacent to v by v_1 and v_2. Let G_1 and G_2 be the two pseudographs obtained by splitting the edge e; i.e., u and v are deleted and the new edges u_1v_1 and u_2v_2 are added to produce G_1, while the new edges u_1v_2 and u_2v_1 are added to produce G_2. By Theorem 12.5, we may assume G_1 to be a connected, bridgeless, cubic multigraph. Since G_1 has order less than that of G, we may express G_1 as a sum of a 1-factor F'_1 and a 2-factor F'_2. We now show, contrary to assumption, that G too can be expressed as the sum of a 1-factor F_1 and a 2-factor F_2. We define every edge of F'_i to be an edge of F_i, $i = 1, 2$. In order to determine to which of F_1 and F_2 the edges u_1u, u_2u, uv, v_1v, and v_2v should belong we distinguish three cases.

CASE 1. *The new edges* u_1v_1 *and* u_2v_2 *belong to* F'_2. Here we assign e to F_1 and the remaining four edges to F_2.

CASE 2. *One of the new edges, say* u_1v_1, *belongs to* F'_2 *and the other belongs to* F'_1. In this case, we assign u_2u and v_2v to F_1 and the remaining three edges to F_2.

In both Cases 1 and 2, it is straightforward to show that the edge-induced subgraphs F_1 and F_2 so defined determine a factorization of G into a 1-factor and a 2-factor.

CASE 3. *The new edges* u_1v_1 *and* u_2v_2 *belong to* F'_1. By Theorem 12.6, the new edge u_1v_1 is on an alternating cycle C of G_1. A new 1-factor of G_1 is produced by defining its edge set to consist of all edges of F'_1 not in C and all edges of F'_2 in C. The remaining edges of G_1 determine a new 2-factor of G_1. This returns us to Case 1 or Case 2, for which the result has already been proved. ■

Theorem 12.7 states that every bridgeless cubic graph G is the sum of a 1-factor and a 2-factor. If the 2-factor is the sum of two 1-factors, then G is 1-factorable. Not every bridgeless cubic graph is 1-factorable, as is illustrated by the (Petersen) graph of Fig. 8.5.

PROBLEM SET 12.2

12.6 Let u_1v_1 and u_2v_2 be two distinct, nonadjacent edges of a multigraph G. Remove these two edges from G, introduce two new vertices w_1 and w_2, and add the five edges u_1w_1, v_1w_1, u_2w_2, v_2w_2, and w_1w_2 to obtain a new multigraph G'. We say G' is obtained from G by *cross-subdividing the edges* u_1v_1 and u_2v_2.

(a) Let G be a connected multigraph with exactly one bridge e. If the components G_1 and G_2 of $G-e$ contain edges e_1 and e_2, respectively, then prove that the multigraph G' obtained by cross-subdividing e_1 and e_2 is bridgeless.

(b) Use (a) to prove that any connected, cubic multigraph with at most one bridge contains a 1-factor.

12.7 Give an example of a cubic multigraph which has exactly two bridges and does not contain a 1-factor.

12.3 The Girth of a Graph

We now switch our attention to minimal regular subgraphs. The problem of determining the smallest order of an r-regular subgraph of a graph for a fixed r is clearly trivial for $r=0$ and $r=1$; while for $r \geq 2$, it is extraordinarily difficult. Indeed, only the minimal 2-regular subgraphs (i.e., the shortest cycles) have been studied to any great extent. This takes us to our next concept.

The length of the shortest cycle of a graph G which is not a forest is called the *girth* of G and is denoted by $g(G)$. It is clear that the girth of a graph G is the minimum of the girths of its components and also the minimum of the girths of its blocks. Thus in determining the girth of a graph G, it is without loss of generality to assume G is a cyclic block. As might be expected, no formula exists for the girth of a graph in general. This, however, has not been the problem which has been considered the most; instead, it has been the following problem: For positive integers $r \geq 2$ and $n \geq 3$, determine the smallest positive integer $f(r,n)$, such that there exists an r–regular graph with girth n having order $f(r,n)$. The cubic graphs of order $f(3,n)$ having girth n have been the object of many investigations; such graphs are called *n–cages*. We introduce the notation $[r,n]$–graph to indicate an r–regular graph having girth n. Thus an n–cage is a $[3,n]$–graph; indeed, it is one of minimum order.

It is clear that $f(2,n)=n$ since the graph C_n has the desired properties and is clearly the $[2,n]$–graph having the smallest order. Likewise, $f(r,3)$ is easily determined; namely, $f(r,3)=r+1$, for

K_{r+1} is the $[r, 3]$-graph having minimum order. This implies that K_4 is the only 3-cage.

That $f(r, n)$ always exists is shown in the following result of Tutte [12, p. 82].

Theorem 12.8 For every pair of positive integers r, $n \geq 3$, the number $f(r, n)$ exists and in fact

$$f(r, n) \leq \left(\frac{r-1}{r-2}\right)[(r-1)^{n-1} + (r-1)^{n-2} + (r-4)].$$

Proof Set the integer

$$\left(\frac{r-1}{r-2}\right)[(r-1)^{n-1} + (r-1)^{n-2} + (r-4)] = p,$$

and let S be the set of all graphs H of order p such that $g(H) = n$ and $\Delta(H) \leq r$. Note that $p \geq n$. The set S is nonempty since the graph consisting of an n-cycle and $p - n$ isolated vertices belongs to S. For each $H \in S$, define

$$Z(H) = \{v \in V(H) \mid \deg v < r\}.$$

If for some $H \in S$, $Z(H) = \emptyset$, then we have the desired result; thus we assume for all $H \in S$, $Z(H) \neq \emptyset$. For $H \in S$, we define $z(H)$ to be the maximum distance between two vertices of $Z(H)$. (We define $d(u_1, u_2) = +\infty$ if u_1 and u_2 are not connected.)

Let S_1 be those graphs of S containing the maximum number of edges, and denote by S_2 those graphs of S_1 for which $|Z(H)|$ is maximum. Now among the graphs of S_2, let G be chosen so that $z(G)$ is maximum.

Let $u, v \in Z(G)$ such that $d(u, v) = z(G)$. Suppose $z(G) \geq n - 1 \geq 2$. By adding the edge uv to G, we obtain a graph G' of order p having $g(G') = n$ and $\Delta(G') \leq r$. Hence $G' \in S$; however, G' has more edges than G, and this produces a contradiction. Therefore, $z(G) \leq n-2$ and $d(u, v) \leq n-2$. (The vertices u and v may not be distinct.)

Denote by W the set of all those vertices w of G such that $d(u, w) \leq n-2$ or $d(v, w) \leq n-1$. From our earlier remark, it follows that $u, v \in W$. The number of vertices different from u and at a distance at most $n-2$ from u cannot exceed

$$\sum_{i=1}^{n-2} (r-1)^i = \left(\frac{r-1}{r-2}\right)[(r-1)^{n-2} - 1],$$

while the number of vertices different from v and at a distance at most $n-1$ from v cannot exceed

$$\sum_{i=1}^{n-1}(r-1)^i=\left(\frac{r-1}{r-2}\right)[(r-1)^{n-1}-1].$$

Hence the number of elements in W is at most

$$\left(\frac{r-1}{r-2}\right)[(r-1)^{n-2}-1]+\left(\frac{r-1}{r-2}\right)[(r-1)^{n-1}-1];$$

however, $[(r-1)/(r-2)][(r-1)^{n-1}+(r-1)^{n-2}-2]=p-r+1<p$. Therefore, there is a vertex $w_1 \in V(G)-W$, so $d(u, w_1) \geq n-1$ and $d(v, w_1) \geq n$.

Since $d(u, w_1) > z(G)$ and $u \in Z(G)$, it follows that $w_1 \notin Z(G)$ and $\deg w_1 = r \geq 3$. Therefore, there exists an edge e incident with w_1 whose removal from G results in a graph having girth n. Suppose $e=w_1w_2$. Clearly, $d(v, w_2) \geq n-1$ so that $w_2 \notin Z(G)$ and $\deg w_2 = r$.

We now add the edge uw_1 to G and delete the edge w_1w_2, producing the graph G_1. The graph G_1 also belongs to S and, in fact, belongs to S_1. The set $Z(G_1)$ contains all the members of $Z(G)$ except possibly u and, in addition, contains w_2. From the manner in which G was chosen, $|Z(G_1)| \leq |Z(G)|$; so that $u \notin Z(G_1)$ and $|Z(G_1)|=|Z(G)|$. Therefore, $\deg u = r$ in G_1, implying that, in G, $\deg u = r-1$.

We now show that u is not the only vertex of $Z(G)$, for suppose it is. Since there are an even number of odd vertices, we must have r and p odd; however, this cannot occur since p is even when r is odd. We conclude that u and v are distinct vertices of $Z(G)$.

The vertices v and w_2 are distinct vertices of $Z(G_1)$. If there exists no v–w_2 path in G_1, then $z(G_1)=+\infty$, and this is a contradiction to the fact that $z(G_1) \leq z(G)$. Thus v and w_2 are connected in G_1. Let P be a shortest v–w_2 path in G_1. If P is also in G, then P has length at least $d(v, w_2)$ in G, but $d(v, w_2) \geq n-1 > z(G)$, which is impossible. If P is not in G, then P contains the edge uw_1 and a u–v path of length $d(u, v)$ as a subpath. Hence $d(v, w_2) > d(u, v) = z(G)$, again a contradiction.

It follows that $Z(G)=\varnothing$ so that G is an r-regular graph of order p having girth n. ■

| We now determine the value of the number $f(r, 4)$. |

Theorem 12.9 For $r \geq 2$, $f(r, 4) = 2r$. Furthermore, there is only one $[r, 4]$-graph of order $f(r, 4)$, namely $K(r, r)$.

Proof Suppose G is an $[r, 4]$-graph, and let $u_1 \in V(G)$. Denote by $v_1, v_2, \ldots, v_r$ the vertices of G adjacent with u_1. Since $g(G) = 4$, v_1 is adjacent to none of the vertices v_i, $2 \leq i \leq r$; hence G contains at least $r - 1$ additional vertices $u_2, u_3, \ldots, u_r$. Therefore $f(r, 4) \geq 2r$. Obviously, the graph $K(r, r)$ is r-regular, has girth 4, and has order $2r$, thus implying $f(r, 4) = 2r$.

To show that $K(r, r)$ is the only r-regular graph having girth 4 with $2r$ vertices, let G be a $[r, 4]$-graph of order $2r$ whose vertices are labeled as above. Since every vertex has degree r and G contains no triangle, each u_i is adjacent to every v_j; therefore, $G = K(r, r)$. ■

By Theorem 12.9, there is a unique 4-cage: the graph $K(3, 3)$. There is no other value of $n > 4$ for which $f(r, n)$ is known for all values of r, nor is there a value of $r > 2$ for which $f(r, n)$ is known for all values of n. In these cases, only bounds have been determined. We illustrate this type of result by establishing a lower bound for $f(r, 5)$. It was shown in **[1]** that $f(r, 5) \geq r^2 + 1$ for all $r \geq 2$. Hoffman and Singleton **[6]** proved that equality holds for $r = 2, 3$, and 7, and possibly 57.

The following result is proved using techniques of Nash-Williams **[8]**.

Theorem 12.10 For $r \geq 2$, $f(r, 5) \geq r^2 + 1$. Furthermore, equality holds only if $r = 2, 3, 7$, or 57.

Proof Let G be an $[r, 5]$-graph, and let $v_1 \in V(G)$. Denote by $v_2, v_3, \ldots, v_{r+1}$ the vertices of G adjacent with v_1. Since $g(G) = 5$, no two vertices v_i and v_j, $1 < i < j \leq r + 1$, are mutually adjacent with a vertex different from v_1. Thus each vertex v_i, $2 \leq i \leq r + 1$, is adjacent with $r - 1$ new vertices. Hence G has at least $r(r - 1) + (r + 1) = r^2 + 1$ vertices, so that $f(r, 5) \geq r^2 + 1$.

Suppose now that G has order $r^2 + 1$ and that the remaining vertices are labeled so that $V(G) = \{v_i \mid i = 1, 2, \ldots, r^2 + 1\}$. Let A be the adjacency matrix of G.

First, we show that

$$A^2 + A = J + (r - 1)I, \tag{12.3}$$

where J is the (r^2+1)–by–(r^2+1) matrix all of whose entries are 1 and I is the (r^2+1)–by–(r^2+1) identity matrix. Since every vertex of G has degree r, each diagonal entry of A^2 is r, and therefore each diagonal entry of A^2+A is r. It is easy to verify that each diagonal entry of $J+(r-1)I$ is r also. Because the (i,j)–entry, $i \neq j$, of $J+(r-1)I$ is 1, it remains to show that each such entry of A^2+A is 1 also. Denote the (i,j)–entry of A by a_{ij} and that of A^2 by $a_{ij}^{(2)}$. By definition, $a_{ij}=1$ if and only if $v_iv_j \in E(G)$, and $a_{ij}=0$ otherwise. By Corollary 4.1a, $a_{ij}^{(2)}$ represents the number of paths of length 2 between v_i and v_j. Since $g(G)=5$, $a_{ij}^{(2)}=0$ or 1. If $a_{ij}=1$, then $a_{ij}^{(2)} \neq 1$; for otherwise, G contains a triangle. Hence in this case $a_{ij}^{(2)}=0$. Suppose next that $a_{ij}=0$. Because G is r–regular and has order r^2+1, no two vertices can have a distance exceeding 2. Thus since $d(v_i, v_j) \neq 1$, we have $d(v_i, v_j)=2$, thereby proving (12.3).

Next we show that r^2+r is an eigenvalue of A^2+A of multiplicity 1 and that $r-1$ is an eigenvalue of A^2+A of multiplicity r^2. Since $A^2+A=J+(r-1)I$, the eigenvalues of A^2+A are the roots of the equation:

$$|A^2+A-\lambda I| = \begin{vmatrix} r-\lambda & 1 & \cdots & 1 \\ 1 & r-\lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & r-\lambda \end{vmatrix} = 0.$$

If we add to the first row all other rows and factor out the common term $r^2+r-\lambda$, we obtain

$$|A^2+A-\lambda I| = [r^2+r-\lambda] \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & r-\lambda & 1 & \cdots & 1 \\ 1 & 1 & r-\lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & r-\lambda \end{vmatrix}.$$

Subtracting the first row from each of the other rows and applying Theorem 4A, we obtain $|A^2+A-\lambda I| = (r^2+r-\lambda)(r-1-\lambda)^{r^2}$, which gives us the desired results.

Let λ_i, $i=1,2,\ldots,r^2+1$, denote the eigenvalues of A. By Theorem 4I, the eigenvalues of $\phi(A)=A^2+A$ are $\phi(\lambda_i)$, $i=1, 2,\ldots,r^2+1$. Let $\phi(\lambda_1)=r^2+r$ and $\phi(\lambda_i)=r-1$ for $2 \leq i \leq r^2+1$.

Necessarily, $\lambda_1 = r$, for if $\alpha = (1, 1, \ldots, 1)$ is a vector all of whose $r^2 + 1$ entries are 1, then $A\alpha^t = r\alpha^t$, where α^t is the transpose of α. This shows that r is an eigenvalue of A.

The remaining r^2 eigenvalues are roots of the equation

$$\lambda^2 + \lambda = r - 1.$$

Hence each λ_i, $2 \le i \le r^2 + 1$, has either the value

$$(-1 + \sqrt{4r - 3})/2 \quad \text{or} \quad (-1 - \sqrt{4r - 3})/2.$$

Assume then that k of the eigenvalues, $0 \le k \le r^2$, are

$$(-1 + \sqrt{4r - 3})/2$$

while the remaining $r^2 - k$ eigenvalues are $(-1 - \sqrt{4r - 3})/2$.

By Theorem 4E, we have

$$r + k(-1 + \sqrt{4r - 3})/2 + (r^2 - k)(-1 - \sqrt{4r - 3})/2 = 0. \quad (12.4)$$

Solving for $2k$ in (12.4), we obtain $2k = (r^2 - 2r)/\sqrt{4r - 3} + r^2$. Since k is a nonnegative integer and $r \ge 2$, either $r = 2$ or $4r - 3$ is the square of an odd positive integer, say $4r - 3 = (2m + 1)^2$, where m is a positive integer. In the latter case, $r = m^2 + m + 1$, which implies that

$$2k = 2m - 1 + \frac{m^2}{4}\left(2m + 3 - \frac{15}{2m + 1}\right) + (m^2 + m + 1)^2.$$

The integers m^2 and $2m + 1$ are relatively prime so $2m + 1$ divides 15; hence $m = 1, 2$, or 7. Then $r = m^2 + m + 1$ implies $r = 3, 7$, or 57 so that r has one of the values $2, 3, 7, 57$. ■

According to Theorem 12.10, $f(3, 5) \ge 10$. It is not difficult to check that the Petersen graph is a 5-cage; in fact, it is the only 5-cage. In order to see this, let v_1 be a vertex of a $[3, 5]$-graph G of order 10, and let v_2, v_3, v_4 be the vertices of G with which v_1 is adjacent. Since $g(G) = 5$, each v_i, $i = 2, 3, 4$, is adjacent to two new vertices of G. Let v_5 and v_6 be adjacent with v_2, v_7 and v_8 with v_3, and v_9 and v_{10} with v_4. Hence $V(G) = \{v_i \mid i = 1, 2, \ldots, 10\}$. The fact that the girth of G is 5 and that every vertex of G has degree 3 implies that v_5 is adjacent with one of v_7 and v_8 and one of v_9 and v_{10}. Without loss of generality, we assume v_5 to be adjacent to v_7 and v_9. We must now have v_6 adjacent to v_8 and v_{10}. The edges

v_7v_{10} and v_8v_9 must now also be present and no others, but this is the Petersen graph.

We note in closing that there is only one 6-cage, referred to as the *Heawood graph*, and this is shown in Fig. 12.4. For more information on this topic, the interested reader is referred to [**12**, Chapter 8].

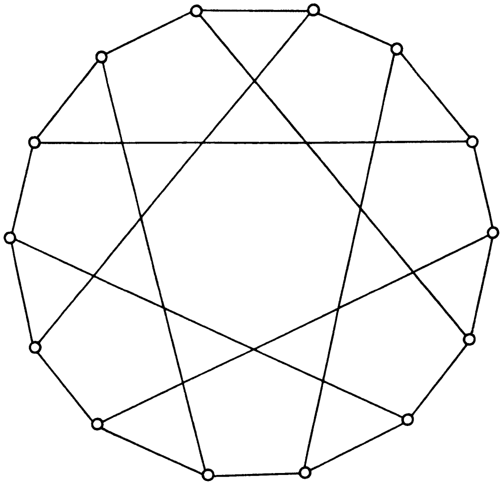

Figure 12.4 The Heawood graph: the unique 6-cage

PROBLEM SET 12.3

12.8 Let G be a connected graph which is not a tree. Show that $g(G) \leq 2 \operatorname{diam}(G) + 1$.

12.9 (a) Prove that $f(3, 6) = 14$.

(b) Prove that the Heawood graph is the only 6-cage.

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13

Graphs and Groups

With every set on which is defined a relation or operation, there exists a group of permutations on that set which preserves the given relation or operation. Graphs offer no exception to this statement. In this chapter, we discuss some well-known groups associated with a graph.

13.1 The Group and Edge-Group of a Graph

An *automorphism* of a graph G is an isomorphism of G with itself, i.e., a permutation on $V(G)$ which preserves adjacency. It is an immediate consequence of the definition that if ϕ is an automorphism of G and $v \in V(G)$, then $\deg \phi v = \deg v$.

It is straightforward to verify that (under the operation of composition) the set of all automorphisms of a graph G forms a group, denoted by $\Gamma(G)$ and referred to as the *vertex-group* or simply *the group* of G. For example, $\Gamma(K_p)$ is the symmetric group S_p of order $p!$, while $\Gamma(C_p)$ is the dihedral group D_p of order $2p$.

It is convenient to indicate that two groups Γ' and Γ'' are isomorphic by writing $\Gamma' \cong \Gamma''$. The first theorem of the chapter is simply a consequence of the definitions, but it is useful.

Theorem 13.1 For any graph G and its complement $\overline{G}$, $\Gamma(G) \cong \Gamma(\overline{G})$.

Proof Every element ϕ of $\Gamma(G)$ is a permutation on $V(G)$ which preserves adjacency in G. However, ϕ preserves adjacency if and only if ϕ preserves nonadjacency. Thus a permutation on $V(G)$ is an automorphism of G if and only if it is an automorphism of $\overline{G}$, implying that $\Gamma(G) \cong \Gamma(\overline{G})$. ■

We have already mentioned that $\Gamma(K_p) \cong S_p$. Certainly, if G is a graph of order p containing adjacent vertices as well as nonadjacent vertices, then $\Gamma(G)$ is isomorphic to a proper subgroup of S_p. Combining this observation with Theorem 13.1 and Lagrange's theorem on the order of subgroups of finite groups, we arrive at the following.

Corollary 13.1a The order $|\Gamma(G)|$ of the group of a graph G of order p is a divisor of $p!$ and equals $p!$ if and only if $G = K_p$ or $G = \overline{K}_p$.

With the aid of the group of a graph G of order p, it is possible to determine the number of non-identical graphs which are isomorphic to G and labeled from the same set of p labels.

Theorem 13.2 Let G be a graph of order p. The number of labelings of G from a set of p labels such that no two resulting graphs are identical is $p!/|\Gamma(G)|$.

Proof Let $\{v_1, v_2, \ldots, v_p\}$ be a set of p labels. Certainly, there exist $p!$ labelings of G without regard to the number of resulting labeled graphs which may be identical. If G_1 and G_2 are two labeled graphs obtained from G, then the relation "G_1 is identical to G_2" is an equivalence relation on the set of labeled graphs obtained from G. For a given labeled graph G_1, each automorphism of G gives rise to a labeled graph which is identical to G_1, and conversely. Hence each equivalence class so determined contains $|\Gamma(G)|$ elements, thus implying there are $p!/|\Gamma(G)|$ equivalence classes in all. This proves the theorem. ■

We turn our attention now to a second group associated with a graph. Two nonempty graphs G and G' are called *edge-isomorphic* if there exists a one-to-one mapping ϕ: $E(G) \to E(G')$ such that two edges e and f of G are adjacent if and only if the edges $\phi(e)$ and $\phi(f)$ of G' are adjacent. In this case, ϕ is called an *edge-isomorphism from* G *to* G$'$.

If G and G' are nonempty isomorphic graphs, then they are edge-isomorphic. In order to see this, let ϕ be an isomorphism from a nonempty graph G to a graph G'. Then $u_1u_2 \in E(G)$ if and only if $\phi(u_1)\phi(u_2) \in E(G')$. Moreover, the edges u_1u_2 and v_1v_2 of G are adjacent if and only if the edges $\phi(u_1)\ \phi(u_2)$ and $\phi(v_1)\ \phi(v_2)$ of G' are adjacent. Hence, each isomorphism from G to G' gives rise to or induces an edge-isomorphism from G to G'. Whenever an edge-isomorphism can be obtained (in this sense) from an isomorphism, we refer to the edge-isomorphism as *induced.* The converse of the above statement is not true in general; i.e., if G and G' are edge-isomorphic, then G and G' need not be isomorphic. This fact is illustrated in Fig. 13.1 with the graphs G_1 and G_2.

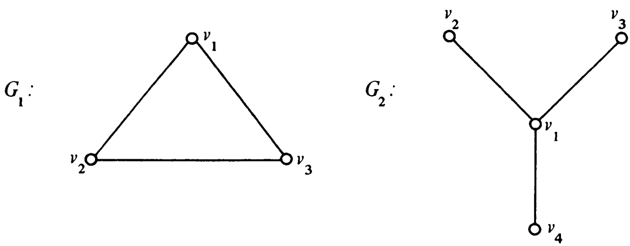

Figure 13.1 Edge-isomorphic graphs which are not isomorphic

An *edge-automorphism* of a nonempty graph G is an edge-isomorphism of G with itself. The set of all edge-automorphisms of G (under composition) forms a group, called the *edge-group* of G and denoted by $\Gamma_1(G)$. As examples, we note that $\Gamma_1(K(1,n)) \cong S_n$ and $\Gamma_1(C_p) \cong D_p$. We have already noted that each isomorphism from G to G' induces an edge-isomorphism from G to G'. In the case that G' also denotes the graph G, then we speak of an *induced edge-automorphism.* The set of all induced edge-automorphisms of G forms a group, called the *induced edge-group* of G and denoted by $\Gamma^*(G)$. Obviously, $\Gamma^*(G)$ is a subgroup of $\Gamma_1(G)$. To see that $\Gamma^*(G)$ may be a proper subgroup of $\Gamma_1(G)$, consider the graphs G_3, G_4, and G_5 shown in Fig. 13.2. Using the labelings as indicated, we observe that each mapping ϕ_i, $i = 3, 4, 5$, given below is an edge-automorphism of G_i which is not an induced edge-automorphism.

$$\phi_3 = \begin{pmatrix} v_1v_2 & v_2v_3 & v_2v_4 & v_3v_4 \\ v_3v_4 & v_2v_3 & v_2v_4 & v_1v_2 \end{pmatrix}$$

$$\phi_4 = \begin{pmatrix} v_1v_2 & v_2v_3 & v_2v_4 & v_3v_4 & v_1v_4 \\ v_2v_3 & v_3v_4 & v_2v_4 & v_1v_4 & v_1v_2 \end{pmatrix}$$

$$\phi_5 = \begin{pmatrix} v_1v_2 & v_2v_3 & v_2v_4 & v_3v_4 & v_1v_4 & v_1v_3 \\ v_3v_4 & v_2v_3 & v_2v_4 & v_1v_2 & v_1v_4 & v_1v_3 \end{pmatrix}.$$

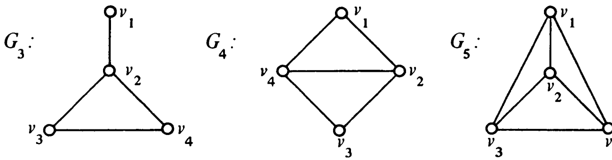

Figure 13.2 Graphs having edge-automorphisms not induced by any automorphism

As an additional observation regarding the graphs G_i of Fig. 13.2, we remark that $|\Gamma_1(G_i)| = 2|\Gamma^*(G_i)|$. We now consider a relationship between the group and the induced edge-group of a graph.

Theorem 13.3 Let G be a nontrivial connected graph. Then $\Gamma(G) \cong \Gamma^*(G)$ if and only if $G \neq K_2$.

Proof Since $\Gamma(K_2) \cong S_2$ while $\Gamma^*(K_2) \cong S_1$, the necessity is clear. For the sufficiency, we assume G to be a connected graph of order at least 3 so that $|E(G)| \geq 2$.

Define a mapping ϕ: $\Gamma(G) \to \Gamma^*(G)$ such that for $\alpha \in \Gamma(G)$, $\phi\alpha = \alpha^*$, where α^* is the edge-automorphism of G induced by α. We show that ϕ is a group isomorphism. By definition, the mapping ϕ is onto $\Gamma^*(G)$.

We next verify that ϕ is one-to-one. Let $\alpha, \beta \in \Gamma(G)$ such that $\alpha \neq \beta$. It is required to show that $\alpha^* \neq \beta^*$, i.e., there exists an edge e of G for which $\alpha^*e \neq \beta^*e$. Let $v \in V(G)$ such that $\alpha v \neq \beta v$, and let u be a vertex of G adjacent with v. If either $\alpha u \neq \beta v$ or $\beta u \neq \alpha v$, then for the edge $e = uv$, we have $\alpha^*e \neq \beta^*e$. Thus, we assume that $\alpha u = \beta v$ and $\beta u = \alpha v$. There exists a vertex w in G adjacent with at least one of u and v; say w is adjacent to v, where

$w \neq u$. If $e_1 = vw$, then $\alpha^* e_1 \neq \beta^* e_1$. If $e_2 = uw$ is an edge of G, then $\alpha^* e_2 \neq \beta^* e_2$. Hence, in any case, ϕ is one-to-one.

It remains to show that ϕ is operation-preserving, i.e., for any $e \in E(G)$, we have $\phi(\alpha\beta)(e) = (\phi\alpha)(\phi\beta)(e)$. Let

$$e = uv \quad \text{and} \quad \beta u = u',\ \beta v = v',\ \alpha u' = u'',\ \alpha v' = v''.$$

Then

$$\phi(\alpha\beta)(e) = \phi(\alpha\beta)(uv) = \alpha\beta(u)\alpha\beta(v) = \alpha u'\alpha v' = u''v''$$

and

$$\begin{aligned}(\phi\alpha)(\phi\beta)(e) &= (\phi\alpha)(\phi\beta)(uv) = (\phi\alpha)(\beta u\beta v) = (\phi\alpha)(u'v') \\ &= \alpha u'\alpha v' = u''v''.\end{aligned}$$

Hence ϕ is operation-preserving and is therefore an isomorphism between the groups $\Gamma(G)$ and $\Gamma^*(G)$. ■

Theorem 13.3 may now be generalized to arbitrary graphs.

Corollary 13.3a For a nontrivial graph G, $\Gamma(G) \cong \Gamma^*(G)$ if and only if G contains neither K_2 as a component nor two or more isolated vertices.

Figure 13.2 shows three graphs G for which $\Gamma_1(G)$ and $\Gamma^*(G)$ are not isomorphic. We now begin an investigation to determine those graphs G such that $\Gamma_1(G) \cong \Gamma^*(G)$. For the purpose of doing this, we present a preliminary result which is due to Whitney [5].

Theorem 13.4 Let ϕ be an edge-isomorphism from a connected graph H_1 onto a connected graph H_2, where H_1 is different from the graphs G_i, $i = 1, 2, 3, 4, 5$, of Figs. 13.1 and 13.2. Then ϕ is induced by an isomorphism from H_1 onto H_2.

Proof We consider two cases.

CASE 1. *Assume* H_1 *has a vertex* v_0 *of degree* $d \geq 4$. Denote by $v_1, v_2, \ldots, v_d$ the vertices of H_1 adjacent with v_0. Let $\phi(v_0v_i) = e_i$, $i = 1, 2, \ldots, d$. Since the edges v_0v_i are mutually adjacent, the edges e_i of H_2 are also mutually adjacent. Since $d \geq 4$, there is a vertex u_0 incident with all edges e_i. Let $e_i = u_0u_i$, $i = 1, 2, \ldots, d$. If v_i and v_j are adjacent vertices in H_1, $i, j \neq 0$, then the edge

$\phi(v_iv_j)$ of H_2 is adjacent with each of u_0u_i and u_0u_j but not with u_0u_k, $k \neq i, j$. This implies that $\phi(v_iv_j) = u_iu_j$. Let $A_1 = \langle\{v_0, v_1, \ldots, v_d\}\rangle$ and $B_1 = \langle\{u_0, u_1, \ldots, u_d\}\rangle$. Then the mapping ψ: $V(A_1) \to V(B_1)$ defined by $\psi(v_i) = u_i$, $i = 0, 1, \ldots, d$, is an isomorphism from A_1 to B_1. The mapping ψ induces an edge-isomorphism ϕ_1 from A_1 to B_1; namely, ϕ_1 is the restriction of ϕ to $E(A_1)$. If $A_1 = H_1$, then the proof is complete.

Assume $A_1 \neq H_1$. Since H_1 is connected, there exists a vertex v_{d+1} of H_1 adjacent with a vertex v_r of A_1. Let v_s be a vertex of A_1 adjacent with v_r (for example, we could choose here $v_s = v_0$). The edge $\phi(v_rv_{d+1})$ is not in B_1, but it is adjacent with the edge $\phi(v_rv_s) = u_ru_s$ in B_1. Thus there exists a vertex u_{d+1} in H_2 not belonging to B_1 such that $\phi(v_rv_{d+1})$ is either u_ru_{d+1} or u_su_{d+1}. Now the edge $\phi(v_rv_{d+1})$ is adjacent with u_ru_s and with every edge of B_1 which is incident with u_r. However, $\phi(v_rv_{d+1})$ is adjacent with no edge of B_1 which is incident with u_s, except u_ru_s. Since at least one of u_r and u_s has degree at least 2 in B_1, it follows that $\phi(v_rv_{d+1}) = u_ru_{d+1}$.

The preceding argument applies to every edge of H_1 incident with v_{d+1}. Hence if v_jv_{d+1} is an edge of H_1, where $1 \leq j \leq d$, then $\phi(v_jv_{d+1}) = u_ju_{d+1}$. Let $A_2 = \langle V(A_1) \cup \{v_{d+1}\}\rangle$ and $B_2 = \langle V(B_1) \cup \{u_{d+1}\}\rangle$. If we extend the aforementioned mapping ψ by defining $\psi(v_{d+1}) = u_{d+1}$, we note that ψ is an isomorphism from the connected graph A_2 onto the connected graph B_2. Furthermore, ψ induces an edge-isomorphism ϕ_2 from A_2 to B_2, and ϕ_2 is the restriction of ϕ to $E(A_2)$. If $A_2 = H_1$, then the desired result follows. Otherwise we proceed inductively until arriving at graphs A_{p-d} and B_{p-d} (where p is the order of H_1), where $A_{p-d} = H_1$, $B_{p-d} = H_2$, and the mapping ψ has been extended to an isomorphism from H_1 to H_2 such that ϕ is induced by ψ.

CASE 2. *Assume that the degree of no vertex of* H_1 *exceeds* 3. We may further assume that H_1 contains a vertex v_0 of degree 3, for the theorem is obvious for paths and cycles. Let v_1, v_2, and v_3 be the vertices of H_1 adjacent with v_0. The subgraph $A_1 = \langle\{v_0, v_1, v_2, v_3\}\rangle$ is either the graph G_2 of Fig. 13.1 or one of the graphs G_i, $i = 3, 4, 5$ of Fig. 13.2. By hypothesis, A_1 is a proper subgraph of H_1. Since every vertex of G_5 has degree 3, $A_1 \neq G_5$. In all other cases, there is at least one other vertex v_4 in H_1 adjacent with some vertex of A_1 different from v_0, say v_1. The edge $\phi(v_1v_4)$ of H_2 is adjacent with $\phi(v_0v_1)$ but adjacent with neither

$\phi(v_0v_2)$ nor $\phi(v_0v_3)$. This implies that the edges $\phi(v_0v_i)$, $i = 1, 2, 3$, do not form a triangle in H_2; however, since these three edges are mutually adjacent they are all incident with a vertex u_0 of H_2. Let $\phi(v_0v_i) = u_0u_i$, $i = 1, 2, 3$. For each edge v_iv_j of A_1, $i, j \neq 0$, the edge $\phi(v_iv_j)$ of H_2 is adjacent with both u_0u_i and u_0u_j but not with the other edge incident with u_0. Hence $\phi(v_iv_j) = u_iu_j$.

Let $B_1 = \langle\{u_0, u_1, u_2, u_3\}\rangle$. Define the function ψ: $V(A_1) \rightarrow V(B_1)$ by $\psi(v_i) = u_i$, $i = 0, 1, 2, 3$. The function ψ is an isomorphism from the connected graph A_1 to the connected graph B_1, and, moreover, induces an edge-isomorphism ϕ_1 from A_1 to B_1, where ϕ_1 is the restriction of ϕ to $E(A_1)$. By employing an argument identical to that used in Case 1, we obtain a proof of the theorem in this case also. ■

We are now in a position to characterize those graphs G for which $\Gamma_1(G) \cong \Gamma^*(G)$.

Theorem 13.5 Let G be a nonempty graph. Then $\Gamma(G) \cong \Gamma^*(G)$ if and only if:

(1) not both G_1 and G_2 (of Fig. 13.1) are components of G and
(2) none of the graphs G_i, $i = 3, 4, 5$ (of Fig. 13.2) is a component of G.

Proof If $\Gamma_1(G) \cong \Gamma^*(G)$, then conditions (1) and (2) must hold. We therefore consider the converse, and assume G to be a graph satisfying (1) and (2). Since $\Gamma^*(G)$ is a subgroup of $\Gamma_1(G)$, it remains only to show that every edge-automorphism of G is induced by an automorphism of G. If G is connected, then Theorem 13.4 immediately implies that $\Gamma_1(G) \cong \Gamma^*(G)$.

Suppose that G is disconnected. Let α be an edge-automorphism of G. For every component H of G, the subgraph $\langle\alpha(E(H))\rangle$ is also a component of G. If $H = G_1$ or $H = G_2$, then since G satisfies (1), we have $H = \langle\alpha(E(H))\rangle$. Therefore, if α is restricted to H, then α is induced by an automorphism of H. If H is different from G_1 and G_2, then by (2), $H \neq G_i$, $i = 1, 2, 3, 4, 5$. In this case, Theorem 13.4 implies that if α is restricted to H, then α is induced by an automorphism of H. Hence by applying the above argument to every component of G, we observe that α is induced by an automorphism of G, so that $\Gamma_1(G) \cong \Gamma^*(G)$. ■

Corollary 13.5a Let G be a nonempty connected graph. Then $\Gamma_1(G) \cong \Gamma^*(G)$ if and only if G is different from the graphs G_i, $i = 3, 4, 5$ (of Fig. 13.2).

Combining Theorem 13.3 and Corollary 13.5a, we obtain the following.

Corollary 13.5b Let G be a connected graph of order $p \geq 3$. Then the groups $\Gamma(G)$, $\Gamma_1(G)$, and $\Gamma^*(G)$ are isomorphic to one another if and only if G is different from the graphs G_i, $i = 3, 4, 5$ (of Fig. 13.2).

PROBLEM SET 13.1

13.1 Let G be a nonempty graph. Prove that $\Gamma_1(G)$ and $\Gamma^*(G)$ are indeed groups.

13.2 Determine $|\Gamma(G)|$, $|\Gamma_1(G)|$, and $|\Gamma^*(G)|$ for $G = P_n$, $n \geq 3$.

13.3 Compute $\Gamma(G_i)$, $\Gamma_1(G_i)$, and $\Gamma^*(G_i)$ for the graphs G_i, $i = 1, 2, 3, 4, 5$, of Figs. 13.1 and 13.2.

13.4 Let G be a nonempty graph. Determine necessary and sufficient conditions for G such that $\Gamma(G) \cong \Gamma_1(G) \cong \Gamma^*(G)$.

13.5 Determine, for as many positive integers n as you can, a graph G_n such that $|\Gamma(G_n)| = n$.

13.2 Graphs with a Given Group

In 1936 the first book on graph theory was published. In this book the author, König [**4**, p. 5], proposed the problem of determining all finite groups Ω for which there exists a graph G such that $\Gamma(G) \cong \Omega$. This problem was solved in 1938 by Frucht [**1**], who proved that every finite group has this property. In this section a proof of Frucht's theorem is presented, and further developments of this problem are discussed.

We begin with a few additional concepts. A digraph D is *complete symmetric* if for every two vertices u and v of D, both (u, v) and (v, u) are arcs of D.

With every finite group $\Omega = \{\omega_1, \omega_2, \ldots, \omega_p\}$ of order p, one can associate a complete symmetric digraph $D(\Omega)$, referred to as the *Cayley color graph* of Ω. The digraph $D(\Omega)$ has vertex set

$\{\omega_1, \omega_2, \ldots, \omega_p\}$ and its arcs are labeled with the non-identity elements of Ω according to the rule: the arc (ω_i, ω_j) is labeled $\omega_i^{-1}\omega_j$. If ω_p denotes the identity of Ω, then, in some sense, one might think of "coloring" the arcs of $D(\Omega)$ with the elements of $\Omega - \{\omega_p\}$.

As an illustration, let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ be the cyclic group of order 4, where ω_4 is the identity and $\omega_2^2 = \omega_4$. The Cayley color graph $D(\Omega)$ is shown in Fig. 13.3.

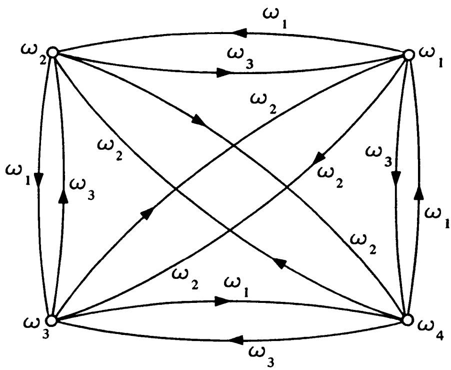

Figure 13.3 The Cayley color graph of the cyclic group of order 4

An *automorphism of a digraph D* is an isomorphism of D with itself. The set of all automorphisms of D forms a group under composition, and we denote this group by $\Gamma(D)$. Let Ω be a finite group. An element $\alpha \in \Gamma(D(\Omega))$ is said to be *color-preserving* if for every arc (ω', ω'') of $D(\Omega)$, the arcs (ω', ω'') and $(\alpha\omega', \alpha\omega'')$ have the same label (or color).

Returning to Fig. 13.3, we note that in this case $(\omega_1\, \omega_2\, \omega_3\, \omega_4)$ is a color-preserving automorphism while $(\omega_1\, \omega_2)(\omega_3)(\omega_4)$ is not.

For a given finite group Ω, it is a routine exercise to prove that the set of all color-preserving automorphisms of $D(\Omega)$ forms a subgroup of $\Gamma(D(\Omega))$. The major significance of this subgroup is contained in the following theorem.

Theorem 13.6 Every finite group Ω is isomorphic to the group of color-preserving automorphisms of $D(\Omega)$.

Proof Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_p\}$, where ω_p is the identity. For $i = 1, 2, \ldots, p$, define α_i: $\Omega \to \Omega$ by $\alpha_i \omega_m = \omega_i \omega_m$ for all m. The label of an arc (ω_r, ω_s) is the same as the label of $(\alpha_i \omega_r, \alpha_i \omega_s)$ since $\omega_r^{-1}\omega_s = (\omega_i\omega_r)^{-1}(\omega_i\omega_s)$; therefore, each α_i is a color-preserving automorphism of $D(\Omega)$. Also, $\alpha_i \neq \alpha_j$ for $i \neq j$.

We now show that if α is a color-preserving automorphism of $D(\Omega)$, then $\alpha = \alpha_i$ for some $i = 1, 2, \ldots, p$. Let $\omega_m \in \Omega$, $m \neq p$, and suppose $\alpha\omega_m = \omega_x$. Further, assume that $\alpha\omega_p = \omega_i$. The label of (ω_p, ω_m) is ω_m; therefore, the label of $(\alpha\omega_p, \alpha\omega_m)$ is ω_m, i.e., $\omega_i^{-1}\omega_x = \omega_m$ so that $\omega_x = \omega_i\omega_m$. Hence $\alpha\omega_m = \omega_i\omega_m$ for all m, thereby implying that $\alpha = \alpha_i$.

Finally, we verify that $\phi(\omega_i) = \alpha_i$ defines an isomorphism from Ω to the group of color-preserving automorphisms of $D(\Omega)$. Clearly, the mapping ϕ is one-to-one and onto. Hence it remains only to show that ϕ is a group homomorphism, i.e., that $\phi(\omega_i\omega_j) = \phi(\omega_i)\phi(\omega_j)$ for all i and j. Let $\omega_i\omega_j = \omega_k$. We must show that $\alpha_i\alpha_j = \alpha_k$ or, equivalently, that $(\alpha_i\alpha_j)(\omega_m) = \alpha_k(\omega_m)$ for each $\omega_m \in \Omega$. However,

$$\begin{aligned}(\alpha_i\alpha_j)(\omega_m) &= \alpha_i(\alpha_j\omega_m) = \alpha_i(\omega_j\omega_m) = \omega_i(\omega_j\omega_m) \\ &= (\omega_i\omega_j)\omega_m = \omega_k\omega_m = \alpha_k(\omega_m). \qquad \blacksquare\end{aligned}$$

We are now in a position to construct a graph G whose group $\Gamma(G)$ is isomorphic to a given finite group $\Omega = \{\omega_1, \omega_2, \ldots, \omega_p\}$, where ω_p is the identity of Ω. We first construct the Cayley color graph $D(\Omega)$ of Ω, which is actually a directed graph, of course. By Theorem 13.6, the group of color-preserving automorphisms of $D(\Omega)$ is isomorphic to Ω. We now transform the digraph $D(\Omega)$ into a graph by the following technique. Let (ω_i, ω_j) be an arbitrary arc of $D(\Omega)$, which is labeled $\omega_i^{-1}\omega_j = \omega_k$. Delete this arc, replacing it with the "graphical" path: $\omega_i, u_{ij}, u'_{ij}, \omega_j$. At the vertex u_{ij} we construct a new path P_{ij} of length $2k-2$, and at the vertex u'_{ij} a path P'_{ij} of length $2k-1$. This construction is now performed with every arc of $D(\Omega)$, and is illustrated in Fig. 13.4 with each of the three types of labeled arcs of the digraph of Fig. 13.3.

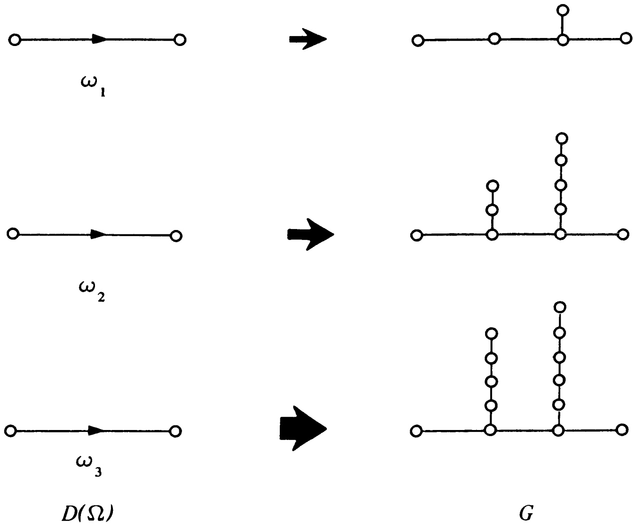

Figure 13.4 Constructing a graph G from a given group Ω

The addition of the paths P_{ij} and P'_{ij} in the formation of G is, in a sense, equivalent to the direction of the arcs in the construction of $D(\Omega)$. It now remains to observe that every color-preserving automorphism of $D(\Omega)$ induces an automorphism of G, and conversely. We state this below.

Theorem 13.7 (Frucht's Theorem) For every finite group Ω, there exists a graph G such that $\Gamma(G) \cong \Omega$.

The condition of having a given prescribed group is not a particularly stringent one for graphs, for it is possible to prescribe a group Ω, a connectivity $\kappa \geq 2$, a degree of regularity $r \geq 3$, and some values of certain other parameters (as well as a combination of these) and then construct a graph possessing all these characteristics [3]. In order to illustrate some of the techniques involved in the construction of such graphs, we devote the remainder of this chapter to the determination of a cubic graph whose group is isomorphic to a given finite cyclic group. Our procedure follows that in [**2**].

Let v be a vertex of a cubic graph G, and suppose that v is incident with the edges e_1, e_2, e_3 of G. Denote by μ_{ij}, $i \neq j$, the length of a shortest cycle of G containing e_i and e_j, where we define $\mu_{ij} = 0$ if e_i and e_j do not lie on a cycle of G. Let μ_1, μ_2, μ_3 be the numbers μ_{ij} in nondecreasing order; then the triple (μ_1, μ_2, μ_3) is called the *type* of the vertex v in G. For example, in Fig. 13.5 the type of the vertex u_1 in G_1 is $(4, 5, 7)$ while that of v_1 in G_2 is $(3, 4, 5)$.

For any vertex v of a cubic graph G and any $\alpha \in \Gamma(G)$, the vertices v and αv have the same type. This trivial observation has some useful consequences, one of which is that if v is the only vertex of a cubic graph G having a given type, then v is fixed by every automorphism of $\Gamma(G)$.

In order to prove the desired result, we first note that the graphs G_1 and G_2 of Fig. 13.5 have groups of order 1 and 2, respectively (which are therefore cyclic, of course). We will prove the second of these remarks.

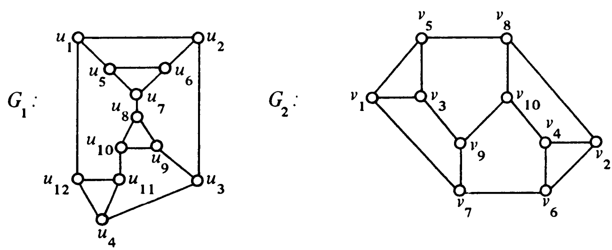

Figure 13.5 Cubic graphs with cyclic groups of order 1 and 2

Theorem 13.8 The group of the cubic graph G_2 of Fig. 13.5 is cyclic of order two.

Proof Denote the identity element of $\Gamma(G_2)$ by α. Due to the symmetry of G_2, the following is also an element of $\Gamma(G_2)$:

$$\beta = (v_1 v_2)(v_3 v_4)(v_5 v_6)(v_7 v_8)(v_9 v_{10}).$$

Our goal, of course, is to show that α and β are the only two elements of $\Gamma(G_2)$. In order to verify this, we let $\delta \in \Gamma(G_2)$ and show that either $\delta = \alpha$ or $\delta = \beta$.

The types of all ten vertices of G_2 are given below:

v_1, v_2, v_3, v_4	type $(3, 4, 5)$
v_5, v_6	type $(3, 5, 6)$
v_7, v_8	type $(4, 5, 6)$
v_9, v_{10}	type $(4, 5, 5)$.

The vertex v_7, for example, must be mapped by δ into either v_7 or v_8. We consider these two cases.

CASE 1. $\delta(v_7) = v_7$. In this case then $\delta(v_8) = v_8$. Since each of the vertices v_1, v_6, and v_9 are adjacent with the fixed vertex v_7 and have distinct types, they must also remain fixed by δ. By the same reasoning, the vertices v_2, v_5, and v_{10} (each adjacent with v_8) are mapped into themselves by δ. This leaves only v_3 and v_4 to investigate. The vertex v_3 is adjacent to the fixed vertices v_1 and v_5; therefore, δ fixes v_3 and hence v_4. Thus $\delta = \alpha$.

CASE 2. $\delta(v_7) = v_8$. We have then $\beta\delta(v_7) = v_7$ so that by Case 1, $\beta\delta = \alpha$. Hence $\delta = \beta^{-1} = \beta$. ■

We are now prepared to treat the more general situation.

Theorem 13.9 For a given finite cyclic group Ω of order $n \geq 3$, there exists a cubic graph G such that $\Gamma(G) \cong \Omega$.

Proof We construct a cubic graph G such that $V(G)$ consists of the $6n$ vertices $u_i, v_i, w_i, x_i, y_i, z_i$, $i = 1, 2, \ldots, n$. We define $E(G)$ in the following manner. First, for each $i = 1, 2, \ldots, n$, the edges given below are elements of $E(G)$:

$$u_iv_i,\ u_iy_i,\ u_iz_i,\ v_iw_i,\ w_ix_i,\ w_iz_i,\ y_iz_i.$$

Secondly, the edge v_ny_1 as well as the edges v_iy_{i+1}, $i = 1, 2, \ldots, n-1$, belong to $E(G)$, and finally, $E(G)$ contains x_1x_n and the edges x_ix_{i+1}, $i = 1, 2, \ldots, n-1$. The types of all $6n$ vertices may now be determined:

$$\begin{array}{ll}
u_i, z_i & \text{type } (3, 4, 5) \\
v_i & \text{type } (4, 7, 9) \\
w_i & \text{type } (4, 7, 7) \\
y_i & \text{type } (3, 7, 8) \\
x_i & \text{type} \begin{cases} (n, 7, 7) & n \leq 7 \\ (7, 7, n) & 7 < n < 11 \\ (7, 7, 11) & n \geq 11. \end{cases}
\end{array}$$

From the manner in which the graph G is constructed, the mapping α, indicated below, is readily seen to be an element of $\Gamma(G)$:

$$\begin{aligned}
\alpha u_n &= u_1, & \alpha u_i &= u_{i+1}\\
\alpha v_n &= v_1, & \alpha v_i &= v_{i+1}\\
\alpha w_n &= w_1, & \alpha w_i &= w_{i+1}\\
\alpha x_n &= x_1, & \alpha x_i &= x_{i+1}\\
\alpha y_n &= y_1, & \alpha y_i &= y_{i+1}\\
\alpha z_n &= z_1, & \alpha z_i &= z_{i+1},
\end{aligned}$$

for all $i = 1, 2, \ldots, n-1$. This implies that $\alpha^i \in \Gamma(G)$ for $i = 1, 2, \ldots, n$. Furthermore, $\alpha^i = \alpha^j$ if and only if $i \equiv j$ (modulo n).

Let $\beta \in \Gamma(G)$. We show that $\beta = \alpha^i$ for some $i = 1, 2, \ldots, n$, thereby proving that $\Gamma(G)$ is generated by α and is a cyclic group of order n. We distinguish two cases.

CASE 1. $\beta v_1 = v_1$. The vertices u_1, w_1, and y_2 are adjacent to v_1 and have distinct types; hence they are fixed by β. Applying the same argument to u_1, we have $\beta y_1 = y_1$ and $\beta z_1 = z_1$. Since u_1 and z_1 are adjacent with the fixed vertex w_1 and are themselves fixed under β, we conclude that $\beta x_1 = x_1$. Hence all vertices with subscript 1 are fixed by β.

The vertices u_1, z_1, and v_n are adjacent with y_1; since u_1, z_1, and y_1 are fixed by β, so too is v_n. Now beginning with v_n, we can show that all vertices having subscript n are fixed by β. From this it now follows that $\beta v_{n-1} = v_{n-1}$. Continuing this procedure, we reach the conclusion that all vertices of G are fixed by β. Hence β is the identity of $\Gamma(G)$, or, equivalently, $\beta = \alpha^n$.

CASE 2. $\beta v_1 \neq v_1$. Due to the type of v_1, we have $\beta v_1 = v_j$ for some $j \neq 1$. Also, we know $\alpha^{j-1} v_1 = v_j$. Hence $(\alpha^{j-1})^{-1}\beta v_1 = v_1$. By Case 1, $(\alpha^{j-1})^{-1}\beta = \alpha^n$ implying that $\beta = \alpha^{j-1}$. This concludes the proof. ■

PROBLEM SET 13.2

13.6 (a) Prove for the graph G_1 of Fig. 13.5 that $|\Gamma(G_1)| = 1$.

(b) Does there exist a cubic graph G of order less than 12 such that $|\Gamma(G)| = 1$?

13.7 Determine the smallest integer $p > 1$ such that there exists a connected graph of order p having the identity group.

13.8 Find a 4-regular graph G such that $|\Gamma(G)| = 1$.

13.9 Determine a cubic graph G with connectivity 1 such that $|\Gamma(G)| = 1$.

13.10 For a given finite group Ω, determine an infinite number of mutually nonisomorphic graphs whose groups are isomorphic to Ω.

13.11 For a given finite group Ω, find two non-homeomorphic graphs whose groups are isomorphic to Ω.

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14

Graph Valued Functions

In Chapter 4 we considered the concept of the nth power G^n of a connected graph G, while the block-cut-vertex graph $BC(G)$ of a nontrivial connected graph G was described in Chapter 5. These are two examples of "graph valued functions" which have been defined on a collection of graphs. Moreover, during the course of our discussions, graphs have been associated with entities which are not themselves graphs. In this chapter we consider other graph valued functions. For the most part, the major emphasis in this area has been focused on three types of problems; namely, if Φ is a graph valued function defined on a subset of the set of graphs, then:

(I) Is Φ a one-to-one mapping, or under what conditions do there exist graphs G_1 and G_2 such that $\Phi(G_1) = \Phi(G_2)$?

(II) Is Φ a mapping onto the set of all graphs, or for which graphs H does there exist a graph G such that $H = \Phi(G)$?

(III) For a given graphical property A, what property must G possess for $\Phi(G)$ to have property A?

We begin with the graph valued function which has received the most attention.

14.1 Line Graphs

With every nonempty graph G one can associate a graph $L(G)$, called the *line graph of* G, with the property that there exists a one-to-one correspondence between $E(G)$ and $V(L(G))$ such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent. (The graph $L(G)$ might be more properly referred to as the "edge graph" of G since it is the edges of G which result in the vertices of the new graph $L(G)$; however, the term "line graph" has been employed most frequently in the literature—particularly, but not exclusively, by those who use the word "line" rather than "edge". The line graph has also been referred to by such terms as "interchange graph", "adjoint", "derived graph", and "derivative".) The line graph function is illustrated in Fig. 14.1.

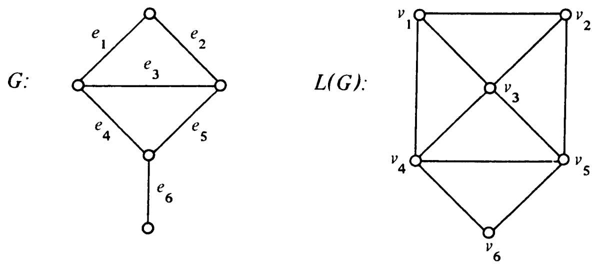


Figure 14.1 A graph and its line graph

It is relatively easy to determine the number of vertices and the number of edges of the line graph $L(G)$ of a graph G in terms of easily computed quantities in G. Indeed, if G is a (p, q) graph whose degree sequence is $d_1, d_2, \ldots, d_p$ and $L(G)$ is a (p', q') graph, then $p' = q$ and

$$q' = \sum_{i=1}^{p} \binom{d_i}{2}.$$

Obviously, two nonempty isomorphic graphs have isomorphic line graphs. This condition is not necessary, however, since the graphs G_1 and G_2 of Fig. 13.1 have G_1 as their line graph. Among the connected graphs, though, these are the only two non-isomorphic graphs with this property. This fact was first discovered by Whitney [9].

Theorem 14.1 Let G_1 and G_2 be two nontrivial connected graphs which are not K_3 and $K(1,3)$. Then $L(G_1)=L(G_2)$ if and only if $G_1=G_2$.

Proof As we have already noted, if $G_1=G_2$ then $L(G_1)=L(G_2)$. Suppose that G_1 and G_2 are two nontrivial connected graphs different from K_3 and $K(1,3)$ such that $L(G_1)=L(G_2)$. By the definition of the line graph function, for $i=1,2$, there exists a mapping from $E(G_i)$ onto $V(L(G_i))$ such that two vertices of $L(G_i)$ are adjacent if and only if the corresponding edges of G_i are adjacent. Since $L(G_1)=L(G_2)$, this implies the existence of a one-to-one mapping α^*: $E(G_1)\to E(G_2)$ with the property that two edges of G_1 are adjacent if and only if the corresponding edges of G_2 are adjacent; i.e., α^* is an edge-isomorphism from G_1 to G_2. If G_1 is different from the graphs G_3, G_4, and G_5 of Fig. 13.2, then it follows directly from Theorem 13.4 that α^* is induced by an isomorphism α from G_1 to G_2. It is a straightforward exercise to verify for $i=3,4,5$ that G_i is the only connected graph whose line graph is $L(G_i)$. Thus in all cases, $G_1=G_2$. ■

The preceding theorem deals with the problem (I) mentioned at the beginning of this chapter. We now turn to the second class of problems. In this connection we present a theorem of van Rooij and Wilf [8]. A few definitions will be needed.

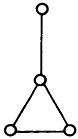
A graph H is called *a line graph* if there exists a graph G such that $H=L(G)$.

A triangle T in a graph H is *even* if every vertex of H is adjacent with an even number (that is, 0 or 2) of vertices of T; otherwise, T is *odd*.

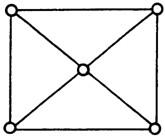
A *clique* of a graph H is a maximal complete subgraph of H, i.e., a clique is a complete subgraph of H which is not properly contained in any other complete subgraph of H.

For later reference the graphs of Fig. 14.2 are given.

F_1:



F_2:



F_3:

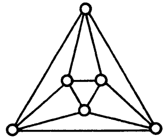


Figure 14.2 Graphs in the proof of Theorem 14.2

Theorem 14.2 A graph H is a line graph if and only if:

(i) $K(1,3)$ is not an induced subgraph of H, and
(ii) if $K_4 - e$ (for any $e \in E(K_4)$) is an induced subgraph of H, then at least one of its two triangles is even.

Proof Suppose, first, that H is a line graph so that there exists a graph G such that $H = L(G)$. Observe that if H' is any induced subgraph of H with $V(H') = V'$ and E' is the corresponding subset of $E(G)$ under the line graph function, then $L(\langle E' \rangle) = H'$. The graph $K(1,3)$ cannot be an induced subgraph of H, for this would imply that there exists an edge of G adjacent with three edges, no two of which are themselves adjacent. This is impossible. Assume $K_4 - e$ is an induced subgraph of H. For the graph F_1 of Fig. 14.2, $L(F_1) = K_4 - e$ so that by Theorem 14.1, F_1 is the only graph with this property. One triangle of $K_4 - e$ is induced by the vertices which correspond to the edges of the triangle of F_1, while the other triangle of $K_4 - e$ is induced by those vertices which correspond to the edges incident with the vertex of degree 3 of F_1. Any edge of G is necessarily adjacent with no edges or two edges of the triangle of F_1; hence, at least one triangle of $K_4 - e$ has the property that any vertex of H is adjacent with none or two of its vertices, i.e., at least one triangle of $K_4 - e$ is even.

Conversely, suppose H is a graph satisfying (i) and (ii). We assume, without loss of generality, that H is connected. We prove that H is a line graph by considering two cases.

Case 1. H *contains two even triangles which have an edge in common.* Let T_1 and T_2 be two such triangles, where $V(T_1) = \{v_1, v_2, v_3\}$ and $V(T_2) = \{v_1, v_2, v_4\}$. The edge v_3v_4 is not present in H since otherwise v_4 is adjacent to each of the vertices of the even triangle T_1. Hence $\langle \{v_1, v_2, v_3, v_4\} \rangle = K_4 - e$. If $H = K_4 - e$, then we have already seen that $L(F_1) = K_4 - e$, so that H is a line graph.

If $H \neq K_4 - e$, then since H is connected there exists a vertex v_5 adjacent with at least one of the vertices v_i, $1 \leq i \leq 4$. By symmetry we must have $v_1v_5 \in E(H)$ or $v_3v_5 \in E(H)$. Assume first that $v_3v_5 \notin E(H)$. Then since T_1 is even, we must have $v_1v_5, v_2v_5 \in E(H)$, which implies $v_4v_5 \notin E(H)$ since T_2 is even. However then $\langle \{v_1, v_3, v_4, v_5\} \rangle = K(1,3)$, which contradicts (i). Therefore $v_3v_5 \in E(H)$. Precisely one of v_1v_5 and v_2v_5 is an edge of H, and, moreover, $v_4v_5 \in E(H)$. By symmetry we may suppose $v_1v_5 \in E(H)$ so that $v_2v_5 \notin E(H)$. No additional edges of the type v_iv_j, $1 \leq i < j \leq 5$, are possible since the triangle T_1, for example, is even. If $H = \langle\{v_1, v_2, v_3, v_4, v_5\}\rangle$, then H is the graph F_2 of Fig. 14.2, which is the line graph of $K_4 - e$.

Suppose $H \neq F_2$. Then H contains F_2 as an induced subgraph together with at least one additional vertex v_6 adjacent with one or more of the vertices v_i, $1 \leq i \leq 5$. We note that because of (ii), at least one of the triangles $T_3 = \langle\{v_1, v_3, v_5\}\rangle$ and $T_4 = \langle\{v_1, v_4, v_5\}\rangle$ is even, say T_3. (A symmetric argument is employed if T_4 is even.) Assume now that $v_1v_6 \in E(H)$. Then v_6 is adjacent with exactly one of v_2 and v_3, say v_2. This implies that $v_5v_6 \in E(H)$, but the evenness of the triangles T_1, T_2, T_3 prevents v_6 from being adjacent with either of v_3 and v_4. However then $\langle\{v_1, v_3, v_4, v_6\}\rangle = K(1,3)$ which contradicts (i). Therefore $v_1v_6 \notin E(H)$. The evenness of T_1, T_2, T_3 now assures us that each of the edges v_iv_6, $2 \leq i \leq 5$, is present in H. Now $\langle\{v_1, v_2, \ldots, v_6\}\rangle = F_3$ (of Fig. 14.2) and $F_3 = L(K_4)$.

Assume $H \neq F_3$ so that there exists a vertex v_7 adjacent to some v_i, $1 \leq i \leq 6$. Now (ii) implies that either (1) triangle $T_5 = \langle\{v_2, v_3, v_6\}\rangle$ is even, or (2) T_5 is not even, but each of the triangles $T_6 = \langle\{v_2, v_4, v_6\}\rangle$ and $T_7 = \langle\{v_3, v_5, v_6\}\rangle$ is even. We divide these into subcases. (See Fig. 14.3.)

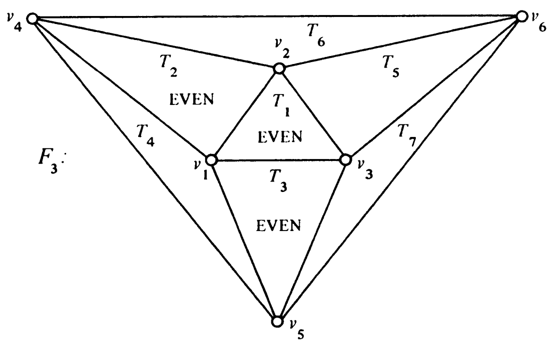


Figure 14.3 The triangles of F_3

SUBCASE 1. *Triangle* T_5 *is even.* Suppose $v_1v_7 \in E(H)$. Then either $v_2v_7 \in E(H)$ or $v_3v_7 \in E(H)$. Without loss of generality, we assume $v_2v_7 \in E(H)$. We must then have v_5v_7, $v_6v_7 \in E(H)$ and v_3v_7, $v_4v_7 \notin E(H)$. However, $\langle\{v_3, v_4, v_6, v_7\}\rangle = K(1,3)$, which is contradictory. Thus $v_1v_7 \notin E(H)$ and similarly $v_2v_7, v_3v_7 \notin E(H)$. If $v_iv_7 \in E(H)$ for some $i = 4, 5, 6$, then the evenness of the triangles T_1, T_2, T_3, T_5 implies that $v_jv_7 \in E(H)$ for some $j = 1, 2, 3$, which is impossible. Hence T_5 is not even.

SUBCASE 2. *Triangles* T_6 *and* T_7 *are even and triangle* T_5 *is not even.* Again suppose that $v_1v_7 \in E(H)$ so that exactly one of v_2v_7 and v_3v_7 is an edge of H, say the former. This, in turn, implies that v_5v_7, $v_6v_7 \in E(H)$ and v_3v_7, $v_4v_7 \notin E(H)$. The subgraph $\langle\{v_3, v_4, v_6, v_7\}\rangle = K(1,3)$ so that $v_1v_7 \notin E(H)$. Assume next that v_2v_7, $v_3v_7 \in E(H)$. We then have v_4v_7, $v_5v_7 \in E(H)$ and $v_6v_7 \notin E(H)$, but $\langle\{v_1, v_5, v_6, v_7\}\rangle = K(1,3)$. Hence $v_iv_7 \notin E(H)$ for $i = 1, 2, 3$. However, v_7 adjacent to v_j, $j = 4, 5, 6$, implies $v_iv_7 \in E(H)$ for some $i = 1, 2, 3$. Thus no such vertex v_7 exists.

CASE 2. *Of every two triangles of* H *having an edge in common, at least one is odd.* We define three types of complete subgraphs of H. A clique of H which is not an even triangle is a subgraph of type S_1. A trivial subgraph whose vertex belongs to a subgraph of type S_1 and which is only adjacent to the other vertices of that subgraph is a subgraph of type S_2. A complete subgraph of order two contained in a single triangle of H is called a subgraph of type S_3 if the triangle is even. A subgraph of type S is one which is of type S_i for some $i = 1, 2, 3$. Note that a subgraph of type S is of type S_i for only one $i = 1, 2, 3$.

Define a graph G such that there exists a one-to-one correspondence between $V(G)$ and the subgraphs of type S of H, where two vertices of G are adjacent if and only if they correspond to subgraphs with a nonempty intersection.

If two subgraphs F' and F'' of type S have a nonempty intersection, then this intersection consists of precisely one vertex. This, of course, is obvious if either F' or F'' is of type S_2 or if both F' and F'' are of type S_3. Suppose F' is of type S_1, F'' is of type S_3, and F'' is contained in the clique F'. However, then F'' belongs to at least two triangles, and this contradicts the fact that F'' is contained in a unique triangle. Suppose next that F' and F'' are both of type S_1 and that $\langle\{u, v\}\rangle$, $u \neq v$, is contained in each of F' and F''. Thus F' has a vertex u_1 adjacent with u and v, and F'' contains a vertex v_1, $v_1 \neq u_1$, adjacent with u and v such that

$v_1 \notin V(F')$; but then $\langle\{u, v, u_1, v_1\}\rangle = K_4 - e$. Now in this case one triangle of $K_4 - e$ is even and the other is odd, say $\langle\{u_1, u, v\}\rangle$ is even. Since F' is not an even triangle, F' contains a fourth vertex adjacent with u, v and u_1. But this implies that $\langle\{u_1, u, v\}\rangle$ is not even.

Next we define a mapping ϕ: $V(L(G)) \rightarrow V(H)$. Let w' be a vertex of $L(G)$, and let $u'u''$ be the edge of G corresponding to w' under the line graph function. Each of u' and u'' corresponds to a subgraph of type S in H, say F' and F'' respectively, where the intersection of F' and F'' consists of a single vertex w. Define $\phi w' = w$. The theorem will be established once it is verified that ϕ is an isomorphism from $L(G)$ to H.

First we show that ϕ is an onto mapping. Let $w_1 \in V(H)$ and suppose w_1 belongs to an even triangle T', where $V(T') = \{w_1, w_2, w_3\}$. If the edge w_1w_2 belongs to no other triangle of H, then w_1w_2 induces a subgraph of type S_3. Suppose, on the other hand, that w_1w_2 belongs to a second triangle T'' where say, $V(T'') = \{w_1, w_2, w_4\}$. The triangle T'' is odd and thus belongs to a subgraph G'' of type S_1 such that $w_3 \notin V(G'')$. Hence in any case, there is a subgraph of type S of H containing w_1 and w_2 but not w_3 and a subgraph of type S containing w_1 and w_3 but not w_2. These two subgraphs have the vertex w_1 in common; therefore, there is a vertex mapped by ϕ into w_1.

Assume next that w_1 belongs to no even triangle of H. Let H' be a clique containing w_1; necessarily H' is a subgraph of type S_1. If w_1 is adjacent only with the vertices of H', then w_1 induces a trivial subgraph H'' of type S_2. On the other hand, if w_1 is adjacent with a vertex w'_1 not in H', let H'' be a clique of H containing w_1 and w'_1. In either case, H' and H'' are subgraphs of type S having the vertex w_1 in common. Here again there is a vertex of $L(G)$ mapped by ϕ into w_1.

In order to verify that ϕ is a one-to-one mapping, it is equivalent to show that three distinct subgraphs of type S of H cannot have a single vertex in common. Assume the contrary. Certainly then at most one of these subgraphs is of type S_2.

Suppose there exist three distinct subgraphs H_1, H_2, and H_3 of type S_1 in H, each subgraph containing the vertex w_0. Since each of the subgraphs H_i, $i = 1, 2, 3$, is of type S_1, every triangle contained in any of these subgraphs is odd. For $i = 1, 2, 3$, let w_0w_i be an edge of H_i. If H contains no edge of the type w_iw_j, $1 \leq i < j \leq 3$, then $\langle\{w_0, w_1, w_2, w_3\}\rangle = K(1, 3)$, which contradicts (i). Suppose then that $w_1w_2 \in E(H)$, and consider the triangle $T = \langle\{w_0, w_1, w_2\}\rangle$. If T is odd, then T is contained in a clique H_4 which is a subgraph of type S_1 distinct from H_1 and H_2. However, H_4 has more than a single vertex in common with H_1, as well as with H_2, and this is impossible. If T is even, then w_3 is adjacent to exactly one of w_1 and w_2, say w_1. However, then $T_1 = \langle\{w_0, w_1, w_3\}\rangle$ is odd since H does not contain two even triangles with an edge in common. But using the same reasoning as with the triangle T, we cannot have the odd triangle T_1. Therefore this situation cannot occur.

If there are three other subgraphs of type S having a vertex in common, then at least one of these subgraphs is of type S_2, for otherwise we contradict (i) immediately. But it is not possible to have exactly one of these three subgraphs of type S_2, for then the definition of such a subgraph is contradicted.

The detail remaining is to show $uv \in E(L(G))$ if and only if $\phi(u)\phi(v) \in E(H)$. Suppose $uv \in E(L(G))$. Then u and v correspond to adjacent edges e and f of G under the line graph function where, say, $e = w_1w_2$ and $f = w_2w_3$. By definition, the vertices w_i, $i = 1, 2, 3$, correspond to subgraphs G_i of type S. The subgraphs G_1 and G_2 have $\phi(u)$ in common while G_2 and G_3 have $\phi(v)$ in common. Thus $\phi(u), \phi(v) \in V(G_2)$, but G_2 is complete so that $\phi(u)\,\phi(v) \in E(H)$. Conversely, suppose $\phi(u)\,\phi(v) \in E(H)$. The vertex $\phi(u)$ is the intersection of two subgraphs G_1 and G_2 of type S while $\phi(v)$ is the intersection of G_3 and G_4. Now any edge incident with $\phi(u)$ must belong to a subgraph of type S; thus, $\phi(v)$ belongs to G_1 or G_2, say G_2. Similarly, $\phi(u)$ belongs to G_3 or G_4, say G_4. Since the mapping ϕ is one-to-one, this implies $G_2 = G_4$ so that u is adjacent to v. ■

With the aid of Theorem 14.2, it is possible to characterize line graphs in terms of the induced subgraphs they cannot contain. These subgraphs were initially found by Beineke [2].

Corollary 14.2a A graph H is a line graph if and only if none of the graphs H_i, $i = 1, 2, \ldots, 9$, of Fig. 14.4 is an induced subgraph of H.

We conclude this section by presenting a result which deals with problem (III) mentioned at the beginning of the chapter, namely a characterization of planar line graphs due to Sedláček [7]. Additional properties of line graphs are considered in the problem set of this section.

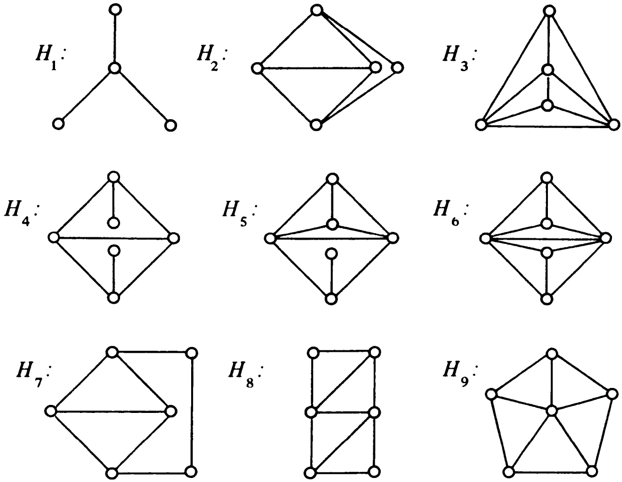


Figure 14.4 The induced subgraphs not contained in any line graph

A cycle in a plane graph G is called *elementary* if it is the boundary of a region of G.

Theorem 14.3 Let G be a planar graph. A necessary and sufficient condition for G to have a planar line graph is that (i) $\Delta(G) \leq 4$, and (ii) if $\deg v = 4$, then v is a cut-vertex of G.

Proof We show first that properties (i) and (ii) are necessary for a graph G to have a planar line graph $L(G)$. That $\Delta(G) \leq 4$ is obvious, for if $\Delta(G) \geq 5$ then $L(G)$ contains K_5 as a subgraph which contradicts Theorem 8.7. To see that (ii) is necessary, assume that G contains a vertex v of degree 4 which is not a cut-vertex, and let $e_i = vv_i$, $1 \leq i \leq 4$, be the edges of G incident with v. Furthermore, denote by w_i the vertex of $L(G)$ which corresponds to e_i. Let $H = \langle \{w_1, w_2, w_3, w_4\} \rangle$, where then $H = K_4$. Assume that $L(G)$ is embedded in the plane, and let A_1, A_2, A_3, A_4 be the four "sections" (not necessarily regions) into which the plane is divided by the edges of H. (See Fig. 14.5.)

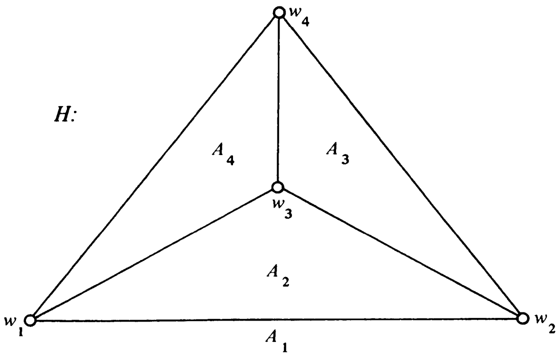


Figure 14.5 The graph H of Theorem 14.3

Since v is not a cut-vertex of G, it is possible to relabel the edges e_i and the vertices v_i, if necessary, so that there exist v_i–v_j paths P_{ij}, for $(i,j)=(1,2)$, $(2,3)$, and $(3,4)$ or $(i,j)=(1,2)$, $(1,3)$, and $(1,4)$, such that P_{ij} contains neither v nor v_k for any $k \neq i, j$. Let us assume the first case, the second case can be treated in an analogous manner. Hence in $L(G)$ there exist w_i–w_j paths Q_{ij}, for $(i,j)=(1,2)$, $(2,3)$, and $(3,4)$, such that Q_{ij} contains no edges of H. Each such path Q_{ij} lies entirely within section A_i or entirely within section A_j. The vertex preceding w_2 in Q_{12} is adjacent with the vertex following w_2 in Q_{23}; thus, Q_{12} and Q_{23} lie in A_2. However, the vertex preceding w_3 in Q_{23} is also adjacent with the vertex following w_3 in Q_{34}. This implies that $L(G)$ is not planar, which is a contradiction.

For the converse we employ mathematical induction. For each positive integer n, define $A(n)$ as follows: Every connected, planar graph G with n blocks such that $\Delta(G) \leq 4$ and each vertex of degree 4 is a cut-vertex has a planar line graph $L(G)$ which can be embedded in the plane so that whenever three edges of one block are incident with a vertex of degree 3 in G, then the triangle of $L(G)$ determined by these edges is an elementary triangle of $L(G)$.

We consider $A(1)$ first. Suppose G is a planar block with $\Delta(G) \leq 3$, and let G be embedded in the plane. For convenience, we assume each edge to be a straight line segment, which is possible by Theorem 7.3. We construct the line graph of G by distinguishing the mid-point of each edge of G as a vertex. This collection of mid-points constitutes the vertex set of $L(G)$. Whenever two edges of G are adjacent, the resulting edge of $L(G)$ can be drawn in the region of G whose boundary contains the two edges in a manner such that a plane graph $L(G)$ results. Furthermore, every vertex of degree 3 in G yields an elementary triangle. (Figure 14.6 illustrates this construction where the vertices of $L(G)$ are shown as solid circles and the edges of $L(G)$ are dashed lines.)

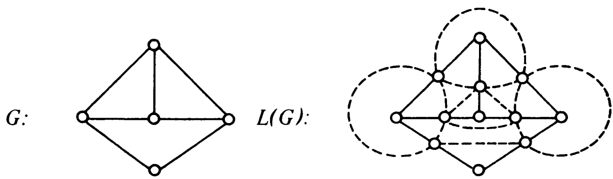


Figure 14.6 Construction of a plane line graph from a plane graph

Assume $A(n)$ to hold for $n \geq 1$, and let G be a connected, planar graph with $n+1$ blocks such that $\Delta(G) \leq 4$ and each vertex of degree 4 is a cut-vertex. Let B be an end-block of G and v the cut-vertex of G belonging to B. Remove from G all vertices of B different from v and denote the resulting graph by H. Since H also has properties (i) and (ii) and has n blocks, we may apply the inductive hypothesis to obtain a planar embedding of $L(H)$ with the property that whenever a triangle of $L(H)$ results from a vertex of degree 3 in H incident with three edges in a single block, then the triangle is elementary.

The vertex v has degree at most 3 in B. If $\deg_B v = 3$, then v is incident with a bridge e in H. As we have just seen in the proof of $A(1)$, the graph $L(B)$ can be embedded in the plane so that whenever a triangle of $L(B)$ results from a vertex of degree 3 in B, then the triangle is elementary. Since v is such a vertex, the triangle in $L(B)$ resulting from the three edges of B incident with v is elementary. We take this region as the exterior region of the plane graph $L(B)$. This embedding of $L(B)$ may then be inserted in a region of the plane graph $L(H)$ whose boundary includes the vertex u corresponding to e under the line graph function. An embedding of $L(G)$ with the desired properties can then be obtained by joining u to the exterior triangle of $L(B)$.

Suppose next that $\deg_B v = 2$. Then v is incident with one or two edges of H. In either case, there is a region R of $L(H)$ whose boundary includes the one or two vertices corresponding to the one or two edges of H incident with v. We may select an embedding of $L(B)$ so that the boundary of the exterior region of $L(B)$ contains the edge resulting from the two edges of B incident with v. By placing the plane graph $L(B)$ in the region R and joining the appropriate vertices, an embedding of $L(G)$ with the suitable properties is produced.

Finally, suppose $\deg_B v = 1$ so that B is an acyclic block. There is no difficulty in arriving at the desired result if v is incident with one or more bridges in H or if $\deg_H v = 2$; thus we assume $\deg_H v = 3$, where all three edges of H incident with v belong to one block. However, by the inductive hypothesis, there exists a planar embedding of $L(H)$ so that every triangle which results from three such edges is elementary. We may then insert the trivial graph $L(B)$ in the appropriate triangular region of $L(H)$ and join it to the three vertices to obtain a desirable embedding of $L(G)$. This verifies $A(n+1)$ and completes the proof. ■

PROBLEM SET 14.1

14.1 Determine a formula for the number of triangles in the line graph $L(G)$ in terms of quantities in G.

14.2 Generalize Theorem 14.1 to arbitrary graphs.

14.3 Prove Corollary 14.2a.

14.4 Find a necessary and sufficient condition for $G = L(G)$.

14.5 Show that the condition that G be planar in Theorem 14.3 is essential to the validity of the theorem.

14.6 Find a necessary and sufficient condition for a graph to have an outerplanar line graph.

14.7 Prove that $L(G)$ is eulerian if G is eulerian.

14.8 Prove that if $L(L(L(G)))$ is eulerian, then $L(L(G))$ is eulerian.

14.9 Show that each of the following conditions is sufficient for a graph G to have a hamiltonian line graph:

(a) G is eulerian.

(b) G is hamiltonian.

14.2 Clique Graphs and Other Graph-Valued Functions

The line graph of a graph G is defined in the terms of the edges of G. There are several variations of this concept, each of which gives rise to a graph-valued function. For example, the line graph of a graph G can be considered as determined by its complete subgraphs of order two. If G contains no triangles, then each such subgraph is a clique. This suggests the following. The *clique graph* $K(G)$ *of a graph* G is that graph whose vertex set can be put in one–to–one correspondence with the cliques of G such that two vertices of $K(G)$ are adjacent if and only if the corresponding cliques of G have a nonempty intersection. A graph H is a *clique graph* if there exists a graph G such that $H = K(G)$. Clique graphs have been characterized by Roberts and Spencer [6]. In order to present this result, an additional concept is useful. A collection S of subgraphs of a graph G is said to have the *intersection property* if whenever the intersection of each pair of elements of a subset T of S is nonempty, then the intersection of all elements of T is nonempty.

Theorem 14.4 A graph H is a clique graph if and only if H has a collection S of complete subgraphs such that (i) every edge of H is contained in at least one element of S, and (ii) S has the intersection property.

Proof Assume H is a connected graph, and let

$$S = \{H_1, H_2, \ldots, H_n\}$$

be a collection of complete subgraphs of H satisfying (i) and (ii). We prove that H is a clique graph. Define a graph G such that there exists a one-to-one mapping ϕ from $V(H) \cup S$ onto $V(G)$ with the property that two vertices of G are adjacent if and only if the intersection of their corresponding elements is nonempty. (Thus if two vertices u and v of G correspond to two vertices of H, then u and v are not adjacent.) It can be shown in a straightforward manner that the graph G is connected. We show that $K(G) = H$.

For $v \in V(H)$, define

$$K(v) = \{\phi v\} \cup \{\phi H_i \mid H_i \text{ contains } v\}.$$

From the manner in which G is defined, the subgraph $\langle K(v)\rangle$ is a clique of G. Furthermore, every clique of G is $\langle K(v)\rangle$ for some $v \in V(H)$, for if F is a nontrivial complete subgraph containing the vertex ϕv, then F is a subgraph of $\langle K(v)\rangle$. Otherwise, there is a subset S_1 of S such that $\phi(S_1) = V(F)$, i.e., $V(F) = \{\phi H_i \mid H_i \in S_1\}$. Since F is complete, every two elements of S_1 have a vertex in common. However, S has the intersection property implying that each element of S_1 contains a vertex u of H. Then $\langle\{\phi(\{u\} \cup S_1)\}\rangle$ is a complete subgraph of G, i.e., F is a subgraph of the clique $\langle K(u)\rangle$.

Define a mapping ψ from $V(H)$ to the set of cliques of G by $\psi v = \langle K(v)\rangle$. We have already observed that this mapping is onto. Also ψ is one-to-one since if $u \neq v$, then $\phi u \notin \langle K(v)\rangle$.

Let $uv \in E(H)$ so that by (*i*), $uv \in E(H_j)$ for some $H_j \in S$. Hence each of $\langle K(u)\rangle$ and $\langle K(v)\rangle$ contains the vertex $\phi(H_j)$. Thus if u and v are adjacent in H, then ψu and ψv are two cliques of G having a nonempty intersection. If, on the other hand, $uv \notin E(H)$, then $\langle K(u)\rangle$ and $\langle K(v)\rangle$ have no vertex in common, for if there were such a vertex, it would be $\phi(H_j)$ for some $H_j \in S$. But then $u, v \in V(H_j)$, which is impossible since H_j is complete. If v' is the vertex of $K(G)$ which corresponds to the clique $\langle K(v)\rangle$ of G, then it follows that the mapping α defined by $\alpha(v) = v'$, $v \in V(H)$, is an isomorphism from H to $K(G)$. This proves the sufficiency of the conditions (i) and (ii).

We now consider the necessity of (i) and (ii) for a graph H to be a clique graph. Suppose then that there exists a graph G such that $K(G) = H$. Let $V(H) = \{u_1, u_2, \ldots, u_p\}$ and $V(G) = \{v_1, v_2, \ldots, v_m\}$, and let $G_1, G_2, \ldots, G_p$ be the cliques of G, so labeled that for each $i = 1, 2, \ldots, p$, u_i corresponds to G_i under the clique graph function. For each $i = 1, 2, \ldots, m$, define $U_i = \{u_j \mid v_i \in V(G_j)\}$, and let $F_i = \langle U_i\rangle$. Observe that each F_i is complete and let $S = \{F_1, F_2, \ldots, F_m\}$. We show that S satisfies (i) and (ii).

Let $u_iu_j \in E(H)$, $i \neq j$. Thus the cliques G_i and G_j of G have a vertex v_k in common. Therefore $u_i, u_j \in U_k$ and $u_iu_j \in E(F_k)$. In order to show that S has the intersection property, let $S' = \{F_{i_1}, F_{i_2}, \ldots, F_{i_r}\}$ be a subset of S such that every two elements of S' have a nonempty intersection. If u_s is a vertex contained in each of F_{i_j} and F_{i_k}, then each of v_{i_j} and v_{i_k} is a vertex of G_s so that $v_{i_j}v_{i_k} \in E(G_s)$. Therefore $v_{i_j}v_{i_k} \in E(G)$, so that

$$\langle\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}\rangle$$

is a complete subgraph of G which is contained in some clique G_t of G. This implies that the vertex u_t belongs to every element of S' and S has the intersection property. ■

To illustrate Theorem 14.4 (in a somewhat negative manner) we consider the graph H of Fig. 14.7. The graph H (as was pointed out in [3]) is not a clique graph. The only nontrivial complete subgraphs of H have order two or three and no collection of these satisfy both properties (i) and (ii) of Theorem 14.4.

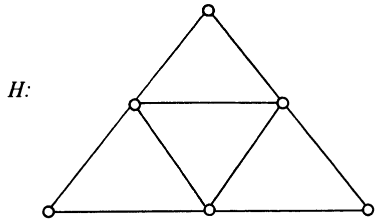


Figure 14.7 A graph which is not a clique graph

Another variation of the line graph function is the total graph function. The *total graph* $T(G)$ *of a graph* G is that graph whose vertex set can be put in one-to-one correspondence with the set $V(G) \cup E(G)$ such that two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are adjacent or incident. A graph and its total graph are shown in Fig. 14.8.

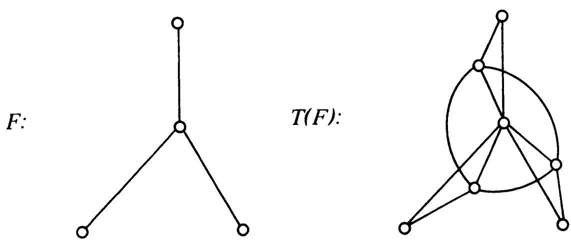


Figure 14.8 A graph and its total graph

A graph-valued function that is intimately related to the total graph is the subdivision graph function. If each edge $e = uv$ of a graph G is replaced by a new vertex w and the new edges uw and vw, then the resulting graph is called the *subdivision graph* of G and is denoted by $S(G)$. Of course, $S(G)$ is a subdivision of G, as defined in Section 8.1. One might observe that the graph H of Fig. 11.4 is the subdivision graph of the graph F of Fig. 14.8 and that $H^2 = T(F)$. Indeed, it is true in general that $[S(G)]^2 = T(G)$. Hence the total graph is the composition of the subdivision graph and square functions. Some of the results on total graphs are considered in Problem Set 14.2. A characterization of total graphs is given in [**1**].

There are numerous graph-valued functions whose domains of definition are not classes of graphs. These functions occur primarily in areas where graph theory is applied. One of the most elementary examples of functions of this type is the intersection graph. Let S be a nonempty set and A a collection of subsets of S. Then the *intersection graph determined by S and A* is that graph G whose vertex set can be put in one-to-one correspondence with the elements of A such that two vertices of G are adjacent if and only if the corresponding elements of A have a nonempty intersection. A graph H is *an intersection graph* if there exists a set S and a collection A of subsets of S such that the intersection graph determined by S and A is isomorphic to H. Intersection graphs and related graph-valued functions are discussed [**4**, pp. 19–21].

PROBLEM SET 14.2

14.10 Determine an infinite class of graphs which have the same clique graph.

14.11 Determine a graph, different from the graph H of Fig. 14.7, which is not a clique graph.

14.12 Show that if G is a graph without triangles, then $K(G) = L(G)$.

14.13 Show that not all graphs are total graphs.

14.14 Determine a necessary and sufficient condition for a total graph to be planar.

14.15 (a) Prove that the total graph of every nontrivial connected graph has a spanning eulerian subgraph.

(b) Prove that if a graph G contains a spanning eulerian subgraph, then $T(G)$ is hamiltonian.

(c) Prove that for any nontrivial connected graph G, $T(G^2)$ is hamiltonian (See [**5**]).

(d) Prove that for any nontrivial connected graph G, $T(T(G))$ is hamiltonian.

14.16 Show that every graph is an intersection graph.

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15

Chromatic Numbers

The graph-theoretic parameter which has received the most attention over the years is the vertex chromatic number. Its prominence in the theory of graphs is undoubtedly due to its involvement with the Four Color Problem, which is considered in the next chapter. In the present chapter we discuss the vertex chromatic number itself as well as two other types of chromatic numbers.

15.1 The (Vertex) Chromatic Number

An assignment of colors (objects of a set) to the vertices of a graph G so that adjacent vertices are assigned different colors is called a *coloring* of G; a coloring in which n colors are used is an *n-coloring*. A graph G is *n-colorable* if there exists an m-coloring of G for some $m \leq n$. It is obvious that if G has order p, then G can be p-colored, so that G is p-colorable.

The minimum n for which a graph G is n-colorable is called the *vertex chromatic number* or simply the *chromatic number* of G and is denoted by $\chi(G)$. If G is a graph for which $\chi(G) = n$, then G is *n-chromatic*. In a given n-coloring of a graph G, the set of vertices which are assigned the same color is referred to as a *color class*.

The chromatic number of G may be defined alternatively, but equivalently, as the minimum number of independent sets of vertices in a partition of $V(G)$. Each such independent set is then a color class in the $\chi(G)$-coloring of G so defined.

For several special classes of graphs, the chromatic number is quite easy to determine. For example,

$$\chi(C_{2n}) = 2,$$
$$\chi(C_{2n+1}) = 3,$$
$$\chi(K(p_1, p_2, \ldots, p_n)) = n,$$

so that $\chi(K_p) = p$; and for every nontrivial tree T, $\chi(T) = 2$. If G is a 2-chromatic graph, then necessarily G is bipartite; for in any 2-coloring of G, the color classes so determined are the defining sets V_1 and V_2 of a bipartite graph. On the other hand, every nonempty bipartite graph is 2-chromatic. By Theorem 2.2, we therefore conclude that a nonempty graph G is 2-chromatic if and only if it contains no odd cycles. From this it follows directly that the even cycles and nontrivial trees are 2-chromatic graphs. The graph G of Fig. 15.1 is 3-chromatic; a 3-coloring of G is indicated, with the colors denoted by the integers 1, 2, 3. This graph G is n-colorable therefore for any $n \geq 3$.

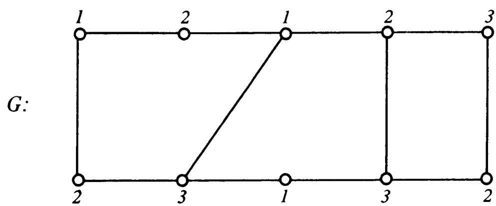

Figure 15.1 A 3-chromatic graph

The chromatic number most properly belongs to the collection of graphical parameters that includes vertex-arboricity and others discussed in Chapter 9. Although the quantity of literature dealing with the chromatic number far surpasses that of these other graph-theoretic parameters, no formula exists here either for the chromatic number of an arbitrary graph. Thus, for the most part, one must be content to supply bounds for the chromatic number of graphs. In order to present such bounds, we now discuss graphs which are critical or minimal with respect to chromatic number.

For an integer $n \geq 2$, we say that a graph G is *critically n-chromatic* if $\chi(G) = n$ and $\chi(G - v) = n - 1$ for all $v \in V(G)$; G is *minimally n-chromatic* if $\chi(G) = n$ and $\chi(G - e) = n - 1$ for all $e \in E(G)$. There are several results dealing with critically n-chromatic graphs and minimally n-chromatic graphs, many of which are due to Dirac [4]. We shall consider here though only one of the more elementary of these.

Every critically n-chromatic graph is connected, while every minimally n-chromatic graph without isolated vertices is connected. Furthermore, every minimally n-chromatic graph (without isolated vertices) is critically n-chromatic. The converse is not true in general, however; for example, the graph of Fig. 15.2 is critically 4-chromatic but not minimally 4-chromatic. For $n = 2$ and $n = 3$, the converse is true; in fact, K_2 is the only critically 2-chromatic graph as well as the only minimally 2-chromatic graph, while the odd cycles are the only critically 3-chromatic and minimally 3-chromatic graphs (disregarding isolated vertices). For $n \geq 4$, neither the critically n-chromatic graphs nor the minimally n-chromatic graphs have been characterized. Although it is quite difficult, in general, to determine whether a given n-chromatic graph G is critical or minimal, G contains both critically n-chromatic graphs and minimally n-chromatic graphs. An n-chromatic subgraph of G of minimum order is critically n-chromatic while an n-chromatic subgraph of G with a minimum number of edges is minimally n-chromatic.

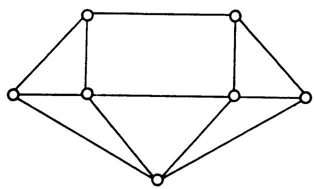

Figure 15.2 A critically 4-chromatic graph which is not minimally 4-chromatic

There is not a great deal known about the structure of critically or minimally n-chromatic graphs. One fact which is known, however, is the following.

Theorem 15.1 Every critically n-chromatic graph, $n \geq 2$, is $(n-1)$-edge connected.

Proof If $n = 2$ or $n = 3$, then G is K_2 or an odd cycle, respectively. Therefore, G is 1-edge connected or 2-edge connected. Assume $n \geq 4$ and that G is not $(n-1)$-edge connected. Hence by Theorem 10.3, there exists a partition of $V(G)$ into subsets V_1 and V_2 such that the set E' of edges joining V_1 and V_2 contains fewer than $n-1$ elements. Since G is critically n-chromatic, the subgraphs $G_1 = \langle V_1 \rangle$ and $G_2 = \langle V_2 \rangle$ are $(n-1)$-colorable. Let each of G_1 and G_2 be colored with at most $n-1$ colors, using the same set of $n-1$ colors. If each edge in E' is incident with vertices of different colors, then G is $(n-1)$-colored. This contradicts the fact that $\chi(G) = n$; hence G is $(n-1)$-edge connected. Thus, suppose there are edges of E' incident with vertices assigned the same color. We show that the colors assigned to the elements of V_1 may be permuted so that each edge in E' joins vertices assigned different colors. This will complete the proof.

Let $\{v_1, v_2, \ldots, v_m\}$ be those vertices of G_1 incident with one or more elements of E'. If v_1 is adjacent only with vertices of G_2 which are assigned colors different from that assigned to v_1, then the coloring of G_1 is not altered. On the other hand, if v_1 is adjacent with a vertex of G_2 which is assigned the same color as that of v_1, then we permute the $n-1$ colors so that in the coloring of G_1 the vertex v_1 is assigned a color different from those assigned to the vertices adjacent with v_1. This is possible since v_1 is adjacent with fewer than $n-1$ elements of G_2. We now consider v_2, if it exists. If v_2 is adjacent with no vertex of G_2 assigned the same color as v_2, then no recoloring is made; otherwise, we permute the $n-2$ colors different from that assigned to v_1 so that in the coloring of G_1, the vertex v_2 is colored differently from the vertices adjacent with it. Again, this is possible since v_2 is adjacent with fewer than $n-2$ vertices of G_2. Continuing this process, we arrive at an $(n-1)$-coloring of G. ■

Since every connected minimally n-chromatic graph is critically n-chromatic, the preceding result has an immediate consequence.

Corollary 15.1a If G is a connected, minimally n-chromatic graph, $n \geq 2$, then G is $(n-1)$-edge connected.

Theorem 15.1 and Corollary 15.1a imply that $\kappa_1(G) \geq n-1$ for every critically n-chromatic graph G or connected minimally n-chromatic graph G. The next corollary now follows directly from Theorem 10.1.

Corollary 15.1b If G is critically n-chromatic or connected and minimally n-chromatic, then $\delta(G) \geq n-1$.

We are now prepared to present bounds for the chromatic number of a graph. We give here three such bounds, beginning with the best known and most applicable of the three. The theorem is due to Brooks [**2**], but the proof here is by Melnikov and Vizing [**8**].

Theorem 15.2 (Brooks' Theorem) For any graph G,

$$\chi(G) \leq 1 + \Delta(G).$$

Furthermore, if G is connected, then equality holds if and only if G is a complete graph or an odd cycle.

Proof If G is a complete graph or an odd cycle, then $\chi(G) = 1 + \Delta(G)$. If G is connected, with $\Delta(G) = 0, 1$, or 2, and $\chi(G) = 1 + \Delta(G)$, then it is obvious that G is K_1, K_2, or an odd cycle, respectively. Assume then that $\Delta(G) \geq 3$ and that G is not complete. We verify that $\chi(G) \leq \Delta(G)$.

Assume this result to be false, i.e., assume there are connected, non-complete graphs G with $\Delta(G) \geq 3$ such that $\chi(G) > \Delta(G)$. Among all such graphs G, let H be one of minimum order. Further, let $\Delta(H) = n \geq 3$. Let u be any vertex of H, and consider $H' = H - u$. It follows that $\chi(H') = n$; for if $\chi(H') > n$ then this contradicts the manner in which H is chosen, while if $\chi(H') < n$ then $\chi(H) \leq n$, which again is a contradiction. This further implies that $\chi(H) = 1 + n$. Also, $\deg u = n$, for if $\deg u < n$ then $\chi(H) \leq n$. Thus we may denote the vertices adjacent with u by $u_1, u_2, \ldots, u_n$. Any n-coloring of H' must assign n different colors to $u_1, u_2, \ldots, u_n$; let an n-coloring of H' be given where u_i is assigned the color i, $1 \leq i \leq n$.

For i and j, $1 \leq i, j \leq n$, $i \neq j$, denote by H_{ij} the subgraph of H' induced by those vertices assigned color i or color j. The vertices u_i and u_j belong to the same component of H_{ij}; for suppose this is not the case. Then we may interchange the colors i and j assigned to the vertices in the component containing u_i.

This produces an n-coloring of H' in which both u_i and u_j are assigned the color j; however, this is impossible as we have seen.

Denote by F_{ij} the component of H_{ij} containing u_i and u_j. We now verify that F_{ij} is a path. First, u_i is adjacent with only one vertex colored j; for suppose it were adjacent to two or more vertices assigned the color j. Then for some t, $1 \leq t \leq n$, u_i is adjacent with no vertex colored t. By assigning u_i the color t, then, instead of i, an n-coloring of H' is produced in which the vertices $u_1, u_2, \ldots, u_n$ are not colored with n colors, which is impossible. Similarly, the vertex u_j is adjacent to only one vertex colored i. Let P be a u_i–u_j path in F_{ij}, say P: $u_i = w_0, w_1, \ldots, w_m = u_j$. If F_{ij} is not the path P, then there exists a least positive integer s such that w_s has degree exceeding 2 in F_{ij}. However, then, w_s is adjacent to three or more vertices colored i or three or more vertices colored j. There is at least one color r with $r \neq i$ or $r \neq j$ such that w_s is not adjacent to any vertex colored r. By coloring w_s with r, an n-coloring of H' results in which u_i and u_j lie in different components of H_{ij}, a contradiction. This proves that each F_{ij} is a path.

Next we show that the paths F_{ij} and F_{ik}, $j \neq k$, have only the vertex u_i in common. If there is a vertex w in common, where $w \neq u_i$, then w is assigned the color i and $\deg w \geq 4$ so that $n \geq 4$. Hence we may assign a color to w different from i, j, or k, thereby obtaining an n-coloring of H' in which u_i and u_j are not joined by a u_i–u_j path, all of whose vertices are colored i or j, a contradiction.

Since $\Delta(H) = n$ and $H \neq K_{n+1}$, the graph H does not contain K_{n+1} as a subgraph. Thus, without loss of generality, we may assume that u_1 and u_2 are not adjacent. Hence the path F_{12} contains a vertex v adjacent to u_1 but different from u_2. By interchanging the colors 1 and 3 assigned to the vertices in the path F_{13}, an n-coloring of H' is obtained. However, with respect to this n-coloring of H', the paths F_{12} and F_{23} contain v as well as u_2, which is a contradiction.

This completes the proof. ∎

This bound for chromatic number is exact for only two classes of graphs. On the other hand, the bound provided for the star graphs $K(1, n)$ differs from its chromatic number by $n - 1$. We shall prove in the next chapter that 5 serves as an upper bound for the chromatic number of all planar graphs; however, Theorem 15.2 gives no bound for the entire class. Hence there are many important classes of graphs for which the bound $\chi(G) \leq 1 + \Delta(G)$ is poor indeed. A better bound in many cases is given by an inequality observed by Szekeres and Wilf [11]. The reader will observe the similarity of this result with Theorem 5.8.

Theorem 15.3 For any graph G,

$$\chi(G) \leq 1 + \max \delta(G'),$$

where the maximum is taken over all induced subgraphs G' of G.

Proof The theorem is obvious if $\chi(G) = 1$; thus we assume G is an n-chromatic graph, where $n \geq 2$. Let H be a critically n-chromatic subgraph of G so that, by Corollary 15.1b, $\delta(H) \geq n - 1$. Since H is an induced subgraph of itself,

$$\delta(H) \leq \max_{H' \prec H} \delta(H'),$$

where, recall $H' \prec H$ indicates that H' is an induced subgraph of H. Moreover, because every induced subgraph of H is also an induced subgraph of G,

$$\max_{H' \prec H} \delta(H') \leq \max_{G' \prec G} \delta(G').$$

Thus we conclude that

$$n - 1 \leq \delta(H) \leq \max_{G' \prec G} \delta(G')$$

so that

$$\chi(G) \leq 1 + \max_{G' \prec G} \delta(G'). \blacksquare$$

Theorem 15.3 gives an upper bound of 2 for the chromatic numbers of the graphs $K(1, n)$, which is exact. Since every planar graph has minimum degree at most 5 (by Corollary 7.2b) and since every subgraph of a planar graph is planar, a bound of 6 is provided for the chromatic number of planar graphs by Theorem 15.3. In each of these two cases, a marked improvement is shown over the result offered by Theorem 15.2. If G is a regular graph of degree r, then both Theorems 15.2 and 15.3 give $r + 1$ as an upper bound for $\chi(G)$; however, this bound is poor for many r-regular graphs, such as $K(r, r)$.

A different type of bound for the chromatic number of a graph is given in the following theorem of Wilf [**13**].

Theorem 15.4 If G is a connected graph, then

$$\chi(G) \leq 1 + \epsilon(G),$$

where $\epsilon(G)$ denotes the maximum eigenvalue of the adjacency matrix of G. Moreover, equality holds if and only if G is a complete graph or an odd cycle.

Proof First we verify the inequality. Let G be a connected n-chromatic graph of order p and, as in the proof of the preceding theorem, let G' denote a critically n-chromatic subgraph of G. The graph G' is necessarily connected and $\delta(G') \geq n-1$.

Let $A(G)=[a_{ij}]$ and $A(G')=[a'_{ij}]$ denote the adjacency matrices of G and G', respectively. Furthermore denote by A^* the p–by–p matrix obtained from $A(G)$ by replacing the rows and columns corresponding to the vertices deleted in obtaining G' by zero rows and columns. This implies that the eigenvalues of A^* are those of $A(G')$ plus an additional p–p' 0's. If $\epsilon(A^*)$ represents the maximum eigenvalue of A^*, then it follows by Theorem 4E that $\epsilon(G')=\epsilon(A^*)$. By Theorem 4F, we have

$$\epsilon(G')=\epsilon(A^*) \leq \epsilon(G). \tag{15.1}$$

Consider the p'-vector $x=(1,1,\ldots,1)$. By Theorem 4H,

$$\epsilon(G')=\max_{y\neq 0}\frac{(yA(G'),y)}{(y,y)} \geq \frac{(xA(G'),x)}{p'}=\frac{\sum_{j=1}^{p'}\left(\sum_{i=1}^{p'} a'_{ij}\right)}{p'}. \tag{15.2}$$

This last expression is the average of the row sums of the matrix $A(G')$. Now the minimum row sum of $A(G')$ certainly cannot exceed this number; however, since $\delta(G') \geq n-1$, the minimum row sum in $A(G')$ is at least $n-1$. Therefore,

$$\epsilon(G') \geq n-1. \tag{15.3}$$

Combining (15.1) and (15.3), we arrive at the desired bound $\chi(G) \leq 1+\epsilon(G)$.

We now turn to the second statement of the theorem. Since $\epsilon(K_p)=p-1$ and $\epsilon(C_{2k+1})=2$ (cf. Exercise 4.13), it follows that $\chi(G)=1+\epsilon(G)$ if $G=K_p$ or $G=C_{2k+1}$.

Suppose next that G is a connected graph such that $\chi(G) = 1 + \epsilon(G) = n$. Now (15.1), (15.2), and (15.3) yield

$$n - 1 = \epsilon(G) \geq \epsilon(G') \geq \frac{\sum_{j=1}^{p'} \left(\sum_{i=1}^{p'} a'_{ij} \right)}{p'} \geq n - 1;$$

therefore, every row sum in $A(G')$ is $n - 1$, implying that G' is an $(n-1)$-regular graph. Thus $\chi(G') = 1 + \Delta(G')$, and by Theorem 15.2, G' is a complete graph or an odd cycle. We consider these two possibilities.

CASE 1. G' *is a complete graph of order* n. We show in this case that G is also complete. If $G' = K_1$, then $n = 1$ so that $\chi(G) = 1$. Since G is connected, $G = K_1$. Hence we assume $G' = K_n$, where $n \geq 2$.

If $p = n$, then the proof is complete; otherwise $n < p$. The matrix $A(G)$ can be expressed (with appropriate labeling of the vertices of G) as a p-by-p matrix such that its upper left n-by-n block is $A(G')$. Since G is connected,

$$\sum_{i=1}^{n} \left(\sum_{j=n+1}^{p} a_{ij} \right) \neq 0.$$

Hence there exists m, where $n < m \leq p$, such that

$$2 \sum_{i=1}^{n} a_{im} > 0$$

so that

$$\frac{2 \sum_{i=1}^{n} a_{im}}{n - 1} > 0.$$

Thus there exists a real number $\gamma > 0$ such that

$$\frac{\left(2 \sum_{i=1}^{n} a_{im} \right)}{(n - 1)} > \gamma.$$

This, in turn, implies that $n + \left(2\gamma \sum_{i=1}^{n} a_{im} \right) \Big/ (n - 1) > n + \gamma^2$ or, equivalently, that $\left(n(n-1) + 2\gamma \left(\sum_{i=1}^{n} a_{im} \right) \right) \Big/ (n + \gamma^2) > n - 1$. Let x be a p-vector, whose first n entries are 1, whose mth entry is γ, and all others of which are 0; i.e., $x=(1,\ldots,1,0,\ldots,0,\gamma,0,\ldots,0)$. Then again by Theorem 4F,

$$\epsilon(G) \geq \frac{(xA(G),x)}{(x,x)} = \frac{n(n-1)+2\gamma\sum_{i=1}^{n} a_{im}}{n+\gamma^2} > n-1,$$

which is a contradiction. Hence $n=p$ and $G=G'$ is a complete graph.

CASE 2. *G' is an odd cycle of order p'.* We show here also that $G=G'$ so that G is an odd cycle. In this case $n=3$. If $p'=p$, then the result follows directly. We henceforth assume $p'<p$. With an appropriate labeling of the vertices of G, we can express $A(G)$ in such a way that its upper left p'-by-p' block is $A(G')$, i.e.,

$$A(G')=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Since G is connected, $\sum_{i=1}^{p'}\left(\sum_{j=n+1}^{p} a_{ij}\right) \neq 0$. Again there exists m, where $p'<m\leq p$ such that $2\sum_{i=1}^{p'} a_{im} > 0$. Hence we select a real number $\gamma>0$ such that $\left(2\sum_{i=1}^{p'} a_{im}\right)\Big/(n-1) > \gamma$. Because $n=3$, we have that

$$\frac{2p'+2\gamma\sum_{i=1}^{p} a_{im}}{p'+\gamma^2} > n-1.$$

Let $x=(1,\ldots,1,0,\ldots,0,\gamma,0,\ldots,0)$ be a p-vector whose first p' entries are 1, whose mth entry is γ, and the remaining entries of which are 0. As before, we have

$$\epsilon(G) \geq \frac{(xA(G),x)}{(x,x)} = \frac{2p'+2\gamma\sum_{i=1}^{p} a_{im}}{p'+\gamma^2} > n-1,$$

which is contrary to hypothesis. Hence $p'=p$ and $G=G'$. ∎

Theorem 15.4 also constitutes an improved bound, in general, for the chromatic number over that provided by Theorem 15.2. With the aid of Theorem 15.4, we can now give an upper bound for $\chi(G)$ in terms of p and q.

Corollary 15.4a For any (p, q) graph G,

$$\chi(G) \leq 1 + \left(\frac{2q(p-1)}{p}\right)^{\frac{1}{2}}.$$

Proof Denote the eigenvalues of G by $\epsilon_1, \epsilon_2, \ldots, \epsilon_p$, where ϵ_p is a maximum eigenvalue. We prove first that

$$\epsilon_p \leq \left(\frac{p-1}{p}\sum_{i=1}^{p} \epsilon_i^2\right)^{\frac{1}{2}}. \tag{15.4}$$

We apply here the Cauchy-Schwarz inequality, which states for any two n-vectors $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ that

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{15.5}$$

By letting $n = p - 1$ in (15.5) as well as $a_i = \epsilon_i$ and $b_i = 1$, $i = 1, 2, \ldots, p-1$, we arrive at

$$\left(\sum_{i=1}^{p-1} \epsilon_i\right)^2 \leq (p-1)\left(\sum_{i=1}^{p-1} \epsilon_i^2\right). \tag{15.6}$$

However, $\sum_{i=1}^{p} \epsilon_i = 0$ so that $\epsilon_p = -\sum_{i=1}^{p-1} \epsilon_i$. Therefore, $\epsilon_p^2 = \left(\sum_{i=1}^{p-1} \epsilon_i\right)^2$ so that (15.6) becomes

$$\epsilon_p^2 \leq (p-1)\left(\sum_{i=1}^{p-1} \epsilon_i^2\right). \tag{15.7}$$

Adding $(p-1)\epsilon_p^2$ to both sides of (15.7), we obtain

$$p\epsilon_p^2 \leq (p-1)\sum_{i=1}^{p} \epsilon_i^2,$$

from which (15.4) follows.

By Theorem 4I, the eigenvalues of $(A(G))^2$ are $\epsilon_1^2, \epsilon_2^2, \ldots, \epsilon_p^2$, and by Corollary 4.1a (ii), the trace of $(A(G))^2$ is $2q$. Hence

$2q = \sum_{i=1}^{p} \epsilon_i^2$. Now applying (15.4) and Theorem 15.4, we have the desired result. ■

We have already noted that the chromatic number of a graph G is bounded below by the order of the largest complete subgraph of G. From our observations regarding upper bounds, one might conjecture (correctly) that this lower bound is not a particularly good one in general. One aspect of this conjecture has been the object of several investigations. Namely, if G is a nonempty graph which contains no triangles, then how large can the integer $\chi(G)$ be? It has been shown by several mathematicians (Tutte [3], Zykov [14], Mycielski [9]) that such graphs may possess arbitrarily large chromatic numbers, and, in fact, a somewhat stronger statement can be made. We give one such proof of this by presenting the following result of Kelly and Kelly [6].

Theorem 15.5 For every positive integer $n \geq 2$, there exists a graph G such that $\chi(G) = n$ and $g(G) > 5$.

Proof The proof is by induction on n. For $n = 2$ the graph C_6 satisfies the theorem.

Next, assume there exists an n-chromatic graph G_n, $n \geq 2$, whose girth exceeds 5. We construct a graph G_{n+1} such that $\chi(G_{n+1}) = n + 1$ and $g(G_{n+1}) > 5$. Suppose G_n is a (p, q) graph. To begin the construction consider a set U consisting of np vertices. There are $\binom{np}{p}$ distinct subsets U_r of U, $1 \leq r \leq \binom{np}{p}$, each of which contains p elements. For $r = 1, 2, \ldots, \binom{np}{p}$, define $H_r = G_n$ such that for $r \neq s$, $V(H_r) \cap V(H_s) = \emptyset$. Now for each such r, we construct a graph F_r so that

$$V(F_r) = U_r \cup V(H_r)$$

and

$$E(F_r) = E(H_r) \cup E',$$

where E' is a set of p edges joining H_r and U_r in a one-to-one manner. Then we define a graph G'_{n+1} by letting $V(G'_{n+1}) = \bigcup_r V(F_r)$ and $E(G'_{n+1}) = \bigcup_r E(F_r)$. The graph G'_{n+1} has

$$p\left[n + \binom{np}{p}\right] \text{ vertices and } (p + q)\binom{np}{p} \text{ edges.}$$

We now show that $\chi(G'_{n+1}) \geq n+1$. Since G'_{n+1} contains G_n as a subgraph, $\chi(G'_{n+1}) \geq n$. Suppose $\chi(G'_{n+1}) = n$, and consider any n-coloring of G'_{n+1}. Hence at most n colors are used to color the elements of U. This implies that some color, say c_0, has been assigned to at least p vertices of U. Let U_i be a subset of p vertices of U, all of which are colored c_0. Since $\chi(H_i) = n$, the color c_0 is assigned to at least one vertex of H_i. However then, F_i, and therefore G'_{n+1}, contains two adjacent vertices colored c_0. This produces a contradiction. Thus $\chi(G'_{n+1}) \geq n+1$.

To show that the girth of G'_{n+1} exceeds 5, we use the inductive hypothesis that no cycle in G_n has length less than 6. Let C be a cycle of G'_{n+1}. If no vertex of C belongs to U, then necessarily C belongs entirely to some H_r and therefore has length at least 6. Otherwise, C contains a vertex v_1 of U, where $v_1, v_2, \ldots, v_k, v_1$ is the cycle C. By the construction of G'_{n+1}, v_2 must belong to some H_r; furthermore, v_3 belongs to H_r. Since v_2 is the only vertex of H_r which is adjacent with v_1, some vertex v_j, $j \geq 4$, must belong to U, but then each of v_{j+1} and v_{j+2} must belong to some H_s. Hence $k \geq j+2$, and C has length at least 6.

Finally, if $\chi(G'_{n+1}) > n+1$, then a subgraph G_{n+1} of G'_{n+1} can be selected so that $\chi(G_{n+1}) = n+1$. Of course, the girth of G_{n+1} cannot be less than that of G'_{n+1}. ■

We mention in concluding this section that Erdös [5] and Lovász [7] have extended this result so that given any two integers $m, n \geq 2$, there exists an n-chromatic graph whose girth exceeds m.

PROBLEM SET 15.1

15.1 Prove that $p/\beta(G) \leq \chi(G) \leq p + 1 - \alpha(G)$.

15.2 Prove a result analogous to Theorem 15.2 for disconnected graphs.

15.3 What bound is given for $\chi(G)$ by Theorems 15.2 and 15.3 in the case that G is (a) a tree, (b) an outerplanar graph?

15.4 Let G be a 4-regular graph of order 10. What bound for $\chi(G)$ is given by (a) Theorem 15.2, (b) Theorem 15.3, and (c) Corollary 15.4a?

15.5 Give an example of a 4-chromatic graph without triangles (a) using Theorem 15.5, (b) having order less than the graph in (a).

15.6 Use Theorem 15.3 to prove for any graph G of order p that $\chi(G) + \chi(\overline{G}) \leq p + 1$.

15.7 Verify the bounds **[10]** for a graph G of order p:

$$2\sqrt{p} \leq \chi(G) + \chi(\overline{G}) \leq p + 1$$
$$p \leq \chi(G) \cdot \chi(\overline{G}) \leq [(p + 1)/2]^2.$$

15.2 The Edge and Total Chromatic Numbers

Each concept introduced at the beginning of the preceding section has an analogue for edges. An assignment of colors to the edges of a nonempty graph G so that adjacent edges are colored differently is an *edge-coloring* of G (an *n-edge coloring* if n colors are used). A nonempty graph G is *n-edge colorable* if there exists an m-edge coloring of G for some $m \leq n$. The minimum n for which a graph G is n-edge colorable is its *edge chromatic number* $\chi_1(G)$. The determination of the edge chromatic number of a graph can be transformed into a problem dealing with chromatic numbers; indeed it is immediate from the definitions that for an arbitrary nonempty graph G,

$$\chi_1(G) = \chi(L(G)),$$

where $L(G)$ is the line graph of G.

It is obvious that $\Delta(G)$ is a lower bound for $\chi_1(G)$; however, it was proved by Vizing [12] that $\chi_1(G)$ never exceeds $\Delta(G)$ by more than one.

Theorem 15.6 (Vizing's Theorem) If G is a nonempty graph, then

$$\chi_1(G) \leq 1 + \Delta(G).$$

Proof Suppose the theorem is not true. Then among the graphs for which the theorem is false, let G_1 be one with a minimum number of edges. Hence G_1 is not $(1 + \Delta)$-edge colorable, where $\Delta = \Delta(G_1)$; however, if $e = uv$ is an edge of G_1, then $G_1 - e$ is $(1 + \Delta)$-edge colorable.

Let there be given a $(1 + \Delta)$-edge coloring of $G_1 - e$; i.e., every edge of G_1 except e is assigned one of $1 + \Delta$ colors so that adjacent edges are colored differently. For each edge $e' = uv'$ of G_1 which is incident with u, we define its *dual color* as any one of the $1 + \Delta$ colors which is not used to color edges incident with v'. Since no vertex of G_1 has degree exceeding Δ, there is at least one color available for the dual color. It may occur that distinct edges have the same dual color.

Let $e=e_0$ have dual color α_1. (The color α_1 is not the color of any edge of G_1 incident with v.) There must be some edge e_1 incident with u which has been assigned the color α_1; for if not, then the edge e could be colored α_1, thereby producing a $(1+\Delta)$–edge coloring of G_1. Let α_2 be the dual color of e_1. If there is an edge incident with u which has been assigned the color α_2, then we denote it by e_2 and call its dual color α_3. In this manner, we construct a sequence $e_0, e_1, \ldots, e_k$, $k \geq 1$, containing a maximum number of distinct edges. The final edge e_k of this sequence is therefore colored α_k and has dual color α_{k+1}.

If there is no edge of G_1 incident with u which is assigned the color α_{k+1}, then we may assign each of the edges $e_0, e_1, \ldots, e_k$, with its dual color and obtain a $(1+\Delta)$–edge coloring of G_1. This, of course, is impossible. Hence we may assume that there exists an edge e_{k+1} of G incident with u which is colored α_{k+1}. Since $e_0, e_1, \ldots, e_k$ is maximum as to the number of distinct edges, we must have $e_{k+1}=e_i$ for some i, $1 \leq i \leq k$, or equivalently, $\alpha_{k+1}=\alpha_i$ for some i, $1 \leq i \leq k$. Now certainly $\alpha_{k+1} \neq \alpha_k$ since the color assigned to e_k cannot be the same as its dual color. Thus $\alpha_{k+1}=\alpha_i$ for some i, $1 \leq i < k$.

We now make some observations which will be important in the remainder of the proof. Since the edge e cannot be assigned any of the $1+\Delta$ colors without producing two adjacent edges having the same color, it follows that for each color α among the $1+\Delta$ colors, there is an edge of G_1 adjacent with e which is colored α. This implies that there must be colors assigned to edges incident with v which are not assigned to any edge incident with u. Let β be one such color. Furthermore, let $e_i=uv_i$, $i=0,1,\ldots,k$, where then $v_0=v$. The color β must be assigned to some edge incident with v_i for each $i=1,2,\ldots,k$; for suppose there is a vertex v_m, $1 \leq m \leq k$, such that no edge incident with v_m is colored β. Then we may change the color of e_m to β and color each e_i, $0 \leq i < m$, with its dual color to obtain a $(1+\Delta)$–edge coloring of G_1.

Suppose $\alpha_{k+1}=\alpha_{t+1}$, $0 \leq t < k-1$; i.e., the edges e_k and e_t have the same dual color. We define two paths P and Q as follows. Let P be a path with initial vertex v_k of maximum length whose edges are alternately colored β and α_{k+1}, while Q is a path with initial vertex v_t having maximum length whose edges are alternately colored β and $\alpha_{t+1}=\alpha_{k+1}$. Suppose P terminates at w and Q

at w'. We consider four cases depending on the choices of w and w'.

CASE 1. $w = v_m$, $0 \le m \le k-1$. In this case, the initial and terminal edges of P are colored β. We note also that unless $v_m = v_t$, v_t is not on P. Interchange the colors β and α_{k+1} of the edges of P. Upon doing this, we have no edge incident with v_m which is assigned the color β; and, moreover, the dual color of each e_i, $i < m$, is not altered. This implies that G_1 is $(1+\Delta)$–edge colorable, which is contradictory.

CASE 2. $w' = v_m$, $0 \le m \le k$, $m \ne t$. Here also, the initial and terminal edges of Q are assigned the color β. Also, Q does not contain v_k unless $v_m = v_k$. If $m < t$, we proceed as in Case 1. If $m > t$, then we interchange the colors β and α_{k+1} of the edges of Q. We then have no edge incident with v_t having the color β and the dual color of each e_i, $i < t$, need not be changed. Once again this implies that G_1 is $(1+\Delta)$–edge colorable, which is impossible.

CASE 3. $w \ne v_m$, $0 \le m \le k-1$, *and* $w \ne u$; *or* $w' \ne v_m$ *for any* $m \ne t$ *and* $w' \ne u$. We consider w only, the conclusion being identical for w'. Observe that by interchanging the colors β and α_{k+1} of P, the color β is assigned to no edge incident with v_k and the dual color of each e_i, $0 \le i < k$, remains the same. This situation, as we have seen, yields a contradiction.

Thus only one other case remains.

CASE 4. $w = u$ *and* $w' = u$. Since u is incident with no edge colored β, the initial edge of both paths P and Q is colored β while each terminal edge is assigned α_{k+1}. If P and Q are edge–disjoint, then u is incident with two distinct edges colored α_{k+1}, which cannot occur. Thus P and Q have an edge in common. But then there is a vertex incident with three edges belonging to P or Q. At least two of these edges are colored either β or α_{k+1}. This is a contradiction. ■

We now discuss briefly the much less studied total chromatic number. A *total coloring* of a graph G is an assignment of colors to the elements (vertices and edges) of G so that adjacent elements and incident elements of G are colored differently. An *n–total coloring* is a total coloring which uses n colors. The minimum n for which a graph G admits an n–total coloring is called the *total chromatic* *number* of G and is denoted $\chi_2(G)$. The inequalities

$$\chi_2(G) \geq \chi(G)$$

and

$$\chi_2(G) \geq \chi_1(G)$$

are obvious. It should also be transparent that for any graph G, $\chi_2(G) = \chi(T(G))$, where $T(G)$ is the total graph of G. Just as it is an immediate observation that $\Delta(G)$ is a lower bound for $\chi_1(G)$, so too does it follow that $1 + \Delta(G)$ is a lower bound for $\chi_2(G)$. No upper bound for $\chi_2(G)$ analogous to that given for $\chi_1(G)$ in Theorem 15.6 is known; indeed, the following (see [1]) is referred to as:

THE TOTAL COLORING CONJECTURE. For any graph G,

$$\chi_2(G) \leq 2 + \Delta(G).$$

PROBLEM SET 15.2

15.8 (a) What bounds for $\chi_1(G)$ are given by Theorem 15.6, if G is an r-regular graph, $r > 0$?
(b) If G is r-regular, $r > 0$, show that $\chi_1(G) = r$ if and only if G is 1-factorable.

15.9 Show that $\chi_1(K_p)$ has the value $p - 1$ if p is even and p if p is odd, $p > 1$.

15.10 Prove that $\chi_1(K(m, n)) = \max(m, n)$.

15.11 Determine $\chi_1(G)$ if G is bipartite.

15.12 Give an example of a cubic planar graph G for which $\chi_1(G) = 4$.

15.13 Give an example of a cubic bridgeless graph G for which $\chi_1(G) = 4$.

15.14 Determine formulas for the total chromatic number of:
(a) K_p,
(b) $K(m, n)$,
(c) C_p.

15.15 Find a necessary and sufficient condition for a graph G to satisfy:

$$\chi(G) + \chi_1(G) = \chi_2(G).$$

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16

The Four Color Problem

Probably the most famous unsolved problem in all mathematics is the Four Color Problem. Although this problem was not originally stated in graph–theoretic terms, there are several equivalent graphical formulations of it. As with all well-known unsolved problems, the attempts to settle the Four Color Problem have been many and unsuccessful, but they have contributed significantly to the growth of graph theory.

16.1 The Origin of the Four Color Problem

It has often been said that even the mapmakers of many centuries past were aware of the "fact" that any map on the plane (or sphere) could be colored with four or fewer colors so that no two adjacent countries were colored alike. Two countries are considered to be *adjacent* if they shared a common boundary (not simply a single point). As was pointed out in [9], however, there has been no indication in ancient atlases, books on cartography, or books on the history of mapmaking that people were familiar with this so-called fact. Evidently, not only was no mention made of this statement, but it was rare when any attempt was made to color maps using a minimum number of colors. (It is quite conceivable, in fact, that those colorings which proved to be minimum colorings were done quite unknowingly.) It is most likely therefore that this, the Four Color Problem, did not have its roots among the ancient mapmakers; indeed, it is probable that it originated and grew in the minds of mathematicians.

It was written by Ball [**1**] in 1892 that "the problem was mentioned by A. F. Möbius in his Lectures in 1840"; however, what Ball was evidently referring to were lectures by Möbius in which he presented and then solved a problem communicated to him by Weiske, namely that it is not possible to have five mutually adjacent regions in any map (on the sphere).

It was mentioned by Ball in the same sentence that "it was not until Francis Guthrie communicated it (the problem) to De Morgan about 1850 that attention was generally called to it." This statement also seems to be open to question, for it was not until 1878 when any printed reference to the Four Color Problem appeared. In the *Proceedings of the London Mathematical Society* of that year, Cayley asked whether the conjecture had been proved. Excitement in the problem then came quickly, and it has been of interest to many people to this very day.

In 1880 Frederick Guthrie wrote to the Royal Society of Edinburgh that some thirty years earlier he had informed Professor De Morgan of the Four Color Problem and that the problem had been shown to him earlier by his brother Francis. He said that Francis had shown him a proof, but Francis was not satisfied with his own solution. This is further substantiated in a letter from De Morgan to Sir William Hamilton on October 23, 1852, where De Morgan speaks of the problem and that it was given to him by a student. Three days later Hamilton replied that he would not likely be attempting the problem very soon. Perhaps Francis Guthrie and Hamilton showed exceptional wisdom when Guthrie questioned whether he had actually solved the problem and Hamilton refused to work on it altogether.

We now consider the graph-theoretic version of "Guthrie's Problem." A thorough investigation of the Four Color Problem is given in Ore [**10**].

16.2 The Four Color Conjecture

The Four Color Problem can be stated quite easily in graphical terms. A plane graph G is said to be *n-region colorable* if the regions of G can be colored with n or fewer colors so that adjacent regions are colored differently. The Four Color Problem is thus the problem of determining whether every plane graph is 4-region colorable. A solution in the affirmative to the Four Color Problem does not require one to show that an arbitrary plane graph is 4-region colorable, since there are several classes of plane graphs for which verification that each member of the class is 4-region colorable implies an affirmative solution to the Four Color Problem. We consider some of these equivalent formulations now.

Define the *region chromatic number* $\chi^*(G)$ of a plane graph G to be the minimum n for which G is n-region colorable. Since $\chi^*(G)$ is the maximum region chromatic number among the components of G, it follows that if $\chi^*(G) \leq 4$ for every connected plane graph G, then the Four Color Problem is solved in the affirmative; hence, one may restrict himself to connected plane graphs when dealing with the Four Color Problem. It is likewise obvious that for any nonempty connected plane graph G, the number $\chi^*(G)$ is the maximum value $\chi^*(B)$ among the blocks B of G. Thus the Four Color Problem is solved affirmatively if $\chi^*(G) \leq 4$ for every plane block G.

It is also interesting to note that the Four Color Problem is solved in the affirmative if the region chromatic number of every cubic plane block does not exceed 4. In order to see this, assume that $\chi^*(G) \leq 4$ for every cubic plane block G, and let H be a plane block. We now construct a cubic plane block H' from H as follows. If H contains a 2-valent vertex v, incident with edges e and f, we subdivide e and f by introducing vertices v_1 and v_2 into e and f, respectively, remove v, and then identify v_1 and v_2, respectively, with the 2-valent vertices of a copy of the graph $K(1,1,2)$. (See Fig. 16.1(a).) If H contains an n-valent vertex u, $n \geq 4$, incident with the consecutive edges $e_1, e_2, \ldots, e_n$, then we subdivide each e_i by inserting a vertex u_i in each e_i, $i = 1, 2, \ldots, n$, removing the vertex u, and identifying each u_i with the corresponding vertex of the n-cycle $u_1, u_2, \ldots, u_n, u_1$. (See Fig. 16.1(b).) By hypothesis, $\chi^*(H') \leq 5$, for the resulting cubic plane block H'; hence there exists a k-region coloring, $k \leq 4$, of H'. However, by identifying all vertices of the graph $K(1,1,2)$ for each 2-valent vertex and by identifying the vertices of the n-cycle for each n-valent vertex, $n \geq 4$, the graph H is reproduced and a k-region coloring of H is induced. Hence H is 4-region colorable, and since H was an arbitrary plane block, the Four Color Problem has an affirmative solution. We summarize these results below.

Theorem 16.1 The Four Color Problem is equivalent to the problem of determining whether every cubic plane block is 4-region colorable.

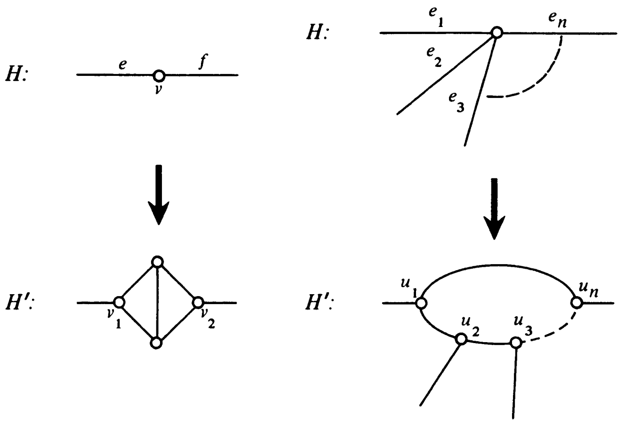

Figure 16.1 Constructing a cubic graph H' from a graph H

In graph theory the Four Color Problem is more often given in terms of coloring the vertices of a graph, i.e., coloring the graph. In this form the Four Color Problem is the problem of settling what is known as:

THE FOUR COLOR CONJECTURE. Every planar graph is 4-colorable.

It is in these terms that the Four Color Problem will henceforth be considered. We now verify that this formulation of the Four Color Problem is indeed equivalent to the original. First, however, we present a concept which will be useful in the proof.

For a given plane graph G, we construct a pseudograph G^* as follows. A vertex is placed in each region of G, and these vertices constitute the vertex set of G^*. Two distinct vertices of G^* are then joined by an edge for each edge which belongs to the boundary of both corresponding regions of G. In addition, one loop is added at a vertex of G^* for each bridge of G which belongs to the boundary of that region. We note that each edge of G^* can be so drawn that it crosses its associated edge of G but no other edge of G and G^*.

The pseudograph G^* is referred to as the *dual* of G. The dual has the properties, therefore, that it is planar and has the same number of edges as G, and that each region of G^* contains a single vertex of G. Also, $(G^*)^* = G$. If each set of multiple edges of G^* is replaced by a single edge and all loops are deleted, the result is a graph, referred to as the *underlying graph* H of G^*. These concepts are illustrated in Fig. 16.2, with the vertices of G^* represented by solid circles.

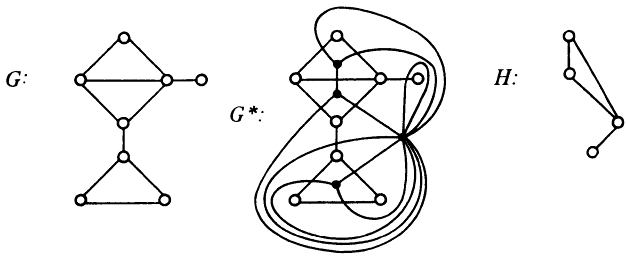

Figure 16.2 The dual (and its underlying graph) of a plane graph

Theorem 16.2 The Four Color Conjecture is true if and only if every plane graph is 4-region colorable.

Proof Suppose the Four Color Conjecture is true, and let G be an arbitrary plane graph. Let H be the underlying graph of the dual of G. Now two regions of G are adjacent if and only if the corresponding vertices of H are adjacent. Since H is planar, it follows, by hypothesis, that H is 4-colorable; thus, G is 4-region colorable.

For the converse, assume that every plane graph is 4-region colorable, and let G be an arbitrary plane graph. As we have noted, the dual G^* of G can be embedded in the plane so that each region of G^* contains one vertex of G. If G^* is not a graph, then it can be converted into a graph G' by inserting two vertices into each loop of G^* and by placing a vertex in all but one edge in each set of multiple edges joining the same two vertices. Two vertices of G are adjacent if and only if the corresponding regions of G' are adjacent. Since G' is 4-region colorable, G is 4-colorable. ■

As with the region-coloring version of the Four Color Problem, the Four Color Conjecture can be shown to hold if all graphs in certain subclasses of planar graphs are shown to be 4-colorable. First, if every connected planar graph is 4-colorable, then the Four Color Conjecture is true. Since the chromatic number of any nonempty connected graph is the maximum of the chromatic numbers of its blocks, we arrive at the following conclusion.

Theorem 16.3 The Four Color Conjecture is true if and only if every planar block is 4-colorable.

It is also interesting to note the following result by Whitney [**14**]: The Four Color Conjecture is true if and only if every hamiltonian planar graph is 4–colorable.

There is no result analogous to Theorem 16.1 involving (vertex-) coloring of graphs. Indeed, by Theorem 15.2, every cubic planar block is known to be 4-colorable.

There is a formulation of the Four Color Conjecture which involves edge–coloring of planar graphs. In proving the next result, it is convenient to make use of the Klein four–group K of algebra. Using $\oplus$ as the binary operation for K and denoting its elements by $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$, we have that

$$(i,j) \oplus (m,n) = (i+m, j+n),$$

where "+" denotes addition modulo two; hence $(0,0)$ is the zero element of K.

Theorem 16.4 The Four Color Conjecture is true if and only if every cubic plane block is 3-edge colorable.

Proof By Theorems 16.1 and 16.2, it is equivalent to show that a cubic plane block G is 4-region colorable if and only if G is 3-edge colorable. It is convenient, though not necessary, to assume the edges of G to be straight line segments.

Assume first that G is 4-region colorable, and let the regions of G be colored with the elements of the Klein four-group K. Since G is a block, each edge of G belongs to the boundary of two (adjacent) regions. Define the color of an edge to be the sum of the colors of those two regions bounded in part, by the edge. Since every element of K is self-inverse, no edge of G is assigned the color $(0,0)$. However, since K is a group, it follows that the three edges incident with a vertex are assigned the colors $(0,1)$, $(1,0)$, and $(1,1)$. Hence G is 3-edge colorable.

Suppose next that G is 3-edge colorable, and let the edges of G be colored with the nonzero elements of K. Let R be some region of G and assign the color $(0,0)$ to it. Let S be some other region of G. We now assign a color (an element of K) according to the following rule. Let A be a continuous curve joining a point of region R with a point of region S such that A passes through no vertex of G. We now define the color of S to be the sum of the colors of those edges crossed by A, where the color of an edge e is counted as many times as e is crossed. In order to show that the color of S is well-defined, we verify that the color assigned to S is independent of the curve A; however, this will be accomplished once it has been shown that if C is any simple closed curve not passing through vertices of G, then the sum of the colors of the edges crossed by C is $(0,0)$. Let C be such a curve. If no vertex of G lies interior to C, then each edge crossed by C is crossed an even number of times; and since each element of K is self-inverse it follows that the sum of the colors of the edges crossed by C is $(0,0)$. If C encloses vertices, then without loss of generality, we may assume that any edge crossed by C is crossed exactly once. We proceed as follows. Let $e_1, e_2, \ldots, e_s$ be those edges crossed by or lying interior to C, and suppose the first r of these edges are crossed by C. Observe that the sum of the colors of the three edges incident with any vertex is $(0,0)$; hence, if we were to total these sums for all vertices lying interior to C we, of course, arrive at $(0,0)$ also. However, this sum also equals

$$c(e_1) + c(e_2) + \cdots + c(e_r) + 2[c(e_{r+1}) + c(e_{r+2}) + \cdots + c(e_s)],$$

where $c(e_i)$ indicates the color of the edge e_i. Therefore, $c(e_1) + c(e_2) + \cdots + c(e_r) = (0,0)$, i.e., the sum of the colors of the edges crossed by C is $(0,0)$.

It now remains to show that this procedure yields a 4-region coloring of G. However, if R_1 and R_2 are two adjacent regions, sharing the edge e in their boundaries, then the colors assigned R_1 and R_2 differ by $c(e) \neq (0,0)$. This completes the proof. ■

The preceding theorem is intimately related to a false proof of the Four Color Conjecture. In 1880 Tait [12] "proved" that every cubic, planar, bridgeless graph is 3-edge colorable. (A 3-edge coloring of a cubic graph is often referred to as a *Tait coloring.*) According to Theorem 16.4, this would constitute an affirmative solution of the Four Color Problem. However, in Tait's "proof" of the Four Color Conjecture, he evidently assumed that every cubic, planar, bridgeless graph is hamiltonian. Tutte [**13**], though, showed Tait's assumption to be incorrect by producing a cubic, planar, bridgeless graph which is not hamiltonian (see Fig. 16.3).

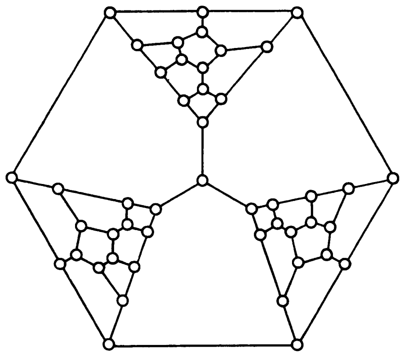

Figure 16.3 A non-hamiltonian, cubic, planar, bridgeless graph

In each formulation of the Four Color Conjecture considered, it is known that the number of colors required (whether it be for regions, vertices, or edges) does not exceed that conjectured by more than one. For example, every cubic planar block (indeed every cubic graph) is 4-edge colorable by Theorem 15.6. Likewise every plane graph is 5-region colorable. By using the technique employed in the proof of Theorem 16.2, one can verify immediately that the truth of either of the two statements implies the truth of the other. We prove the second of these, which is known as the Five Color Theorem.

Theorem 16.5 Every planar graph is 5-colorable.

Proof The proof is by induction on the order p of the graph. For $p = 1$, the result is obvious.

Assume that all planar graphs with $p-1$ vertices, $p > 1$, are

5-colorable, and let G be a plane graph of order p. By Corollary 7.2b, G contains a vertex v of degree 5 or less. By deleting v from G, we obtain the plane graph $G-v$. Since $G-v$ has order $p-1$, it is 5-colorable by the inductive hypothesis. Let there be given a 5-coloring of $G-v$, denoting the colors by 1, 2, 3, 4, and 5. If some color is not used in coloring the vertices adjacent with v, then v may be assigned that color, producing a 5-coloring of G itself. Otherwise, $\deg v = 5$ and all five colors are used for the vertices adjacent with v.

Without loss of generality, we assume that v_1, v_2, v_3, v_4, v_5 are the five vertices adjacent with and arranged cyclically about v and that v_i is assigned the color i, $1 \leq i \leq 5$. Consider now any two colors assigned non-consecutive vertices v_i, say 1 and 3, and let H be the subgraph of $G-v$ induced by all those vertices colored 1 or 3. If v_1 and v_3 belong to different components of H, then by interchanging the colors assigned vertices in the component of H containing v_1, for example, a 5-coloring of $G-v$ is produced in which no vertex adjacent with v is assigned the color 1. Thus if we color v with 1, a 5-coloring of G results.

Suppose then that v_1 and v_3 belong to the same component of H, so that there exists a v_1—v_3 path P all of whose vertices are colored 1 or 3. The path P together with the path v_3, v, v_1 produces a cycle C in G which encloses v_2, or v_4 and v_5. Hence there exists no v_2—v_4 path in G, all of whose vertices are colored 2 or 4. Denote by F the subgraph of G induced by all those vertices colored 2 or 4. Interchanging the colors of the vertices in the component of F containing v_2, we arrive at a 5-coloring of $G-v$ in which no vertex adjacent with v is assigned the color 2. If we color v with 2, a 5-coloring of G results. ■

The Five Color Theorem was first proved by Heawood [**3**], who also presented a counter-example to an intended proof of the Four Color Conjecture by Kempe [**5**]. Kempe's article appeared in 1878, and the Four Color Problem was considered solved for 11 years until Heawood discovered the error which destroyed Kempe's proof. Over the years there has been an extraordinarily large number of people (not all mathematicians) who have led themselves into believing that they had unlocked the mystery of the Four Color Problem, only to eventually learn that they had not. It is interesting to note that many of these so-called proofs have used essentially the same technique employed by Kempe. Evidently, Kempe's method is natural and his error is somewhat subtle. We invite the reader to follow through a variation of Kempe's proof of the Four Color Conjecture and to locate the mistake for himself.

"Theorem" 16.6 "The Four Color Conjecture is true."

Incorrect Proof The proof given here is very much like that of Theorem 16.5. We prove that every planar graph is 4-colorable by using induction on the order p of the graph. For $p = 1$, the result is immediate.

Assume that all planar graphs of order $p-1$, $p > 1$, are 4-colorable, and let G' be a plane graph of order p. Add sufficiently many edges to G' to obtain a maximal plane graph G. If it can be shown that G is 4-colorable, then it follows immediately that G' too is 4-colorable. By Corollary 7.2b, G contains a vertex v of degree 5 or less. The plane graph $G - v$ has order $p-1$ and is therefore 4-colorable by the induction hypothesis. Let there be given a 4-coloring of $G - v$, denoting the colors by $1, 2, 3$, and 4. If one of these colors is not used in coloring the vertices adjacent with v, then v may be assigned this color yielding a 4-coloring of G. Hence we may now assume that all four colors are used in coloring the vertices adjacent with v. Two cases now arise depending on whether $\deg v = 4$ or $\deg v = 5$.

Case 1. $\deg v = 4$. We may assume here without loss of generality that the vertices v_1, v_2, v_3, v_4 are adjacent with v and arranged cyclically about v such that v_i is colored i, $1 \leq i \leq 4$. Continuing as in the proof of Theorem 16.5, we let H denote the subgraph of $G - v$ induced by those vertices colored 1 or 3. If v_1 and v_3 lie in different components of H, then by interchanging the colors of those vertices belonging to the component of H containing v_1, we obtain a 4-coloring of $G - v$ in which no vertex adjacent with v is colored 1. We may now assign 1 to v and produce a 4-coloring of G.

If, on the other hand, v_1 and v_3 belong to the same component of H, then there exists a v_1–v_3 path in $G - v$ all of whose vertices are colored 1 or 3. This path together with the path v_1, v, v_3 produce a cycle enclosing v_2 or v_4. Hence there exists no v_2–v_4 path in $G - v$, all of whose vertices are colored 2 or 4. If we let F denote the subgraph of $G - v$ induced by those vertices colored 2 or 4, then v_2 and v_4 belong to different components of F. Hence we may interchange the colors of those vertices in the component of F containing v_2 and obtain a 4-coloring of $G - v$ with the added property that no vertex adjacent with v is colored 2. We may then assign the color 2 to v and produce a 4-coloring of G.

CASE 2. $\deg v = 5$. Suppose here that the vertices adjacent with v are v_1, v_2, v_3, v_4, v_5 and are arranged cyclically about v. Since G is a maximal plane graph, it follows that C: $v_1, v_2, v_3, v_4, v_5, v_1$ is a cycle of length 5 in G. Since no two consecutive vertices of C are assigned the same color in the 4-coloring of $G - v$ and since four colors are used in coloring the vertices of C, we may assume, without loss of generality, that v_i has been assigned the color i, where $1 \leq i \leq 4$, and v_5 has been colored 2.

Let H denote the subgraph of $G - v$ induced by those vertices colored 1 or 3. If v_1 and v_3 belong to different components of H, then by interchanging the colors of those vertices in the component of H containing v_1, we obtain a 4-coloring of $G - v$ in which no vertex adjacent with v is assigned the color 1. A 4-coloring of G may now be produced by assigning 1 to v. Otherwise, there exists a v_1–v_3 path in $G - v$ all of whose vertices are colored 1 or 3. This path together with the path v_1, v, v_3 produce a cycle in G enclosing v_2 or enclosing both v_4 and v_5.

Consider next the subgraph F of $G - v$ induced by those vertices colored 1 or 4. If v_1 and v_4 belong to different components of F, then we may interchange the colors of those vertices belonging to the component of F containing v_1 to produce a 4-coloring of $G - v$ in which no vertex adjacent with v is assigned the color 1. By coloring v with 1, we now arrive at a 4-coloring of G. If, however, v_1 and v_4 belong to the same component of F, then $G - v$ contains a v_1–v_4 path all of whose vertices are colored 1 or 4. This path and the path v_1, v, v_4 produce a cycle in G which encloses v_5 or encloses both v_2 and v_3.

The two aforementioned cycles imply that there exists neither a v_2–v_4 path in $G - v$ all of whose vertices are colored 2 or 4 nor a v_3–v_5 path in $G - v$ all of whose vertices are colored 2 or 3. Let H_1 be the subgraph of $G - v$ induced by those vertices colored 2 or 4, and let F_1 be the subgraph of $G - v$ induced by those vertices colored 2 or 3. Hence v_2 and v_4 lie in different components of H_1, and v_3 and v_5 belong to different components of F_1. Next we interchange the colors of those vertices in the component of H_1 containing v_2 and interchange the colors of those vertices in the component of F_1 containing v_5. By doing this, we obtain a 4-coloring of $G - v$ in which no vertex adjacent with v

is colored 2. Thus the color 2 may be assigned to v to produce a 4-coloring of G.

This completes the "proof." ■

PROBLEM SET 16.2

16.1 Prove that every planar graph is 5-colorable if and only if every plane graph is 5-region colorable.

16.2 Prove that the Four Color Conjecture is true if and only if every cubic planar block is 1-factorable.

16.3 Prove that every hamiltonian cubic graph is 3-edge colorable.

16.4 Find an error in the "proof" of "Theorem" 16.6.

16.5 Use a proof similar to that in Theorem 16.5 or Theorem 16.6 that will show $a(G) \leq 3$ for a planar graph G.

16.3 The Heawood Map Coloring Theorem

The *chromatic number of a surface* (a compact orientable 2-manifold) S_n of genus n, denoted $\chi(S_n)$, is the maximum chromatic number among all graphs which can be embedded on S_n. The surface S_0 is the sphere and the Four Color Conjecture states that $\chi(S_0) = 4$. In Theorem 16.5, we proved that $\chi(S_0) \leq 5$. Because there are planar graphs G for which $\chi(G) = 4$, it therefore follows that $\chi(S_0) = 4$ or $\chi(S_0) = 5$. Heawood [3], who proved the Five Color Theorem, also showed that $\chi(S_1) = 7$, i.e., the chromatic number of a torus is 7. Moreover, Heawood was under the impression that he had proved

$$\chi(S_n) = \left[\frac{7 + \sqrt{1 + 48n}}{2}\right]$$

for all $n > 0$. However, Heffter [4] pointed out that Heawood had only established the upper bound:

$$\chi(S_n) \leq \left[\frac{7 + \sqrt{1 + 48n}}{2}\right]. \tag{16.1}$$

The statement that $\chi(S_n) = [(7 + \sqrt{1 + 48n})/2]$ for all $n > 0$ eventually became known as the Heawood Map Coloring Conjecture. In 1968 Ringel and Youngs [11] completed a remarkable proof of the conjecture, which has involved a number of people. This result is now known as the Heawood Map Coloring Theorem.

Theorem 16.7 (The Heawood Map Coloring Theorem) For every positive integer n,

$$\chi(S_n) = \left[\frac{7 + \sqrt{1 + 48n}}{2}\right].$$

The proof of the Heawood Map Coloring Theorem is considerably too involved to present here (or practically anywhere else), but a few words involving the idea of proof are in order. For $n = 1$ the inequality (16.1) states that $\chi(S_1) \leq 7$. In order to prove that $\chi(S_1) = 7$, it is necessary to produce an example of a graph G with $\chi(G) = 7$ such that G can be embedded on the torus. Certainly, the most obvious graph with the chromatic number 7 is K_7. Fortunately, K_7 can be embedded on S_1 so that $\chi(S_1) = 7$. (An embedding of K_7 on the torus is shown in Fig. 16.4. The torus is obtained from the given rectangle by identifying opposite sides.)

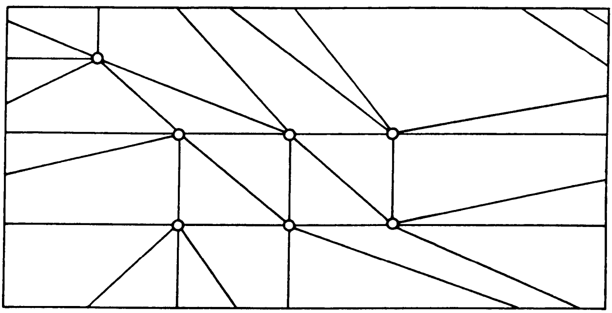

Figure 16.4 An embedding of K_7 on the torus

The method used in showing $\chi(S_1) = 7$ is precisely the technique employed in proving $\chi(S_n) = [(7 + \sqrt{1 + 48n})/2]$ for all $n > 0$; namely, it is shown that the complete graph K_p, where

$$p = \left[\frac{7 + \sqrt{1 + 48n}}{2}\right], \tag{16.2}$$

can be embedded in S_n. Solving (16.2) for n in terms of p, we find that $n = \{(p - 3)(p - 4)/12\}$. Hence, the Heawood Map-Coloring Theorem was proved by showing

$$\gamma(K_p) = \left\{\frac{(p-3)(p-4)}{12}\right\}, \qquad p \geq 3.$$

This result was previously mentioned in Chapter 9.

PROBLEM SET 16.3

16.6 Assuming the formula for the genus of the complete graph and the inequality (16.1), prove the Heawood Map Coloring Theorem.

16.7 Determine an embedding of $K(4, 4)$ on the torus.

16.4 k–Degenerate Graphs

There are classes of graphs which are closely related to many of the concepts and results we have discussed thus far. In the present section we introduce these classes and indicate many of the relationships.

A graph G is said to be *k–degenerate*, $k \geq 0$, if for every induced subgraph H of G, $\delta(H) \leq k$. The 0–degenerate graphs are obviously the empty graphs, and by Exercise 5.4, the 1–degenerate graphs are precisely the forests (acyclic graphs). By Corollary 7.2b, every planar graph is 5–degenerate. The reason for the term "k–degenerate" lies in the fact that a graph G of order p is k–degenerate if and only if the vertices of G may be ordered $v_1, v_2, \ldots, v_p$ such that the degree of v_i is at most k in the subgraph $\langle\{v_i, v_{i+1}, \ldots, v_p\}\rangle$ for $i = 1, 2, \ldots, p$; i.e., G can be reduced to the trivial (or degenerate) graph K_1 by the successive removal of vertices having degree at most k.

The *vertex partition number* $\rho_k(G)$, $k \geq 0$, of a graph G is the minimum number of subsets into which $V(G)$ can be partitioned so that each subset induces a k–degenerate subgraph of G. Hence $\rho_0(G)$ is the chromatic number of G (considered in Chapter 15) and $\rho_1(G)$ is the vertex-arboricity of G (discussed in Chapter 5). Therefore, the parameters ρ_k may be considered as a generalization of the chromatic number and vertex-arboricity. For example, $\rho_0(K_p) = \chi(K_p) = p$ and $\rho_1(K_p) = a(K_p) = \{p/2\}$, while for all positive integers p and nonnegative integers k,

$$\rho_k(K_p) = \left\{\frac{p}{k+1}\right\}. \tag{16.3}$$

A graph G is said to be *n-critical with respect to ρ_k*, $n \geq 2$, if $\rho_k(G) = n$ and $\rho_k(G - v) = n - 1$ for every $v \in V(G)$. If $\rho_k(G) = n$ and H is an induced subgraph of minimum order for which $\rho_k(H) = n$, then H is n-critical with respect to ρ_k; i.e., every graph G with $\rho_k(G) = n$ contains an induced subgraph which is n-critical with respect to ρ_k.

The following result by Lick and White [7] generalizes Theorem 5.7 and Corollary 15.1b.

Theorem 16.8 If G is a graph which is n-critical with respect to the vertex partition number ρ_k, then

$$\delta(G) \geq (k + 1)(n - 1).$$

Proof Assume that G is n-critical with respect to ρ_k and that G contains a vertex v such that $\deg v < (k + 1)(n - 1)$. Now $\rho_k(G - v) = n - 1$ so that there exists a partition $V_1, V_2, \ldots, V_{n-1}$ of $V(G - v)$ such that $\langle V_i \rangle$ is k-degenerate for $i = 1, 2, \ldots, n - 1$. Since $\deg_G v < (k + 1)(n - 1)$, at least one of the subsets $V_1, V_2, \ldots, V_{n-1}$ contains at most k vertices adjacent to v. Let V_j be such a subset; hence $\langle V_j \cup \{v\} \rangle$ is a k-degenerate subgraph of G. However, then, $V_1, V_2, \ldots, V_j \cup \{v\}, \ldots, V_{n-1}$ is a partition of $V(G)$ such that each subset induces a k-degenerate subgraph of G. This implies that $\rho_k(G) < n$, contradicting the fact that G is n-critical with respect to ρ_k and thereby completing the proof. ■

The *vertex partition number $\rho_k(S_n)$ of a compact orientable 2-manifold S_n* is defined as the maximum value of $\rho_k(G)$ among all graphs G which can be embedded in S_n. The numbers $\rho_0(S_n) = \chi(S_n)$ are, of course, precisely what we considered in the previous section. The next result is also due to Lick and White [**8**].

Theorem 16.9 The vertex partition numbers ρ_k, $k \geq 0$, for a compact orientable 2-manifold S_n, $n \geq 0$, are given by the formula

$$\rho_k(S_n) = \left[\frac{(2k + 7) + \sqrt{1 + 48n}}{(2k + 2)} \right]$$

except that (i) $\rho_0(S_0) = 4$ or 5, (ii) $\rho_1(S_0) = 3$, (iii) $\rho_3(S_0) = 2$, and (iv) $\rho_4(S_0) = 2$.

Proof For notational convenience, we let

$$f_k(t) = \frac{(2k+7) + \sqrt{1+48t}}{(2k+2)}$$

for $k \geq 0$ and $t \geq 0$.

First we verify that $\rho_k(S_n) \leq [f_k(n)]$ if $k \geq 0$ and $n > 0$. Let G be a graph of genus $n(>0)$ embedded in S_n, and suppose $\rho_k(G) = m$. Let H be an induced (p, q) subgraph of G which is m-critical with respect to ρ_k. Necessarily, $\gamma(H) = n_1 \leq n$. By Theorem 16.8, $\delta(H) \geq (k+1)(m-1)$ so that $2q \geq (k+1)(m-1)p$.

The fact that H is m-critical with respect to ρ_k assures us that H is connected. Hence we may apply Corollary 9.3b and obtain

$$\gamma(H) = n_1 \geq \frac{q}{6} - \frac{p}{2} + 1$$

and

$$\frac{(k+1)(m-1)}{2} \leq \frac{q}{p} \leq 3 + \frac{6(n_1 - 1)}{p},$$

so that

$$m - 1 \leq \frac{6}{k+1} + \frac{12(n_1 - 1)}{p(k+1)}.$$

Since $\rho_k(H) = \rho_k(G) = m$, we have $p \geq (k+1)(m-1) + 1$. Now, if $n_1 > 0$, then

$$m - 1 \leq \frac{6}{(k+1)} + \frac{12(n_1 - 1)}{(k+1)[(k+1)(m-1)+1]}.$$

Solving this inequality for m, we find that $\rho_k(G) = m \leq [f_k(n_1)] \leq [f_k(n)]$. Suppose that $n_1 = 0$, i.e., suppose that H is planar. It then follows from Corollary 7.2b and Theorem 16.8 that

$$5 \geq \delta(H) \geq (k+1)(m-1). \tag{16.4}$$

For the finitely many pairs m and k satisfying (16.4), we can immediately derive $m \leq [1 + 6/(k+1)] \leq [f_k(n)]$. Hence for an arbitrary graph G embedded in S_n, $n > 0$, it follows that $\rho_k(G) \leq [f_k(n)]$, so that $\rho_k(S_n) \leq [f_k(n)]$ for all $k \geq 0$.

Next we show that $\rho_k(S_n) \geq [f_k(n)]$ for all $k \geq 0$ and $n \geq 0$. For each nonnegative integer n, there exists a positive integer p such that $\gamma(K_p) \leq n < \gamma(K_{p+1})$. Since $f_k(t)$ is a strictly increasing function of t, we have

$$f_k(\gamma(K_p)) \le f_k(n) < f_k(\gamma(K_{p+1})).$$

By (9.2), $\gamma(K_p) = \{(p-3)(p-4)/12\}$ for $p \ge 3$. Now for $p \ge 4$,

$$\frac{p+k}{k+1} = f_k\left(\frac{(p-3)(p-4)}{12}\right) \le f_k(\gamma(K_p)) \le f_k(n).$$

Furthermore, $n \le \gamma(K_{p+1}) - 1 < (p-2)(p-3)/12$. Thus, because $f_k(t)$ is a strictly increasing function of t,

$$f_k(n) < f_k\left(\frac{(p-2)(p-3)}{12}\right) = \frac{p+k+1}{k+1}$$

so that $(p+k)/(k+1) \le f_k(n) < (p+k+1)/(k+1)$. Therefore, $[f_k(n)] = [(p+k)/(k+1)]$. However, by (16.3),

$$\rho_k(K_p) = \left\{\frac{p}{k+1}\right\} = \left[\frac{p+k}{k+1}\right]$$

and so $\rho_k(K_p) = [f_k(n)]$. Since $\gamma(K_p) \le n$, the complete graph K_p can be embedded in S_n so that $\rho_k(K_p) \le \rho_k(S_n)$. Hence $[f_k(n)] \le \rho_k(S_n)$ for $k \ge 0$ and $n \ge 0$. Therefore, if $k \ge 0$ and $n > 0$, then

$$\rho_k(S_n) = [f_k(n)] = \left[\frac{(2k+7) + \sqrt{1+48n}}{(2k+2)}\right].$$

We now consider the case $n = 0$. As we have already noted, $\rho_0(S_0) = 4$ or 5. By Exercise 16.5, $\rho_1(S_0) \le 3$. However, the dual of the graph in Fig. 16.3 has vertex-arboricity 3 (see [2]) so that $\rho_1(S_0) = 3$.

We next prove that $\rho_2(S_0) \le 2$ by verifying that $\rho_2(G) \le 2$ for an arbitrary planar graph G. We proceed by induction on p, noting that $\rho_2(K_1) \le 2$. Assume that $\rho_2(G) \le 2$ for all planar graphs G having order less than p, $p \ge 2$, and let H be a planar graph of order p. Let v be a vertex of H such that $\deg v \le 5$. By the induction hypothesis, $\rho_2(H - v) \le 2$. If $\rho_2(H-v) = 1$, then $H - v$ and $\langle \{v\} \rangle$ are 2-degenerate subgraphs of H so that $\rho_2(H) \le 2$. Suppose that $\rho_2(H - v) = 2$ and that $V(H - v)$ is partitioned into subsets V_1 and V_2 such that $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are 2-degenerate. Since v has degree at most 5 in H, v is adjacent to at most two vertices in one of the sets V_1 and V_2, say V_1. Therefore, each of $\langle V_1 \cup \{v\} \rangle$ and $\langle V_2 \rangle$ is 2-degenerate so that $\rho_2(H) \le 2$. Hence $\rho_2(S_0) \le 2$. Because $\rho_2(K_4) = 2$, we have $\rho_2(S_0) = 2$.

Similarly, induction on p may be used to prove that $\rho_3(S_0) \le 2$ and $\rho_4(S_0) \le 2$; the graphs of the octahedron and icosahedron, respectively, show that both equalities hold. Furthermore, since every planar graph is 5-degenerate, it follows that $\rho_k(S_0) = 1$ for $k \geq 5$. ■

Theorem 16.9 constitutes a generalization of the Heawood Map Coloring Theorem, although, of course, it makes use of the theorem in its proof. For $k=1$ and $n>0$ the result had originally been obtained by Kronk [6]. It is interesting to note that among all $k \geq 0$ and $n \geq 0$, the only unsettled case is what appears to be the simplest, namely $k = 0$ and $n = 0$. This returns us to the Four Color Problem.

PROBLEM SET 16.4

16.8 (a) Prove that if G is a planar 4-degenerate graph, then $\chi(G) \leq 4$.
(b) How is the result in (a) related to the Four Color Conjecture?

16.9 (a) Obtain a generalization of $\chi(G) \leq 1 + \Delta(G)$ and $a(G) \leq 1 + [\Delta(G)/2]$ for the vertex partition numbers ρ_k, $k \geq 0$.
(b) Obtain a generalization of Theorem 15.3 and Theorem 5.8 for the vertex partition numbers ρ_k, $k \geq 0$.

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17

Extremal Problems

17.1 Turán's Theorem

In 1941, Turán [**8**] proposed and then solved the following problem: For given positive integers p and n, determine the minimum integer $T(p, n)$ such that every $(p, T(p, n))$ graph contains a subgraph isomorphic to K_n. This problem is the first of several related problems which make up the area known as extremal graph theory. We prove here Turán's theorem as well as elaborate on some generalizations of this problem.

In order to establish Turán's theorem, we first prove a theorem which is complementary to Turán's result ([**6**], p. 214).

Theorem 17.1 For integers $p \geq 2$ and $n \geq 3$, with $n \leq p + 1$, let G be a graph of order p having the minimum number $T'(p, n)$ of edges such that $\beta(G) < n$. Then G is the $(p, t(p - n + 1 + r)/2)$ graph having $n - 1$ components, r of which are K_{t+1} and $n - 1 - r$ of which are K_t, where

$$p = t(n - 1) + r, \qquad 0 \leq r < n - 1.$$

Proof The theorem can be readily verified for $n = p$ and $n = p + 1$, so that the result follows for $p = 2$ and $p = 3$. We employ induction on p and assume $3 \leq n < p$, where $p \geq 4$. Thus suppose the result follows for all graphs of order less than p satisfying the hypotheses, and let G be a graph of order p which satisfies the hypotheses of the theorem.

Since the removal of any edge of G results in a graph with independence number at least n and since the removal of an edge from a graph can increase the independence number by at most one, it follows that $\beta(G) = n - 1$. Let

$$U = \{u_1, u_2, \ldots, u_{n-1}\}$$

be an independent set of vertices of G. Every element of $V(G) - U$ is adjacent with at least one element of U. Since $\beta(G) = n - 1$, the independence number of the subgraph $G' = \langle V(G) - U \rangle$ cannot exceed $n - 1$, so that its number of edges is at least $T'(p - n + 1, n)$; consequently,

$$T'(p, n) \geq T'(p - n + 1, n) + p - n + 1. \tag{17.1}$$

Because $p = t(n - 1) + r$, $0 \leq r < n - 1$, it follows that

$$p - n + 1 = (t - 1)(n - 1) + r, \qquad 0 \leq r < n - 1. \tag{17.2}$$

Applying the inductive hypothesis, we arrive at $T'(p - n + 1, n) = (t - 1)(p - 2n + 2 + r)/2$. Substituting this value into (17.1), we obtain

$$T'(p, n) \geq t(p - n + 1 + r)/2. \tag{17.3}$$

Suppose that equality holds in (17.3). This can occur if and only if equality holds in (17.1), and there is equality in (17.1) if and only if the following two conditions hold:

(1) Each vertex of $V(G) - U$ is adjacent with precisely one vertex of U.
(2) The graph G' has $T'(p - n + 1, n)$ edges and $\beta(G') < n$; thus by the inductive hypothesis, G' contains components $G'_1, G'_2, \ldots, G'_{n-1}$, r of which are isomorphic to K_t and $n - 1 - r$ of which are isomorphic to K_{t-1}.

Suppose there exists a vertex $u \in U$ such that u is adjacent with both v'_1 and v'_2, where $v'_i \in V(G'_i)$, $i = 1, 2$. Then $U \cup \{v'_1, v'_2\} - \{u\}$ is an independent set having n elements which, of course, cannot occur. Hence each element of U, say v_i, can be adjacent only to vertices in the same component, say G'_i, of G'. This implies that there exists a one-to-one mapping ϕ from

$\{G'_1, G'_2, \ldots, G'_{n-1}\}$ onto U where, say, $\phi(G'_i) = v_i$. Thus for $i = 1, 2, \ldots, n-1$, each vertex of G'_1 is adjacent to v_i and only to v_i. Hence G has $n-1$ components, r of which are K_{t+1} and $n-1-r$ of which are K_t, so that G is of the desired type. This completes the proof. ■

We now present the theorem with which we are primarily concerned here.

Theorem 17.2 (Turán's Theorem) For positive integers p and n, with $3 \le n \le p$, let $T(p, n)$ be the smallest positive integer such that every $(p, T(p, n))$ graph contains a subgraph isomorphic to K_n. Then

$$T(p, n) = \binom{p}{2} + 1 - t(p - n + 1 + r)/2,$$

where $p = t(n-1) + r$, $0 \le r < n-1$. Furthermore, the only $(p, T(p, n) - 1)$ graph which fails to contain K_n as a subgraph is the complete $(n-1)$-partite graph $K(p_1, p_2, \ldots, p_{n-1})$, where $p_1 = p_2 = \cdots = p_r = t + 1$ and $p_{r+1} = p_{r+2} = \cdots = p_{n-1} = t$. (If $r = 0$, then $p_i = t$ for all i.)

Proof By hypothesis, every $(p, T(p, n))$ graph contains K_n as a subgraph, while there exists a $(p, T(p, n) - 1)$ graph G which fails to contain K_n as a subgraph. Accordingly, by considering complementary graphs, every $(p, \binom{p}{2} - T(p, n))$ graph has independence number at least n while $\beta(\overline{G}) < n$. Hence by Theorem 17.1,

$$\binom{p}{2} - T(p, n) + 1 = t(p - n + 1 + r)/2,$$

where $p = t(n-1) + r$, $0 \le r < n-1$; or, equivalently, $T(p, n) = \binom{p}{2} + 1 - t(p - n + 1 + r)/2$. Furthermore, by Theorem 17.1, the graph $\overline{G}$ is unique, consisting of $n-1$ components, r of which are K_{t+1} and $n-1-r$ of which are K_t. Therefore, G is unique and $G = K(p_1, p_2, \ldots, p_{n-1})$, where $p_1 = \cdots = p_r = t + 1$ and $p_{r+1} = \cdots = p_{n-1} = t$ if $r > 0$, and where $p_i = t$ for all i if $r = 0$. ■

The special case of Turán's Theorem in which $n = 3$ is of added interest.

Corollary 17.2a For $p \geq 3$, the smallest positive integer $T(p, 3)$ such that every $(p, T(p, 3))$ graph contains a triangle is given by

$$T(p, 3) = 1 + \left[\frac{p^2}{4}\right].$$

Moreover, the only graph of order $p \geq 3$ having $[p^2/4]$ edges which fails to contain a triangle is the complete bipartite graph $K\left(\left[\frac{p}{2}\right], \left\{\frac{p}{2}\right\}\right)$.

Turán also proposed the following general problem, which encompasses a great number of problems belonging to extremal graph theory. Determine the minimum number q so that for a fixed integer p, every (p, q) graph contains a prescribed subgraph H. Accordingly, then, there exists at least one $(p, q-1)$ graph G which fails to contain H; the graph G is often referred to as an extremal graph of order p with respect to the subgraph H. Turán's Theorem therefore qualifies as a solution to this type of problem. Many of the results in extremal graph theory have been obtained by Erdös [**2, 3, 4**]. The next theorem [**3**] is due to Pósa, a protegé of Erdös.

For a vertex v of a graph G, the *neighborhood* $N(v)$ of v consists of the vertices of G adjacent with v, and the *closed neighborhood* $\overline{N}(v) = N(v) \cup \{v\}$.

Theorem 17.3 For $p \geq 6$, the smallest integer $s(p)$ so that every $(p, s(p))$ graph contains two disjoint cycles is

$$s(p) = 3p - 5.$$

Proof First we use induction on p to show that $s(p) \leq 3p - 5$. There are only two $(6, 13)$ graphs—one obtained by removing two independent edges from K_6 and the other obtained by removing two adjacent edges from K_6. In both cases, the graph has two disjoint triangles. Thus $s(6) \leq 13$.

For $n \geq 7$, we assume that $s(p) \leq 3p - 5$ for all $p \leq n - 1$ and let G be an $(n, 3n - 5)$ graph. Since

$$\sum_{v \in V(G)} \deg v = 6n - 10,$$

there exists a vertex v_0 of G such that $\deg v_0 \leq 5$. Assume first $\deg v_0 = 5$, and $N(v_0) = \{v_i | i = 1, 2, \ldots, 5\}$. If $\langle \overline{N}(v_0) \rangle$ contains 13 or more edges, then we have already noted that $\langle \overline{N}(v_0) \rangle$ has two disjoint cycles implying that G has two disjoint cycles. If, on the other hand, $\langle \overline{N}(v_0) \rangle$ contains 12 or fewer edges, then, since $\deg v_0 = 5$, some vertex, say v_1, is not adjacent with two other elements of $N(v_0)$, say v_2 and v_3. Add to G the edges $v_1 v_2$ and $v_1 v_3$

and delete the vertex v_0, obtaining the graph G', i.e., $G' = G + v_1v_2 + v_1v_3 - v_0$. The graph G' is a $(n-1, 3n-8)$ graph and, by the inductive hypothesis, contains two disjoint cycles C_1 and C_2. At least one of these cycles, say C_1, does not contain the vertex v_1 and thus contains neither the edge v_1v_2 nor the edge v_1v_3. Hence C_1 is a cycle of G. If C_2 also fails to contain either v_1v_2 or v_1v_3, then C_1 and C_2 are disjoint cycles of G. If C_2 contains v_1v_2 but not v_1v_3, then by removing v_1v_2 and adding v_0, v_0v_1, and v_0v_2, we produce a cycle of G which is disjoint from C_1. The procedure is similar if C_2 contains v_1v_3 but not v_1v_2. If C_2 contains both v_1v_2 and v_1v_3, then by removing v_1 from C_2 and adding v_0, v_0v_2, and v_0v_3, a cycle of G disjoint from C_1 is produced.

Suppose next that $\deg v_0 = 4$, where $N(v_0) = \{v_1, v_2, v_3, v_4\}$. If $\langle \overline{N}(v_0)\rangle$ is not complete, then some two vertices of $N(v_0)$ are not adjacent, say v_1 and v_2. By adding v_1v_2 to G and deleting v_0, we obtain a $(n-1,\ 3n-8)$ graph G', which by hypothesis contains two disjoint cycles. We may proceed as before to show now that G has two disjoint cycles. Assume then that $\langle \overline{N}(v_0)\rangle$ is a complete graph of order 5. If some vertex of $V(G) - \overline{N}(v_0)$ is adjacent with two or more elements of $N(v_0)$, then G contains two disjoint cycles. Hence we may suppose that no element of $V(G) - \overline{N}(v_0)$ is adjacent with more than one element of $N(v_0)$. Remove the vertices v_0, v_1, and v_2 from G, and note that the resulting graph G'' has order $n-3$ and contains at least $(3n-5)-(n-5)-9 = 2n-9$ edges. However, $n \geq 6$ implies that $2n-9 \geq n-3$, so that G'' contains at least one cycle C. The cycle C and the cycle v_0, v_1, v_2, v_0 are disjoint and belong to G.

Finally, we assume that $\deg v_0 \leq 3$. The graph $G - v_0$ is an $(n-1, q)$ graph, where $q \geq 3n-8$. Hence by the inductive hypothesis, $G - v_0$ (and therefore G) contains two disjoint cycles.

This establishes the fact that $s(p) \leq 3p-5$. To prove that $s(p) = 3p-5$, we need only observe that for each $p \geq 6$, the graph $K(1, 1, 1, p-3)$ is a $(p, 3p-6)$ graph which fails to contain two disjoint triangles. This follows because each triangle of $K(1, 1, 1, p-3)$ contains at least two of the three vertices having degree $p-1$. ■

PROBLEM SET 17.1

17.1 Use Theorem 17.2 to give a proof of Corollary 17.2a.

17.2 Extend Theorem 17.3 to find the smallest positive integer $s'(p)$ such that every $(p, s'(p))$ graph contains three disjoint cycles.

17.3 Prove that for $p \geq 4$, every $(p, 2p-2)$ graph contains a subgraph homeomorphic with K_4. Furthermore, show that the number $2p-2$ is minimum with respect to this property.

17.2 The Ramsey Numbers

In any gathering of six people, there are three mutual acquaintances or three mutual strangers. Such a conclusion does not follow, however, for groups of five (or fewer) people. There is an immediate graph-theoretic interpretation of these observations; namely, a graph of order 6 or its complement contains a triangle, while there exist graphs of order 5 for which this is not the case. The preceding statement has a natural generalization, and this brings us to probably the best known problem in extremal graph theory.

For positive integers s and t, the (*vertex*) *Ramsey number* $R(s,t)$ is the smallest positive integer n such that every graph G of order n contains s mutually adjacent vertices or t mutually nonadjacent vertices; i.e., either G contains K_s as a subgraph or $\overline{G}$ contains K_t as a subgraph. The Ramsey number is named for Frank Ramsey [7], who studied this parameter in a set theoretic framework. In accordance with our earlier remarks, we may now write $R(3,3)=6$. As further illustrations, we note that $R(1,t) = R(s,1)=1$, $R(2,t)=t$, and $R(s,2)=s$. The degree of difficulty in determining the value of other Ramsey numbers increases sharply. Before proceeding, however, we verify that the Ramsey numbers exist and, at the same time, establish an upper bound for $R(s,t)$.

Theorem 17.4 For every two positive integers s and t,

$$R(s,t) \leq \binom{s+t-2}{s-1}.$$

Proof We proceed by induction on $k=s+t$. Observe for $s=1$ and t arbitrary, or $t=1$ or 2 and s arbitrary, that equality holds. Hence the upper bound given above is correct for $k \leq 4$.

Assume for any two positive integers s' and t', where $s'+t'<k$, $k>4$, that

$$R(s',t') \leq \binom{s'+t'-2}{s'-1},$$

and let s and t be two positive integers such that $s+t=k$. We may assume also, without loss of generality, that $s \geq 3$ and

$t \geq 3$. By applying the inductive hypothesis to $(s-1)+t$ as well as to $s+(t-1)$, we obtain

$$R(s-1,\, t) \leq \binom{s+t-3}{s-2}$$

and

$$R(s,\, t-1) \leq \binom{s+t-3}{s-1}.$$

However,

$$\binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}$$

so that it is sufficient to prove that

$$R(s\,,t) \leq R(s-1,\, t) + R(s\,,t-1).$$

Let G be any graph having order $\binom{s+t-2}{s-1}$. Since $s > 2$, $\binom{s+t-2}{s-1} > t$. Thus if G is empty, G certainly contains an independent set of t vertices. Hence we may assume that G contains edges. Let $u \in V(G)$ such that $\deg u > 0$, and let $W = V(G) - \overline{N}(u)$.

Case 1. $\deg u \geq R(s-1, t)$. Then $\langle N(u) \rangle$ contains $s-1$ mutually adjacent vertices or t mutually nonadjacent vertices. If the latter occurs, then G also contains t mutually nonadjacent vertices. If this is not the case, then necessarily $\langle \overline{N}(u) \rangle$ contains K_s as a subgraph, as does G. Hence in this case G contains K_s as a subgraph or $\overline{G}$ contains K_t as a subgraph.

Case 2. $\deg u < R(s-1,\, t)$. Since

$$R(s-1,\, t) + R(s,\, t-1) \leq \binom{s+t-2}{s-1},$$

it follows that $|W| \geq R(s,\, t-1)$. If $\langle W \rangle$ contains s mutually adjacent vertices, then G does also; if not, then $\langle W \rangle$ contains $t-1$ mutually nonadjacent vertices, so that $\langle W \cup \{u\} \rangle$ contains t mutually nonadjacent vertices, as does G. In this case also, G contains K_s as a subgraph or $\overline{G}$ contains K_t as a subgraph.

The result now follows by mathematical induction. ■

As we have already noted, the bound given for $R(s\,,t)$ is exact if one of s and t is 1 or 2. The bound is also exact for $R(3\,,3)$. There are very few other pairs (s, t) for which the number $R(s, t)$ is known, although there have been other upper bounds established for the Ramsey numbers (for example, see [5]). Considerable attention has been paid to the numbers $R(3, t)$. By Theorem 17.4,

$$R(3, t) \leq \frac{t^2 + t}{2}.$$

An improved bound for $R(3, t)$ is now presented.

Theorem 17.5 For every positive integer $t \geq 2$,

$$R(3, t) \leq \frac{t^2 + 3}{2}. \tag{17.4}$$

Proof First we verify the following:

$$R(3, t) \leq R(3, t-1) + t. \tag{17.5}$$

By definition, there exists a graph G of order $R(3, t) - 1$ which contains neither a triangle nor an independent set of t vertices. Let $v \in V(G)$. Since G contains no triangles, no two vertices adjacent with v are adjacent with each other; i.e., the subgraph $\langle N(v) \rangle$ is empty. Thus $d = \deg v = |N(v)| \leq t - 1$. Define $G_0 = G - \overline{N}(v)$; the graph G_0 therefore has order $R(3, t) - d - 2 = p_0$. Since $d \leq t - 1$,

$$p_0 \geq R(3, t) - t - 1.$$

Certainly G_0 has no triangles. Also, G_0 cannot have an independent set of $t - 1$ vertices, for if it did, then this set together with the vertex v would produce an independent set of t vertices in G. Hence $R(3, t-1) > p_0$, so that the inequality (17.5) holds.

We complete the proof of (17.4) by employing induction on $t \geq 2$. For $t = 2$, $R(3, t) = 3$, while $(t^2 + 3)/2 > 3$; so that (17.4) holds if $t = 2$. Assume (17.4) holds for $t = n - 1$, $n \geq 3$, and consider $t = n$.

Assume n is odd so that $n = 2k + 1$. By (17.5), $R(3, 2k + 1) \leq R(3, 2k) + (2k + 1)$ so that, by the inductive hypothesis, $R(3, 2k + 1) \leq (4k^2 + 3)/2 + 2k + 1$. Hence,

$$R(3, 2k + 1) \leq 2k^2 + 2k + 2 = \frac{(2k + 1)^2 + 3}{2},$$

which is the desired result.

Assume now that n is even, i.e., $n=2k$. Again by (17.5), $R(3,2k) < R(3, 2k-1)+2k$. Applying the inductive hypothesis, we obtain

$$R(3,2k) \leq \frac{(2k-1)^2+3}{2} + 2k = 2k^2+2. \tag{17.6}$$

To complete the proof of the theorem, it is sufficient to show that the inequality given in (17.6) is strict. Suppose $R(3,2k)=2k^2+2$. Then there exists a graph H of order $2k^2+1$ which contains neither a triangle nor an independent set of $2k$ vertices. Necessarily, then, $\Delta(H) < 2k$. Since H has odd order, not every vertex of H can have the odd degree $2k-1$, i.e., H contains a vertex v such that $\deg v = d \leq 2k-2$. Define $H_0 = H - \overline{N}(v)$. The order p_0 of H_0 satisfies

$$p_0 = 2k^2 - d \geq \frac{(2k-1)^2+3}{2} \geq R(3,2k-1).$$

However, this implies that H_0 contains a triangle or an independent set of $2k-1$ vertices, which, in turn, implies that H contains a triangle or an independent set of $2k$ vertices, which produces a contradiction. ■

According to Theorem 17.5, $R(3,4) \leq 9$ and $R(3,5) \leq 14$. Actually, equality holds in both these cases. For example, the equality $R(3,5)=14$ follows since there exists a graph of order 13 containing neither a triangle nor an independent set of 5 vertices. The graph G of Fig. 17.1 has both these properties.

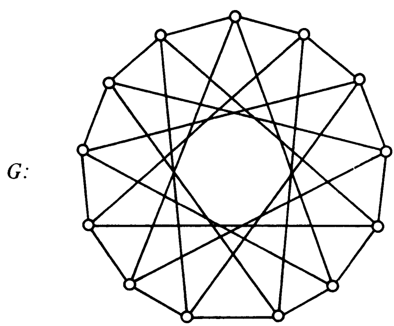

Figure 17.1 An extremal graph showing $R(3,5)=14$

PROBLEM SET 17.2

17.4 Show that $R(s, t) = R(t, s)$.

17.5 (a) Without using Theorems 17.4 and 17.5, prove that $R(3, 3) = 6$.
(b) Prove that $R(3, 4) = 9$.

17.6 For positive integers s and t, define $R^*(s, t)$ as the smallest positive integer such that every connected graph of order $R^*(s, t)$ contains s mutually adjacent vertices or t mutually nonadjacent vertices. Prove that $R^*(s, t) = R(s, t)$ if and only if neither s nor t is 2.

17.7 For $s, t \geq 3$, define $C(s, t)$ to be the least integer p such that for any graph G of order p, either G contains C_s or $\overline{G}$ contains C_t. Determine $C(s, t)$ for $3 \leq s \leq t \leq 5$.

17.3 The Edge Ramsey Numbers

There is an edge analogue of the Ramsey numbers, which we discuss in the present section. For positive integers s and t, the *edge Ramsey number* $R_1(s, t)$ is the smallest positive integer n such that every connected graph of order n contains s mutually adjacent edges or t mutually nonadjacent edges. Our first result on edge Ramsey numbers permits us to restrict our attention to trees.

Theorem 17.6 Let $R'(s, t)$ denote the smallest positive integer n such that every tree of order n contains s mutually adjacent edges or t mutually nonadjacent edges. Then $R'(s, t) = R_1(s, t)$.

Proof Certainly $R'(s, t) \leq R_1(s, t)$; thus it suffices to show that $R_1(s, t) \leq R'(s, t)$. By the definition of $R_1(s, t)$, there exists a connected graph G of order $R_1(s, t) - 1$ which has neither s mutually adjacent edges nor an independent set of t edges. Let G_0 be a spanning tree of G. Hence G_0 fails to contain either s mutually adjacent edges or an independent set of t edges. This implies that $R'(s, t) > R_1(s, t) - 1$ or, equivalently, $R_1(s, t) \leq R'(s, t)$. ■

We consider a special class of trees which prove to be relevant to the number $R_1(s, t)$. For $s > 2$ and $t > 1$, let G_i, $i = 1, 2, \ldots, t-1$, be the star graph of order s, where $V(G_i) \cap V(G_j) = \varnothing$ for $i \neq j$. For each $i = 1, 2, \ldots, t-2$, identify an end-vertex of G_i with an end-vertex of G_{i+1} so that no end-vertex is involved in more than one identification. Thus we obtain a tree T. As an illustration see Fig.

17.2 for the case where $s=6$ and $t=5$. This tree has order $(s-1)(t-1)+1$ and contains neither s mutually adjacent edges nor t mutually nonadjacent edges. Hence the tree T serves to show that

$$R_1(s,t) \geq (s-1)(t-1)+2. \tag{17.7}$$

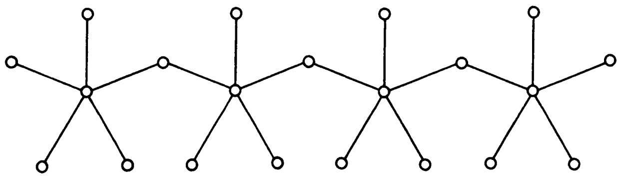

Figure 17.2 A tree which shows that $R_1(6,5) \geq 22$.

The tree of Fig. 17.2 is an element of a special class $\mathscr{B}(s,t)$ of trees which we now describe. For a fixed $s>2$, the class $\mathscr{B}(s,2)$ consists of the single tree $K(1,s-1)$. Having constructed $\mathscr{B}(s,t-1)$, we define $\mathscr{B}(s,t)$ as the set of all trees T obtained by identifying a vertex of degree less than $s-1$ of some element of $\mathscr{B}(s,t-1)$ with an end-vertex of $K(1,s-1)$. Hence the class $\mathscr{B}(s,t)$ is defined for all $s>2$ and $t>1$. Each tree of $\mathscr{B}(s,t)$ has order $(s-1)(t-1)+1$. The class $\mathscr{B}(6,5)$ is shown in Fig. 17.3.

It is not difficult to convince oneself that each tree in $\mathscr{B}(s,t)$ has neither s mutually adjacent edges nor t mutually nonadjacent edges. Of more importance, however, is the fact that among the trees of order $(s-1)(t-1)+1$ (or more) only the elements of $\mathscr{B}(s,t)$ have neither s mutually adjacent edges nor t mutually nonadjacent edges. We verify this now.

Theorem 17.7 If T is a tree of order p, where

$$p \geq (s-1)(t-1)+1, \qquad s>2, t>1,$$

such that T has neither s mutually adjacent edges nor t mutually nonadjacent edges, then T is a member of $\mathscr{B}(s,t)$.

Proof We use induction on t, considering the value $t=2$ first. If T is a tree of order s (or more) containing neither a vertex whose degree exceeds $s-1$ nor two independent edges, then $T=K(1,s-1)$, which is the only member of $\mathscr{B}(s,2)$.

Assume the theorem to be true for $t=k-1$, $k\geq 3$, and let T

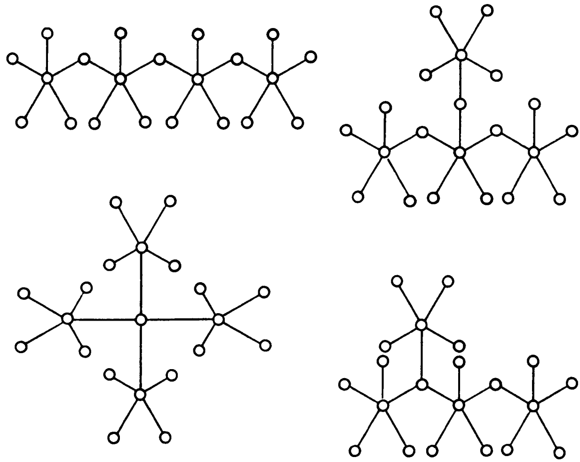

Figure 17.3 The elements of $\mathscr{B}(6,5)$.

be a tree of order p, $p \geq (s-1)(k-1)+1$, having neither s mutually adjacent edges nor k mutually nonadjacent edges. Let v be a vertex of T which is not an end-vertex but which is adjacent with at most one vertex w which is not an end-vertex. (See Exercise 5.2.) Since $\deg v \leq s-1$, the removal of v and the end-vertices adjacent with it results in a tree T_0 of order p_0, where $p_0 \geq p-s+1$. Because $p \geq (s-1)(k-1)+1$, it follows that $p_0 \geq (s-1)(k-2)+1$. Certainly, T_0 contains no vertex of degree s or more. Also T_0 does not contain an independent set of $k-1$ edges, for if it did, then T would contain k mutually nonadjacent edges, which is contradictory. Hence T_0 is a member of $\mathscr{B}(s,k-1)$ by the inductive hypothesis so that $p_0 = (s-1)(k-2)+1$. Therefore,

$$(s-1)(k-2)+1 \geq p-s+1$$

implying that $(s-1)(k-1)+1 \geq p$. However, $p \geq (s-1)(k-1)+1$ implies $p=(s-1)(k-1)+1$. Thus $\deg v = s-1$. Therefore T satisfies the definition of an element of $\mathscr{B}(s,k)$. ■

We are now in a position to give a complete solution [1] to the problem of determining the edge Ramsey numbers.

Corollary 17.7a For $s > 2$ and $t > 1$,

$$R_1(s,t) = (s-1)(t-1) + 2.$$

Proof By (17.7), $R_1(s,t) \geq (s-1)(t-1)+2$. Let T be a tree of order $(s-1)(t-1)+2$. By Theorem 17.7, T contains s mutually adjacent edges or t mutually nonadjacent edges. Hence $R_1(s,t) \leq (s-1)(t-1)+2$, giving the desired result. ■

PROBLEM SET 17.3

17.8 Show that $R_1(s,t)$ is not defined if the word "connected" is deleted from the definition.

17.9 Determine $R_1(s,t)$ for (a) $s = 1$, (b) $s = 2$, and (c) $t = 1$.

17.10 Under what conditions does $R_1(s,t) = R_1(t,s)$?

17.11 Determine the elements of $\mathscr{B}(5,7)$.

17.12 Define $R^*_1(s,t)$ to be the smallest positive integer such that every graph of order $R^*_1(s,t)$ having no isolated vertices contains s mutually adjacent edges or t mutually nonadjacent edges. Derive a formula for $R^*_1(s,t)$.

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18

Enumeration of Graphs

In this concluding chapter, we present a brief discussion of enumeration, a subject which most properly belongs to the field of combinatorial mathematics. However, its applications to graph counting problems are so numerous that even an introductory course in graph theory must contain the basic elements of this topic. In the first section we provide the background material needed for a proof of an important theorem due to Pólya. The applications of this theorem to graphical enumeration are illustrated in the second section.

18.1 Pólya's Theorem

Let A be a group of permutations on a finite set D, where $|D| = n \geq 1$. (There is no loss in generality to assume $D = \{1, 2, \ldots, n\}$.) It is well-known that every permutation $\alpha \in A$ can be expressed uniquely (except for order) as a product of disjoint (permutation) cycles. Let $j_k(\alpha)$ denote the number of cycles in α of length k. The following equation therefore holds:

$$\sum_{k=1}^{n} k \cdot j_k(\alpha) = n.$$

Let $x_1, x_2, \ldots, x_n$ be indeterminates. The *cycle index* $Z(A;\, x_1, x_2, \ldots, x_n)$ of the permutation group A in $x_1, x_2, \ldots, x_n$ is defined by

$$Z(A;\ x_1, x_2, \ldots, x_n) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{i=1}^{n} x_i^{j_i(\alpha)}.$$

For a function $f(x)$ (ordinarily a polynomial or rational function) in an indeterminate x with integer coefficients, we define $Z(A;\, f(x))$ by

$$\begin{aligned} Z(A; f(x)) &= Z(A; f(x), f(x^2), \ldots, f(x^n)) \\ &= \frac{1}{|A|} \sum_{\alpha \in A} \prod_{i=1}^{n} [f(x^i)]^{j_i(\alpha)}. \end{aligned} \tag{18.1}$$

We illustrate the concepts introduced thus far with an example.

Example 18.1 Let A be the Klein 4-group considered as a permutation group on $D = \{1, 2, 3, 4\}$. Thus $n = 4$. Let $A = \{\alpha, \beta, \gamma, \delta\}$, where

$$\begin{aligned} \alpha &= (1)(2)(3)(4), \\ \beta &= (12)(34), \\ \gamma &= (13)(24), \text{ and} \\ \delta &= (14)(23). \end{aligned}$$

The cycle index of this group A is $Z(A;\ x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1^4 + 3x_2^2)$. Suppose $f(x) = 1 + x$. Then

$$\begin{aligned} Z(A;\ f(x)) &= \tfrac{1}{4}((1 + x)^4 + 3(1 + x^2)^2) \\ &= 1 + x + 3x^2 + x^3 + x^4. \end{aligned}$$

At this point a few group-theoretic concepts are needed. Again we let A denote a group of permutations on a finite nonempty set D. If for $x, y \in D$, there exists $\alpha \in A$ such that $\alpha x = y$, then we say x *is related to* y and write $x \sim y$. The relation $\sim$ is an equivalence relation on D, the resulting equivalence classes being referred to as the *orbits* of A. Of course, the orbits of A partition the set D. The orbit containing an element $d \in D$ is denoted by $O(d)$. We note that $O(y) = O(z)$ if and only if y and z belong to the same orbit of A.

The *stabilizer* A_d of an element $d \in D$ is the subgroup of A which consists of all permutations α in A for which $\alpha d = d$. We now verify the identity

$$|A_d|\,|O(d)| = |A| \text{ for each } d \in D. \tag{18.2}$$

In order to prove (18.2), it suffices to show that the number of left cosets of A_d in A is $|O(d)|$. Denote the set of all left cosets by L. We define a mapping ϕ from $O(d)$ into L by $\phi y = \alpha A_d$ for $y \in O(d)$, where $\alpha \in A$ has the property that $\alpha d = y$. It must first be justified that ϕ is indeed a mapping. Suppose $\phi y = \alpha A_d$ and $\phi y = \beta A_d$; we show $\alpha A_d = \beta A_d$. It is equivalent to show $\alpha^{-1}\beta \in A_d$. Since $\alpha d = y$ and $\beta d = y$, $\alpha^{-1}\beta d = d$ implying that $\alpha^{-1}\beta \in A_d$.

To show ϕ is onto L, let $\beta A_d \in L$. Suppose $\beta d = z$; then $\phi z = \beta A_d$, proving that ϕ is onto L. Suppose next that $\phi y = \phi z$, where $\phi y = \alpha A_d$ and $\phi z = \beta A_d$. Therefore $\alpha A_d = \beta A_d$, implying that $\alpha^{-1}\beta \in A_d$. Thus $\alpha^{-1}\beta d = d$ so that $\alpha d = \beta d$ and $y = z$. Hence ϕ is a 1–1 mapping from $O(d)$ onto L, producing $|O(d)| = |A|/|A_d|$.

By a *weight function on the set D* we shall mean a mapping ω from D into a ring which contains the rational numbers such that $\omega(d) = \omega(\alpha d)$ for all $d \in D$ and $\alpha \in A$; the number $\omega(d)$ is called the *weight* of d. (Ordinarily, the range of ω will be a subset of N_0, the set of nonnegative integers.) The *weight* $\omega(O)$ *of an orbit* O of A is defined by $\omega(O) = \omega(d)$, where $d \in O$. If no permutation group is prescribed, it is assumed to be the identity group. The following theorem is known as the weighted form of Burnside's Theorem [**1**, p. 191].

Theorem 18.1 If A is a group of permutations on a finite set D such that A has the orbits $O_1, O_2, \ldots, O_m$, and ω is a weight function on D, then

$$\sum_{i=1}^{m} \omega(O_i) = \frac{1}{|A|} \sum_{\alpha \in A} \sum_{\{d \in D | d = \alpha d\}} \omega(d).$$

Proof For each i, $1 \le i \le m$,

$$|O_i| \cdot \omega(O_i) = \sum_{d \in O_i} \omega(d).$$

Let $x, y \in O_i$. We show that $|A_x| = |A_y|$. By (18.2),

$$|A_x|\,|O(x)| = |A| \qquad \text{and} \qquad |A_y|\,|O(y)| = |A|,$$

so that $|A_x|\,|O(x)| = |A_y|\,|O(y)|$. Since $|O(x)| = |O(y)|$, $|A_x| = |A_y|$.

We now conclude that

$$\sum_{d \in O_i} |A_d|\omega(d) = |A_d| \sum_{d \in O_i} \omega(d)$$
$$= |A_d|\,|O_i| \cdot \omega(O_i) = |A| \cdot \omega(O_i),$$

the last equality again following by (18.2). Thus

$$\omega(O_i) = \frac{1}{|A|} \sum_{d \in O_i} |A_d|\omega(d).$$

Summing over all orbits of A, we obtain

$$\sum_{i=1}^{m} \omega(O_i) = \frac{1}{|A|} \sum_{i=1}^{m} \sum_{d \in O_i} |A_d|\omega(d) = \frac{1}{|A|} \sum_{d \in D} |A_d|\omega(d).$$

For an element $d_1 \in D$, the term $|A_{d_1}|\omega(d_1)$ is contributed to the sum $\sum_{d \in D} |A_d|\omega(d)$. Now A_{d_1} consists of all permutations $\alpha_1 \in A$ such that $d_1 = \alpha_1 d_1$. Hence $\omega(d_1)$ is summed as many times in $|A_{d_1}|\omega(d_1)$ as there are $\alpha_1 \in A$ which fix d_1. Therefore, in general,

$$\sum_{d \in D} |A_d|\omega(d) = \sum_{\alpha \in A} \sum_{\{d \in D \mid d = \alpha d\}} \omega(d),$$

so that

$$\frac{1}{|A|} \sum_{d \in D} |A_d|\omega(d) = \frac{1}{|A|} \sum_{\alpha \in A} \sum_{\{d \in D \mid d = \alpha d\}} \omega(d). \blacksquare$$

Let D and R be nonempty finite sets, referred to as the *domain* and *range*, respectively, and let R^D denote the set of all functions from D to R. Let A be a group of permutations on D. For each $\alpha \in A$, we define a mapping $\alpha^*: R^D \to R^D$ by $\alpha^*\phi(d) = \phi(\alpha d)$ for $d \in D$. Furthermore, let $A^* = \{\alpha^* \mid \alpha \in A\}$. It is not difficult to verify that A^* is a permutation group. (See Exercise 18.3(c).) For ϕ, $\psi \in R^D$, the function ϕ is said to be related to ψ if there exists $\alpha^* \in A^*$ such that $\alpha^*\phi = \psi$. This relation is an equivalence relation on R^D. As before, we refer to the equivalence classes of R^D as the *orbits* of A^*.

We now define a weight function on R^D. Let ω be a weight function on R. Then ω induces a function from R^D to $\{x^i \mid i \in N_0\}$, which we also denote by ω and define as follows: $\omega(\phi) = x^i$, where $i = \sum_{d \in D} \omega(\phi d)$ for $\phi \in R^D$. The function ω so defined is referred to as a *weight function* on R^D; in general, it will be clear by context as to whether ω denotes the weight of an element of D or the weight of a function in R^D. Returning to the equivalence relation just described, we now show that equivalent functions have the same weight. Suppose ϕ is related to ψ so that $\alpha^*\phi = \psi$ for some $\alpha^* \in A^*$.

The weight of ϕ is given by

$$\omega(\phi) = x^{\sum_{d\in D} \omega(\phi(d))} = x^{\sum_{d\in D} \omega(\phi(\alpha d))}, \tag{18.3}$$

the last equality in (18.3) following because α is a permutation on D. According to the definition of $\alpha^*\phi$, we have

$$\omega(\phi) = x^{\sum_{d\in D} \omega(\alpha^*\phi(d))} = x^{\sum_{d\in D} \omega(\psi(d))} = \omega(\psi).$$

This allows us to define *the weight of an orbit* of A^* as the weight of any element in the orbit. In a manner completely analogous to that used in establishing Theorem 18.1, one can now verify the following theorem.

Theorem 18.2 Let A be a permutation group acting on a domain set D and let R be a range set, with the group A^* having orbits $O_1, O_2, \ldots, O_m$. If ω denotes a weight function on R, then

$$\sum_{i=1}^{m} \omega(O_i) = \frac{1}{|A^*|} \sum_{\alpha^*\in A^*} \sum_{\{\phi\in R^D|\phi=\alpha^*\phi\}} \omega(\phi).$$

At this point we introduce two important power series in an indeterminate x, which we denote by $c(x)$ and $C(x)$. If c_i is the number of elements of R of weight i, then we define

$$c(x) = \sum_{i=0}^{+\infty} c_i x^i. \tag{18.4}$$

If C_i is the number of orbits of A^* of weight x^i, then we define

$$C(x) = \sum_{i=0}^{+\infty} C_i x^i. \tag{18.5}$$

Before proceeding further, we consider an example.

Example 18.2 Let the set $D = \{1,2,3,4\}$ and group $A = \{\alpha,\beta,\gamma,\delta\}$ be as given in Example 18.1, and let the range $R = \{a,b\}$. We now define a weight function ω on R as follows:

$$\omega(a) = 1, \quad \omega(b) = 0.$$

The set R^D consists of the 16 elements φ_i, $0 \le i \le 15$, given below.

$$\begin{aligned}
\varphi_0 &= \{(1,a),(2,a),(3,a),(4,a)\}\\
\varphi_1 &= \{(1,a),(2,a),(3,a),(4,b)\}\\
\varphi_2 &= \{(1,a),(2,a),(3,b),(4,a)\}\\
\varphi_3 &= \{(1,a),(2,a),(3,b),(4,b)\}\\
\varphi_4 &= \{(1,a),(2,b),(3,a),(4,a)\}\\
\varphi_5 &= \{(1,a),(2,b),(3,a),(4,b)\}\\
\varphi_6 &= \{(1,a),(2,b),(3,b),(4,a)\}\\
\varphi_7 &= \{(1,a),(2,b),(3,b),(4,b)\}\\
\varphi_8 &= \{(1,b),(2,a),(3,a),(4,a)\}\\
\varphi_9 &= \{(1,b),(2,a),(3,a),(4,b)\}\\
\varphi_{10} &= \{(1,b),(2,a),(3,b),(4,a)\}\\
\varphi_{11} &= \{(1,b),(2,a),(3,b),(4,b)\}\\
\varphi_{12} &= \{(1,b),(2,b),(3,a),(4,a)\}\\
\varphi_{13} &= \{(1,b),(2,b),(3,a),(4,b)\}\\
\varphi_{14} &= \{(1,b),(2,b),(3,b),(4,a)\}\\
\varphi_{15} &= \{(1,b),(2,b),(3,b),(4,b)\}.
\end{aligned}$$

The group A^* has the four elements α^*, β^*, γ^*, and δ^*. There is one orbit of A^* consisting solely of the function φ_0. The orbit containing φ_1 also contains φ_2, φ_4, and φ_8 since $\alpha^*\varphi_1 = \varphi_1$, $\beta^*\varphi_1 = \varphi_2$, $\gamma^*\varphi_1 = \varphi_4$, $\delta^*\varphi_1 = \varphi_8$. There are seven orbits of A^* in all, namely:

$$\begin{aligned}
O_1 &= \{\varphi_0\},\\
O_2 &= \{\varphi_1, \varphi_2, \varphi_4, \varphi_8\},\\
O_3 &= \{\varphi_3, \varphi_{12}\},\\
O_4 &= \{\varphi_5, \varphi_{10}\},\\
O_5 &= \{\varphi_6, \varphi_9\},\\
O_6 &= \{\varphi_7, \varphi_{11}, \varphi_{13}, \varphi_{14}\},\\
O_7 &= \{\varphi_{15}\}.
\end{aligned}$$

The weight of each orbit (and accordingly the weight of any function within an orbit) can now be given. For example,

$$\omega(O_1) = x^{\sum_{d \in D} \omega(\varphi_0(d))} = x^{4\omega(a)} = x^4.$$

The weights of all seven orbits are shown below:

$$\begin{aligned}
\omega(O_1) &= x^4,\\
\omega(O_2) &= x^3,\\
\omega(O_3) &= \omega(O_4) = \omega(O_5) = x^2,\\
\omega(O_6) &= x,\\
\omega(O_7) &= 1.
\end{aligned}$$

For this example, (18.4) becomes $c(x) = 1 + x$. Thus, by Example 18.1,

$$Z(A;\ c(x)) = 1 + x + 3x^2 + x^3 + x^4.$$

Hence we observe here that $C(x) = Z(A;\ c(x))$. This is no coincidence; indeed, this example serves to illustrate the following theorem of Pólya [3]. We follow the proof employed in [2, p. 24].

Theorem 18.3 (Pólya's Theorem) For any group A of permutations on a finite nonempty set,

$$C(x) = Z(A;\ c(x)).$$

Proof We first establish some notation. Let D and R denote domain set and range set, respectively, where A acts on D. Let ω denote a weight function on R as well as the induced weight function on R^D. By (18.5),

$$C(x) = \sum_{i=0}^{+\infty} C_i x^i,$$

where C_i is the number of orbits of the group A^* having weight x^i. Thus,

$$\sum_{i=0}^{+\infty} C_i x^i = \sum_{i=1}^{m} \omega(O_i),$$

where $O_1, O_2, \ldots, O_m$ are the orbits of A^*. By Theorem 18.2, however, we have the following:

$$C(x) = \sum_{i=1}^{m} \omega(O_i) = \frac{1}{|A^*|} \sum_{\alpha^* \in A^*} \sum_{\{\phi \in R^D | \phi = \alpha^* \phi\}} \omega(\phi).$$

By (18.1),

$$Z(A;\ c(x)) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{i=1}^{n} [c(x^i)]^{j_i(\alpha)},$$

where $n = |D|$. Since $|A| = |A^*|$, it suffices to prove that

$$\sum_{\{\phi \in R^D | \phi = \alpha^* \phi\}} \omega(\phi) = \prod_{i=1}^{n} [c(x^i)]^{j_i(\alpha)}.$$

Let $\alpha \in A$ so that α^* is the corresponding permutation in A^*. If $\phi = \alpha^* \phi$, where $\phi \in R^D$, then we know that $\phi(d) = \phi(\alpha d)$ for all $d \in D$. Let z be a cycle of length h in the unique disjoint cycle decomposition of α. Then for each d in the representation of z, we have $\phi(d) = \phi(zd)$. Thus all h elements of D permuted by z have the same image under ϕ. Conversely, suppose $\phi(d_1) = \phi(d_2)$ for every two elements $d_1, d_2 \in D$ belonging to the same permutation-cycle of α; then $\phi = \alpha^*\phi$. Thus all functions in R^D which are fixed by α^* can be obtained by selecting an element $r \in R$ for each cycle z of α and setting $\phi(d) = r$ for each $d \in D$ in the representation of z. The cycle z therefore produces the factor

$$\sum_{r \in R} (x^{\omega(r)})^h$$

in the sum

$$\sum_{\{\phi \in R^D | \phi = \alpha^* \phi\}} \omega(\phi).$$

However,

$$\sum_{r \in R} (x^{\omega(r)})^h = c(x^h),$$

so that

$$\sum_{\{\phi \in R^D | \phi = \alpha^* \phi\}} \omega(\phi) = \prod_{i=1}^{n} [c(x^i)]^{j_i(\alpha)}. \blacksquare$$

We conclude this section with an example to illustrate the concepts and results considered thus far in the chapter.

Example 18.3 Let $D = \{1, 2, 3\}$ and $A = \{I = (1)(2)(3), (12) = (12)(3)\}$. Then $j_1(I) = 3$ and $j_i(I) = 0$ for all $i \neq 1$; $j_1((12)) = 1$, $j_2((12)) = 1$, and $j_i((12)) = 0$ for all $i \neq 1, 2$. Hence $Z(A; x_1, x_2, x_3) = \frac{1}{2}(x_1^{\,3} + x_1 x_2)$ and

$$Z(A; f(x)) = \tfrac{1}{2}([f(x)]^3 + f(x) \cdot f(x^2)).$$

If, for example, $f(x) = x^2 + x^7$, then

$$\begin{aligned} Z(A; x^2 + x^7) &= \tfrac{1}{2}[(x^2 + x^7)^3 + (x^2 + x^7)(x^4 + x^{14})] \\ &= x^6 + 2x^{11} + 2x^{16} + x^{21}. \end{aligned}$$

Now $O(1) = O(2) = \{1, 2\}$ and $O(3) = \{3\}$, while $A_1 = A_2 = \{I\}$ and $A_3 = \{1, (12)\}$.

Next we illustrate Theorem 18.1. Thus we define a weight function ω on D and hence on the orbits of D. Let $\omega(1) = 3$, $\omega(2) = 3$, and $\omega(3) = 5$. Then $\omega(O(1)) = \omega(O(2)) = 3$ and $\omega(O(3)) = 5$.

By Theorem 18.1,

$$\sum_{i=1}^{m} \omega(O_i) = \frac{1}{|A|} \sum_{\alpha\in A} \sum_{\{d\in D|d=\alpha d\}} \omega(d), \tag{18.6}$$

where $O_1, O_2, \ldots, O_m$ are the orbits of D. In this example, there are two (distinct) orbits, namely $O(1)$ and $O(3)$. Hence the left-hand side of (18.6) is $\omega(O(1)) + \omega(O(3)) = 3 + 5 = 8$.

Next we consider the right-hand side of (18.6). Here we have

$$\begin{aligned}\sum_{\alpha\in A} \sum_{\{d\in D|d=\alpha d\}} \omega(d) &= \sum_{\{d\in D|d=Id\}} \omega(d) + \sum_{\{d\in D|d=(12)d\}} \omega(d)\\ &= [\omega(1) + \omega(2) + \omega(3)] + [\omega(3)] = 16.\end{aligned}$$

Since $\frac{1}{|A|}(16) = \frac{1}{2}(16) = 8$, we have illustrated Theorem 18.1.

We next consider all functions from D to R, where $R = \{a, b\}$. We denote these eight functions by φ_i, $i = 1, 2, \ldots, 8$, which we define as follows:

	$\varphi_i(1)$	$\varphi_i(2)$	$\varphi_i(3)$
φ_1	a	a	a
φ_2	a	a	b
φ_3	a	b	a
φ_4	a	b	b
φ_5	b	a	a
φ_6	b	a	b
φ_7	b	b	a
φ_8	b	b	b

Also $A^* = \{I^*, (12)^*\}$, where

$$I^* = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 & \varphi_7 & \varphi_8 \\ \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 & \varphi_7 & \varphi_8 \end{pmatrix}$$

and

$$(12)^* = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 & \varphi_7 & \varphi_8 \\ \varphi_1 & \varphi_2 & \varphi_5 & \varphi_6 & \varphi_3 & \varphi_4 & \varphi_7 & \varphi_8 \end{pmatrix} = (\varphi_3\varphi_5)(\varphi_4\varphi_6).$$

Then A^* acting on R^D yields the following orbits:

$$\begin{aligned} O_1 &= \{\varphi_1\}, \\ O_2 &= \{\varphi_2\}, \\ O_3 &= \{\varphi_3, \varphi_5\}, \\ O_4 &= \{\varphi_4, \varphi_6\}, \\ O_5 &= \{\varphi_7\}, \\ O_6 &= \{\varphi_8\}. \end{aligned}$$

A weight function ω is defined on R by setting $\omega(a) = 2$ and $\omega(b) = 7$. This induces a weight function on R^D as follows:

$$\begin{aligned} &\omega(\varphi_1) = x^6, \\ &\omega(\varphi_2) = \omega(\varphi_3) = \omega(\varphi_5) = x^{11}, \\ &\omega(\varphi_4) = \omega(\varphi_6) = \omega(\varphi_7) = x^{16}, \\ &\omega(\varphi_8) = x^{21}. \end{aligned}$$

Next we illustrate Theorem 18.2 by verifying that:

$$\sum_{i=1}^{m} \omega(O_i) = \frac{1}{|A^*|} \sum_{\alpha^* \in A^*} \sum_{\{\phi \in R^D | \phi = \alpha^* \phi\}} \omega(\phi). \tag{18.7}$$

The left-hand side of (18.7) is $\omega(O_1) + \omega(O_2) + \cdots + \omega(O_6) = x^6 + x^{11} + x^{11} + x^{16} + x^{16} + x^{21} = x^6 + 2x^{11} + 2x^{16} + x^{21}$. In order to calculate the right-hand side of (18.7), we note that

$$\sum_{\alpha^* \in A^*} \sum_{\{\phi \in R^D | \phi = \alpha^* \phi\}} \omega(\phi) = \sum_{\{\phi \in R^D | \phi = I^* \phi\}} \omega(\phi) + \sum_{\{\phi \in R^D | \phi = (12)^* \phi\}} \omega(\phi)$$

$$\begin{aligned} &= [\omega(\varphi_1) + \omega(\varphi_2) + \cdots + \omega(\varphi_8)] + [\omega(\varphi_1) + \omega(\varphi_2) + \omega(\varphi_7) + \omega(\varphi_8)] \\ &= 2[\omega(\varphi_1) + \omega(\varphi_2) + \omega(\varphi_7) + \omega(\varphi_8)] + [\omega(\varphi_3) + \omega(\varphi_4) + \omega(\varphi_5) + \omega(\varphi_6)] \\ &= 2[x^6 + x^{11} + x^{16} + x^{21}] + [x^{11} + x^{16} + x^{11} + x^{16}] \\ &= 2x^6 + 4x^{11} + 4x^{16} + 2x^{21}. \end{aligned}$$

Since $|A^*| = 2$, the right-hand side of (18.7) also yields $x^6 + 2x^{11} + 2x^{16} + x^{21}$.

The series $c(x)$ and $C(x)$ are now considered. Here we have that $c_2 = 1$ and $c_7 = 1$ and $c_i = 0$, $i \neq 2, 7$ so that $c(x) = x^2 + x^7$. Similarly, $C_6 = 1$, $C_{11} = 2$, $C_{16} = 2$, $C_{21} = 1$, and $C_i = 0$, if $i \neq 6, 11, 16, 21$. Hence $C(x) = x^6 + 2x^{11} + 2x^{16} + x^{21}$.

Finally we illustrate Pólya's Theorem. Since $Z(A; c(x)) = Z(A; x^2 + x^7)$ and we have already seen that $Z(A; x^2 + x^7) = x^6 + 2x^{11} + 2x^{16} + c^{21}$, it follows that $Z(A; c(x)) = C(x)$.

PROBLEM SET 18.1

18.1 Define all terms as in Examples 18.1 and 18.2, except let A be the cyclic group of order 4; i.e., let $A = \{\alpha_1, \beta_1, \gamma_1, \delta_1\}$, where $\alpha_1 = (1)(2)(3)(4)$, $\beta_1 = (1\,2\,3\,4)$, $\gamma_1 = (13)(24)$, and $\delta_1 = (1\,4\,3\,2)$, Find all those quantities which are determined in these examples.

18.2 Prove Theorem 18.2.

18.3 Let A and B be permutation groups acting on finite nonempty sets D and R, respectively. Define the *power group*

$$B^A = \{(\alpha;\, \beta) | \alpha \in A,\, \beta \in B\}$$

as follows: for each $\phi \in R^D$, $(\alpha;\, \beta)(\phi) = \beta(\phi\alpha)$; i.e., for each $d \in D$, $(\alpha;\, \beta)(\phi(d)) = \beta\phi(\alpha d)$.

(a) Prove that the power group B^A is indeed a permutation group on R^D.

(b) Determine $|B^A|$ and $|R^D|$ in terms of $|A|$, $|B|$, $|D|$, and $|R|$.

(c) Prove that the group A^* (defined in the text) is isomorphic to E^A, where E is the identity permutation group on R.

18.2 Applications of Pólya's Theorem to Graphical Enumeration

For a finite set D, with $|D| = n \geq 2$, we define $D^{(2)}$ to be the set of all 2-element subsets of D; i.e.,

$$D^{(2)} = \{\{i, j\} \mid i, j \in D, \quad i \neq j\}.$$

If A is a group of permutations on D, then by the *pair group* $A^{(2)}$ we mean the permutation group induced by A and acting on $D^{(2)}$ which is defined according to the following rule. For each $\alpha \in A$, there exists $\alpha' \in A^{(2)}$ such that $\alpha'\{i, j\} = \{\alpha i, \alpha j\}$. It is not difficult to verify that if A is not the symmetric group S_2 on two objects, then the groups A and $A^{(2)}$ are isomorphic.

We consider an example.

Example 18.4 Let D and A be as given in Example 18.1. Then

$$D^{(2)} = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}.$$

The pair group $A^{(2)} = \{\alpha', \beta', \gamma', \delta'\}$, where

$$\begin{aligned}
\alpha' &= (\{1,2\})(\{1,3\})(\{1,4\})(\{2,3\})(\{2,4\})(\{3,4\}),\\
\beta' &= (\{1,2\})(\{3,4\})(\{1,3\}\{2,4\})(\{1,4\}\{2,3\}),\\
\gamma' &= (\{1,2\}\{3,4\})(\{1,3\})(\{1,4\}\{2,3\})(\{2,4\}),\\
\delta' &= (\{1,2\}\{3,4\})(\{1,3\}\{2,4\})(\{1,4\})(\{2,3\}).
\end{aligned}$$

For positive integers p and q, let g_{pq} denote the number of nonisomorphic (p, q) graphs. Then for a fixed positive integer p, the *counting polynomial $g_p(x)$ for graphs* of order p is the following polynomial in the indeterminate x:

$$g_p(x) = \sum_{q=0}^{\binom{p}{2}} g_{pq} x^q.$$

For example, $g_3(x) = 1 + x + x^2 + x^3$ while $g_4(x) = 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6$.

We now employ Pólya's Theorem to establish a formula for $g_p(x)$ for an arbitrary positive integer p.

Theorem 18.4 For every positive integer p, the counting polynomial for graphs of order p is

$$g_p(x) = Z(S_p^{(2)};\ 1 + x).$$

Proof Let $D = \{v_1, v_2, \ldots, v_p\}$ and $R = \{0, 1\}$. Then $D^{(2)} = \{\{v_i, v_j\} | v_i, v_j \in D,\ i \neq j\}$. For each function $\phi' \in R^{D^{(2)}}$ there corresponds a graph $G_{\phi'}$ with $V(G_{\phi'}) = D$ such that $v_i v_j \in E(G_{\phi'})$ if and only if $\phi'(\{v_i, v_j\}) = 1$. Conversely, for every graph G with $V(G) = D$ there exists a function $\phi' \in R^{D^{(2)}}$ such that $G_{\phi'} = G$.

Next we define a weight function ω on R by $\omega(i) = i$; $i = 0, 1$. Hence $c(x) = 1 + x$. For $\phi' \in R^{D^{(2)}}$,

$$\omega(\phi') = x^{\Sigma\omega(\phi'(\{v_i, v_j\}))},$$

where the sum is taken over all elements $\{v_i, v_j\}$ of $D^{(2)}$. Thus the number $\Sigma\omega(\phi'(\{v_i, v_j\}))$ is the number of edges in the graph $G_{\phi'}$.

Now we consider the groups $S_p^{(2)}$ and $(S_p^{(2)})^*$ acting on $D^{(2)}$ and $R^{D^{(2)}}$, respectively. By Pólya's Theorem, $Z(S_p^{(2)};\ 1 + x) = C(x)$. By definition, however,

$$C(x) = \sum_{i=0}^{+\infty} C_i x^i,$$

where C_i is the number of orbits of the group $(S_p^{(2)})^*$ having weight x^i. We consider next the number C_q, for $0 \leq q \leq \binom{p}{2}$. Let $\phi', \psi' \in R^{D^{(2)}}$ such that $|E(G_{\phi'})| = |E(G_{\psi'})| = q$. If ϕ' and ψ'

belong to the same orbit of $(S_p^{(2)})^*$, then there exists $(\alpha')^* \in (S_p^{(2)})^*$ such that $(\alpha')^*\phi' = \psi'$. Hence for each $\{v_i, v_j\} \in D^{(2)}$,

$$\psi'(\{v_i, v_j\}) = (\alpha')^*\phi'(\{v_i, v_j\}) = \phi'(\alpha'\{v_i, v_j\}) = \phi'(\{\alpha v_i, \alpha v_j\}).$$

Therefore v_i is adjacent to v_j if and only if αv_i is adjacent to αv_j. This implies that $\alpha \in S_p$ is an isomorphism between $G_{\phi'}$ and $G_{\psi'}$. On the other hand, if ϕ' and ψ' belong to different orbits of $(S_p^{(2)})^*$, then for every $\alpha \in S_p$, we have $(\alpha')^*\phi' \neq \psi'$ implying that $G_{\phi'} \neq G_{\psi'}$. Hence the number of orbits of $(S_p^{(2)})^*$ having weight x^q is the number of non-isomorphic (p, q) graphs; i.e., $C_q = g_{pq}$. Therefore,

$$C(x) = g_p(x),$$

completing the proof. ■

Next for a positive integer p, we define the *counting series* $m_p(x)$ *for multigraphs* of order p as

$$m_p(x) = \sum_{q=0}^{+\infty} m_{pq} x^q,$$

where m_{pq} is the number of non-isomorphic (p, q) multigraphs. If, in the proof of Theorem 18.4, we let $R = N_0$ (the set of nonnegative integers) and define $\omega(i) = i$ for all $i \in N_0$, we establish the following result.

Theorem 18.5 The counting series for multigraphs of order p is

$$m_p(x) = Z\left(S_p^{(2)}; \frac{1}{1-x}\right).$$

We illustrate the preceding two theorems with an example.

Example 18.5 The cycle index of $S_3^{(2)}$ in the indeterminates x_1, x_2, and x_3 is $Z(S_3^{(2)}; x_1, x_2, x_3) = \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3)$. This implies that $Z(S_3^{(2)}; 1 + x) = g_3(x) = 1 + x + x^2 + x^3$. Thus there is one $(3, i)$ graph for each $i = 0, 1, 2, 3$ and no $(3, i)$ graph for $i > 3$. Also,

$$Z\left(S_3^{(2)}; \frac{1}{1-x}\right) = m_3(x) = \frac{1}{1 - x - x^2 + x^4 + x^5 - x^6}.$$

By direct calculation, one sees that $m_3(x) = 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + \cdots$. Thus, for example, there are seven non-isomorphic $(3, 6)$ multigraphs. These are shown in Fig. 18.1.

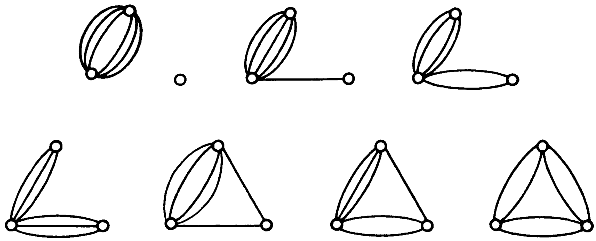

Figure 18.1 The (3, 6) multigraphs

We have thus developed formulas for the number of non-isomorphic graphs and multigraphs of order p in terms of the cycle index of the pair group $S_p^{(2)}$. Because the determination of the cycle index $Z(S_p^{(2)};\ x_1, x_2, \ldots, x_p)$ entails the calculations of quantities not easily calculated, there may be some question as to whether Theorem 18.4 constitutes a solution to the problem of enumerating graphs. However, in a certain sense it does, for it is possible to express this cycle index in an alternative manner. We conclude by giving this expression. Some additional notation is needed first.

By a partition of the positive integer p we mean a summation Σp_i, p_i a positive integer, whose value is p (and in which the order of the summands is of no consequence). For example, the 11 partitions of $p = 6$ are:

$1 + 1 + 1 + 1 + 1 + 1$,
$1 + 1 + 1 + 1 + 2$,
$1 + 1 + 2 + 2$,
$1 + 1 + 1 + 3$,
$1 + 2 + 3$,
$2 + 2 + 2$,
$3 + 3$,
$1 + 1 + 4$,
$2 + 4$,
$1 + 5$,
6.

In a given partition of p, we denote by j_i the number of summands of the partition having value i. Obviously, $\Sigma i \cdot j_i = p$ for any partition of p. Finally, let $d(r, s)$ and $m(r, s)$ denote the greatest common divisor and least common multiple respectively of integers

r and s. We state the following (a verification of which appears in [2, Chapter 6]).

$$Z(S_p^{(2)}; x_1, x_2, \ldots, x_p) = \frac{1}{p!} \sum \frac{p!}{\prod_{k=1}^{p} j_k! k^{j_k}} \cdot \prod_{k=1}^{[p/2]} (x_k x_{2k}^{k-1})^{j_{2k}} \cdot \prod_{k=1}^{[p/2]} x_k^{k\binom{j_k}{2}}$$

$$\cdot \prod_{k=0}^{[(p-1)/2]} x_{2k+1}^{k \cdot j_{2k+1}} \cdot \prod_{1 \le r < s \le p-1} x_{m(r,s)}^{d(r,s) j_r j_s},$$

where the sum is taken over all partitions of p.

PROBLEM SET 18.2

18.4 (a) Determine all quantities found in Examples 18.1 through 18.5, where the group A is taken to be the symmetric group S_4 and all other terms are defined as before.

(b) Draw all graphs of order 4 having q edges, $0 \le q \le 6$, and check your result with the results obtained in (a).

18.5 (a) Define $m_p^{(n)}(x)$ to be the number of non-isomorphic multigraphs of order p such that between every two vertices there are at most n edges. Determine a formula for $m_p^{(n)}(x)$ in terms of the cycle index of $S_p^{(2)}$.

(b) Use your result in (a) to calculate $m_4^{(3)}(x)$ and draw all corresponding multigraphs.

18.6 Show that if A is a permutation group (not the symmetric group S_2) on a set D, $|D| \ge 2$, then $A \cong A^{(2)}$.

18.7 Show that $S_p^{(2)} \cong \Gamma^*(K_p)$.

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Index of Definitions

Index of Symbols

This index is divided into three categories: (1) Roman letters, (2) Greek letters, (3) special symbols.

ROMAN LETTERS

GREEK LETTERS

SPECIAL SYMBOLS