Functional Correctness

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1 Introduction

Here we are interested in determining the functional correctness of the software as written. The function nonnegdef should return a symmetric, non-negative definite square matrix with eigenvalues on the unit circle.

The degree to which the generated matrix, $\mathbf{M}$, differs from its Hermitian adjoint–in terms of simple subtraction–should tell us something about the correctness. By inspection of $\mathbf{M}_\epsilon = \mathbf{M} - \mathbf{M}^H$, we can determine the cumulative effects of bounded numerical accuracy–as well as algorithmic correctness. The matrix is claimed to be symmetric and non-negative definite, we thus naturally start with $\mathbf{M}_\epsilon$. $\mathbf{M}_\epsilon^U$ is not free in the sense that each element is necessarily identical in magnitude to its counterpart in $\mathbf{M}_\epsilon^L$ (and opposite in phase).

The Frobenius norm of the difference will be a good indication of how far it differs from the ideal result, $\mathbf{0}$. As indicated above, restricting the evaluation to the lower triangular matrix can be done without loss of information.

$$s(\mathbf{A}) = \left\|\mathbf{A}^L\right\|_F \tag{1}$$

More explicitly, the computation is

$$s = \left[\sum_{\substack{j \leq i \\ i}} |a_{i,j}|^2\right]^{\frac{1}{2}} \tag{2}$$

Now as for the domain

$$\mathrm{Dom}(s) = \{\mathbf{A} \mid \mathbf{A} \in \mathbf{M}_{n,n}^{\mathbb{C}}\} \tag{3}$$

In words s is the root of the element-wise sum of squared moduli of the lower triangular matrix of $\mathbf{A}$.

2 The Norm of $\mathbf{M} - \mathbf{M}^H$

The best we can hope to do is understand the expectation value over multiple executions, particularly if we consider the limit as the number of executions increases without bound.

$$\mathbf{M}_\epsilon = \mathbf{M} - \mathbf{M}^H \tag{4}$$

The value of interest is a scalar, v, as defined below

$$v = \mathrm{E}[s(\mathbf{M}_\epsilon)] \tag{5}$$

Consider the ratio of the number of non-zero elements in a strictly upper triangular square matrix to the number of total elements in the same. For an n x n matrix, the first non-trivial value is 1/4, for n=2. The triangular numbers sum to (n)(n+1)/2, but we can simply re-index this to account for the strictness– which implies zeros on the diagonal. Then we have (n)(n-1)/2 for the maximum number of non-zero elements, and of course n^2 for the total element count.

We can simplify this ratio by removing the common factor of n. This leaves (n-1)/2n. Now consider this in the limit as n increases without bound. By inspection we have clearly 1/2 in the leading terms. More rigor would bear this out, but any attempt to regurgitate L'Hôpital's Rule verbatim is omitted here.

By n=20 (as in the example code) we already have a value of 19/40.