Notes on Asymmetric Cloning

1. Universal asymmetric cloning machines

In this section we develop the universal quantum cloning machine (UQCM) following the presentation in [2]. We restrict our attention to the 2-dimensional case, i.e. to qubits.

1.1. **Universal Cloning Machines.** Here we consider a unitary transformation

$$|i\rangle_A |O\rangle_B |\Sigma\rangle_X \to \mu |i\rangle_A |i\rangle_B |i\rangle_X + \nu \sum_{j\neq i} \Big(|i\rangle_A |j\rangle_B + |j\rangle_A |i\rangle_B \Big) |j\rangle_X .$$

Here A refers to the input qubit, B is a blank qubit, and X is an ancilla. The ancilla is initially in some fixed state, say $|\Sigma\rangle$. In particular, the unitary can be expressed in terms of the basis states $|0\rangle$ and $|1\rangle$:

$$|0\rangle_A |O\rangle_B |\Sigma\rangle_X \to \mu |0\rangle_A |0\rangle_B |0\rangle_X + \nu \Big(|0\rangle_A |1\rangle_B |1\rangle_X + |1\rangle_A |0\rangle_B |0\rangle_X \Big)$$
$$|1\rangle_A |O\rangle_B |\Sigma\rangle_X \to \mu |1\rangle_A |1\rangle_B |1\rangle_X + \nu \Big(|1\rangle_A |0\rangle_B |0\rangle_X + |0\rangle_A |1\rangle_B |0\rangle_X \Big).$$

We point out that the parameters, μ and ν, can be taken to be real parameters (imaginary terms can be absorbed into the ancilla). We impose the following restrictions on the output of the cloner:

(1) the fidelity of the copies, $F = \langle\psi|\rho^{(\text{out})}|\psi\rangle$ does not depend on the particular state which is being copied;
(2) the outputs are symmetric, meaning that $\rho_A^{(\text{out})} = \rho_B^{(\text{out})}$.

These restrictions yield the following relations:

$$\begin{aligned}
\rho_A^{(\text{out})} &= \eta |\psi\rangle_A \langle\psi| + \frac{1-\eta}{2}\mathbf{1}_A \\
\rho_B^{(\text{out})} &= \eta |\psi\rangle_A \langle\psi| + \frac{1-\eta}{2}\mathbf{1}_B \\
\mu^2 &= 2\mu\nu \\
\mu^2 &= \frac{2}{3} \\
\nu^2 &== \frac{1}{6} \\
\eta = \mu^2 &= \frac{2}{3}.
\end{aligned}$$

Here $\mathbf{1}_A$ is the identity operator on the Hilbert space $\mathcal{H}_A$ and $\eta = 2F - 1$ is called the shrinking factor (recall that F is the fidelity as defined above). In the case of qubits we see that the fidelity is $F = 5/6$.

Detailed Calculations. Need to fill in the calculations for the previous relations. □

1.2. **Asymmetric Universal Cloning Machines.** Notice that in the definition of the cloning machine given above the symmetry of the outputs ($\rho_A^{(\mathrm{out})} = \rho_B^{(\mathrm{out})}$) is a consequence of the equality of the coefficients of the terms $|i\rangle_A |j\rangle_B |j\rangle_X$ and $|j\rangle_A |i\rangle_B |j\rangle_X$. To develop an asymmetric cloning machine, then, we give different contributions to these terms. In particular, we define

$$|0\rangle_A |O\rangle_B |\Sigma\rangle_X \to \mu\, |0\rangle_A |0\rangle_B |0\rangle_X + \nu\, |0\rangle_A |1\rangle_B |1\rangle_X + \xi\, |1\rangle_A |0\rangle_B |0\rangle_X$$
$$|1\rangle_A |O\rangle_B |\Sigma\rangle_X \to \mu\, |1\rangle_A |1\rangle_B |1\rangle_X + \nu\, |1\rangle_A |0\rangle_B |0\rangle_X + \xi\, |0\rangle_A |1\rangle_B |0\rangle_X .$$

If a state in the form $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ is given as the input this machine, then the state of the output copy A is

{Aclone}
$$\rho_A^{(\mathrm{out})} = 2\mu\nu\, |\psi\rangle_A \langle\psi| + \xi^2 \mathbf{1}_A + (\mu^2 + \nu^2 - \xi^2 - 2\mu\nu)\Big(|\alpha_0|^2 |0\rangle\langle 0| + |\alpha_1|^2 |1\rangle\langle 1|\Big), \tag{1}$$

with the corresponding output in B is

{Bclone}
$$\rho_B^{(\mathrm{out})} = 2\mu\xi\, |\psi\rangle_A \langle\psi| + \nu^2 \mathbf{1}_A + (\mu^2 + \xi^2 - \nu^2 - 2\mu\xi)\Big(|\alpha_0|^2 |0\rangle\langle 0| + |\alpha_1|^2 |1\rangle\langle 1|\Big). \tag{2}$$

Observe that $rho_A^{(\mathrm{out})}$ and $\rho_B^{(\mathrm{out})}$ are similar; the B-case is obtained from the A-case by swapping the roles of ν and ξ.

Detailed Calculations. Need to fill in the calculations for the reduced density operators above. □

Notice that the last terms in (1) and (2) are state-dependent. By imposing the requirement that the cloner be independent of the input state we require

$$\mu^2 + \nu^2 - \xi^2 - 2\mu\nu = 0$$
$$\mu^2 + \xi^2 - \nu^2 - 2\mu\xi = 0.$$

Adding these equations yields

$$\mu^2 - \mu\xi - \mu\nu = 0$$

from which we conclude that $\mu = \nu + \xi$. Since we require the output of the cloner to be normalized, we require that

{normalization}
$$\mu^2 + \nu^2 + \xi^2 = 1 \tag{3}$$

Also from (1) we find that

$$\eta_A = 2\mu\nu \quad \text{and} \quad \frac{1 - \eta_A}{2} = \xi^2,$$

while from (2) we see that

$$\eta_B = 2\mu\xi, \quad \text{and} \quad \frac{1 - \eta_B}{2} = \nu^2.$$

Recalling that the fidelity, F, is related to the shrinking factor η by $\eta = 2F-1$, we see that these calculations yield fidelities for the A and B copies:

$$F_A = \frac{1}{2}(2\mu\nu + 1) = 1 - \xi^2$$
$$F_B = \frac{1}{2}(2\mu\xi + 1) = 1 - \nu^2.$$

1.3. **Asymmetric Phase-Covariant Cloning Machine.** Consider an input state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\phi}\,|1\rangle \Big).$$

In this case we find that the final term in (1) and (2) is

$$\left|\frac{1}{\sqrt{2}}\right|^2 |0\rangle\langle 0| + \left|\frac{e^{i\phi}}{\sqrt{2}}\right|^2 |1\rangle\langle 1| = \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1| \Big) = \frac{1}{2}\mathbf{1},$$

meaning that the last term is no longer dependent on the input state. In particular we find that the outputs reduce to

$$\rho_A^{(\text{out})} = 2\mu\nu\,|\psi\rangle_A\langle\psi| + \left(\xi^2 + \frac{\mu^2+\nu^2-\xi^2-2\mu\nu}{2}\right)\mathbf{1}_A \tag{4}$$ {PCAclone}

and

$$\rho_B^{(\text{out})} = 2\mu\xi\,|\psi\rangle_A\langle\psi| + \left(\nu^2 + \frac{\mu^2+\xi^2-\nu^2-2\mu\xi}{2}\right)\mathbf{1}_B. \tag{5}$$ {PCBclone}

We are thus lead to the following formulas for the shrinking factors:

$$\eta_A = 2\mu\nu = 2\nu\sqrt{1-(\nu^2+\xi^2)} \tag{6}$$ {Ashrink}

$$\eta_B = 2\mu\xi = 2\xi\sqrt{1-(\nu^2+\xi^2)}. \tag{7}$$ {Bshrink}

This cloning machine is optimal if, whenever we fix the quality of one of the clones, say A, the quality of the other clone is as high as possible. Since the quality of the clone A can be expressed in terms of η_A, η_B, we focus on the trade-off in the shrinking factors. For a fixed value of η_A we solve (6) for ξ in terms of ν and insert this into (7) to see that

$$\eta_B(\nu) = \frac{\eta_A}{\nu}\sqrt{1-\nu^2-\frac{\eta_A^2}{4\nu^2}}.$$

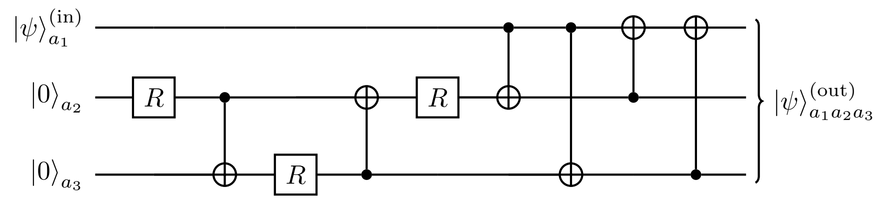
Thus given a value of η_A we can determine a value of $\nu \in (0,1]$ that maximizes the value of η_B. Need to check the domain for eta_B. In general dimesions there is no formula for the corresponding value of ν, but in dimension 2 there might be a formula. Right now I have reduced this to the following equation for ν:

$$4\nu^5 - 4\nu^4 + 4\nu^2 - 3\eta^2 = 0.$$

In general, fifth order equations are hopeless, but this specific form might admit a nice solution. Check with Mathematica?

2. Implementation

The following circuit is drawn from [1]. We write $|\psi\rangle_{a_1}^{(\text{in})}$ for the qubit we are trying to clone. The circuit below aims to produce two copies of the input qubit. In their initial state we write $|0\rangle_{a_2}, |0\rangle_{a_3}$ for these qubits. The first part of the circuit prepares the target qubits (a_2 and a_3) in a state which is useful for the cloning operation. The second component of the circuit (which involves $|\psi\rangle_{a_1}^{(\text{in})}$) is the piece of the circuit that handles the actual copying.



Here the gate $R = R(\theta)$ is a rotation gate defined by

$$R|0\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$
$$R|1\rangle = -\sin(\theta)|0\rangle + \cos(\theta)|0\rangle .$$

The first part of this circuit involves only the a_2 and a_3 qubits; this is a preparation component of the circuit. The output of this portion of the circuit is of the form

$$|\psi\rangle_{a_2 a_3}^{(\text{out})} = C_1 |0\rangle_{a_2} |0\rangle_{a_3} + C_2 |0\rangle_{a_2} |1\rangle_{a_3} + C_3 |1\rangle_{a_2} |0\rangle_{a_3} + C_4 |1\rangle_{a_2} |1\rangle_{a_3} .$$

Following the circuit above we find that the coefficients $C_j, j = 1, 2, 3, 4$ are given by

$$\begin{aligned}
C_1 &= \cos(\theta_1)\cos(\theta_2)\cos(\theta_3) + \sin(\theta_1)\sin(\theta_2)\sin(\theta_3)\\
C_2 &= \sin(\theta_1)\cos(\theta_2)\cos(\theta_3) - \cos(\theta_1)\sin(\theta_2)\sin(\theta_3)\\
C_3 &= \cos(\theta_1)\cos(\theta_2)\sin(\theta_3) - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\\
C_4 &= \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) + \sin(\theta_1)\cos(\theta_2)\cos(\theta_3).
\end{aligned}$$

Consider an input $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$. The output from the circuit above is

$$\begin{aligned}
|\psi\rangle^{(out)} = {} & \alpha_0 C_1 |000\rangle + \alpha_0 C_2 |101\rangle + \alpha_0 C_3 |110\rangle + \alpha_0 C_4 |011\rangle\\
& + \alpha_1 C_1 |111\rangle + \alpha_1 C_2 |010\rangle + \alpha_1 C_3 |001\rangle + \alpha_1 C_4 |100\rangle .
\end{aligned}$$

The output state of the cloning machine in the preceding section for this input is

$$\begin{aligned}
|\psi\rangle^{(out)} = {} & \alpha_0\mu|000\rangle + \alpha_0\nu|011\rangle + \alpha_0\xi|101\rangle\\
& + \alpha_1\mu|111\rangle + \alpha_1\nu|100\rangle + \alpha_1\xi|101\rangle .
\end{aligned}$$

By comparing coefficients we see that we require

$$C_1 = \mu, \quad C_2 = \xi, \quad C_3 = 0, \quad C_4 = \nu.$$

This means that, in the notation of [2], we have

$$\begin{aligned}
\mu &= \cos(\theta_1)\cos(\theta_2)\cos(\theta_3) + \sin(\theta_1)\sin(\theta_2)\sin(\theta_3)\\
\xi &= \sin(\theta_1)\cos(\theta_2)\cos(\theta_3) - \cos(\theta_1)\sin(\theta_2)\sin(\theta_3)\\
\nu &= \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) + \sin(\theta_1)\cos(\theta_2)\cos(\theta_3),
\end{aligned}$$

together with the restriction that

$$\cos(\theta_1)\cos(\theta_2)\sin(\theta_3) - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3) = 0.$$

Observe that since the coefficients μ, ξ, ν satisfy the normalization condition (3), this restriction is automatic.

References

[1] V. Bužek, S. L. Braunstein, M. Hillery, and D. Bruß. Quantum copying: A network. *Phys. Rev. A*, 56:3446–3452, Nov 1997.

[2] A.T. Rezakhani, S. Siadatnejad, and A.H. Ghaderi. Separability in asymmetric phase-covariant cloning. *Physics Letters A*, 336(4):278–289, 2005.