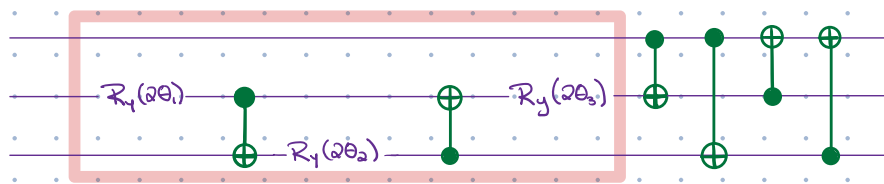


Original:



We know $\theta_1 = \frac{\pi}{4} \Rightarrow R_y(2\theta_1) = R_y(\frac{\pi}{2})$

* Observe the general relationship: $R_y(\chi) \equiv R_x(\frac{\pi}{2}) - R_z(\chi) - R_x(-\frac{\pi}{2}) \equiv SX - R_z(\chi) - SX - X$

IBM Basis Gates: X, SX, R_z, CX

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad SX = \sqrt{X} = \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \quad CNOT \equiv CX$$

* See calculations on next page

Note $R_x(\frac{\pi}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = e^{-i\pi/4} \sqrt{X}$, so $R_x(\frac{\pi}{2}) \equiv SX$

$$R_x(\pi) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -iX, \text{ so } R_x(\pi) \equiv X$$

$$R_y(\frac{\pi}{2}) \equiv R_x(-\frac{\pi}{2}) - R_z(-\frac{\pi}{2}) - R_x(\frac{\pi}{2}) \quad \text{note } R_x(-\frac{\pi}{2}) \equiv R_x(\frac{3\pi}{2}) \equiv X - SX$$

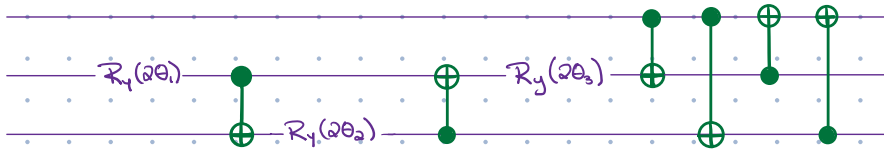
$$\equiv X - SX - R_z(-\frac{\pi}{2}) - SX$$

$$H \equiv R_y(\frac{\pi}{2}) - R_x(-\pi)$$

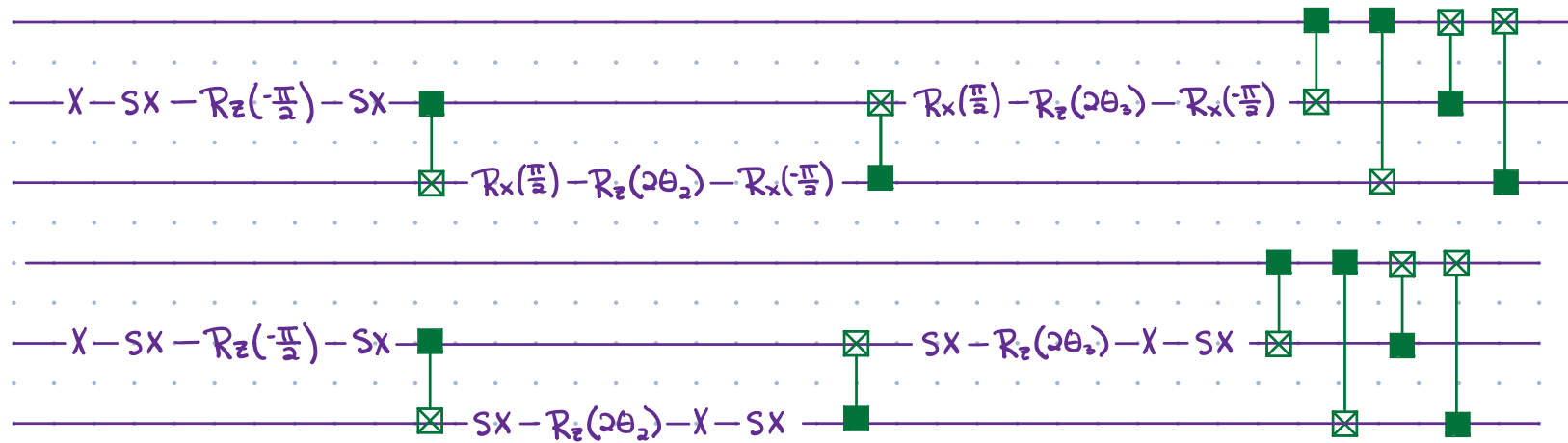
$$\equiv X - SX - R_z(-\frac{\pi}{2}) - SX - X$$

Verify by multiplying the representative matrices (see next page)
* from Maslov

We also need H gate for prep: $H \equiv X - SX - R_z(-\frac{\pi}{2}) - SX - X$



* CNOT \rightarrow CX(control, target) + * relations:



to Basis Gates

$$R_x(\frac{\pi}{2}) \equiv SX$$

$$R_x(-\frac{\pi}{2}) \equiv X - SX$$

$$* R_y\left(\frac{\pi}{2}\right) \equiv R_x\left(\frac{\pi}{2}\right) R_z\left(-\frac{\pi}{2}\right) R_x\left(-\frac{\pi}{2}\right)$$

As circuit: $R_x(-\frac{\pi}{2}) - R_z(-\frac{\pi}{2}) - R_x(\frac{\pi}{2})$

$$R_y\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

$$\begin{pmatrix} \cos \frac{\pi}{4} & -i \sin \frac{\pi}{4} \\ -i \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \begin{pmatrix} \cos(\pi/4) & -i \sin(\pi/4) \\ -i \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1+i & i-1 \\ i+1 & 1-i \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{pmatrix} 1+i+1-i & i-1-i-1 \\ -i+1+i+1 & 1+i+1-i \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = R_y\left(\frac{\pi}{2}\right)$$

In general, $R_y(\delta) \equiv R_x(\frac{\pi}{2}) R_z(\delta) R_x(-\frac{\pi}{2})$:

$$\underbrace{\begin{pmatrix} \cos(\pi/4) & -i \sin(\pi/4) \\ -i \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix} \begin{pmatrix} \cos \frac{\pi}{4} & -i \sin \frac{\pi}{4} \\ -i \sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix}}_{R_x(-\frac{\pi}{2}) \cdot R_z(\delta) \cdot R_x(\frac{\pi}{2})} = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{-i\delta/2} & -ie^{-i\delta/2} \\ -ie^{i\delta/2} & e^{i\delta/2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-i\delta/2} + e^{i\delta/2} & -ie^{-i\delta/2} + ie^{i\delta/2} \\ ie^{-i\delta/2} - ie^{i\delta/2} & e^{-i\delta/2} + e^{i\delta/2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$a: e^{-i\delta/2} + e^{i\delta/2} = \cos\left(\frac{\delta}{2}\right) + i \sin\left(\frac{\delta}{2}\right) + \cos\left(\frac{\delta}{2}\right) + i \sin\left(\frac{\delta}{2}\right) = 2 \cos \frac{\delta}{2}$$

$$b: -i \cos\left(\frac{\delta}{2}\right) + \sin\left(\frac{\delta}{2}\right) + i \cos\left(\frac{\delta}{2}\right) - \sin\left(\frac{\delta}{2}\right) = -2 \sin \frac{\delta}{2}$$

$$c: i \cos\left(\frac{\delta}{2}\right) - \sin\left(\frac{\delta}{2}\right) - i \cos\left(\frac{\delta}{2}\right) + \sin\left(\frac{\delta}{2}\right) = 2 \sin \frac{\delta}{2}$$

$$d: \cos\left(\frac{\delta}{2}\right) + i \sin\left(\frac{\delta}{2}\right) + \cos\left(\frac{\delta}{2}\right) + i \sin\left(\frac{\delta}{2}\right) = 2 \cos \frac{\delta}{2}$$

$$\begin{pmatrix} \cos \frac{\delta}{2} & -\sin \frac{\delta}{2} \\ \sin \frac{\delta}{2} & \cos \frac{\delta}{2} \end{pmatrix} = \underline{R_y(\delta)}$$

$$\text{H gate} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{aligned} SX R_z(-\frac{\pi}{2}) SX &= \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \begin{pmatrix} e^{i\pi/4} & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix} \begin{pmatrix} e^{i\pi/4} + ie^{-i\pi/4} & e^{i\pi/4} - ie^{-i\pi/4} \\ e^{-i\pi/4} - ie^{i\pi/4} & e^{-i\pi/4} + ie^{i\pi/4} \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} (1+i)^2 e^{i\pi/4} + (1-i)^2 e^{-i\pi/4} & (1+i)(1-i)(e^{i\pi/4} + e^{-i\pi/4}) \\ (1+i)(1-i)(e^{i\pi/4} + e^{-i\pi/4}) & (1+i)^2 e^{-i\pi/4} + (1-i)^2 e^{i\pi/4} \end{pmatrix} \\ &\stackrel{*}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ : mult by } X \end{aligned}$$

in front: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = H \checkmark$

$$\stackrel{*}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$a: (1+i)^2 e^{i\pi/4} + (1-i)^2 e^{-i\pi/4} = 2i(e^{i\pi/4} - e^{-i\pi/4})$$

$$d: 2i(e^{-i\pi/4} - e^{i\pi/4})$$

$$b, c: (1+i)(e^{i\pi/4} + e^{-i\pi/4}) = 2(e^{i\pi/4} + e^{-i\pi/4})$$

$$e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i) \quad \therefore \quad e^{-i\pi/4} = \frac{1}{\sqrt{2}}(1-i)$$

$$\text{so } a = i\sqrt{2}(1+i - (1-i)) = i\sqrt{2}(2i) = -2\sqrt{2}$$

$$d = i\sqrt{2}(1-i - (1+i)) = 2\sqrt{2}$$

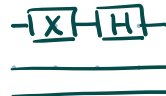
$$b, c: \sqrt{2}(1+i + 1-i) = 2\sqrt{2}$$

Alice Send Prep:

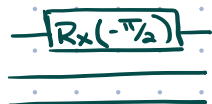
basis X is bitval 0:



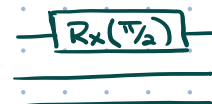
basis X is bitval 1:



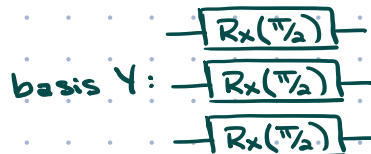
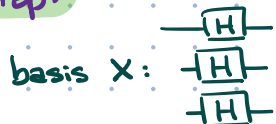
basis Y is bitval 0:



basis Y is bitval 1:



Measurement Prep:



IBM Qiskit transpile (feed basis gates ['cx', 'rz', 'i', 'sx', 'x'])

$$H \rightarrow \boxed{R_z(\pi/2)} \boxed{\sqrt{X}} \boxed{R_z(\pi/2)}$$

vs

$$R_y(\pi/2) \rightarrow \boxed{R_z(-\pi/2)} \boxed{\sqrt{X}} \boxed{R_z(\pi/2)}$$

vs

$$R_y(\theta) \rightarrow \boxed{R_z(-\pi)} \boxed{\sqrt{X}} \boxed{R_z(\theta)} \boxed{\sqrt{X}}$$

vs

$$\boxed{X} \boxed{H} \rightarrow \boxed{R_z(\pi/2)} \boxed{\sqrt{X}} \boxed{R_z(-\pi/2)}$$

vs

$$\boxed{R_x(\pi/2)} \rightarrow \boxed{\sqrt{X}}$$

vs

$$\boxed{R_x(-\pi/2)} \rightarrow \boxed{R_z(-\pi)} \boxed{\sqrt{X}} \boxed{R_z(-\pi)}$$

vs

Agree

Ours

$$H \rightarrow \boxed{X} \boxed{\sqrt{X}} \boxed{R_z(-\pi/2)} \boxed{\sqrt{X}} \boxed{X}$$

$$R_y(\pi/2) \rightarrow \boxed{R_x(-\pi/2)} \boxed{R_z(-\pi/2)} \boxed{R_x(\pi/2)}$$

$$R_y(\theta) \rightarrow \boxed{\sqrt{X}} \boxed{R_z(\theta)} \boxed{\sqrt{X}} \boxed{X}$$

$$\boxed{X} \boxed{H} \rightarrow \boxed{\sqrt{X}} \boxed{R_z(-\pi/2)} \boxed{\sqrt{X}} \boxed{X}$$

$$\boxed{R_x(\pi/2)} \rightarrow \boxed{\sqrt{X}}$$

$$\boxed{R_x(-\pi/2)} \rightarrow \boxed{\sqrt{X}} \boxed{X}$$

IBM transpile also lists global phase.