

Notes on Asymmetric Cloning

1. ASYMMETRIC CLONING MACHINES

In this section we develop the universal quantum cloning machine (UQCM) following the presentation in [2]. We restrict our attention to the 2-dimensional case, i.e. to qubits.

1.1. Universal Cloning Machines. Here we consider a unitary transformation

$$|i\rangle_A |O\rangle_B |\Sigma\rangle_X \rightarrow \mu |i\rangle_A |i\rangle_B |i\rangle_X + \nu \sum_{j \neq i} \left(|i\rangle_A |j\rangle_B + |j\rangle_A |i\rangle_B \right) |j\rangle_X.$$

Here A refers to the input qubit, B is a blank qubit, and X is an ancilla. The ancilla is initially in some fixed state, say $|\Sigma\rangle$. In particular, the unitary can be expressed in terms of the basis states $|0\rangle$ and $|1\rangle$:

$$\begin{aligned} |0\rangle_A |O\rangle_B |\Sigma\rangle_X &\rightarrow \mu |0\rangle_A |0\rangle_B |0\rangle_X + \nu \left(|0\rangle_A |1\rangle_B |1\rangle_X + |1\rangle_A |0\rangle_B |1\rangle_X \right) \\ |1\rangle_A |O\rangle_B |\Sigma\rangle_X &\rightarrow \mu |1\rangle_A |1\rangle_B |1\rangle_X + \nu \left(|1\rangle_A |0\rangle_B |0\rangle_X + |0\rangle_A |1\rangle_B |0\rangle_X \right). \end{aligned} \tag{1} \quad \{\text{UCM}\}$$

We point out that the parameters, μ and ν , can be taken to be real parameters (imaginary terms can be absorbed into the ancilla). We impose the following restrictions on the output of the cloner:

- (1) the fidelity of the copies, $F = \langle \psi | \rho^{(\text{out})} | \psi \rangle$ does not depend on the particular state which is being copied;
- (2) the outputs are symmetric, meaning that $\rho_A^{(\text{out})} = \rho_B^{(\text{out})}$.

These restrictions yield the following relations:

$$\begin{aligned} \rho_A^{(\text{out})} &= \eta |\psi\rangle_A \langle \psi| + \frac{1-\eta}{2} \mathbf{1}_A \\ \rho_B^{(\text{out})} &= \eta |\psi\rangle_A \langle \psi| + \frac{1-\eta}{2} \mathbf{1}_B \\ \mu^2 &= 2\mu\nu \\ \mu^2 &= \frac{2}{3} \\ \nu^2 &= \frac{1}{6} \\ \eta &= \mu^2 = \frac{2}{3}. \end{aligned}$$

Here $\mathbf{1}_A$ is the identity operator on the Hilbert space \mathcal{H}_A and $\eta = 2F - 1$ is called the shrinking factor (recall that F is the fidelity as defined above). In the case of qubits we see that the fidelity is $F = 5/6$.

Detailed Calculations. For brevity we will write $|ijk\rangle$ in place of $|i\rangle_A |j\rangle_B |k\rangle_X$.

First, since we require the output to be normalized, we find that

$$\mu^2 + 2\nu^2 = 1.$$

Consider input $|\psi\rangle_A = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ with $|\alpha_0|^2 + |\alpha_1|^2 = 1$. The output of the machine described by (1) is

$$\mu\alpha_0 |000\rangle + \alpha_0\nu(|011\rangle + |101\rangle) + \mu\alpha_1 |111\rangle + \nu\alpha_1(|100\rangle + |010\rangle).$$

The associated density operator is

$$\begin{aligned} \rho &= \left(\mu\alpha_0 |000\rangle + \alpha_0\nu(|011\rangle + |101\rangle) + \mu\alpha_1 |111\rangle + \nu\alpha_1(|100\rangle + |010\rangle) \right) \\ &\quad \left(\mu\alpha_0^* \langle 000| + \alpha_0^*\nu(\langle 011| + \langle 101|) + \mu\alpha_1^* \langle 111| + \nu\alpha_1^*(\langle 100| + \langle 010|) \right) \\ &= \mu^2 |\alpha_0|^2 |000\rangle \langle 000| + \mu\nu |\alpha_0|^2 (|000\rangle \langle 011| + |000\rangle \langle 101|) \\ &\quad + \mu^2 \alpha_0 \alpha_1^* |000\rangle \langle 111| + \mu\nu \alpha_0 \alpha_1^* (|000\rangle \langle 100| + |000\rangle \langle 010|) + \dots \end{aligned}$$

To determine the density operator $\rho_A^{(\text{out})}$ we now trace out the qubits from B and X :

$$\begin{aligned} \rho_A^{(\text{out})} &= (|\alpha_0|^2 \mu^2 + |\alpha_0|^2 \nu^2 + |\alpha_1|^2 \nu^2) |0\rangle \langle 0| \\ &\quad + 2\mu\nu \alpha_0 \alpha_1^* \mu\nu |0\rangle \langle 1| + 2\mu\nu \alpha_0^* \alpha_1 |1\rangle \langle 0| \\ &\quad + (|\alpha_1|^2 \mu^2 + |\alpha_0|^2 \nu^2 + |\alpha_1|^2 \nu^2) |1\rangle \langle 1| \end{aligned}$$

Recalling that $|\alpha_0|^2 + |\alpha_1|^2 = 1$, the $|0\rangle \langle 0|$ and $|1\rangle \langle 1|$ entries simplify and we are left with

$$\begin{aligned} \rho_A^{(\text{out})} &= (\mu^2 |\alpha_0|^2 |0\rangle \langle 0| + 2\mu\nu \alpha_0 \alpha_1^* |0\rangle \langle 1| + 2\mu\nu \alpha_0^* \alpha_1 |1\rangle \langle 0| + \mu^2 |\alpha_1|^2 |1\rangle \langle 1|) \\ &\quad + \nu^2 (|0\rangle \langle 0| + |1\rangle \langle 1|) \end{aligned}$$

Since $\mu^2 + 2\nu^2 = 1$ we have $\nu^2 = (1 - \mu^2)/2$. An analogous calculation yields

$$|\psi\rangle_A \langle \psi| = |\alpha_0|^2 |0\rangle \langle 0| + \alpha_0 \alpha_1^* |0\rangle \langle 1| + \alpha_0^* \alpha_1 |1\rangle \langle 0| + |\alpha_1|^2 |1\rangle \langle 1|.$$

With these calculations in hand, if we write

$$\rho_A^{(\text{out})} = \eta |\psi\rangle_A \langle \psi| + \frac{1 - \eta}{2} \mathbf{1}_A,$$

then we must have

$$\eta = \mu^2 \quad \text{and} \quad \mu^2 = 2\mu\nu.$$

From this last equality we have that $\mu = 0$ or $\mu = 2\nu$. Substituting this equality into $\mu^2 + 2\nu^2 = 1$ yields $6\nu^2 = 1$, whence $\nu^2 = 1/6$. It follows that $\mu^2 = 4\nu^2 = 2/3$.

□

1.2. Asymmetric Universal Cloning Machines. Notice that in the definition of the cloning machine given above the symmetry of the outputs ($\rho_A^{(\text{out})} = \rho_B^{(\text{out})}$) is a consequence of the equality of the coefficients of the terms $|i\rangle_A |j\rangle_B |j\rangle_X$ and $|j\rangle_A |i\rangle_B |j\rangle_X$. To develop an asymmetric cloning machine, then, we give different contributions to these terms. In particular, we define

$$\begin{aligned} |0\rangle_A |0\rangle_B |\Sigma\rangle_X &\rightarrow \mu |0\rangle_A |0\rangle_B |0\rangle_X + \nu |0\rangle_A |1\rangle_B |1\rangle_X + \xi |1\rangle_A |0\rangle_B |0\rangle_X \\ |1\rangle_A |0\rangle_B |\Sigma\rangle_X &\rightarrow \mu |1\rangle_A |1\rangle_B |1\rangle_X + \nu |1\rangle_A |0\rangle_B |0\rangle_X + \xi |0\rangle_A |1\rangle_B |0\rangle_X. \end{aligned}$$

If a state in the form $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ is given as the input this machine, then the state of the output copy A is

$$\rho_A^{(\text{out})} = 2\mu\nu |\psi\rangle_A \langle\psi| + \xi^2 \mathbf{1}_A + (\mu^2 + \nu^2 - \xi^2 - 2\mu\nu) \left(|\alpha_0|^2 |0\rangle \langle 0| + |\alpha_1|^2 |1\rangle \langle 1| \right), \quad (2) \quad \{\mathbf{Aclone}\}$$

with the corresponding output in B is

$$\rho_B^{(\text{out})} = 2\mu\xi |\psi\rangle_A \langle\psi| + \nu^2 \mathbf{1}_A + (\mu^2 + \xi^2 - \nu^2 - 2\mu\xi) \left(|\alpha_0|^2 |0\rangle \langle 0| + |\alpha_1|^2 |1\rangle \langle 1| \right). \quad (3) \quad \{\mathbf{Bclone}\}$$

Observe that $\rho_A^{(\text{out})}$ and $\rho_B^{(\text{out})}$ are similar; the B -case is obtained from the A -case by swapping the roles of ν and ξ .

Detailed Calculations. The calculations here are very similar to those from Section 1.1. \square

Notice that the last terms in (2) and (3) are state-dependent. By imposing the requirement that the cloner be independent of the input state we require

$$\begin{aligned} \mu^2 + \nu^2 - \xi^2 - 2\mu\nu &= 0 \\ \mu^2 + \xi^2 - \nu^2 - 2\mu\xi &= 0. \end{aligned}$$

Adding these equations yields

$$\mu^2 - \mu\xi - \mu\nu = 0$$

from which we conclude that $\mu = \nu + \xi$. Since we require the output of the cloner to be normalized, we require that

$$\mu^2 + \nu^2 + \xi^2 = 1 \quad (4) \quad \{\mathbf{normalization}\}$$

Also from (2) we find that

$$\eta_A = 2\mu\nu \quad \text{and} \quad \frac{1 - \eta_A}{2} = \xi^2,$$

while from (3) we see that

$$\eta_B = 2\mu\xi, \quad \text{and} \quad \frac{1 - \eta_B}{2} = \nu^2.$$

Recalling that the fidelity, F , is related to the shrinking factor η by $\eta = 2F - 1$, we see that these calculations yield fidelities for the A and B copies:

$$F_A = \frac{1}{2}(2\mu\nu + 1) = 1 - \xi^2$$

$$F_B = \frac{1}{2}(2\mu\xi + 1) = 1 - \nu^2.$$

1.3. Asymmetric Phase-Covariant Cloning Machine. Consider an input state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle).$$

In this case we find that the final term in (2) and (3) is

$$\left|\frac{1}{\sqrt{2}}\right|^2 |0\rangle\langle 0| + \left|\frac{e^{i\phi}}{\sqrt{2}}\right|^2 |1\rangle\langle 1| = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\mathbf{1},$$

meaning that the last term is no longer dependent on the input state. In particular we find that the outputs reduce to

$$\{\text{PCAClone}\} \quad \rho_A^{(\text{out})} = 2\mu\nu |\psi\rangle_A \langle\psi| + \left(\xi^2 + \frac{\mu^2 + \nu^2 - \xi^2 - 2\mu\nu}{2}\right) \mathbf{1}_A \quad (5)$$

and

$$\{\text{PCBClone}\} \quad \rho_B^{(\text{out})} = 2\mu\xi |\psi\rangle_A \langle\psi| + \left(\nu^2 + \frac{\mu^2 + \xi^2 - \nu^2 - 2\mu\xi}{2}\right) \mathbf{1}_B. \quad (6)$$

We are thus lead to the following formulas for the shrinking factors:

$$\{\text{Ashrink}\} \quad \eta_A = 2\mu\nu = 2\nu\sqrt{1 - (\nu^2 + \xi^2)} \quad (7)$$

$$\{\text{Bshrink}\} \quad \eta_B = 2\mu\xi = 2\xi\sqrt{1 - (\nu^2 + \xi^2)}. \quad (8)$$

This cloning machine is optimal if, whenever we fix the quality of one of the clones, say A , the quality of the other clone is as high as possible. Since the quality of the clone A can be expressed in terms of η_A, η_B , we focus on the trade-off in the shrinking factors. For a fixed value of η_A we solve (7) for ξ in terms of ν and insert this into (8) to see that

$$\eta_B(\nu) = \frac{\eta_A}{\nu} \sqrt{1 - \nu^2 - \frac{\eta_A^2}{4\nu^2}}.$$

Thus given a value of η_A we can determine a value of ν that maximizes the value of η_B . We note that the domain of η_B has two components:

$$\text{Domain } \eta_B = \left[-\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \eta_A^2}}, -\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \eta_A^2}} \right]$$

$$\cup \left[\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \eta_A^2}}, \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \eta_A^2}} \right].$$

Since we expect the scaling coefficient η_B to be nonnegative, we seek a solution from the positive component of the domain. That is, we require

$$\nu \in \left[\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \eta_A^2}}, \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \eta_A^2}} \right].$$

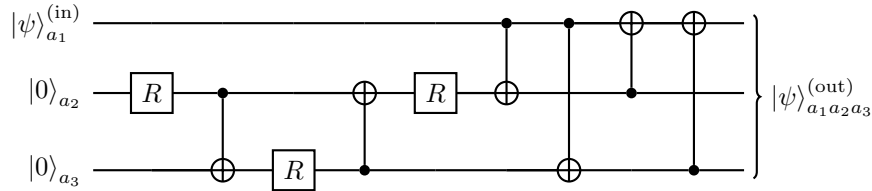
In general dimesions there is no formula for the corresponding value of ν , but in dimension 2 there might be a formula. Right now I have reduced this to the following equation for ν :

$$4\nu^5 - 4\nu^4 + 4\nu^2 - 3\eta^2 = 0.$$

Can we show that there is a unique value of ν that corresponds to the maximum?

2. IMPLEMENTATION

The following circuit is drawn from [1]. We write $|\psi\rangle_{a_1}^{(\text{in})}$ for the qubit we are trying to clone. The circuit below aims to produce two copies of the input qubit. In their initial state we write $|0\rangle_{a_2}, |0\rangle_{a_3}$ for these qubits. The first part of the circuit prepares the target qubits (a_2 and a_3) in a state which is useful for the cloning operation. The second component of the circuit (which involves $|\psi\rangle_{a_1}^{(\text{in})}$) is the piece of the circuit that handles the actual copying.



Here the gate $R = R(\theta)$ is a rotation gate defined by

$$\begin{aligned} R|0\rangle &= \cos(\theta)|0\rangle + \sin(\theta)|1\rangle \\ R|1\rangle &= -\sin(\theta)|0\rangle + \cos(\theta)|1\rangle. \end{aligned}$$

The first part of this circuit involves only the a_2 and a_3 qubits; this is a preparation component of the circuit. The output of this portion of the circuit is of the form

$$|\psi\rangle_{a_2 a_3}^{(\text{out})} = C_1 |0\rangle_{a_2} |0\rangle_{a_3} + C_2 |0\rangle_{a_2} |1\rangle_{a_3} + C_3 |1\rangle_{a_2} |0\rangle_{a_3} + C_4 |1\rangle_{a_2} |1\rangle_{a_3}.$$

Following the circuit above we find that the coefficients $C_j, j = 1, 2, 3, 4$ are given by

$$\begin{aligned} C_1 &= \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) + \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \\ C_2 &= \sin(\theta_1) \cos(\theta_2) \cos(\theta_3) - \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \\ C_3 &= \cos(\theta_1) \cos(\theta_2) \sin(\theta_3) - \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) \\ C_4 &= \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) + \sin(\theta_1) \cos(\theta_2) \cos(\theta_3). \end{aligned}$$

Consider an input $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$. The output from the circuit above is

$$\begin{aligned} |\psi\rangle^{(out)} &= \alpha_0 C_1 |000\rangle + \alpha_0 C_2 |101\rangle + \alpha_0 C_3 |110\rangle + \alpha_0 C_4 |011\rangle \\ &\quad + \alpha_1 C_1 |111\rangle + \alpha_1 C_2 |010\rangle + \alpha_1 C_3 |001\rangle + \alpha_1 C_4 |100\rangle. \end{aligned}$$

The output state of the cloning machine in the preceding section for this input is

$$\begin{aligned} |\psi\rangle^{(out)} &= \alpha_0 \mu |000\rangle + \alpha_0 \nu |011\rangle + \alpha_0 \xi |101\rangle \\ &\quad + \alpha_1 \mu |111\rangle + \alpha_1 \nu |100\rangle + \alpha_1 \xi |101\rangle. \end{aligned}$$

By comparing coefficients we see that we require

$$C_1 = \mu, \quad C_2 = \xi, \quad C_3 = 0, \quad C_4 = \nu.$$

This means that, in the notation of [2], we have

$$\begin{aligned} \mu &= \cos(\theta_1) \cos(\theta_2) \cos(\theta_3) + \sin(\theta_1) \sin(\theta_2) \sin(\theta_3) \\ \xi &= \sin(\theta_1) \cos(\theta_2) \cos(\theta_3) - \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) \\ \nu &= \cos(\theta_1) \sin(\theta_2) \sin(\theta_3) + \sin(\theta_1) \cos(\theta_2) \cos(\theta_3), \end{aligned}$$

together with the restriction that

$$\cos(\theta_1) \cos(\theta_2) \sin(\theta_3) - \sin(\theta_1) \sin(\theta_2) \cos(\theta_3) = 0.$$

Observe that since the coefficients μ, ξ, ν satisfy the normalization condition (4), this restriction is automatic. Nonetheless, we can rewrite this restriction as

$$\{\text{theta3formula}\} \quad \tan(\theta_3) = \tan(\theta_1) \tan(\theta_2), \quad (9)$$

meaning that once θ_1 and θ_2 are chosen, the value of θ_3 can be determined. In fact, we can use this formula to reduce the equations above to involve only θ_1 and θ_2 . From (9) we find that

$$\theta_3 = \arctan \left(\tan(\theta_1) \tan(\theta_2) \right),$$

from which one calculates

$$\cos(\theta_3) = \frac{1}{\sqrt{1 + \tan^2(\theta_1) \tan^2(\theta_2)}} \quad \text{and} \quad \sin(\theta_3) = \frac{\tan(\theta_1) \tan(\theta_2)}{\sqrt{1 + \tan^2(\theta_1) \tan^2(\theta_2)}}$$

We now find that

$$\begin{aligned}\mu(\theta_1, \theta_2) &= \frac{\cos(\theta_1) \cos(\theta_2) + \sin(\theta_1) \sin(\theta_2) \tan(\theta_1) \tan(\theta_2)}{\sqrt{1 + \tan^2(\theta_1) \tan^2(\theta_2)}} \\ \xi(\theta_1, \theta_2) &= \frac{\sin(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2) \tan(\theta_2)}{\sqrt{1 + \tan^2(\theta_1) \tan^2(\theta_2)}} \\ \nu(\theta_1, \theta_2) &= \frac{\sin(\theta_1) \sin(\theta_2) \tan(\theta_2) + \sin(\theta_1) \cos(\theta_2)}{\sqrt{1 + \tan^2(\theta_1) \tan^2(\theta_2)}}.\end{aligned}$$

The values of the factors η_A and η_B can now be rewritten in terms of $\theta_1, \theta_2, \theta_3$. Following some algebraic simplifications these become

$$\begin{aligned}\eta_A(\theta_1, \theta_2, \theta_3) &= \frac{1}{2} \sin(2\theta_2) \sin(2\theta_3) [1 + \sin(2\theta_1) \cos^2(\theta_2 + \theta_3)] \\ \eta_B(\theta_1, \theta_2, \theta_3) &= \sin(2\theta_1) \cos(\theta_2 + \theta_3) \cos(\theta_2 - \theta_3) - \frac{1}{4} \cos(2\theta_1) \sin(2\theta_2) \sin(2\theta_3)\end{aligned}$$

I believe these formulas can now be used to calculate the information gain and the disturbance generated by Eve (who would keep one of the copies, say the B copy) in terms of θ_1, θ_2 , and θ_3 .

2.1. Circuit Output. The output of the circuit depicted above with input $|\psi\rangle_{a_1}^{(\text{in})} = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ is

$$\begin{aligned}|\psi\rangle_{a_1 a_2 a_3}^{(\text{out})} &= \alpha_0 (C_1 |000\rangle + C_2 |101\rangle + C_3 |110\rangle + C_4 |011\rangle) \\ &\quad + \alpha_1 (C_1 |111\rangle + C_2 |010\rangle + C_3 |001\rangle + C_4 |100\rangle).\end{aligned}$$

Calculations similar to those developed in Section 1.1 yield the following reduced density operators (in the a_1 and a_2 qubits):

$$\begin{aligned}\rho_{a_1}^{(\text{our})} &= \left(|\alpha_0|^2 (C_1^2 + C_4^2) + |\alpha_1|^2 (C_2^2 + C_3^2) \right) |0\rangle \langle 0| \\ &\quad + \left(2\alpha_0 \alpha_1^* C_1 C_4 + 2\alpha_0^* \alpha_1 C_2 C_3 \right) |0\rangle \langle 1| \\ &\quad + \left(2\alpha_0 \alpha_1^* C_2 C_3 + 2\alpha_0^* \alpha_1 C_1 C_4 \right) |1\rangle \langle 0| \\ &\quad + \left(|\alpha_0|^2 (C_2^2 + C_3^2) + |\alpha_1|^2 (C_1^2 + C_4^2) \right) |1\rangle \langle 1|,\end{aligned}$$

and

$$\begin{aligned}
\rho_{a_2}^{(\text{out})} = & \left(|\alpha_0|^2 (C_1^2 + C_2^2) + |\alpha_1|^2 (C_3^2 + C_4^2) \right) |0\rangle \langle 0| \\
& + \left(2\alpha_0 \alpha_1^* C_1 C_2 + 2\alpha_0^* \alpha_1 C_3 C_4 \right) |0\rangle \langle 1| \\
& + \left(2\alpha_0 \alpha_1^* C_3 C_4 + 2\alpha_0^* \alpha_1 C_1 C_2 \right) |1\rangle \langle 0| \\
& + \left(|\alpha_0|^2 (C_3^2 + C_4^2) + |\alpha_1|^2 (C_1^2 + C_2^2) \right) |1\rangle \langle 1|.
\end{aligned}$$

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