Notes on Asymmetric Cloning

1. Asymmetric cloning machines

In this section we develop the universal quantum cloning machine (UQCM) following the presentation in [3]. We restrict our attention to the 2-dimensional case, i.e. to qubits.

1.1. **Universal Cloning Machines.** Here we consider a unitary transformation

$$|i\rangle_A |O\rangle_B |\Sigma\rangle_X \to \mu |i\rangle_A |i\rangle_B |i\rangle_X + \nu \sum_{j\neq i} \Big(|i\rangle_A |j\rangle_B + |j\rangle_A |i\rangle_B \Big) |j\rangle_X .$$

Here A refers to the input qubit, B is a blank qubit, and X is an ancilla. The ancilla is initially in some fixed state, say $|\Sigma\rangle$. In particular, the unitary can be expressed in terms of the basis states $|0\rangle$ and $|1\rangle$:

$$\begin{aligned} |0\rangle_A |O\rangle_B |\Sigma\rangle_X &\to \mu |0\rangle_A |0\rangle_B |0\rangle_X + \nu\Big(|0\rangle_A |1\rangle_B |1\rangle_X + |1\rangle_A |0\rangle_B |1\rangle_X \Big) \\ |1\rangle_A |O\rangle_B |\Sigma\rangle_X &\to \mu |1\rangle_A |1\rangle_B |1\rangle_X + \nu\Big(|1\rangle_A |0\rangle_B |0\rangle_X + |0\rangle_A |1\rangle_B |0\rangle_X \Big). \end{aligned} \tag{1}$$ {UCM}

We point out that the parameters, μ and ν, can be taken to be real parameters (imaginary terms can be absorbed into the ancilla). We impose the following restrictions on the output of the cloner:

(1) the fidelity of the copies, $F = \langle\psi|\rho^{(\mathrm{out})}|\psi\rangle$ does not depend on the particular state which is being copied;
(2) the outputs are symmetric, meaning that $\rho_A^{(\mathrm{out})} = \rho_B^{(\mathrm{out})}$.

These restrictions yield the following relations:

$$\begin{aligned} \rho_A^{(\mathrm{out})} &= \eta |\psi\rangle_A \langle\psi| + \frac{1-\eta}{2}\mathbf{1}_A \\ \rho_B^{(\mathrm{out})} &= \eta |\psi\rangle_A \langle\psi| + \frac{1-\eta}{2}\mathbf{1}_B \\ \mu^2 &= 2\mu\nu \\ \mu^2 &= \frac{2}{3} \\ \nu^2 &= \frac{1}{6} \\ \eta = \mu^2 &= \frac{2}{3}. \end{aligned}$$

Here $\mathbf{1}_A$ is the identity operator on the Hilbert space $\mathcal{H}_A$ and $\eta = 2F-1$ is called the shrinking factor (recall that F is the fidelity as defined above). In the case of qubits we see that the fidelity is $F = 5/6$.

Detailed Calculations. For brevity we will write $|ijk\rangle$ in place of $|i\rangle_A |j\rangle_B |k\rangle_X$.

First, since we require the output to be normalized, we find that

$$\mu^2 + 2\nu^2 = 1.$$

Consider input $|\psi\rangle_A = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ with $|\alpha_0|^2 + |\alpha_1|^2 = 1$. The output of the machine described by (1) is

$$\mu\alpha_0 |000\rangle + \alpha_0\nu\Big(|011\rangle + |101\rangle \Big) + \mu\alpha_1 |111\rangle + \nu\alpha_1\Big(|100\rangle + |010\rangle \Big).$$

The associated density operator is

$$\begin{aligned}\rho = &\Big(\mu\alpha_0 |000\rangle + \alpha_0\nu\Big(|011\rangle + |101\rangle \Big) + \mu\alpha_1 |111\rangle + \nu\alpha_1\Big(|100\rangle + |010\rangle \Big)\Big)\\ &\Big(\mu\alpha_0^* \langle 000| + \alpha_0^*\nu\Big(\langle 011| + \langle 101| \Big) + \mu\alpha_1^* \langle 111| + \nu\alpha_1^*\Big(\langle 100| + \langle 010| \Big)\Big)\\ = &\ \mu^2|\alpha_0|^2 |000\rangle\langle 000| + \mu\nu|\alpha_0|^2\Big(|000\rangle\langle 011| + |000\rangle\langle 101| \Big)\\ &+ \mu^2\alpha_0\alpha_1^* |000\rangle\langle 111| + \mu\nu\alpha_0\alpha_1^*\Big(|000\rangle\langle 100| + |000\rangle\langle 010| \Big) + \ldots .\end{aligned}$$

To determine the density operator $\rho_A^{(\mathrm{out})}$ we now trace out the qubits from B and X:

$$\begin{aligned}\rho_A^{(\mathrm{out})} = &\Big(|\alpha_0|^2\mu^2 + |\alpha_0|^2\nu^2 + |\alpha_1^2\nu^2\Big) |0\rangle\langle 0|\\ &+2\mu\nu\alpha_0\alpha_1^*\mu\nu |0\rangle\langle 1| + 2\mu\nu\alpha_0^*\alpha_1 |1\rangle\langle 0|\\ &+\Big(|\alpha_1|^2\mu^2 + |\alpha_0|^2\nu^2 + \alpha_1|^2\nu^2\Big) |1\rangle\langle 1|\end{aligned}$$

Recalling that $|\alpha_0|^2 + |\alpha_1|^2 = 1$, the $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ entries simplify and we are left with

$$\begin{aligned}\rho_A^{(\mathrm{out})} = &\Big(\mu^2|\alpha_0|^2 |0\rangle\langle 0| + 2\mu\nu\alpha_0\alpha_1^* |0\rangle\langle 1| + 2\mu\nu\alpha_0^*\alpha_1 |1\rangle\langle 0| + \mu^2|\alpha_1|^2\Big)\\ &+ \nu^2\Big(|0\rangle\langle 0| + |1\rangle\langle 1| \Big)\end{aligned}$$

Since $\mu^2 + 2\nu^2 = 1$ we have $\nu^2 = (1-\mu^2)/2$. An analogous calculation yields

$$|\psi\rangle_A \langle\psi| = |\alpha_0|^2 |0\rangle\langle 0| + \alpha_0\alpha_1^* |0\rangle\langle 1| + \alpha_0^*\alpha_1 |1\rangle\langle 0| + |\alpha_1|^2 |1\rangle\langle 1| .$$

With these calculations in hand, if we write

$$\rho_A^{(\mathrm{out})} = \eta |\psi\rangle_A \langle\psi| + \frac{1-\eta}{2}\mathbf{1}_A,$$

then we must have

$$\eta = \mu^2 \qquad \text{and} \qquad \mu^2 = 2\mu\nu.$$

From this last equality we have that $\mu = 0$ or $\mu = 2\nu$. Substituting this equality into $\mu^2 + 2\nu^2 = 1$ yields $6\nu^2 = 1$, whence $\nu^2 = 1/6$. It follows that $\mu^2 = 4\nu^2 = 2/3$.

□

1.2. **Asymmetric Universal Cloning Machines.** Notice that in the definition of the cloning machine given above the symmetry of the outputs ($\rho_A^{(\mathrm{out})} = \rho_B^{(\mathrm{out})}$) is a consequence of the equality of the coefficients of the terms $|i\rangle_A |j\rangle_B |j\rangle_X$ and $|j\rangle_A |i\rangle_B |j\rangle_X$. To develop an asymmetric cloning machine, then, we give different contributions to these terms. In particular, we define

$$\begin{aligned}
|0\rangle_A |O\rangle_B |\Sigma\rangle_X &\to \mu\, |0\rangle_A |0\rangle_B |0\rangle_X + \nu\, |0\rangle_A |1\rangle_B |1\rangle_X + \xi\, |1\rangle_A |0\rangle_B |0\rangle_X \\
|1\rangle_A |O\rangle_B |\Sigma\rangle_X &\to \mu\, |1\rangle_A |1\rangle_B |1\rangle_X + \nu\, |1\rangle_A |0\rangle_B |0\rangle_X + \xi\, |0\rangle_A |1\rangle_B |0\rangle_X .
\end{aligned}$$

If a state in the form $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ is given as the input this machine, then the state of the output copy A is

$$\rho_A^{(\mathrm{out})} = 2\mu\nu\, |\psi\rangle_A \langle\psi| + \xi^2 \mathbf{1}_A + (\mu^2+\nu^2-\xi^2-2\mu\nu)\Big(|\alpha_0|^2 |0\rangle\langle 0| + |\alpha_1|^2 |1\rangle\langle 1|\Big), \tag{2}$$ {Aclone}

with the corresponding output in B is

$$\rho_B^{(\mathrm{out})} = 2\mu\xi\, |\psi\rangle_A \langle\psi| + \nu^2 \mathbf{1}_A + (\mu^2+\xi^2-\nu^2-2\mu\xi)\Big(|\alpha_0|^2 |0\rangle\langle 0| + |\alpha_1|^2 |1\rangle\langle 1|\Big). \tag{3}$$ {Bclone}

Observe that $\rho_A^{(\mathrm{out})}$ and $\rho_B^{(\mathrm{out})}$ are similar; the B-case is obtained from the A-case by swapping the roles of ν and ξ.

Detailed Calculations. The calculations here are very similar to those from Section 1.1. □

Notice that the last terms in (2) and (3) are state-dependent. By imposing the requirement that the cloner be independent of the input state we require

$$\begin{aligned}
\mu^2 + \nu^2 - \xi^2 - 2\mu\nu &= 0 \\
\mu^2 + \xi^2 - \nu^2 - 2\mu\xi &= 0.
\end{aligned}$$

Adding these equations yields

$$\mu^2 - \mu\xi - \mu\nu = 0$$

from which we conclude that $\mu = \nu + \xi$. Since we require the output of the cloner to be normalized, we require that

$$\mu^2 + \nu^2 + \xi^2 = 1 \tag{4}$$ {normalization}

Also from (2) we find that

$$\eta_A = 2\mu\nu \quad \text{and} \quad \frac{1-\eta_A}{2} = \xi^2,$$

while from (3) we see that

$$\eta_B = 2\mu\xi, \quad \text{and} \quad \frac{1-\eta_B}{2} = \nu^2.$$

Recalling that the fidelity, F, is related to the shrinking factor η by $\eta = 2F - 1$, we see that these calculations yield fidelities for the A and B copies:

$$F_A = \frac{1}{2}(2\mu\nu + 1) = 1 - \xi^2$$
$$F_B = \frac{1}{2}(2\mu\xi + 1) = 1 - \nu^2.$$

1.3. **Asymmetric Phase-Covariant Cloning Machine.** Consider an input state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\phi}|1\rangle \Big).$$

In this case we find that the final term in (2) and (3) is

$$\left|\frac{1}{\sqrt{2}}\right|^2 |0\rangle\langle 0| + \left|\frac{e^{i\phi}}{\sqrt{2}}\right|^2 |1\rangle\langle 1| = \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1| \Big) = \frac{1}{2}\mathbf{1},$$

meaning that the last term is no longer dependent on the input state. In particular we find that the outputs reduce to

{PCAclone}
$$\rho_A^{(\text{out})} = 2\mu\nu\,|\psi\rangle_A\langle\psi| + \left(\xi^2 + \frac{\mu^2 + \nu^2 - \xi^2 - 2\mu\nu}{2}\right)\mathbf{1}_A \tag{5}$$

and

{PCBclone}
$$\rho_B^{(\text{out})} = 2\mu\xi\,|\psi\rangle_A\langle\psi| + \left(\nu^2 + \frac{\mu^2 + \xi^2 - \nu^2 - 2\mu\xi}{2}\right)\mathbf{1}_B. \tag{6}$$

We are thus lead to the following formulas for the shrinking factors:

{Ashrink}
$$\eta_A = 2\mu\nu = 2\nu\sqrt{1 - (\nu^2 + \xi^2)} \tag{7}$$

{Bshrink}
$$\eta_B = 2\mu\xi = 2\xi\sqrt{1 - (\nu^2 + \xi^2)}. \tag{8}$$

This cloning machine is optimal if, whenever we fix the quality of one of the clones, say A, the quality of the other clone is as high as possible. Since the quality of the clone A can be expressed in terms of η_A, η_B, we focus on the trade-off in the shrinking factors. For a fixed value of η_A we solve (7) for ξ in terms of ν and insert this into (8) to see that

{etaB_optimal}
$$\eta_B(\nu) = \frac{\eta_A}{\nu}\sqrt{1 - \nu^2 - \frac{\eta_A^2}{4\nu^2}}. \tag{9}$$

Thus given a value of η_A we can determine a value of ν that maximizes the value of η_B. We note that the domain of η_B has two components:

$$\text{Domain } \eta_B = \left[-\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \eta_A^2}}, -\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \eta_A^2}}\right]$$
$$\bigcup \left[\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1 - \eta_A^2}}, \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 - \eta_A^2}}\right].$$

Since we expect the scaling coefficient η_B to be nonnegative, we seek a solution from the positive component of the domain. That is, we require

$$\nu \in \left[\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1-\eta_A^2}}, \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1-\eta_A^2}} \right].$$

1.4. **Optimization.** We seek to optimize the value of η_B given η_A. Since the fidelity of a clone is given by $F = (1+\eta)/2$, this amounts to optimizing the fidelity of the B-clone once the fidelity of the A-clone has been set. Recalling that $\eta_A = 2\mu\nu$ and $\eta_B = 2\mu\xi$ with $\mu^2 + \nu^2 + \xi^2 = 1$, we notice that one approach is to consider this as a constrained optimization problem to which we apply the method of Lagrange multipliers. As such we seek values of $\mu, \nu, \xi, \lambda_1, \lambda_2$ so that

$$\begin{aligned} \nabla \eta_B &= \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2, \\ g_1 &= 0, \\ g_2 &= 0, \end{aligned} \tag{10}$$ {lagrange1}

where

$$g_1(\mu,\nu,\xi) = 2\mu\nu - \eta_A \quad \text{and} \quad g_2(\mu,\nu,\xi) = \mu^2 + \nu^2 + \xi^2 - 1.$$

The equations (10) yield the following system of (nonlinear) equations:

$$\xi = \lambda_1\nu + \lambda_2\mu \tag{11}$$ {lagrangesys1}

$$0 = \lambda_1\mu + \lambda_2\nu \tag{12}$$ {lagrangesys2}

$$\mu = \lambda_2\xi \tag{13}$$ {lagrangesys3}

$$2\mu\nu = \eta_A \tag{14}$$ {lagrangesys4}

$$1 = \mu_2 + \nu^2 + \xi^2. \tag{15}$$ {lagrangesys5}

We view equations (11) and (12) as a linear system of equations in μ and ν (viewing λ_1 and λ_2 as coefficients), the solution of which is given by

$$\mu = \frac{\lambda_2}{\lambda_2^2 - \lambda_1^2}\xi \quad \text{and} \quad \nu = -\frac{\lambda_1}{\lambda_2^2 - \lambda_1^2}\xi. \tag{16}$$

In light of equation (13) together with the solution to (11) and (12) we find that

$$\lambda_2^2 - \lambda_1^2 = 1,$$

meaning that

$$\mu = \lambda_2\xi \quad \text{and} \quad \nu = -\lambda_1\xi.$$

These facts can now be inserted into (15) to yield

$$\xi^2(\lambda_2^2 + \lambda_1^2 + 1) = 1.$$

The fact that $\xi_2^2 - \xi_1^2 = 1$ allows us to simplify this to read

$$2\lambda_2^2\xi^2 = 1.$$

Once again we use equation (13) to see that

$$2\mu^2 = 1,$$

whence $\mu = \pm\sqrt{2}/2$. We now use (14) to find that

$$\nu = \pm\frac{\eta_A}{\sqrt{2}}.$$

Returning our formulas for μ, ν to equation (15) yields

$$\xi = \pm\sqrt{\frac{1-\eta_A^2}{2}}.$$

It follows that the optimal value of η_B is given by

$$\eta_B = 2\left(\frac{\sqrt{2}}{2}\right)\sqrt{\frac{1-\eta_A^2}{2}} = \sqrt{1-\eta_A^2}.$$

This yields the circle relation

$$\eta_A^2 + \eta_B^2 = 1,$$

which agrees with the optimization result in [3].

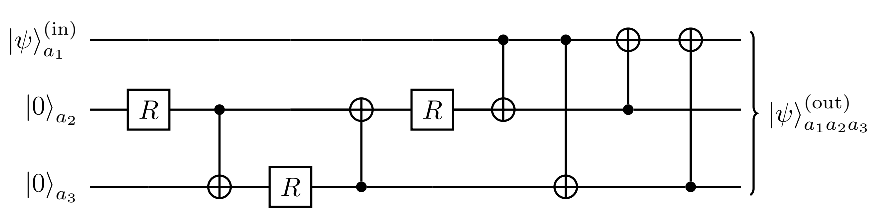
We point out that in the symmetric case in which $\eta_A = \eta_B$ we have $\eta_A = 1/\sqrt{2}$ which yields the optimal fidelity

$$F = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{\sqrt{8}},$$

in agreement with [2].

2. Implementation

The following circuit is drawn from [1]. We write $|\psi\rangle_{a_1}^{(\mathrm{in})}$ for the qubit we are trying to clone. The circuit below aims to produce two copies of the input qubit. In their initial state we write $|0\rangle_{a_2}, |0\rangle_{a_3}$ for these qubits. The first part of the circuit prepares the target qubits (a_2 and a_3) in a state which is useful for the cloning operation. The second component of the circuit (which involves $|\psi\rangle_{a_1}^{(\mathrm{in})}$) is the piece of the circuit that handles the actual copying.



Here the gate $R = R(\theta)$ is a rotation gate defined by

$$R|0\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$$
$$R|1\rangle = -\sin(\theta)|0\rangle + \cos(\theta)|0\rangle .$$

The first part of this circuit involves only the a_2 and a_3 qubits; this is a preparation component of the circuit. The output of this portion of the circuit is of the form

$$|\psi\rangle^{(\mathrm{out})}_{a_2 a_3} = C_1 |0\rangle_{a_2} |0\rangle_{a_3} + C_2 |0\rangle_{a_2} |1\rangle_{a_3} + C_3 |1\rangle_{a_2} |0\rangle_{a_3} + C_4 |1\rangle_{a_2} |1\rangle_{a_3} .$$

Following the circuit above we find that the coefficients $C_j, j = 1, 2, 3, 4$ are given by

$$\begin{aligned} C_1 &= \cos(\theta_1)\cos(\theta_2)\cos(\theta_3) + \sin(\theta_1)\sin(\theta_2)\sin(\theta_3) \\ C_2 &= \sin(\theta_1)\cos(\theta_2)\cos(\theta_3) - \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) \\ C_3 &= \cos(\theta_1)\cos(\theta_2)\sin(\theta_3) - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3) \\ C_4 &= \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) + \sin(\theta_1)\cos(\theta_2)\cos(\theta_3). \end{aligned}$$

Consider an input $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$. The output from the circuit above is

$$\begin{aligned} |\psi\rangle^{(out)} &= \alpha_0 C_1 |000\rangle + \alpha_0 C_2 |101\rangle + \alpha_0 C_3 |110\rangle + \alpha_0 C_4 |011\rangle \\ &\quad + \alpha_1 C_1 |111\rangle + \alpha_1 C_2 |010\rangle + \alpha_1 C_3 |001\rangle + \alpha_1 C_4 |100\rangle . \end{aligned}$$

The output state of the cloning machine in the preceding section for this input is

$$\begin{aligned} |\psi\rangle^{(out)} &= \alpha_0 \mu |000\rangle + \alpha_0 \nu |011\rangle + \alpha_0 \xi |101\rangle \\ &\quad + \alpha_1 \mu |111\rangle + \alpha_1 \nu |100\rangle + \alpha_1 \xi |101\rangle . \end{aligned}$$

By comparing coefficients we see that we require

$$C_1 = \mu, \quad C_2 = \xi, \quad C_3 = 0, \quad C_4 = \nu.$$

This means that, in the notation of [3], we have

$$\begin{aligned} \mu &= \cos(\theta_1)\cos(\theta_2)\cos(\theta_3) + \sin(\theta_1)\sin(\theta_2)\sin(\theta_3) \\ \xi &= \sin(\theta_1)\cos(\theta_2)\cos(\theta_3) - \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) \\ \nu &= \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) + \sin(\theta_1)\cos(\theta_2)\cos(\theta_3), \end{aligned}$$

together with the restriction that

$$\cos(\theta_1)\cos(\theta_2)\sin(\theta_3) - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3) = 0.$$

Observe that since the coefficients μ, ξ, ν satisfy the normalization condition (4), this restriction is automatic. Nonetheless, we can rewrite this restriction as

$$\tan(\theta_3) = \tan(\theta_1)\tan(\theta_2), \tag{17}$$ {theta3formula}

meaning that once θ_1 and θ_2 are chosen, the value of θ_3 can be determined. In fact, we can use this formula to reduce the equations above to involve only θ_1 and θ_2. From (17) we find that

$$\theta_3 = \arctan\Big(\tan(\theta_1)\tan(\theta_2)\Big),$$

from which one calculates

$$\cos(\theta_3) = \frac{1}{\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}} \quad \text{and} \quad \sin(\theta_3) = \frac{\tan(\theta_1)\tan(\theta_2)}{\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}}$$

We now find that

$$\begin{aligned}
\mu(\theta_1,\theta_2) &= \frac{\cos(\theta_1)\cos(\theta_2)+\sin(\theta_1)\sin(\theta_2)\tan(\theta_1)\tan(\theta_2)}{\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}} \\
\xi(\theta_1,\theta_2) &= \frac{\sin(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)\tan(\theta_2)}{\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}} \\
\nu(\theta_1,\theta_2) &= \frac{\sin(\theta_1)\sin(\theta_2)\tan(\theta_2)+\sin(\theta_1)\cos(\theta_2)}{\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}}.
\end{aligned}$$

These formulas for the coefficients of the cloner in terms of the circuit angles can be further simplified:

$$\begin{aligned}
\mu(\theta_1,\theta_2) &= \cos(\theta_1)\cos(\theta_2)\left(\frac{1+\tan^2(\theta_1)\tan^2(\theta_2)}{\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}}\right) \\
&= \cos(\theta_1)\cos(\theta_2)\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}; \\
\xi(\theta_1,\theta_2) &= \sin(\theta_1)\cos(\theta_2)\left(\frac{1-\tan^2(\theta_2)}{\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}}\right); \\
\nu(\theta_1,\theta_2) &= \frac{\sin(\theta_1)\cos(\theta_2)(\tan^2(\theta_2)+1)}{\sqrt{1+\tan^2(\theta_1)\tan^2(\theta)}} \\
&= \frac{\sin(\theta_1)\cos(\theta_2)\sec^2(\theta_2)}{\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}} \\
&= \frac{\sin(\theta_1)}{\cos(\theta_2)\sqrt{1+\tan^2(\theta_1)\tan^2(\theta_2)}}.
\end{aligned}$$

The values of the factors η_A and η_B can now be rewritten in terms of θ_1 and θ_2. The simplified formulas provided above yield particularly convenient formulas:

$$\eta_A(\theta_1,\theta_2) = 2\sin(\theta_1)\cos(\theta_1) = \sin(2\theta_1)$$

and

$$\eta_B(\theta_1,\theta_2) = 2\sin(\theta_1)\cos(\theta_1)\cos^2(\theta_2)\Big(1-\tan^2(\theta_2)\Big)$$
$$= \sin(2\theta_1)\cos(2\theta_2).$$

Of significance here is the fact that η_A is independent of θ_1. Note that the problem of choosing an optimal cloning scenario was spelled out at the end of the last section. Once one selects θ_1 the value of η_A is determined. The optimal cloning problem amounts to selecting a value of θ_2 so that

$$\eta_B = \frac{\eta_A}{\nu}\sqrt{1-\nu^2-\frac{\eta_A^2}{4\nu^2}}$$

is maximized. This can now be rewritten as an optimization problem in the variable θ_2.

2.0.1. *Fidelity.* It is common to evaluate the quality of the clones using the measure of fidelity, defined by

$$F_A = \langle\psi|\,\rho_A^{(\text{out})}\,|\psi\rangle \qquad \text{and} \qquad F_B = \langle\psi|\,\rho_B^{(\text{out})}\,|\psi\rangle\,,$$

where $|\psi\rangle$ is the input state that we were attempting to clone. Having written

$$\rho_A^{(\text{out})} = \eta_A\,\langle\psi|\,|\psi\rangle + \frac{1-\eta_A}{2}\mathbf{1}, \quad \text{and} \quad \rho_B^{(\text{out})} = \eta_B\,\langle\psi|\,|\psi\rangle + \frac{1-\eta_B}{2}\mathbf{1}$$

the fidelities are given by

$$F_A = \frac{1}{2}(1+\eta_A) \qquad \text{and} \qquad F_B = \frac{1}{2}(1+\eta_B).$$

In particular, with the aid of the formulas for η_A and η_B developed above we see that

$$F_A(\theta_1,\theta_2) = \frac{1}{2}\Big(1+\sin(2\theta_1)\Big) \quad \text{and} \quad F_B(\theta_1,\theta_2) = \frac{1}{2}\Big(1+\sin(2\theta_1)\cos(\theta_2)\Big).$$

Note that these calculations do not reflect the relationship that exists between η_A and η_B that are a reflection of the normalization feature of the coefficients (4). The optimal cloning situation is determined by selecting a value of θ_1 and then determining the value of θ_2 that optimizes η_B as in (9). Indeed if we write

$$\eta_B^{\text{opt}} = \max_{\theta_2}\left[\frac{\eta_A(\theta_1,\theta_2)}{\nu(\theta_1,\theta_2)}\sqrt{1-\nu^2(\theta_1,\theta_2)-\frac{\eta_A^2(\theta_1,\theta_2)}{4\nu^2(\theta_1,\theta_2)}}\right]$$

then the optimal cloning situation has the fidelities

$$F_A = \frac{1}{2}(1+\sin(2\theta_1)) \qquad \text{and} \qquad F_B = \frac{1}{2}(1+\eta_B^{\text{opt}}).$$

Note that by having optimized over θ_2 to obtain η_B^{opt}, these fidelities depend only on θ_1.

To determine the angle θ_2 that yields the optimal fidelity in the B-clone we use the optimization arguments introduced earlier. The shrinking coefficients satisfy

$$\eta_A^2 + \eta_B^2 = 1.$$

Since $\eta_A = \sin(2\theta_1)$ and $\eta_B = \sin(2\theta_1)\cos(2\theta_2)$ we find that we achieve the optimal fidelity for the B-clone provided

$$\theta_2 = \frac{1}{2}\arccos(\cot(2\theta_1)).$$

This formula imposes some restrictions on the values of θ_1. First we note that we require $\theta_1 \neq n\pi$ for any $n \in \mathbf{Z}$ in order to remain in the domain of the cotangent function. In order to satisfy the domain requirements for the inverse cosine function we further require that

$$\frac{\pi}{8} \leq \theta_1 \leq \frac{3\pi}{8}.$$

2.1. **Circuit Output.** The output of the circuit depicted above with input $|\psi\rangle_{a_1}^{(\text{in})} = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ is

$$\begin{aligned} |\psi\rangle_{a_1 a_2 a_3}^{(\text{out})} =& \alpha_0\Big(C_1 |000\rangle + C_2 |101\rangle + C_3 |110\rangle + C_4 |011\rangle\Big) \\ &+\alpha_1\Big(C_1 |111\rangle + C_2 |010\rangle + C_3 |001\rangle + C_4 |100\rangle\Big). \end{aligned}$$

Calculations similar to those developed in Section 1.1 yield the following reduced density operators (in the a_1 and a_2 qubits):

$$\begin{aligned} \rho_{a_1}^{(\text{our})} =& \Big(|\alpha_0|^2(C_1^2 + C_4^2) + |\alpha_1|^2(C_2^2 + C_3^2)\Big) |0\rangle\langle 0| \\ &+\Big(2\alpha_0\alpha_1^* C_1 C_4 + 2\alpha_0^*\alpha_1 C_2 C_3\Big) |0\rangle\langle 1| \\ &+\Big(2\alpha_0\alpha_1^* C_2 C_3 + 2\alpha_0^*\alpha_1 C_1 C_4\Big) |1\rangle\langle 0| \\ &+\Big(|\alpha_0|^2(C_2^2 + C_3^2) + |\alpha_1|^2(C_1^2 + C_4^2)\Big) |1\rangle\langle 1|, \end{aligned}$$

and

$$\begin{aligned} \rho_{a_2}^{(\text{out})} =& \Big(|\alpha_0|^2(C_1^2 + C_2^2) + |\alpha_1^2(C_3^2 + C_4^2)\Big) |0\rangle\langle 0| \\ &+\Big(2\alpha_0\alpha_1^* C_1 C_2 + 2\alpha_0^*\alpha_1 C_3 C_4\Big) |0\rangle\langle 1| \\ &+\Big(2\alpha_0\alpha_1^* C_3 C_4 + 2\alpha_0^*\alpha_1 C_1 C_2\Big) |1\rangle\langle 0| \\ &+\Big(|\alpha_0|^2(C_3^2 + C_4^2) + |\alpha_1|^2(C_1^2 + C_2^2)\Big) |1\rangle\langle 1|. \end{aligned}$$

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