Notes on Asymmetric Cloning

1. Asymmetric cloning machines

In this section we develop the universal quantum cloning machine (UQCM) following the presentation in [2]. We restrict our attention to the 2-dimensional case, i.e. to qubits.

1.1. **Universal Cloning Machines.** Here we consider a unitary transformation

$$|i\rangle_A |O\rangle_B |\Sigma\rangle_X \to \mu |i\rangle_A |i\rangle_B |i\rangle_X + \nu \sum_{j \neq i} \Big(|i\rangle_A |j\rangle_B + |j\rangle_A |i\rangle_B \Big) |j\rangle_X .$$

Here A refers to the input qubit, B is a blank qubit, and X is an ancilla. The ancilla is initially in some fixed state, say $|\Sigma\rangle$. In particular, the unitary can be expressed in terms of the basis states $|0\rangle$ and $|1\rangle$:

$$\begin{aligned} |0\rangle_A |O\rangle_B |\Sigma\rangle_X &\to \mu |0\rangle_A |0\rangle_B |0\rangle_X + \nu \Big(|0\rangle_A |1\rangle_B |1\rangle_X + |1\rangle_A |0\rangle_B |1\rangle_X \Big) \\ |1\rangle_A |O\rangle_B |\Sigma\rangle_X &\to \mu |1\rangle_A |1\rangle_B |1\rangle_X + \nu \Big(|1\rangle_A |0\rangle_B |0\rangle_X + |0\rangle_A |1\rangle_B |0\rangle_X \Big). \end{aligned} \tag{1}$$ {UCM}

We point out that the parameters, μ and ν, can be taken to be real parameters (imaginary terms can be absorbed into the ancilla). We impose the following restrictions on the output of the cloner:

(1) the fidelity of the copies, $F = \langle\psi|\rho^{(\mathrm{out})}|\psi\rangle$ does not depend on the particular state which is being copied;
(2) the outputs are symmetric, meaning that $\rho_A^{(\mathrm{out})} = \rho_B^{(\mathrm{out})}$.

These restrictions yield the following relations:

$$\begin{aligned} \rho_A^{(\mathrm{out})} &= \eta |\psi\rangle_A \langle\psi| + \frac{1-\eta}{2} \mathbf{1}_A \\ \rho_B^{(\mathrm{out})} &= \eta |\psi\rangle_A \langle\psi| + \frac{1-\eta}{2} \mathbf{1}_B \\ \mu^2 &= 2\mu\nu \\ \mu^2 &= \frac{2}{3} \\ \nu^2 &= \frac{1}{6} \\ \eta = \mu^2 &= \frac{2}{3}. \end{aligned}$$

Here $\mathbf{1}_A$ is the identity operator on the Hilbert space $\mathcal{H}_A$ and $\eta = 2F - 1$ is called the shrinking factor (recall that F is the fidelity as defined above). In the case of qubits we see that the fidelity is $F = 5/6$.

Detailed Calculations. For brevity we will write $|ijk\rangle$ in place of $|i\rangle_A |j\rangle_B |k\rangle_X$.

First, since we require the output to be normalized, we find that

$$\mu^2 + 2\nu^2 = 1.$$

Consider input $|\psi\rangle_A = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ with $|\alpha_0|^2 + |\alpha_1|^2 = 1$. The output of the machine described by (1) is

$$\mu\alpha_0 |000\rangle + \alpha_0\nu\Big(|011\rangle + |101\rangle \Big) + \mu\alpha_1 |111\rangle + \nu\alpha_1\Big(|100\rangle + |010\rangle \Big).$$

The associated density operator is

$$\begin{aligned}
\rho = &\Big(\mu\alpha_0 |000\rangle + \alpha_0\nu\Big(|011\rangle + |101\rangle \Big) + \mu\alpha_1 |111\rangle + \nu\alpha_1\Big(|100\rangle + |010\rangle \Big)\Big) \\
&\Big(\mu\alpha_0^* \langle 000| + \alpha_0^*\nu\Big(\langle 011| + \langle 101| \Big) + \mu\alpha_1^* \langle 111| + \nu\alpha_1^*\Big(\langle 100| + \langle 010| \Big)\Big) \\
= &\mu^2|\alpha_0|^2 |000\rangle\langle 000| + \mu\nu|\alpha_0|^2\Big(|000\rangle\langle 011| + |000\rangle\langle 101| \Big) \\
&+ \mu^2\alpha_0\alpha_1^* |000\rangle\langle 111| + \mu\nu\alpha_0\alpha_1^*\Big(|000\rangle\langle 100| + |000\rangle\langle 010| \Big) + \ldots .
\end{aligned}$$

To determine the density operator $\rho_A^{(\text{out})}$ we now trace out the qubits from B and X:

$$\begin{aligned}
\rho_A^{(\text{out})} = &\Big(|\alpha_0|^2\mu^2 + |\alpha_0|^2\nu^2 + |\alpha_1^2\nu^2\Big) |0\rangle\langle 0| \\
&+ 2\mu\nu\alpha_0\alpha_1^*\mu\nu |0\rangle\langle 1| + 2\mu\nu\alpha_0^*\alpha_1 |1\rangle\langle 0| \\
&+ \Big(|\alpha_1|^2\mu^2 + |\alpha_0|^2\nu^2 + \alpha_1|^2\nu^2\Big) |1\rangle\langle 1|
\end{aligned}$$

Recalling that $|\alpha_0|^2 + |\alpha_1|^2 = 1$, the $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ entries simplify and we are left with

$$\begin{aligned}
\rho_A^{(\text{out})} = &\Big(\mu^2|\alpha_0|^2 |0\rangle\langle 0| + 2\mu\nu\alpha_0\alpha_1^* |0\rangle\langle 1| + 2\mu\nu\alpha_0^*\alpha_1 |1\rangle\langle 0| + \mu^2|\alpha_1|^2\Big) \\
&+ \nu^2\Big(|0\rangle\langle 0| + |1\rangle\langle 1| \Big)
\end{aligned}$$

Since $\mu^2 + 2\nu^2 = 1$ we have $\nu^2 = (1 - \mu^2)/2$. An analogous calculation yields

$$|\psi\rangle_A \langle\psi| = |\alpha_0|^2 |0\rangle\langle 0| + \alpha_0\alpha_1^* |0\rangle\langle 1| + \alpha_0^*\alpha_1 |1\rangle\langle 0| + |\alpha_1|^2 |1\rangle\langle 1| .$$

With these calculations in hand, if we write

$$\rho_A^{(\text{out})} = \eta |\psi\rangle_A \langle\psi| + \frac{1-\eta}{2}\mathbf{1}_A,$$

then we must have

$$\eta = \mu^2 \qquad \text{and} \qquad \mu^2 = 2\mu\nu.$$

From this last equality we have that $\mu = 0$ or $\mu = 2\nu$. Substituting this equality into $\mu^2 + 2\nu^2 = 1$ yields $6\nu^2 = 1$, whence $\nu^2 = 1/6$. It follows that $\mu^2 = 4\nu^2 = 2/3$.

□

1.2. **Asymmetric Universal Cloning Machines.** Notice that in the definition of the cloning machine given above the symmetry of the outputs ($\rho_A^{(\mathrm{out})} = \rho_B^{(\mathrm{out})}$) is a consequence of the equality of the coefficients of the terms $|i\rangle_A |j\rangle_B |j\rangle_X$ and $|j\rangle_A |i\rangle_B |j\rangle_X$. To develop an asymmetric cloning machine, then, we give different contributions to these terms. In particular, we define

$$\begin{aligned}
|0\rangle_A |O\rangle_B |\Sigma\rangle_X &\to \mu |0\rangle_A |0\rangle_B |0\rangle_X + \nu |0\rangle_A |1\rangle_B |1\rangle_X + \xi |1\rangle_A |0\rangle_B |0\rangle_X \\
|1\rangle_A |O\rangle_B |\Sigma\rangle_X &\to \mu |1\rangle_A |1\rangle_B |1\rangle_X + \nu |1\rangle_A |0\rangle_B |0\rangle_X + \xi |0\rangle_A |1\rangle_B |0\rangle_X .
\end{aligned}$$

If a state in the form $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ is given as the input this machine, then the state of the output copy A is

$$\rho_A^{(\mathrm{out})} = 2\mu\nu |\psi\rangle_A \langle\psi| + \xi^2 \mathbf{1}_A + (\mu^2 + \nu^2 - \xi^2 - 2\mu\nu)\Big(|\alpha_0|^2 |0\rangle\langle 0| + |\alpha_1|^2 |1\rangle\langle 1| \Big), \tag{2}$$ {Aclone}

with the corresponding output in B is

$$\rho_B^{(\mathrm{out})} = 2\mu\xi |\psi\rangle_A \langle\psi| + \nu^2 \mathbf{1}_A + (\mu^2 + \xi^2 - \nu^2 - 2\mu\xi)\Big(|\alpha_0|^2 |0\rangle\langle 0| + |\alpha_1|^2 |1\rangle\langle 1| \Big). \tag{3}$$ {Bclone}

Observe that $\rho_A^{(\mathrm{out})}$ and $\rho_B^{(\mathrm{out})}$ are similar; the B-case is obtained from the A-case by swapping the roles of ν and ξ.

Detailed Calculations. The calculations here are very similar to those from Section 1.1. □

Notice that the last terms in (2) and (3) are state-dependent. By imposing the requirement that the cloner be independent of the input state we require

$$\begin{aligned}
\mu^2 + \nu^2 - \xi^2 - 2\mu\nu &= 0 \\
\mu^2 + \xi^2 - \nu^2 - 2\mu\xi &= 0.
\end{aligned}$$

Adding these equations yields

$$\mu^2 - \mu\xi - \mu\nu = 0$$

from which we conclude that $\mu = \nu + \xi$. Since we require the output of the cloner to be normalized, we require that

$$\mu^2 + \nu^2 + \xi^2 = 1 \tag{4}$$ {normalization}

Also from (2) we find that

$$\eta_A = 2\mu\nu \quad \text{and} \quad \frac{1-\eta_A}{2} = \xi^2,$$

while from (3) we see that

$$\eta_B = 2\mu\xi, \quad \text{and} \quad \frac{1-\eta_B}{2} = \nu^2.$$

Recalling that the fidelity, F, is related to the shrinking factor η by $\eta = 2F-1$, we see that these calculations yield fidelities for the A and B copies:

$$F_A = \frac{1}{2}(2\mu\nu + 1) = 1 - \xi^2$$
$$F_B = \frac{1}{2}(2\mu\xi + 1) = 1 - \nu^2.$$

1.3. **Asymmetric Phase-Covariant Cloning Machine.** Consider an input state of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle + e^{i\phi}|1\rangle \Big).$$

In this case we find that the final term in (2) and (3) is

$$\left|\frac{1}{\sqrt{2}}\right|^2 |0\rangle\langle 0| + \left|\frac{e^{i\phi}}{\sqrt{2}}\right|^2 |1\rangle\langle 1| = \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1| \Big) = \frac{1}{2}\mathbf{1},$$

meaning that the last term is no longer dependent on the input state. In particular we find that the outputs reduce to

{PCAclone} $$\rho_A^{(\mathrm{out})} = 2\mu\nu\,|\psi\rangle_A\langle\psi| + \left(\xi^2 + \frac{\mu^2+\nu^2-\xi^2-2\mu\nu}{2}\right)\mathbf{1}_A \tag{5}$$

and

{PCBclone} $$\rho_B^{(\mathrm{out})} = 2\mu\xi\,|\psi\rangle_A\langle\psi| + \left(\nu^2 + \frac{\mu^2+\xi^2-\nu^2-2\mu\xi}{2}\right)\mathbf{1}_B. \tag{6}$$

We are thus lead to the following formulas for the shrinking factors:

{Ashrink} $$\eta_A = 2\mu\nu = 2\nu\sqrt{1-(\nu^2+\xi^2)} \tag{7}$$

{Bshrink} $$\eta_B = 2\mu\xi = 2\xi\sqrt{1-(\nu^2+\xi^2)}. \tag{8}$$

This cloning machine is optimal if, whenever we fix the quality of one of the clones, say A, the quality of the other clone is as high as possible. Since the quality of the clone A can be expressed in terms of η_A, η_B, we focus on the trade-off in the shrinking factors. For a fixed value of η_A we solve (7) for ξ in terms of ν and insert this into (8) to see that

$$\eta_B(\nu) = \frac{\eta_A}{\nu}\sqrt{1-\nu^2-\frac{\eta_A^2}{4\nu^2}}.$$

Thus given a value of η_A we can determine a value of ν that maximizes the value of η_B. We note that the domain of η_B has two components:

$$\text{Domain } \eta_B = \left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\eta_A^2}}, -\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-\eta_A^2}}\right] \bigcup \left[\sqrt{\frac{1}{2}-\frac{1}{2}\sqrt{1-\eta_A^2}}, \sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\eta_A^2}}\right].$$

Since we expect the scaling coefficient η_B to be nonnegative, we seek a solution from the positive component of the domain. That is, we require

$$\nu \in \left[\sqrt{\frac{1}{2} - \frac{1}{2}\sqrt{1-\eta_A^2}}, \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1-\eta_A^2}}\right].$$

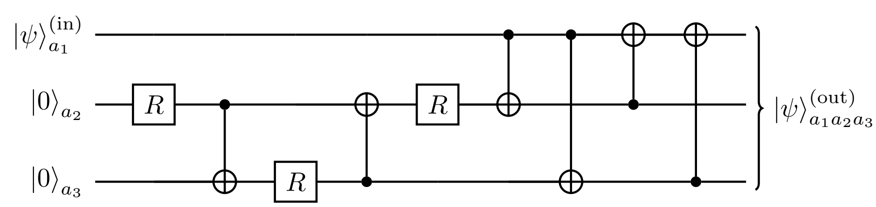
In general dimesions there is no formula for the corresponding value of ν, but in dimension 2 there might be a formula. Right now I have reduced this to the following equation for ν:

$$4\nu^5 - 4\nu^4 + 4\nu^2 - 3\eta^2 = 0.$$

Can we show that there is a unique value of ν that corresponds to the maximum?

2. Implementation

The following circuit is drawn from [1]. We write $|\psi\rangle_{a_1}^{(\mathrm{in})}$ for the qubit we are trying to clone. The circuit below aims to produce two copies of the input qubit. In their initial state we write $|0\rangle_{a_2}, |0\rangle_{a_3}$ for these qubits. The first part of the circuit prepares the target qubits (a_2 and a_3) in a state which is useful for the cloning operation. The second component of the circuit (which involves $|\psi\rangle_{a_1}^{(\mathrm{in})}$) is the piece of the circuit that handles the actual copying.



Here the gate $R = R(\theta)$ is a rotation gate defined by

$$\begin{aligned} R\,|0\rangle &= \cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle \\ R\,|1\rangle &= -\sin(\theta)\,|0\rangle + \cos(\theta)\,|0\rangle\,. \end{aligned}$$

The first part of this circuit involves only the a_2 and a_3 qubits; this is a preparation component of the circuit. The output of this portion of the circuit is of the form

$$|\psi\rangle_{a_2a_3}^{(\mathrm{out})} = C_1\,|0\rangle_{a_2}|0\rangle_{a_3} + C_2\,|0\rangle_{a_2}|1\rangle_{a_3} + C_3\,|1\rangle_{a_2}|0\rangle_{a_3} + C_4\,|1\rangle_{a_2}|1\rangle_{a_3}\,.$$

Following the circuit above we find that the coefficients $C_j, j = 1, 2, 3, 4$ are given by

$$
\begin{aligned}
C_1 &= \cos(\theta_1)\cos(\theta_2)\cos(\theta_3) + \sin(\theta_1)\sin(\theta_2)\sin(\theta_3)\\
C_2 &= \sin(\theta_1)\cos(\theta_2)\cos(\theta_3) - \cos(\theta_1)\sin(\theta_2)\sin(\theta_3)\\
C_3 &= \cos(\theta_1)\cos(\theta_2)\sin(\theta_3) - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3)\\
C_4 &= \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) + \sin(\theta_1)\cos(\theta_2)\cos(\theta_3).
\end{aligned}
$$

Consider an input $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$. The output from the circuit above is

$$
\begin{aligned}
|\psi\rangle^{(out)} &= \alpha_0 C_1 |000\rangle + \alpha_0 C_2 |101\rangle + \alpha_0 C_3 |110\rangle + \alpha_0 C_4 |011\rangle\\
&\quad + \alpha_1 C_1 |111\rangle + \alpha_1 C_2 |010\rangle + \alpha_1 C_3 |001\rangle + \alpha_1 C_4 |100\rangle .
\end{aligned}
$$

The output state of the cloning machine in the preceding section for this input is

$$
\begin{aligned}
|\psi\rangle^{(out)} &= \alpha_0 \mu |000\rangle + \alpha_0 \nu |011\rangle + \alpha_0 \xi |101\rangle\\
&\quad + \alpha_1 \mu |111\rangle + \alpha_1 \nu |100\rangle + \alpha_1 \xi |101\rangle .
\end{aligned}
$$

By comparing coefficients we see that we require

$$C_1 = \mu, \quad C_2 = \xi, \quad C_3 = 0, \quad C_4 = \nu.$$

This means that, in the notation of [2], we have

$$
\begin{aligned}
\mu &= \cos(\theta_1)\cos(\theta_2)\cos(\theta_3) + \sin(\theta_1)\sin(\theta_2)\sin(\theta_3)\\
\xi &= \sin(\theta_1)\cos(\theta_2)\cos(\theta_3) - \cos(\theta_1)\sin(\theta_2)\sin(\theta_3)\\
\nu &= \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) + \sin(\theta_1)\cos(\theta_2)\cos(\theta_3),
\end{aligned}
$$

together with the restriction that

$$\cos(\theta_1)\cos(\theta_2)\sin(\theta_3) - \sin(\theta_1)\sin(\theta_2)\cos(\theta_3) = 0.$$

Observe that since the coefficients μ, ξ, ν satisfy the normalization condition (4), this restriction is automatic. Nonetheless, we can rewrite this restriction as

$$\tan(\theta_3) = \tan(\theta_1)\tan(\theta_2),$$

meaning that once θ_1 and θ_2 are chosen, the value of θ_3 can be determined.

The values of the factors η_A and η_B can now be rewritten in terms of $\theta_1, \theta_2, \theta 3$. Following some algebraic simplfications these become

$$\eta_A(\theta_1, \eta_2, \eta_3) = \frac{1}{2}\sin(2\theta_2)\sin(2\theta_3)\left[1 + \sin(2\theta_1)\cos^2(\theta_2 + \theta_3)\right]$$

$$\eta_B(\theta_1, \theta_2, \theta_3) = \sin(2\theta_1)\cos(\theta_2 + \theta_3)\cos(\theta_2 - \theta_3) - \frac{1}{4}\cos(2\theta_1)\sin(2\theta_2)\sin(2\theta_3)$$

I believe these formulas can now be used to calculate the information gain and the disturbance generated by Eve (who would keep one of the copies, say the B copy) in terms of θ_1, θ_2, and θ_3.

References

[1] V. Bužek, S. L. Braunstein, M. Hillery, and D. Bruß. Quantum copying: A network. *Phys. Rev. A*, 56:3446–3452, Nov 1997.

[2] A.T. Rezakhani, S. Siadatnejad, and A.H. Ghaderi. Separability in asymmetric phase-covariant cloning. *Physics Letters A*, 336(4):278–289, 2005.