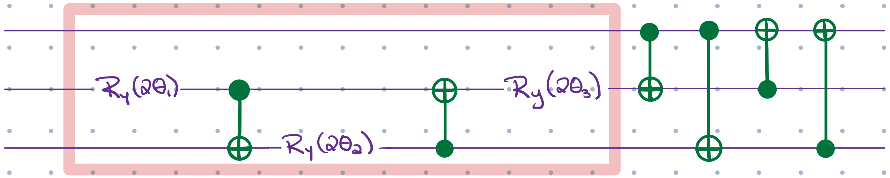
Original:



We know $\theta_1 = \frac{\pi}{4} \rightsquigarrow R_y(2\theta_1) = R_y(\pi/2)$

*Observe the general relationship: $R_y(\gamma) \equiv R_x(\frac{\pi}{2}) - R_z(\gamma) - R_x(-\frac{\pi}{2}) \equiv SX - R_z(\gamma) - SX - X$

Note $R_x(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = e^{-\pi i/4}\sqrt{X}$, so $RX(\frac{\pi}{2}) \equiv SX$

$RX(\pi) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -iX$, so $RX(\pi) \equiv X$

*$R_y(\frac{\pi}{2}) = R_x(-\frac{\pi}{2}) - R_z(-\frac{\pi}{2}) - R_x(\frac{\pi}{2})$; note $R_x(-\frac{\pi}{2}) \equiv R_x(\frac{3\pi}{2}) \equiv X - SX$

$\equiv X - SX - R_z(-\frac{\pi}{2}) - SX$

We also need H gate for prep: $H \equiv X - SX - R_z(-\frac{\pi}{2}) - SX - X$

IBM Basis Gates: X, SX, Rz, CX

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad SX = \sqrt{X} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$

$Rz(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$ $\quad CNOT \equiv CX$

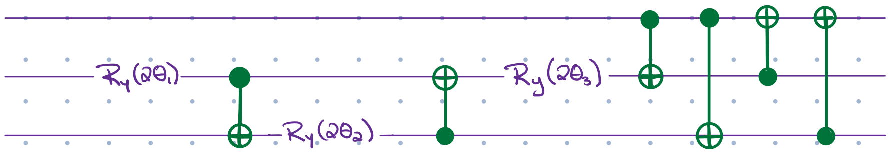
*See calculations on next page

$H \overset{*}{\equiv} R_y(\frac{\pi}{2}) - R_x(-\pi)$

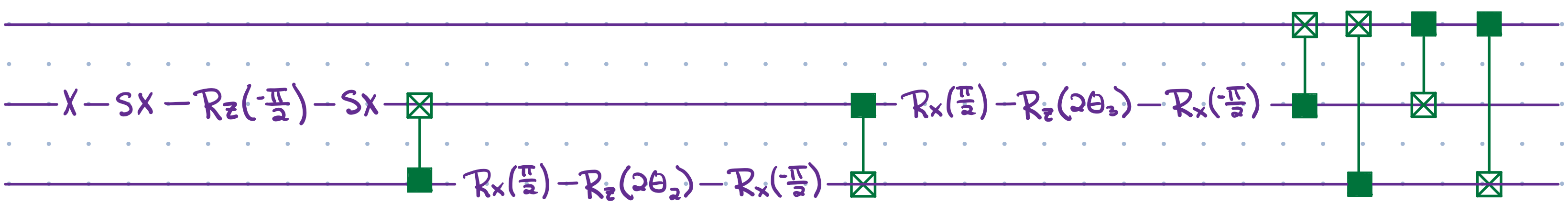
$\equiv X - SX - R_z(-\frac{\pi}{2}) - SX - X$

Verify by multiplying the representative matrices (see next page)

* from Maslov



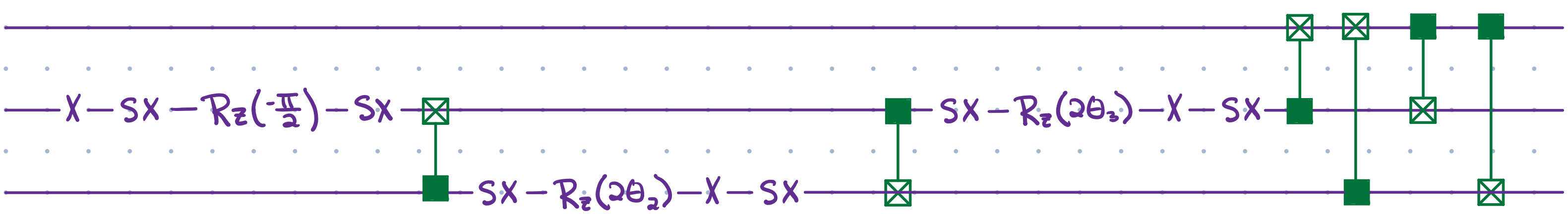
*CNOT → CX(control, target) + *relations:



to Basis Gates

$R_x(\frac{\pi}{2}) \equiv SX$

$R_x(-\frac{\pi}{2}) \equiv X - SX$



* Note: IBM's CX: is equivalent to CNOT: , the representation is simply flipped

* $R_y\left(\frac{\pi}{2}\right) \equiv R_x\left(\frac{\pi}{2}\right) R_z\left(-\frac{\pi}{2}\right) R_x\left(-\frac{\pi}{2}\right)$

As circuit: $R_x\left(-\frac{\pi}{2}\right) - R_z\left(-\frac{\pi}{2}\right) - R_x\left(\frac{\pi}{2}\right)$

$R_y\left(\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\begin{pmatrix} \cos\frac{\pi}{4} & -i\sin\frac{\pi}{4} \\ -i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}\begin{pmatrix} e^{\pi/4} & 0 \\ 0 & e^{-\pi/4} \end{pmatrix}\begin{pmatrix} \cos(-\pi/4) & -i\sin(-\pi/4) \\ -i\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1+i & i-1 \\ i+1 & 1-i \end{pmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{pmatrix} 1+i+1-i & i-1-i-1 \\ -i+1+i+1 & 1+i+1-i \end{pmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^3 \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = R_y\left(\frac{\pi}{2}\right)$$

In general, $R_y(\gamma) \equiv R_x\left(\frac{\pi}{2}\right) - R_z(\gamma) - R_x\left(-\frac{\pi}{2}\right)$:

$$\underbrace{\begin{pmatrix} \cos(-\pi/4) & -i\sin(-\pi/4) \\ -i\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{4} & -i\sin\frac{\pi}{4} \\ -i\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}}_{R_x\left(-\frac{\pi}{2}\right) \cdot R_z(\gamma) \cdot R_x\left(\frac{\pi}{2}\right)} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} e^{-i\gamma/2} & -ie^{-i\gamma/2} \\ -ie^{i\gamma/2} & e^{i\gamma/2} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\gamma/2} + e^{i\gamma/2} & -ie^{-i\gamma/2} + ie^{i\gamma/2} \\ ie^{-i\gamma/2} - ie^{i\gamma/2} & e^{-i\gamma/2} + e^{i\gamma/2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

a: $e^{-i\gamma/2} + e^{i\gamma/2} = \cos\left(-\frac{\gamma}{2}\right) + \cancel{i\sin\left(-\frac{\gamma}{2}\right)} + \cos\frac{\gamma}{2} + \cancel{i\sin\frac{\gamma}{2}} = 2\cos\frac{\gamma}{2}$

b: $\cancel{-i\cos\left(-\frac{\gamma}{2}\right)} + \sin\left(-\frac{\gamma}{2}\right) + \cancel{i\cos\frac{\gamma}{2}} - \sin\frac{\gamma}{2} = -2\sin\frac{\gamma}{2}$

c: $\cancel{i\cos\left(-\frac{\gamma}{2}\right)} - \sin\left(-\frac{\gamma}{2}\right) - \cancel{i\cos\frac{\gamma}{2}} + \sin\frac{\gamma}{2} = 2\sin\gamma/2$

d: $\cos\left(-\frac{\gamma}{2}\right) + \cancel{i\sin\left(-\frac{\gamma}{2}\right)} + \cos\frac{\gamma}{2} + \cancel{i\sin\frac{\gamma}{2}} = 2\cos\gamma/2$

$$= \begin{pmatrix} \cos\frac{\gamma}{2} & -\sin\frac{\gamma}{2} \\ \sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} \end{pmatrix} = R_y(\gamma)$$

H gate $= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$SX \cdot R_z\left(-\frac{\pi}{2}\right) \cdot SX = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}\begin{pmatrix} e^{\pi i/4} & 0 \\ 0 & e^{-\pi i/4} \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}\begin{pmatrix} e^{\pi i/4} + ie^{\pi i/4} & e^{\pi i/4} - ie^{\pi i/4} \\ e^{-\pi i/4} - e^{-\pi i/4} & e^{-\pi i/4} + ie^{-\pi i/4} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} (1+i)^2 e^{\pi i/4} + (1-i)^2 e^{-\pi i/4} & (1+i)(1-i)(e^{\pi i/4} + e^{-\pi i/4}) \\ (1+i)(1-i)(e^{\pi i/4} + e^{-\pi i/4}) & (1+i)^2 e^{-\pi i/4} + (1-i)^2 e^{\pi i/4} \end{pmatrix}$$

$\overset{*}{=} \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}}$ $\cdot X = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ & mult by X in front: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H$ ✓

$\overset{*}{=} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

a: $(1+2i-1)e^{\pi i/4} + (1-2i-1)e^{-\pi i/4} = 2i(e^{\pi i/4} - e^{-\pi i/4})$

d: $2i(e^{-\pi i/4} - e^{\pi i/4})$

b & c: $(1+1)(e^{\pi i/4} + e^{-\pi i/4}) = 2(e^{\pi i/4} + e^{-\pi i/4})$

$e^{\pi i/4} = \frac{1}{\sqrt{2}}(1+i)$ & $e^{-\pi i/4} = \frac{1}{\sqrt{2}}(1-i)$

so $a = i\sqrt{2}(1+i-(1-i)) = i\sqrt{2}(2i) = -2\sqrt{2}$

$d = i\sqrt{2}(1-i-1-i) = 2\sqrt{2}$

b & c: $\sqrt{2}(1+i+1-i) = 2\sqrt{2}$