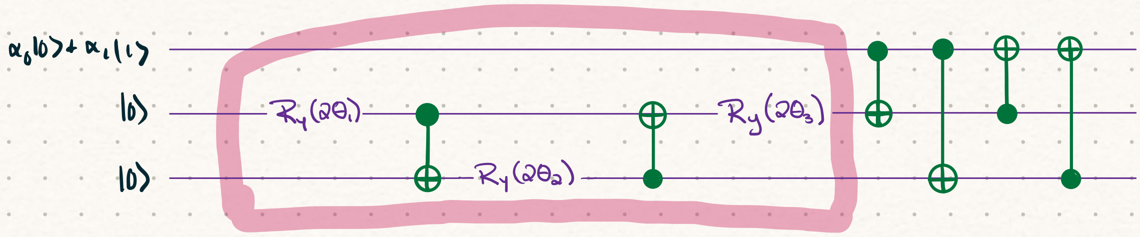
$P_{kl}|0\rangle_k|0\rangle_l = |0\rangle_k|0\rangle_l$ $\quad P_{kl}|0\rangle_k|1\rangle_l = |0\rangle_k|1\rangle_l$

$P_{kl}|1\rangle_k|0\rangle_l = |1\rangle_k|1\rangle_l$ $\quad P_{kl}|1\rangle_k|1\rangle_l = |1\rangle_k|0\rangle_l$

$|\,\rangle_k$ is control qubit so $|\,\rangle_l$ changes if we have $|1\rangle_k$

$R_j(\theta)|0\rangle_j = \cos\theta|0\rangle_j + \sin\theta|1\rangle_j$

$R_j(\theta)|1\rangle_j = -\sin\theta|0\rangle_j + \cos\theta|1\rangle_j$



Prep

$$R_2(\theta_1)|0\rangle = \cos\theta_1|0\rangle + \sin\theta_1|1\rangle$$

$$P_{23}(\cos\theta_1|0\rangle+\sin\theta_1|1\rangle)|0\rangle_3 = \cos\theta_1|0\rangle|0\rangle+\sin\theta_1|1\rangle|1\rangle$$

$$R_3(\theta_2)(\cos\theta_1|0\rangle|0\rangle+\sin\theta_1|1\rangle|1\rangle) = \cos\theta_1|0\rangle(\cos\theta_2|0\rangle+\sin\theta_2|1\rangle)+\sin\theta_1|1\rangle(-\sin\theta_2|0\rangle+\cos\theta_2|1\rangle)$$

$$= \cos\theta_1\cos\theta_2|0\rangle|0\rangle+\cos\theta_1\sin\theta_2|0\rangle|1\rangle-\sin\theta_1\sin\theta_2|1\rangle|0\rangle+\sin\theta_1\cos\theta_2|1\rangle|1\rangle$$

$$P_{32}(\cos\theta_1\cos\theta_2|0\rangle|0\rangle+\cos\theta_1\sin\theta_2|0\rangle|1\rangle-\sin\theta_1\sin\theta_2|1\rangle|0\rangle+\sin\theta_1\cos\theta_2|1\rangle|1\rangle)$$

$$= \cos\theta_1\cos\theta_2|0\rangle|0\rangle+\cos\theta_1\sin\theta_2|1\rangle|1\rangle-\sin\theta_1\sin\theta_2|1\rangle|0\rangle+\sin\theta_1\cos\theta_2|0\rangle|1\rangle$$

$$R_2(\theta_3)(\cos\theta_1\cos\theta_2|0\rangle|0\rangle+\cos\theta_1\sin\theta_2|1\rangle|1\rangle-\sin\theta_1\sin\theta_2|1\rangle|0\rangle+\sin\theta_1\cos\theta_2|0\rangle|1\rangle)$$

$$\cos\theta_1\cos\theta_2(\cos\theta_3|0\rangle+\sin\theta_3|1\rangle)|0\rangle+\cos\theta_1\sin\theta_2(-\sin\theta_3|0\rangle+\cos\theta_3|1\rangle)|1\rangle$$

$$-\sin\theta_1\sin\theta_2(-\sin\theta_3|0\rangle+\cos\theta_3|1\rangle)|0\rangle+\sin\theta_1\cos\theta_2(\cos\theta_3|0\rangle+\sin\theta_3|1\rangle)|1\rangle$$

So out from prep is

$$\cos\theta_1\cos\theta_2\cos\theta_3|00\rangle+\cos\theta_1\cos\theta_2\sin\theta_3|10\rangle-\cos\theta_1\sin\theta_2\sin\theta_3|01\rangle$$

$$+\cos\theta_1\sin\theta_2\cos\theta_3|11\rangle+\sin\theta_1\sin\theta_2\sin\theta_3|00\rangle-\sin\theta_1\sin\theta_2\cos\theta_3|10\rangle$$

$$+\sin\theta_1\cos\theta_2\cos\theta_3|01\rangle+\sin\theta_1\cos\theta_2\sin\theta_3|11\rangle$$

$$C_1|00\rangle+C_2|01\rangle+C_3|10\rangle+C_4|11\rangle$$

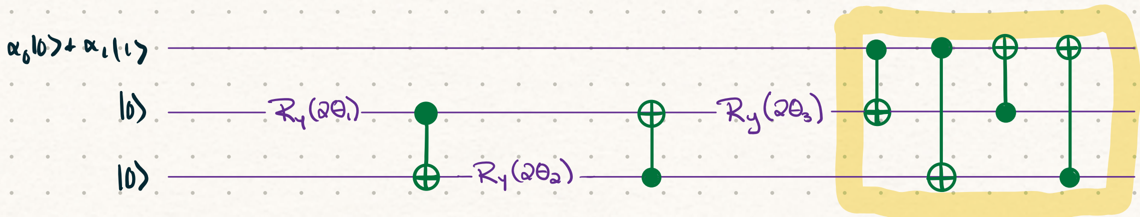
Prep output verified ✓

$C_1 = \cos\theta_1\cos\theta_2\cos\theta_3+\sin\theta_1\sin\theta_2\sin\theta_3$

$C_2 = \sin\theta_1\cos\theta_2\cos\theta_3-\cos\theta_1\sin\theta_2\sin\theta_3$

$C_3 = \cos\theta_1\cos\theta_2\sin\theta_3-\sin\theta_1\sin\theta_2\cos\theta_3$

$C_4 = \cos\theta_1\sin\theta_2\cos\theta_3+\sin\theta_1\cos\theta_2\sin\theta_3$



Cloning $\alpha_0|0\rangle+\alpha_1|1\rangle$

$$P_{12}\big((\alpha_0|0\rangle+\alpha_1|1\rangle)(c_1|00\rangle+c_2|01\rangle+c_3|10\rangle+c_4|11\rangle)\big)$$

$$\alpha_0c_1|0\rangle|00\rangle+\alpha_0c_2|0\rangle|01\rangle+\alpha_0c_3|0\rangle|10\rangle+\alpha_0c_4|0\rangle|11\rangle$$
$$+\alpha_1c_1|1\rangle|10\rangle+\alpha_1c_2|1\rangle|11\rangle+\alpha_1c_3|1\rangle|00\rangle+\alpha_1c_4|1\rangle|01\rangle$$

$$\Big\downarrow P_{13}$$

$$\alpha_0c_1|0\rangle|00\rangle+\alpha_0c_2|0\rangle|01\rangle+\alpha_0c_3|0\rangle|10\rangle+\alpha_0c_4|0\rangle|11\rangle$$
$$+\alpha_1c_1|1\rangle|11\rangle+\alpha_1c_2|1\rangle|10\rangle+\alpha_1c_3|1\rangle|01\rangle+\alpha_1c_4|1\rangle|00\rangle$$

$$\Big\downarrow P_{21}$$

$$\alpha_0c_1|0\rangle|00\rangle+\alpha_0c_2|0\rangle|01\rangle+\alpha_0c_3|1\rangle|10\rangle+\alpha_0c_4|1\rangle|11\rangle$$
$$+\alpha_1c_1|0\rangle|11\rangle+\alpha_1c_2|0\rangle|10\rangle+\alpha_1c_3|1\rangle|01\rangle+\alpha_1c_4|1\rangle|00\rangle$$

$$\Big\downarrow P_{31}$$

$$\alpha_0c_1|0\rangle|00\rangle+\alpha_0c_2|1\rangle|01\rangle+\alpha_0c_3|1\rangle|10\rangle+\alpha_0c_4|0\rangle|11\rangle$$
$$+\alpha_1c_1|1\rangle|11\rangle+\alpha_1c_2|0\rangle|10\rangle+\alpha_1c_3|0\rangle|01\rangle+\alpha_1c_4|1\rangle|00\rangle$$

$$=\boxed{\begin{array}{l}\alpha_0c_1|000\rangle+\alpha_0c_2|101\rangle+\alpha_0c_3|110\rangle+\alpha_0c_4|011\rangle\\ +\alpha_1c_1|111\rangle+\alpha_1c_2|010\rangle+\alpha_1c_3|001\rangle+\alpha_1c_4|100\rangle\end{array}}$$

Cloner output ✓

Note that running cloner backwards switches c_2 & c_3

Rezakhani: $|i\rangle_A |0\rangle_B |\Sigma\rangle_X \rightarrow \mu |i\rangle_A |i\rangle_B |i\rangle_X + \nu |i\rangle_A |j\rangle_B |j\rangle_X + \xi |j\rangle_A |i\rangle_B |j\rangle_X$

$i=0: \quad \mu|000\rangle + \nu|011\rangle + \xi|101\rangle$

$i=1: \quad \mu|111\rangle + \nu|100\rangle + \xi|010\rangle$

$$\alpha_0 C_1|000\rangle + \alpha_0 C_2|101\rangle + \alpha_0 C_3|110\rangle + \alpha_0 C_4|011\rangle$$
$$+\alpha_1 C_1|111\rangle + \alpha_1 C_2|010\rangle + \alpha_1 C_3|001\rangle + \alpha_1 C_4|100\rangle$$

$\therefore \quad C_1 = \mu \quad C_2 = \xi \quad C_3 = 0 \quad C_4 = \nu$

(α_i absorbed into qubit)

Checking μ, ξ, ν relations & resulting η_A, η_B :

$$\left.\begin{aligned} C_1 &= \cos\theta_1\cos\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3 \\ C_2 &= \sin\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_2\sin\theta_3 \\ C_3 &= \cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3 \\ C_4 &= \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\cos\theta_2\sin\theta_3 \end{aligned}\right\} \quad C_1 = \mu \quad C_2 = \xi \quad C_3 = 0 \quad C_4 = \nu$$

so we have $\cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3 = 0$

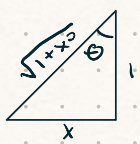
Assuming $\theta_1, \theta_2, \theta_3 \neq \frac{k\pi}{2}$, $k = 2l+1$, $l \in \mathbb{Z}$, we divide by $\cos\theta_1\cos\theta_2\cos\theta_3$ to get

$$\frac{\sin\theta_3}{\cos\theta_3} - \frac{\sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2} = 0 \iff \tan\theta_3 = \tan\theta_1\tan\theta_2$$

$\hookrightarrow$ $\theta_3 = \arctan(\tan\theta_1\tan\theta_2)$

Note range of arctan is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ & Domain is $(-\infty, \infty)$

& $\theta_1, \theta_2 \in \left(\frac{k\pi}{2}, \frac{(k+2)\pi}{2}\right)$, $k = 2l+1$, $l \in \mathbb{Z}$



$$\cos\theta_3 = \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

$$\text{\& } \sin\theta_3 = \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$$

$x = \tan\theta_1\tan\theta_2$

We use these values to rewrite $\mu, \xi,$ & ν in terms of θ_1 & θ_2

$$\mu = \cos\theta_1\cos\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3$$

$$= \frac{\cos\theta_1\cos\theta_2}{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}} + \frac{\sin\theta_1\sin\theta_2\tan\theta_1\tan\theta_2}{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}} \cdot \frac{\cos\theta_1\cos\theta_2}{\cos\theta_1\cos\theta_2}$$

$$= \frac{\cos\theta_1\cos\theta_2(1+\tan^2\theta_1\tan^2\theta_2)}{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}}$$

$$\mu = \cos\theta_1\cos\theta_2\sqrt{1+\tan^2\theta_1\tan^2\theta_2}$$

$$\xi = \sin\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_2\sin\theta_3$$

$$= \frac{\sin\theta_1\cos\theta_2 - \cancel{\cos}\theta_1\sin\theta_2\overset{\sin\theta_1}{\cancel{\tan\theta_1}}\tan\theta_2}{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}} \cdot \frac{\cos\theta_2}{\cos\theta_2}$$

$$\xi = \frac{\sin\theta_1\cos\theta_2(1-\tan^2\theta_2)}{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}}$$

$$\nu = \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\cos\theta_2\sin\theta_3$$

$$= \frac{\cos\theta_1\sin\theta_2 + \sin\theta_1\cancel{\cos\theta_2}\tan\theta_1\overset{\sin\theta_2}{\cancel{\tan\theta_2}}}{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}} \cdot \frac{\cos\theta_1}{\cos\theta_1}$$

$$= \frac{\cos\theta_1\sin\theta_2(\underbrace{1+\tan^2\theta_1}_{\sec^2\theta_1})}{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}}$$

$$\nu = \frac{\sin\theta_2}{\cos\theta_1\sqrt{1+\tan^2\theta_1\tan^2\theta_2}}$$

$\boxed{\eta_A = 2\mu\nu}$ & $\boxed{\eta_B = 2\mu\xi}$

$\boxed{\eta_A = 2\mu\nu}$

$$= 2\cancel{\cos\theta_1}\cos\theta_2\cancel{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}} \cdot \frac{\sin\theta_2}{\cancel{\cos\theta_1}\cancel{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}}}$$

$$= 2\sin\theta_2\cos\theta_2$$

$$= \boxed{\sin 2\theta_2}$$

$\boxed{\eta_B = 2\mu\xi}$

$$= 2\cos\theta_1\cos\theta_2\cancel{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}} \cdot \frac{\sin\theta_1\cos\theta_2\left(1 - \overset{\frac{\sin^2\theta_2}{\cos^2\theta_2}}{\tan^2\theta_2}\right)}{\cancel{\sqrt{1+\tan^2\theta_1\tan^2\theta_2}}}$$

$$= 2\sin\theta_1\cos\theta_1(\cos^2\theta_2 - \sin^2\theta_2)$$

$$= \boxed{\sin(2\theta_1)\cos(2\theta_2)}$$

Circle Relation : $\boxed{\eta_A^2 + \eta_B^2 = 1}$ $\rightsquigarrow$ $\sin^2 2\theta_2 + \sin^2 2\theta_1\cos^2 2\theta_2 = 1$

$$\iff \sin^2 2\theta_1 \cancel{\cos^2 2\theta_2} = \cancel{\cos^2 2\theta_2}$$

Comes from optimization (eq6 §2.4)

$F = \frac{1+\eta}{2}$

$$\iff \sin^2 2\theta_1 = 1$$

$$\iff \boxed{\theta_1 = \frac{k\pi}{4}, \ k = 2l+1, \ l \in \mathbb{Z}}$$

Compared to the Fidelity & version (pg 9/10) using the $\eta_B = \frac{\eta_A}{\nu}\sqrt{1 - \nu^2 - \frac{\eta_A^2}{4\nu^2}}$ relation & plugging in eqs in terms of θ_1 & θ_2

Fix η_A & optimize η_B, but that would mean we fix θ_2 to optimize θ_1? though the circle relation requires $\theta_1 = \frac{k\pi}{4}$

Should we be fixing θ_1 to optimize both? concurrently??

$C_1 = \cos\theta_1\cos\theta_2\cos\theta_3 + \sin\theta_1\sin\theta_2\sin\theta_3$
$C_2 = \sin\theta_1\cos\theta_2\cos\theta_3 - \cos\theta_1\sin\theta_2\sin\theta_3$
$C_3 = \cos\theta_1\cos\theta_2\sin\theta_3 - \sin\theta_1\sin\theta_2\cos\theta_3$
$C_4 = \cos\theta_1\sin\theta_2\cos\theta_3 + \sin\theta_1\cos\theta_2\sin\theta_3$

$\theta_1 = \frac{\pi}{4}, \ \theta_2 = \theta_3 =: \theta$:

$C_1 = \frac{1}{\sqrt{2}}(\cos^2\theta + \sin^2\theta) = \boxed{\frac{1}{\sqrt{2}}}$

$C_2 = \frac{1}{\sqrt{2}}(\cos^2\theta - \sin^2\theta)$
$= \boxed{\frac{1}{\sqrt{2}}\cos 2\theta}$

$C_3 = \frac{1}{\sqrt{2}}(\cancel{\cos\theta\sin\theta - \sin\theta\cos\theta}) \ \boxed{= 0}$

$C_4 = \frac{1}{\sqrt{2}}(\sin\theta\cos\theta + \cos\theta\sin\theta)$
$= \boxed{\frac{1}{\sqrt{2}}\sin 2\theta}$