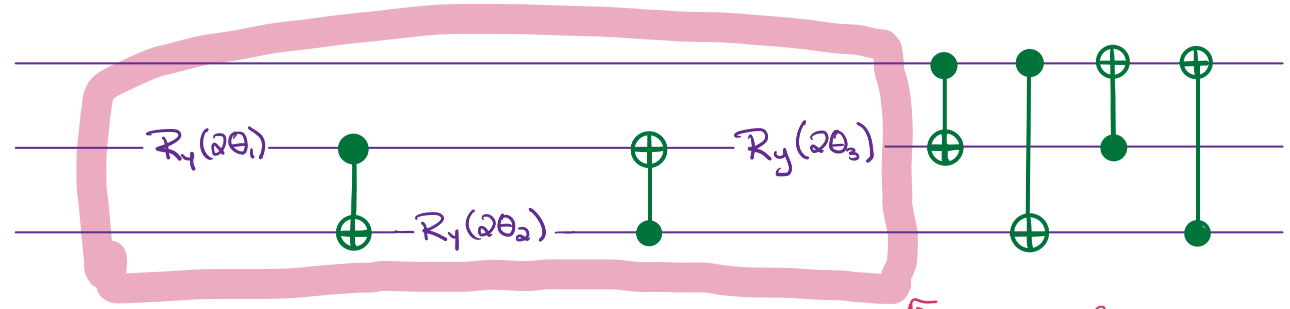
Original:



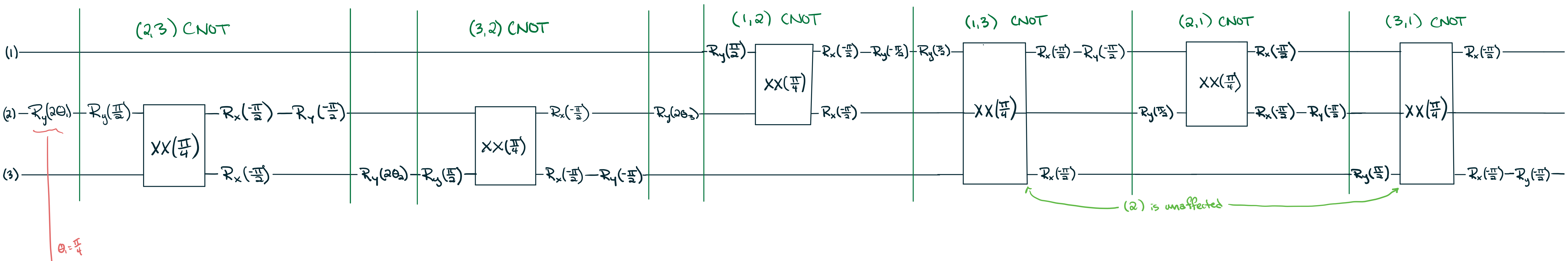
this part is prep (Buzek)

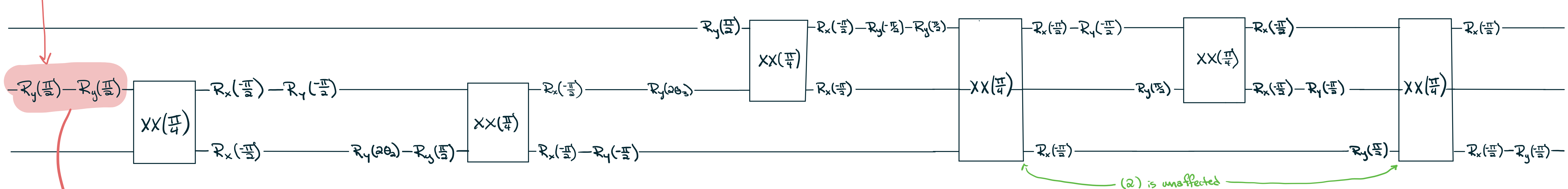
we know $\theta_1 = \frac{\pi}{4} \rightsquigarrow R_y(2\theta_1) = R_y(\pi/2)$

which should be simply GPi2(.25)

$\theta_2 = \theta_3 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

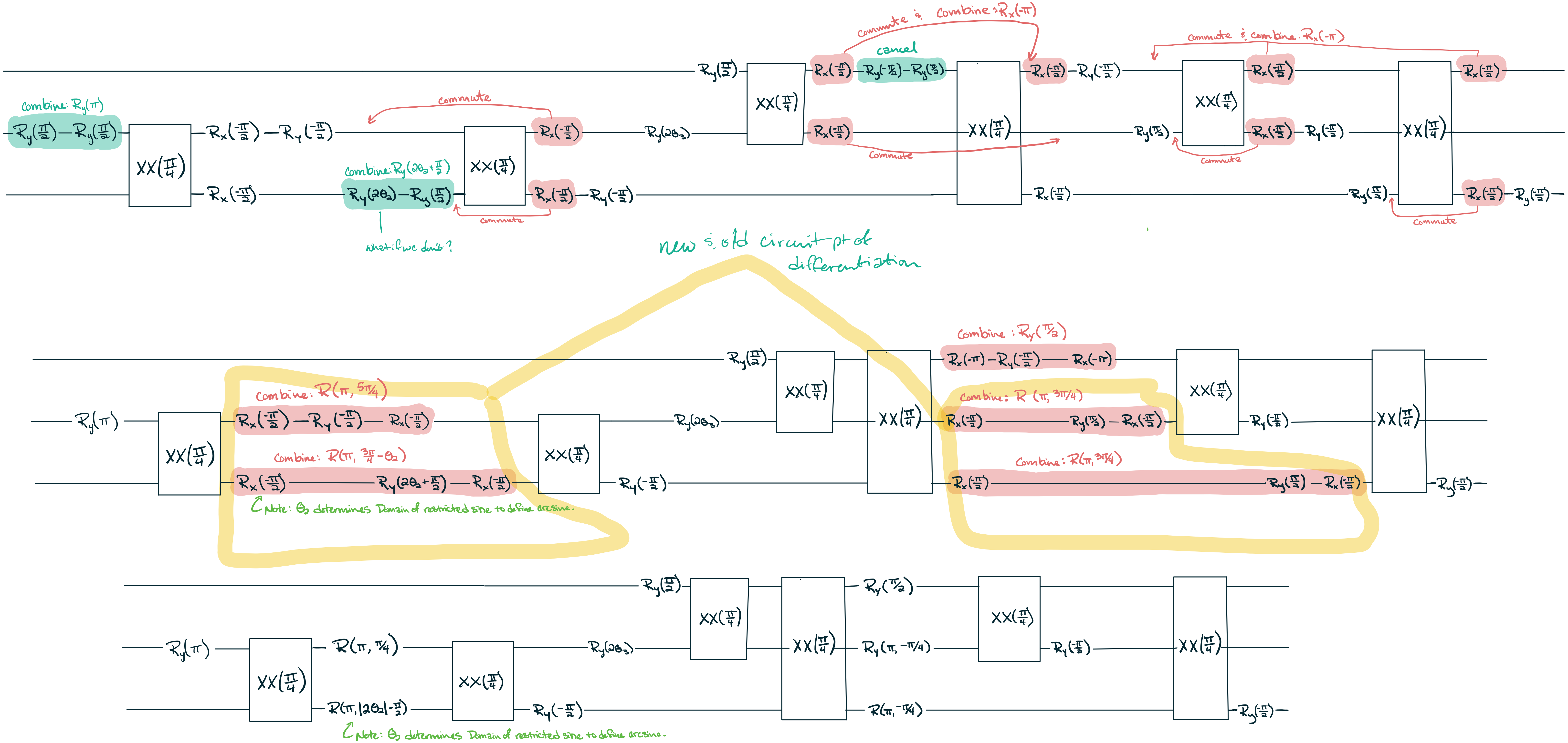
Converting CNOT to XX, R_x, & R_y





$R_y(\pi) \rightsquigarrow GPi(\pi/2) = GPi(.25)$ by global phase equiv $(i = e^{\pi i/2})$

Simplifying using R(π,θ) & Maslov Identities



$R_x(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$ $R_y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$

$\cos\frac{-\pi}{4} = \frac{1}{\sqrt{2}}$ $\sin\frac{-\pi}{4} = -\frac{1}{\sqrt{2}}$ $\cos\frac{-\pi}{2} = 0$, $\sin\frac{-\pi}{2} = -1$

$R_x(a) - R_y(b) - R_x(a) - R(c, d-\pi) \equiv$ —

$c = 2\arccos\left(\cos(a)\cos\left(\frac{b}{2}\right)\right)$

$$d = \begin{cases} \arcsin\left(\frac{\sin(b/2)}{\sqrt{1-\cos^2(a)\cos^2(b/2)}}\right) & a>0 \\ \pi - \arcsin\left(\frac{\sin(b/2)}{\sqrt{1-\cos^2(a)\cos^2(b/2)}}\right) & a<0 \end{cases}$$

$$R(\theta,\phi) = \begin{bmatrix} \cos\theta/2 & -ie^{-i\phi}\sin\theta/2 \\ -ie^{i\phi}\sin\theta/2 & \cos\frac{\theta}{2} \end{bmatrix}$$

combine: $R(\pi, \pi/4)$

$R_x(\frac{-\pi}{2}) - R_y(\frac{-\pi}{2}) - R_x(\frac{-\pi}{2})$ —

combine: $R(\pi, |2\theta_2| - \frac{\pi}{2})$

$R_x(\frac{-\pi}{2})$ — $R_y(2\theta_2 + \frac{\pi}{2}) - R_x(\frac{-\pi}{2})$ —

Note: θ_2 determines Domain of restricted sine to define arcsine.

define restricted sin on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

Combine: $R_y(\pi/2)$

— $R_x(-\pi) - R_y(\frac{-\pi}{2})$ — $R_x(-\pi)$ —

$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \begin{bmatrix} i & i \\ i & -i \end{bmatrix}$

$= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} = -R_y(\pi/2)$

phase equiv to $R_y(\pi/2)$ b/c mult by $e^{-\pi i}$

combine: $R(\pi, -\pi/4)$

$R_x(\frac{-\pi}{2})$ — $R_y(\frac{\pi}{2}) - R_x(\frac{-\pi}{2})$

Combine: $R(\pi, -\pi/4)$

— $R_x(\frac{-\pi}{2})$ — $R_y(\frac{\pi}{2}) - R_x(\frac{-\pi}{2})$

these should be $R(\pi, 3\pi/4)$

$c = 2\arccos(\underbrace{0}_{\frac{\pi}{2}}) = \pi$

$a = -\frac{\pi}{2} = b$

$d = \pi - \underbrace{\arcsin\left(\frac{-1/\sqrt{2}}{\sqrt{1-0}}\right)}_{-\frac{\pi}{4}} = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ → $d - \pi = \frac{\pi}{4}$

$a = -\frac{\pi}{2}$ & $b = 2\theta + \frac{\pi}{2}$

$d = \pi - \arcsin\left(\frac{\sin(2\theta + \pi/2)}{\sqrt{1-0}}\right) = \pi - \arcsin\left(\sin\left(2\theta + \frac{\pi}{2}\right)\right) = \frac{\pi}{2} + |2\theta|$

→ $d - \pi = |2\theta_2| - \frac{\pi}{2}$

$$\begin{cases} \pi - \left(2\theta + \frac{\pi}{2}\right) = \frac{\pi}{2} - 2\theta & \text{if } \cancel{2\theta \in [-\pi, 0]} \quad 2\theta \in [-\frac{\pi}{2}, 0] \\ \pi - \left(\pi - \left(2\theta + \frac{\pi}{2}\right)\right) = \frac{\pi}{2} + 2\theta & \text{if } \cancel{2\theta \in (0, \pi]} \quad 2\theta \in (0, \frac{\pi}{2}] \end{cases}$$

$a = -\frac{\pi}{2}$ & $b = \frac{\pi}{2}$

$d = \pi - \underbrace{\arcsin\left(\frac{1/\sqrt{2}}{\sqrt{1-0}}\right)}_{\pi/4} = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

↓

$d - \pi = -\frac{\pi}{4}$

As def. in notes $= 2\theta_2$ here

$2\theta = \boxed{\theta_2} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow 2\theta + \frac{\pi}{2} \in [0, \pi]$

where $\theta = \theta_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

so $2\theta \in (-\pi, \pi)$

$\Rightarrow 2\theta + \frac{\pi}{2} \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$2\theta + \frac{\pi}{2} \in \left[0, \frac{\pi}{2}\right]$

arcsine gives $2\theta + \frac{\pi}{2}$

for $2\theta + \frac{\pi}{2} \in \left(\frac{\pi}{2}, \pi\right]$

we get $\pi - \left(2\theta + \frac{\pi}{2}\right)$ from arcsine

$= \frac{\pi}{2} - 2\theta$

if $2\theta \in [0, \pi]$

then $\frac{\pi}{2}, \frac{3\pi}{2}$

$\uparrow \quad \uparrow$

$1 \quad -1$

$\downarrow \quad \downarrow$ arcsin

$\frac{\pi}{2} \quad -\frac{\pi}{2}$

So $\pi - 2\theta$

Combine: $R(\pi, \frac{3\pi}{4}-\theta_2)$

$R_x(-\frac{\pi}{2})$ ———— $R_y(2\theta_2+\frac{\pi}{2})$ — $R_x(-\frac{\pi}{2})$

Combine: $R(\pi, 3\pi/4)$

— $R_x(-\frac{\pi}{2})$ ———— $R_y(\frac{\pi}{2})$ — $R_x(-\frac{\pi}{2}$

Combine: $R(\pi, 3\pi/4)$

— $R_x(-\frac{\pi}{2})$ ———————— $R_y(\frac{\pi}{2})$ — $R_x(-\frac{\pi}{2})$

$$\begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}\cos(\theta_2+\frac{\pi}{4}) & -\sin(\theta_2+\frac{\pi}{4}) \\ \sin(\theta_2+\frac{\pi}{4}) & \cos(\theta_2+\frac{\pi}{4})\end{bmatrix}$$

$$R_x(\theta) = \begin{bmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix} \quad R_y(\theta) = \begin{bmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix}$$

$$R_x(-\tfrac{\pi}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & i \\ i & 1\end{bmatrix}, \quad R_y(\tfrac{\pi}{2}) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1 \\ 1 & 1\end{bmatrix}$$

$$\underbrace{R_x(-\tfrac{\pi}{2}) - R_y(\tfrac{\pi}{2}) - R_x(-\tfrac{\pi}{2})}_{GP_i(-\frac{\pi}{4}) \text{ up to global phase of } e^{\pi i/2}} = \left(\frac{1}{\sqrt{2}}\right)^3\begin{bmatrix}1 & i \\ i & 1\end{bmatrix}\begin{bmatrix}1 & -1 \\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & i \\ i & 1\end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3\begin{bmatrix}1 & i \\ i & 1\end{bmatrix}\begin{bmatrix}1-i & i-1 \\ 1+i & i+1\end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3\begin{bmatrix}1-i+i-1 & i-1-1+i \\ i+1+1+i & -1-i+i+1\end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix}0 & 2(i-1) \\ 2(1+i) & 0\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -(1-i) \\ 1+i & 0\end{bmatrix} = \begin{bmatrix}0 & -e^{-\pi i/4} \\ e^{\pi i/4} & 0\end{bmatrix}$$

$$R(\theta,\phi) = \begin{bmatrix}\cos\frac{\theta}{2} & -ie^{-i\phi}\sin\frac{\theta}{2} \\ -ie^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix} = \begin{bmatrix}0 & -e^{-\pi i/4} \\ e^{\pi i/4} & 0\end{bmatrix} \rightsquigarrow \theta = \pi$$

then
$$\left.\begin{aligned} -ie^{-i\phi} &= -e^{-\pi i/4} \\ \underbrace{-i}_{e^{-\pi i/2}}e^{i\phi} &= e^{\pi i/4}\end{aligned}\right\} \to \begin{aligned} -i\phi &= -\pi i/4 + \pi i/2 = \pi i/4 \\ i\phi &= \pi i/4 + \pi i/2 = 3\pi i/4\end{aligned}$$

these don't add up... replace one $-i$ with $e^{3\pi i/2}$? ...

$$GP_i(-\tfrac{1}{8}) = GP_i(-\tfrac{\pi}{4}) = \begin{bmatrix}0 & e^{\pi i/4} \\ e^{-\pi i/4} & 0\end{bmatrix} \cdot e^{x} = \begin{bmatrix}0 & -e^{-\pi i/4} \\ e^{\pi i/4} & 0\end{bmatrix}$$

$$\frac{\pi i}{4} + x = \pm\pi - \frac{\pi i}{4} \rightsquigarrow \frac{\pi i}{2} + x = \pm\pi \quad \text{so } x = \frac{\pi i}{2} \checkmark$$

$$-\frac{\pi i}{4} + x = \frac{\pi i}{4} \rightsquigarrow x = \frac{\pi i}{2}$$

$$e^{\pi i/2} = i = e^{-3\pi i/2}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \iff e^{i\theta+\pi i/2} = i\cos\theta - \sin\theta$$

$$e^{\pi i/4}\begin{bmatrix}0 & e^{i\theta_2} \\ e^{-i\theta_2-\pi i/2} & 0\end{bmatrix} = e^{\pi i/4}\begin{bmatrix}0 & e^{i\theta_2} \\ ie^{-i\theta_2} & 0\end{bmatrix}$$

$$\| $$

$$GP_i(-\tfrac{\pi}{4}-\theta_2) = \begin{bmatrix}0 & e^{\pi i/4+i\theta_2} \\ e^{-\frac{\pi i}{4}-\theta_2 i} & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & e^{i\alpha+\pi i/2} \\ e^{-i\alpha+\pi i/2} & 0\end{bmatrix}, \quad \pi i/2 = -\tfrac{3\pi i}{2}$$

$$\frac{\pi}{4}+\theta_2 = \alpha + \frac{\pi}{2} \; ; \; -\frac{\pi}{4} - \theta_2 = -\alpha - \frac{3\pi}{2}$$

$$\alpha = \theta_2 - \frac{\pi}{4} \qquad \alpha = \theta_2 - \frac{5\pi}{4}$$

$$e^{-i(\frac{\pi}{4}+\theta_2)} = \cos(-\tfrac{\pi}{4}-\theta_2) + i\sin(-\tfrac{\pi}{4}-\theta_2)$$
$$= \cos(\tfrac{\pi}{4}+\theta_2) - i\sin(\tfrac{\pi}{4}+\theta_2) =$$

$$\left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix}1 & i \\ i & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}1 & i \\ i & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & i \\ i & 1\end{bmatrix}\begin{bmatrix}\cos\alpha - i\sin\alpha & i\cos\alpha - \sin\alpha \\ \sin\alpha + i\cos\alpha & i\sin\alpha + \cos\alpha\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\cancel{\cos\alpha} - \cancel{i\sin\alpha} + \cancel{i\sin\alpha} - \cancel{\cos\alpha} & i\cos\alpha - \sin\alpha - \sin\alpha + i\cos\alpha \\ i\cos\alpha + \sin\alpha + \sin\alpha + i\cos\alpha & -\cancel{\cos\alpha} - \cancel{i\sin\alpha} + \cancel{i\sin\alpha} + \cancel{\cos\alpha}\end{bmatrix}$$

$$= \begin{bmatrix}0 & i\cos\alpha - \sin\alpha \\ i\cos\alpha + \sin\alpha & 0\end{bmatrix} = \begin{bmatrix}0 & e^{i\alpha+\pi i/2} \\ e^{-i\alpha+\pi i/2} & 0\end{bmatrix}, \quad \pi i/2 = -3\pi i/2$$

$$-ie^{-i\theta} = -i(\cos(-\theta) + i\sin(-\theta)) = -i\cos\theta - \sin\theta \xleftrightarrow{\text{diff of } e^{\pi i}} -i(\cos(-\alpha) - \sin(-\alpha))$$
$$= \cos\alpha + i\sin(-\alpha)$$
$$= \cos\alpha - i\sin\alpha$$

$$i\cos(-\alpha) - \sin(-\alpha) = e^{-i\alpha+\pi i/2} = e^{\pi i/2}\cos(-\alpha) + i\sin(-\alpha)e^{\pi i/2} = i\cos(-\alpha) - \sin(-\alpha) = i\cos\alpha + \sin\alpha \checkmark$$

$$e^{-i\alpha+\pi i/2} = e^{-i\alpha-3\pi i/2}$$

Rabbit Hole

Check $R_x(-\frac{\pi}{2})R_y(-\frac{\pi}{2})R_x(-\frac{\pi}{2}) = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1+i & 1+i \\ -1+i & 1-i \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{bmatrix} 1+i-i-1 & 1+i+i+1 \\ i-1-1+i & i-1+1-i \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 0 & 2(i+1) \\ 2(i-1) & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & e^{\pi i/4} \\ -e^{-\pi i/4} & 0 \end{bmatrix} = e^{\pi i/4}\begin{bmatrix} 0 & 1 \\ -e^{-\pi i/2} & 0 \end{bmatrix}$

$e^{-\pi i/4} = \frac{1}{\sqrt{2}}(1-i)$

$\therefore e^{\pi i/4} = \frac{1}{\sqrt{2}}(1+i)$, $e^{-\pi i/2} = -i$

$-e^{\pi i/4}e^{x} = -e^{-\pi i/4} \rightsquigarrow x = -\pi i/2$

$e^{i\theta} = \cos\theta + i\sin\theta$

so $e^{-\pi i/4} = \cos(\frac{-\pi}{4}) + i\sin(\frac{-\pi}{4})$

Check if this $= R(\pi, \frac{\pi}{4})$

$= \begin{bmatrix} \cos\frac{\pi}{2} & -ie^{-i\frac{\pi}{4}}\sin\frac{\pi}{2} \\ -ie^{i\frac{\pi}{4}}\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix}$

$= \begin{bmatrix} 0 & -ie^{-\pi i/4} \\ -ie^{\pi i/4} & 0 \end{bmatrix}$

$= e^{-\pi i/2}\begin{bmatrix} 0 & e^{-\pi i/4} \\ e^{\pi i/4} & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & e^{-3\pi i/4} \\ e^{-\pi i/4} & 0 \end{bmatrix} = e^{\pi i}\begin{bmatrix} 0 & e^{\pi i/4} \\ -e^{-\pi i/4} & 0 \end{bmatrix}$

up to global phase, yes

$e^{-3\pi i/4} = \cos(\frac{-3\pi}{4}) + i\sin(\frac{-3\pi}{4})$

$= -\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)$

$= -\frac{\sqrt{2}}{2}(1+i) = -e^{\pi i/4}$

Check $R_x(-\frac{\pi}{2})R_y(\frac{\pi}{2})R_x(-\frac{\pi}{2}) = \underbrace{\left(\frac{1}{\sqrt{2}}\right)^3}_{GP;2(\pi)} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1-i & i-1 \\ 1+i & 1+i \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{bmatrix} 1-i+i-1 & i-1+i-1 \\ i+1+1+i & -1-i+1+i \end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 0 & 2(i-1) \\ 2(1+i) & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & -e^{-\pi i/4} \\ e^{\pi i/4} & 0 \end{bmatrix}$

these don't match

$a = -\pi$ & $b = -\frac{\pi}{2}$

Not Necessary

$e^{i\phi} = -1 = e^{-i\phi} \rightsquigarrow \phi = \pi$

$R(\pi, \frac{-\pi}{4})$

$= \begin{bmatrix} 0 & -ie^{\pi i/4} \\ -ie^{-\pi i/4} & 0 \end{bmatrix}$

$= e^{-\pi i/2}\begin{bmatrix} 0 & e^{\pi i/4} \\ e^{-\pi i/4} & 0 \end{bmatrix}$

$-ie^{-i\phi} = -e^{-\pi i/4}$

$\therefore -ie^{i\phi} = e^{\pi i/4}$

$e^{\pi i/2}e^{-i\phi} = e^{-\pi i/4}$

$\therefore e^{-\pi i/2}e^{i\phi} = e^{\pi i/4}$

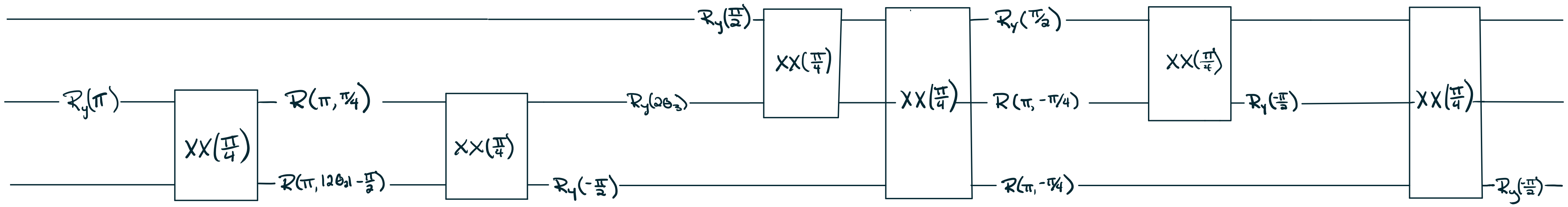
$c = 2\arccos(\underbrace{\cos(-\frac{\pi}{4})}_{\pi/4}) = \frac{\pi}{2}$

$d = \pi - \arcsin\left(\frac{-1/\sqrt{2}}{\underbrace{\sqrt{1-(1/\sqrt{2})^2}}_{\frac{1}{\sqrt{2}}}}\right) = \pi - \arcsin(-1) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$

$\phi - \frac{\pi}{2} = \frac{\pi}{4}$ & $\phi - \frac{\pi}{2} = \pi/4$

So $\phi = 3\pi/4$

Converting to IonQ Native Gates



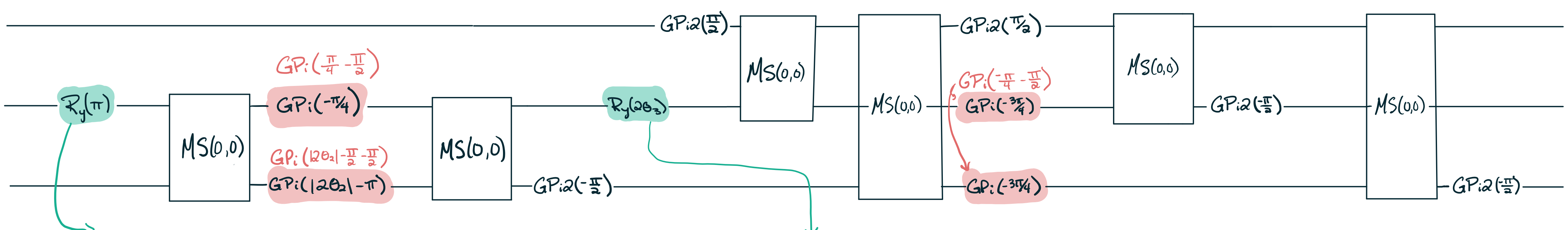
$XX(\frac{\pi}{4}) \longrightarrow MS(0,0)$

$R(\pi, \theta) \longrightarrow GPi(\theta - \frac{\pi}{2})$

$R_y(\frac{\pi}{2}) \longrightarrow GPi2(\pi/2)$

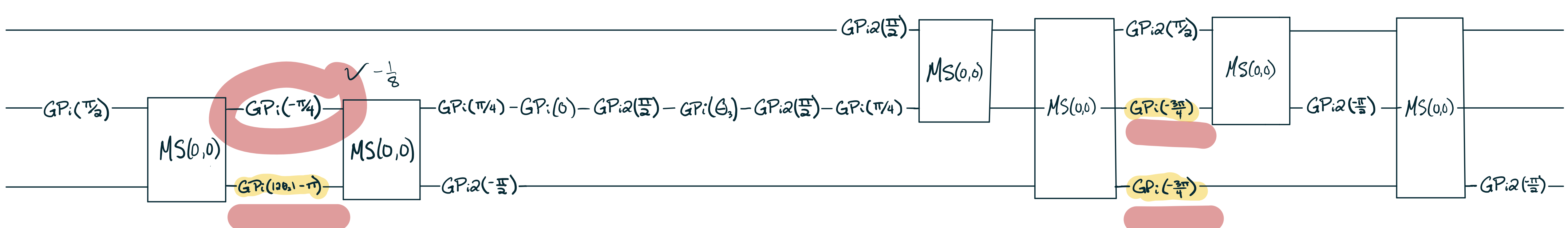
$R_y(-\frac{\pi}{2}) \longrightarrow GPi2(-\pi/2)$

$R_y(\theta) \longrightarrow GPi(\pi/4) - GPi(0) - GPi2(\pi/2) - GPi(\theta/2) - GPi2(\pi/2) - GPi(\pi/4)$



$R_y(\pi) \rightsquigarrow GPi(\pi/2) = GPi(.25)$ by global phase equiv ($i = e^{i\pi/2}$)

$GPi(\pi/4) - GPi(0) - GPi2(\frac{\pi}{2}) - GPi(\frac{2\theta_3}{2}) - GPi2(\frac{\pi}{2}) - GPi(\pi/4)$



vs $-\frac{\pi}{4} - \theta_2$

$(2\theta_2)$

$GPi(-\frac{\pi}{4} - \theta_2)$

Can't seem to verify

$-\frac{1}{8}$ but I have $-3/8$ ...

$GPi(-\frac{1}{8}) = GPi(-\frac{\pi}{4}) \leftarrow$ verified this