1 Solving a differential equation with discrete Fourier transforms

We start with the 1-d equation

$$\frac{d^2}{dz^2}\phi(z) = f(z).$$

Next, we divide the domain $z : [0 - L]$ into N discrete points and apply the discrete Fourier operator

$$\mathcal{F}_m = \sum_{m=0}^{N-1} e^{2\pi imn/N}$$

where

$$z = nL/N,$$

so

$$\mathcal{F}_m = \sum_{m=0}^{N-1} e^{2\pi imz/L}$$

to get

$$-\left(\frac{2\pi m}{L}\right)^2 \phi(m) = f(m),$$

where

$$\phi(m) \equiv \mathcal{F}_m \phi(z) = \sum_{m=0}^{N-1} \phi(z) e^{2\pi imz/L}$$

and

$$f(m) \equiv \mathcal{F}_m f(z) = \sum_{m=0}^{N-1} f(z) e^{2\pi imz/L}.$$

Then

$$\phi(z) = \widetilde{\mathcal{F}}_z \left[-\left(\frac{L}{2\pi m}\right)^2 f(m) \right] = \frac{1}{N} \sum_{m=0}^{N-1} -\left(\frac{L}{2\pi m}\right)^2 f(m) e^{-2\pi imz/L},$$

where we have used the inverse discrete Fourier operator

$$\widetilde{\mathcal{F}}_z = \frac{1}{N} \sum_{m=0}^{N-1} e^{-2\pi imz/L}.$$

2 A simple case

As a test, we take THE SIMPLEST FREAKIN' CASE I CAN IMAGINE,

$$\phi(z) = \phi_1(z) = cos\left(\frac{2\pi}{L}z\right),$$

with the corresponding right hand side given by

$$f(z) = f_1(z) = -\left(\frac{2\pi}{L}\right)^2 cos\left(\frac{2\pi}{L}z\right).$$

Our goal is to use the procedure above to solve our 1-d equation for $\phi(z)$ given $f_1(z)$.

3 Solving the Poisson equation in a cylindrical conducting pipe

We start with the Poisson equation,

$$\nabla^2\phi = -\frac{1}{\epsilon}\rho.$$

We use cylindrical coordinates (r, θ, z), where

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}.$$

We apply the two-dimensional Fourier transform

$$\int e^{-il\theta - imz}d\theta dz$$

to each side of the Poisson equation to obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi^{lm}}{\partial r}\right) - \frac{l^2}{r^2}\phi^{lm} - m^2\phi^{lm} = -\frac{1}{\epsilon}\rho^{lm},$$

where

$$\phi^{lm} = \int \phi e^{-il\theta - imz}d\theta dz$$

and

$$\rho^{lm} = \int \rho e^{-il\theta - imz}d\theta dz.$$

4 A test distribution for Poisson solvers

We want to test solvers to the equation

$$\nabla^2\phi = -\frac{1}{\epsilon}\rho.$$

We use cylindrical coordinates, where

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

The example charge distributions for which there are simple, analytical expressions for ϕ are typically those which exhibit a very large degree of symmetry, reducing the problem to one, or possibly two, dimensions. These examples are poor tests for a solver because they do not equally test all three dimensions.

Instead, we work backwards, using a truly three dimensional $\phi(r,\theta,z)$ and differentiating to find the corresponding $\rho(r,\theta,z)$. For our closed pipe solver, we have the following boundary conditions:

$$\phi(r = r_0, \theta, z) = 0$$

$$\phi(r, \theta = 0, z) = \phi(r, \theta = 2\pi, z)$$
$$\phi(r, \theta, z = -z_0) = \phi(r, \theta, z = +z_0)$$

These conditions are satisfied by the following field

$$\phi(r,\theta,z) = \left(-\frac{r^2}{r_0^2} + 1\right)\sin^2(3\theta)\cos^2\left(\pi\frac{z}{z_0}\right),$$

which has non-trivial dependence on all coordinates.

The maxima program

```
phi:(-r^2/r0^2+1)*sin(3*theta)^2*cos(%pi*z/z0)^2;
phir:1/r*diff(r*diff(phi,r),r);
phitheta:1/r^2*diff(phi,theta,2);
phiz:diff(phi,z,2);
rho:-ratsimp(phir+phitheta+phiz);
load("cformat.lisp");
cformat(rho);
```

provides the C++-formatted version of the resulting charge density.