July 2, 2015

**Abstract**

# 1 The Hamiltonian of a charged particle in electromagnetic field

The Lagrangian for a relativistic particle of mass $m$ with an electric charge $e$ in most general time-dependent electromagnetic field is given by

$$\mathcal{L}[\mathbf{r}, \dot{\mathbf{r}}; t] = -m\, c^2 \sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}} - e\, \Phi(\mathbf{r}, t) + e\, (\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}, t)) \,,$$

where $\mathbf{r} = (x, y, z)$ is a position vector of a particle in three dimensional right handed Cartesian coordinate system $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}$ and $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$ being matching generalized velocities, the action of dot differential operator is the derivative with respect to time $(\dot{\ldots}) \equiv \frac{\mathrm{d}}{\mathrm{d}t}$, $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are the electric scalar and magnetic vector potentials respectively, and, $c$ is the speed of light in vacuum.

The Hamiltonian is constructed using the Legendre transformation of $\mathcal{L}$

$$\mathcal{H} = \sum_{q=\{x,y,z\}} \dot{q}\, p_q - \mathcal{L},$$

where $p_q$ are components of the particle's canonical momentum

$$\mathbf{p} \equiv \frac{\partial\, \mathcal{L}}{\partial\, \dot{\mathbf{r}}} = \frac{m\, \dot{\mathbf{r}}}{\sqrt{1 - \frac{\dot{\mathbf{r}}^2}{c^2}}} + e\, \mathbf{A}(\mathbf{r}, t),$$

which gives

$$\mathcal{H}[\mathbf{p}, \mathbf{q}; t] = c\sqrt{m^2 c^2 + (\mathbf{p} - e\, \mathbf{A}(\mathbf{r}, t))^2} + e\, \Phi(\mathbf{r}, t).$$

## 2 R-elements

## 2.1 Electrostatic Lenses

$$\mathcal{H}[p_x, p_y, p_s; x, y, s; t] = c\sqrt{m^2c^2 + \mathbf{p}^2} + e\,\Phi(x, y).$$

## 2.2 Magnetostatic Lenses

$$\mathcal{H}[p_x, p_y, p_s; x, y, s; t] = c\sqrt{m^2c^2 + p_x^2 + p_y^2 + (p_s - e\,A_s(x, y))^2}.$$

Introducing an extended Hamiltonian with a new time parameter, $\tau$, where the old independent variable and old Hamiltonian with a negative sign will be treated as an additional pair of canonically conjugated coordinates, $(-\mathcal{H}, t)$, one have:

$$\mathcal{O}[p_x, p_y, p_s, -\mathcal{H}; x, y, s, t; \tau] \equiv 0 = c\sqrt{m^2c^2 + p_x^2 + p_y^2 + (p_s - e\,A_s(x, y))^2} - \mathcal{H}.$$

Integration of additional equations of motion gives

$$\frac{\partial\,\mathcal{O}}{\partial t} = -\frac{\mathrm{d}(-\mathcal{H})}{\mathrm{d}\tau}, \qquad \rightarrow \qquad \mathcal{H} = \mathrm{inv}, \tag{1}$$

$$\frac{\partial\,\mathcal{O}}{\partial(-\mathcal{H})} = \frac{\mathrm{d}\,t}{\mathrm{d}\tau}, \qquad \rightarrow \qquad t = \tau + C_0, \tag{2}$$

where we can set a constant of integration $C_0 = 0$.

As a next step we will use longitudinal coordinate, $s$, as an independent variable and $-p_s$ as a new Hamiltonian, reducing number of degrees of freedom back up to three:

$$\mathcal{K}[p_x, p_y, -\mathcal{H}; x, y, t; s] \equiv -p_s = -\sqrt{\left(\frac{\mathcal{H}}{c}\right)^2 - m^2c^2 - p_x^2 - p_y^2} - e\,A_s(x, y).$$

The use of generating function

$$G_2(t, -p) = -t\sqrt{p^2c^2 + (m\,c^2)^2}$$

will allow to use the full momentum $-p$ of a particle instead of $-\mathcal{H}$ as one of canonical momentums:

$$\mathcal{K}[p_x, p_y, -p; x, y, l; s] \equiv -p_s = -\sqrt{p^2 - p_x^2 - p_y^2} - e\,A_s(x, y),$$

where corresponding canonical coordinate is a particle's traversed path

$$l = \frac{\partial\,G_2(t, -p)}{\partial(-p)} = \beta c\,t.$$

Since the Hamiltonian do not explicitly depends on $l$, full momentum $p$ is conserved and we can exclude associated degree of freedom using the further renormalization of the Hamiltonian $\mathcal{K} \to K \equiv \mathcal{K}/p$, which can be achieved by re-normalizing all canonical momentums $P_i \to P_i' = P_i/p$:

$$\mathrm{K}[p_x', p_y'; x, y; s] \equiv -\frac{p_s}{p} = -\sqrt{1 - p_x'^2 - p_y'^2} - \frac{e}{p}\,A_s(x, y).$$

Finally, introducing momentum deviation $\delta = \frac{p - p_{\rm eq}}{p_{\rm eq}}$ one have

$$\mathrm{K}[p'_x, p'_y; x, y; s] = -\sqrt{1 - p'^2_x - p'^2_y} - \frac{e\, A_s(x, y)}{p_{\rm eq}(1 + \delta)},$$

which in paraxial approximation, $p_s \gg p_{x,y}$, and $\delta \ll 1$ gives:

$$\mathrm{K}[p'_x, p'_y; x, y; s] \approx -1 + \frac{p'^2_x}{2} + \frac{p'^2_y}{2} - (1 - \delta)\frac{e\, A_s(x, y)}{p_{\rm eq}},$$

$$\Omega(\mathcal{Z}) = G_n \mathcal{Z}^n,$$

where $\mathcal{Z} = x + i\, y$ and

$$B_y(x, 0) = \sum_{n=0}^{\infty} \frac{\overline{G}_n x^n}{n!}, \qquad \overline{G}_n = \left.\frac{\partial^n B_y}{\partial x^n}\right|_{x=0, y=0}$$

$$B_x(0, y) = \sum_{n=0}^{\infty} \frac{\underline{\mathrm{G}}_n y^n}{n!}, \qquad \underline{\mathrm{G}}_n = \left.\frac{\partial^n B_x}{\partial y^n}\right|_{x=0, y=0}$$

## 2.3 Drift space

The Hamiltonian does not explicitly depend on time-parameter

$$\mathcal{H}[p_x,p_y,p_s;x,y,s;t] \quad = c\sqrt{m_0^2c^2+\mathbf{p}^2}, \tag{3a}$$

$$\mathcal{K}[p_x,p_y,-\mathcal{H};x,y,t;s] = -\sqrt{\left(\frac{\mathcal{H}}{c}\right)^2 - m_0^2c^2 - p_x^2 - p_y^2}, \tag{3b}$$

so it is left invariant:

$$\frac{\mathrm{d}\,\mathcal{H}}{\mathrm{d}t} = 0 \qquad \to \qquad \mathcal{H} \equiv E = \gamma\, m_0\, c^2 = \mathrm{inv}, \tag{4a}$$

$$\frac{\mathrm{d}\,\mathcal{K}}{\mathrm{d}s} = 0 \qquad \to \qquad \mathcal{K} \equiv -p_s = \mathrm{inv}. \tag{4b}$$

First three equations of motion show that all canonical momentums are left invariant as well

$$\begin{cases} \mathrm{d}\,p_x/\mathrm{d}t = -\partial\,\mathcal{H}/\partial x = 0 & \to \quad p_x = \mathrm{inv}, \\ \mathrm{d}\,p_y/\mathrm{d}t = -\partial\,\mathcal{H}/\partial y = 0 & \to \quad p_y = \mathrm{inv}, \\ \mathrm{d}\,p_s/\mathrm{d}t = -\partial\,\mathcal{H}/\partial s = 0 & \to \quad p_s = \mathrm{inv}, \end{cases} \tag{5a}$$

$$\begin{cases} \mathrm{d}\,p_x/\mathrm{d}s = -\partial\,\mathcal{K}/\partial x = 0 & \to \quad p_x = \mathrm{inv}, \\ \mathrm{d}\,p_y/\mathrm{d}s = -\partial\,\mathcal{K}/\partial y = 0 & \to \quad p_y = \mathrm{inv}, \\ \mathrm{d}\,\mathcal{H}/\mathrm{d}s = \quad \partial\,\mathcal{K}/\partial t = 0 & \to \quad \mathcal{H} = \mathrm{inv}. \end{cases} \tag{5b}$$

Another three equations of motion

$$\begin{cases} \mathrm{d}\,x/\mathrm{d}t = \partial\,\mathcal{H}/\partial p_x = c\,p_x/\sqrt{m_0^2c^2+\mathbf{p}^2}, \\ \mathrm{d}\,y/\mathrm{d}t = \partial\,\mathcal{H}/\partial p_y = c\,p_y/\sqrt{m_0^2c^2+\mathbf{p}^2}, \\ \mathrm{d}\,s/\mathrm{d}t = \partial\,\mathcal{H}/\partial p_s = c\,p_s/\sqrt{m_0^2c^2+\mathbf{p}^2}, \end{cases} \tag{6a}$$

$$\begin{cases} \mathrm{d}\,x/\mathrm{d}s = \quad \partial\,\mathcal{K}/\partial p_x \quad = \quad p_x/\sqrt{\left(\frac{\mathcal{H}}{c}\right)^2 - m_0^2c^2 - p_x^2 - p_y^2}, \\ \mathrm{d}\,y/\mathrm{d}s = \quad \partial\,\mathcal{K}/\partial p_y \quad = \quad p_y/\sqrt{\left(\frac{\mathcal{H}}{c}\right)^2 - m_0^2c^2 - p_x^2 - p_y^2}, \\ \mathrm{d}\,t/\mathrm{d}s = \partial\,\mathcal{K}/\partial(-\mathcal{H}) = (\mathcal{H}/c^2)/\sqrt{\left(\frac{\mathcal{H}}{c}\right)^2 - m_0^2c^2 - p_x^2 - p_y^2}, \end{cases} \tag{6b}$$

are easy to integrate using invariants of motion[1]:

$$\begin{cases} x(t) = x_0 + p_x c^2 t/E, \\ y(t) = y_0 + p_y c^2 t/E, \\ s(t) = z_0 + p_s c^2 t/E, \end{cases} \tag{7a}$$

$$\begin{cases} x(s) = x_0 + p_x s/p_s, \\ y(s) = y_0 + p_y s/p_s, \\ t(s) \ = t_0 \ + \mathcal{H}\, s/c^2 p_s. \end{cases} \tag{7b}$$

[1] For the reference particle we setting $t_0, s_0 = 0$ and $\forall\, t$ one simply have $x, y, p_x, p_y(t) = 0$, and,

$$p_{s,\mathrm{eq}} = |\mathbf{p}_{\mathrm{eq}}| = c^{-1}\sqrt{E_{\mathrm{eq}}^2 - m_0^2c^4} = \mathrm{inv}, \qquad s_{\mathrm{eq}}(t) = \frac{p_{\mathrm{eq}}c^2}{E_{\mathrm{eq}}} t = v_{\mathrm{eq}}\, t.$$

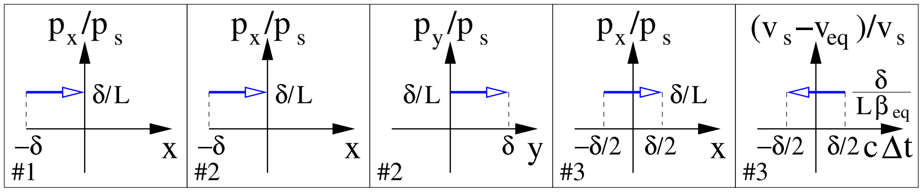


Figure 1: Schematic draw of phase-space flow in a drift space for test particles.

Thus for the drift space of the length $L$ we can write transformation as

$$\begin{bmatrix} x \\ p_x/p_s \\ y \\ p_y/p_s \\ c\,\Delta t \\ (v_s - v_{\rm eq})/v_s \end{bmatrix}_{s=L} = \begin{bmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{L}{\beta_{\rm eq}} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ p_x/p_s \\ y \\ p_y/p_s \\ c\,\Delta t \\ (v_s - v_{\rm eq})/v_s \end{bmatrix}_{s=0},$$

where $c\,\Delta t(s)$ defined as $c(t(s) - t_{\rm eq}(s))$. Examples of initial condition for test particles are listed below in Table 1. Corresponding phase space flow (except trivial cases) is shown in Fig. 1.

| | Particle #1 | | Particle #2 | | Particle #3 | |
|---|---|---|---|---|---|---|
| | ini | fin | ini | fin | ini | fin |
| $x$ | $-q_{\rm off}$ | $0$ | $-q_{\rm off}$ | $0$ | $-q_{\rm off}/2$ | $q_{\rm off}/2$ |
| $p_x/p_s$ | $q_{\rm off}/L$ | $q_{\rm off}/L$ | $q_{\rm off}/L$ | $q_{\rm off}/L$ | $q_{\rm off}/L$ | $q_{\rm off}/L$ |
| $\{p_x/p_{\rm eq}\}^*$ | $\frac{q_{\rm off}}{L}\frac{p_s}{p_{\rm eq}}$ | $\frac{q_{\rm off}}{L}\frac{p_s}{p_{\rm eq}}$ | $\frac{q_{\rm off}}{L}\frac{p_s}{p_{\rm eq}}$ | $\frac{q_{\rm off}}{L}\frac{p_s}{p_{\rm eq}}$ | $\frac{q_{\rm off}}{L}\frac{p_s}{p_{\rm eq}}$ | $\frac{q_{\rm off}}{L}\frac{p_s}{p_{\rm eq}}$ |
| $y$ | $0$ | $0$ | $0$ | $q_{\rm off}$ | $0$ | $0$ |
| $p_y/p_s$ | $0$ | $0$ | $q_{\rm off}/L$ | $q_{\rm off}/L$ | $0$ | $0$ |
| $\{p_y/p_{\rm eq}\}^*$ | $0$ | $0$ | $\frac{q_{\rm off}}{L}\frac{p_s}{p_{\rm eq}}$ | $\frac{q_{\rm off}}{L}\frac{p_s}{p_{\rm eq}}$ | $0$ | $0$ |
| $c\,\Delta t$ | $0$ | $0$ | $0$ | $0$ | $v_{\rm rel}L/2\,\beta_{\rm eq}$ | $-v_{\rm rel}L/2\,\beta_{\rm eq}$ |
| $(v_s - v_{\rm eq})/v_s$ | $0$ | $0$ | $0$ | $0$ | $v_{\rm rel}$ | $v_{\rm rel}$ |
| $\{(p - p_{\rm eq})/p_{\rm eq}\}^*$ | $0$ | $0$ | $0$ | $0$ | $p_{\rm rel}$ | $p_{\rm rel}$ |

Table 1: Initial and final sets of coordinates (including the *Synergia*'s fixed-z representation, starred variables) for three test particles in drift space, where $p_{\rm rel} + 1 = \sqrt{\left(\frac{p_x}{p_{\rm eq}}\right)^2 + \left(\frac{p_y}{p_{\rm eq}}\right)^2 + \left(\frac{E}{E_{\rm eq}}\right)^2 \frac{1}{(1 - v_{\rm rel})^2}}$.

## 2.4 Uniform field (dipole)

The Hamiltonian does not explicitly depend on time-parameter

$$\mathcal{H}[p_x, p_y, p_s; x, y, s; t] \quad = c\sqrt{m_0^2 c^2 + \mathbf{p}^2}, \tag{8a}$$

$$\mathcal{K}[p_x, p_y, -\mathcal{H}; x, y, t; s] = -\sqrt{\left(\frac{\mathcal{H}}{c}\right)^2 - m_0^2 c^2 - p_x^2 - p_y^2} - e\, B_0 x, \tag{8b}$$