Nonlinear Lens Tracking in the IOTA Complex Potential

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1 Potentials and Fields in a Nonlinear Lens Segment

The (2D) magnetic vector potential and magnetic scalar potential for the nonlinear lens are defined so that $\vec{B} = \nabla \times \vec{A} = -\nabla\psi$. When describing the lens in the complex plane, the principal branch is used for all functions (Fig. 1). Table 1 provides a summary of physical parameters.

Results are given in terms of the following dimensionless quantities:

$$\begin{aligned} F &= \frac{A_s + i\psi}{B\rho}, \quad z = \frac{x+iy}{c\sqrt{\beta}}, \quad p = \frac{\Delta p_x + i\Delta p_y}{p^0}, \\ \mathcal{B} &= \left(\frac{c\sqrt{\beta}}{B\rho}\right)(B_x + iB_y), \quad \sigma = \frac{\Delta s}{c\sqrt{\beta}}, \quad \tilde{t} = \frac{\tau c^2}{\beta}. \end{aligned} \tag{1}$$

The potentials, fields, and momentum kick through the nonlinear lens segment are given by:

$$F(z) = \left(\frac{\tilde{t}z}{\sqrt{1-z^2}}\right)\arcsin(z), \quad \mathcal{B}^* = i\frac{dF}{dz}, \quad p^* = \sigma\frac{dF}{dz}. \tag{2}$$

The power series for the complex potential F about the origin takes the form:

$$F(z) = \tilde{t}\sum_{n=1}^{\infty}\frac{2^{2n-1}n!(n-1)!}{(2n)!}z^{2n}, \quad |z| < 1. \tag{3}$$

The domain of definition for (2) and circle of convergence for (3) are shown in Fig. 1.

The subroutine "NonlinearLensPropagatorCmplx" computes the momentum kick across a thin nonlinear lens associated with a single segment of the nonlinear magnetic insert using (2). The arguments are identical to those required by the MAD-X element "NLLENS". Input arguments are:

- knll - integrated strength of the lens segment [m] $= \tau c^2\Delta s/\beta$

- cnll - distance of the singularities from the origin [m] $= c\sqrt{\beta}$
- coord - array of coordinates (x [m], p_x/p^0, y [m], p_y/p^0)

Output is the array "coord" with updated momenta. This subroutine requires the function "Fderivative" for computing dF/dz in (2), together with functions "carcsin", and "croot".

2 Invariants of Motion

Normalized dimensionless coordinates within the segment are defined by:

$$\begin{pmatrix} X_N \\ P_{xN} \end{pmatrix} = \begin{pmatrix} 1/c\sqrt{\beta} & 0 \\ \alpha/c\sqrt{\beta} & \sqrt{\beta}/c \end{pmatrix} \begin{pmatrix} x \\ p_x/p^0 \end{pmatrix}, \qquad \begin{pmatrix} Y_N \\ P_{yN} \end{pmatrix} = \begin{pmatrix} 1/c\sqrt{\beta} & 0 \\ \alpha/c\sqrt{\beta} & \sqrt{\beta}/c \end{pmatrix} \begin{pmatrix} y \\ p_y/p^0 \end{pmatrix}. \tag{4}$$

The dimensionless invariants of motion are given by:

$$H_N = \frac{1}{2}(P_{xN}^2 + P_{yN}^2 + X_N^2 + Y_N^2) - \tau U(X_N, Y_N), \qquad U = \mathcal{R}e\left(\frac{z}{\sqrt{1-z^2}}\arcsin(z)\right), \tag{5a}$$

$$I_N = (X_N P_{yN} - Y_N P_{xN})^2 + P_{xN}^2 + X_N^2 - \tau W(X_N, Y_N), \quad W = \mathcal{R}e\left(\frac{z+z^*}{\sqrt{1-z^2}}\arcsin(z)\right). \tag{5b}$$

The subroutine "InvariantPotentials" computes the potentials U and W in (5), which are the contributions to the two invariants due to the presence of the nonlinear insert. Input and output arguments are:

- x, y (input) - normalized dimensionless coordinates X_N and Y_N
- Hinv (output) - dimensionless function U appearing in the first invariant
- Iinv (output) - dimensionless function W appearing in the second invariant

This subroutine uses the functions "croot" and "carcsin".

References

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[2] K. Halbach, Nucl. Instrum. and Meth. A 169, 1 (1980).

[3] F. O'Shea *et al*, "Measurement of Non-Linear Insert Magnets," in Proc. PAC2013, Pasadena, CA, p. 922 (2013).

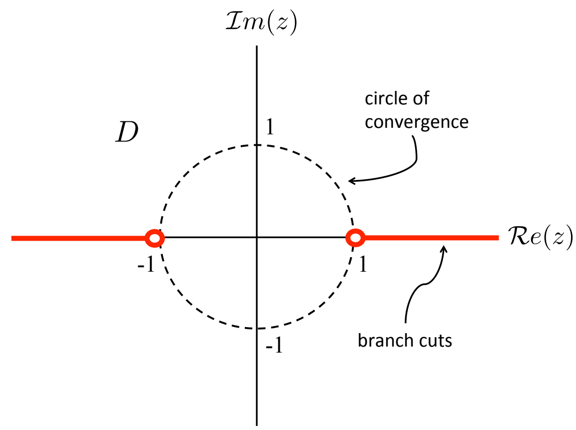

Figure 1: The domain of definition D for the complex potential F. (Red lines are excluded.) Singularities occur at $z = \pm 1$, and $\psi = \mathcal{I}m(F)$ is discontinuous across the two branch cuts. The multipole series converges within the dashed circle.

Table 1: Physical parameters used in the definitions (1-2).

Symbol	Description	Unit
p^0	design momentum	eV/c
$B\rho$	magnetic rigidity	T-m
τ	dimensionless strength of NLI	n/a
c	scale parameter for NLI	$\mathrm{m}^{1/2}$
B_x, B_y	magnetic field of NLI	T
A_s, ψ	vector, scalar potential of NLI	T-m
β	local betatron amplitude	m
α	local Twiss alpha	n/a
Δs	length of segment of NLI	m