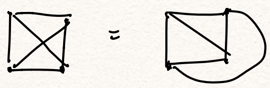
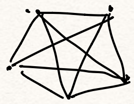
Plabic Graphs

- Introduced in 2006 in a paper called "Total positivity, Grassmannians and networks" (by A. Postnikov), which has been cited 400+ times according to Google.

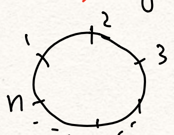
- Some applications outside of math (according to Wikipedia): quantum physics, computer vision (face and shape recognition), a data-visualization technique called grand tour.

Def A graph is planar if it can be drawn in the plane in such a way that the edges don't cross.

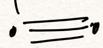
E.g. K_4  is planar

K_5  is non planar

Def A plabic (planar bicolored) graph is a graph

- drawn inside a disk 

- has n boundary vertices on the boundary of the disk, labeled $1, 2, \ldots, n$ in clockwise order
- all internal vertices are colored using 2 colors (shaded / black and empty / white)

Assume simple graph (no multiple edges, no loops)  

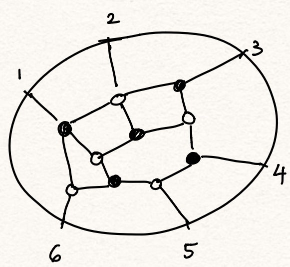
Assume no vertices of the same color are adjacent

Assume connected graph

Assume each boundary vertex i is adjacent to a single internal vertex.

Assume no leaf except for the boundary vertices.
degree 1 vertex

E.g.

$D =$ 

Def "Rules of the Road" Turn (maximally) right at black vertices ●
Turn (maximally) left at white vertices ○

Def Given a plabic graph, a <u>trip</u> T is a path from a boundary vertex i which follows the "rules of the road" until it reaches a boundary vertex j. Refer to this trip as $T_{i\to j}$.

Claim 1 If $T_{i\to j}$ is a trip, then $i \neq j$

Def A permutation on $[n] = \{1, \dots, n\}$ is a bijection $[n] \to [n]$.

2-row notation $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}$

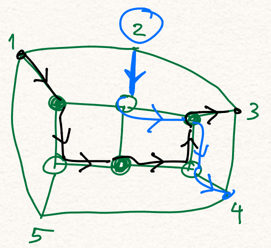
1-row notation $f = 3\ 4\ 5\ 1\ 2$, $g = 1\ 4\ 3\ 2\ 5$

Def Given a plabic graph G, define its <u>trip permutation</u>

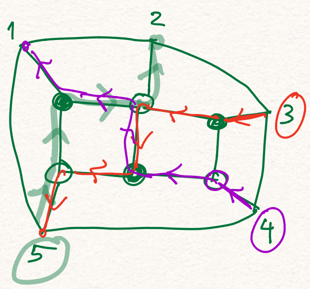
$\pi_G = \pi(1) \cdots \pi(n)$ where $\pi(i) = j$ for each trip of G

E.g.

G_1

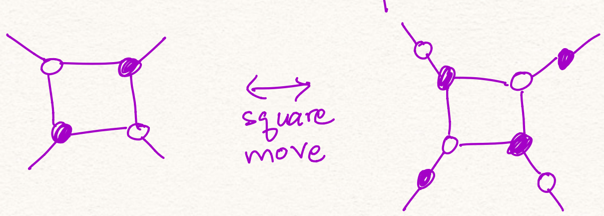


Let's compute $\pi_{G_1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$, the trip permutation of G_1.

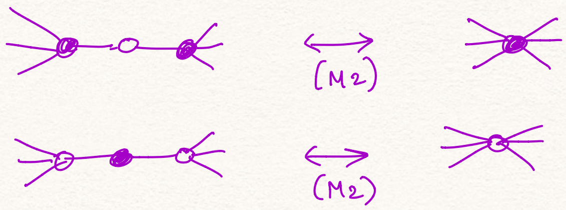


Def Local moves on plabic graphs

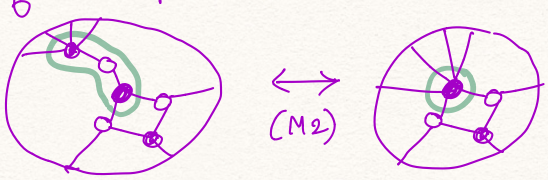
(M1) Square move: If G has a square formed by four degree 3 vertices, then we can switch the colors of these four vertices (and add some degree 2 vertices to preserve the bipartiteness of the graph)



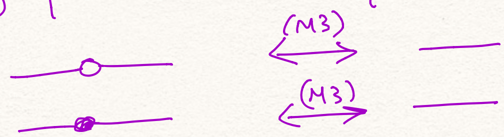
(M2) Contract / expand a vertex: Any degree 2 vertex v not adjacent to a boundary vertex can be deleted, and the two vertices adjacent to v merge into one. This operation can be reversed



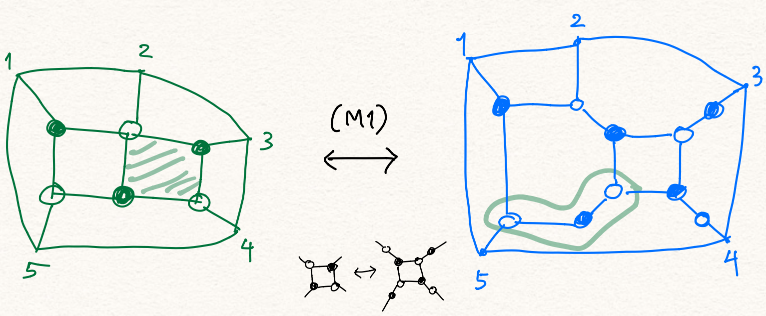
Remark (M2) can be used to change any square face of G into a square face whose four vertices are degree 3 vertices

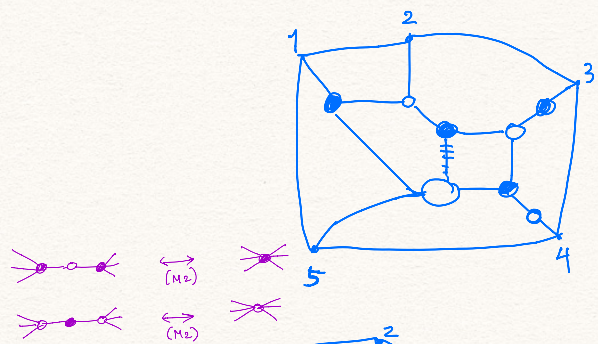


(M3) Middle vertex insertion / removal.
We can remove or add degree 2 vertices, as long as the graph remains bipartite.

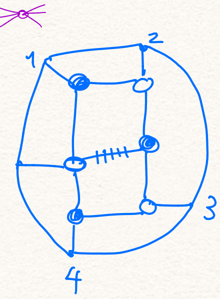


E.g.
$G_1 =$





$G_2 =$

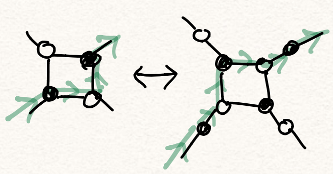


Prop 2 The trip permutation is preserved by (M1), (M2), (M3)

E.g. $\pi_{G_2} = \pi_{G_1} = 3\,4\,5\,1\,2$

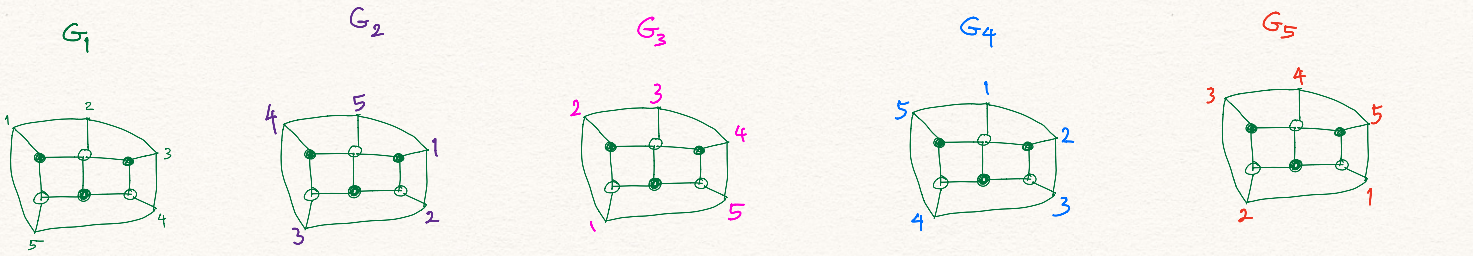
Check (M2) and (M3)

Pf (M1)



Thm [Postnikov, Thm 13.4] If two plabic graphs G, H have the same trip permutation, then we can get from G to H by applying a sequence of the local moves (M1) – (M3).

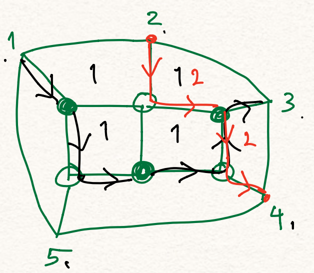
Ex. These are the five plabic graphs with trip permutation 34512

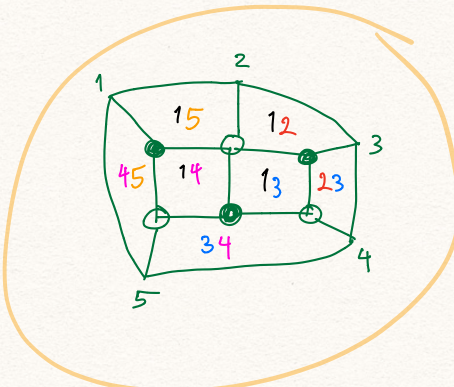


Def A (source) face labeling of G is the following map from the faces of G to the set of subsets of $[n]=\{1,2,\dots,n\}$. For each trip $T_{i\to j}$, place the label i in every face which is to the left of $T_{i\to j}$.

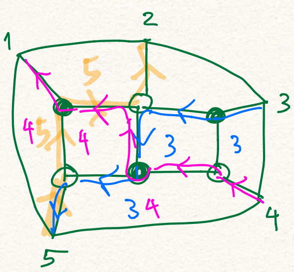
E.x

G_1



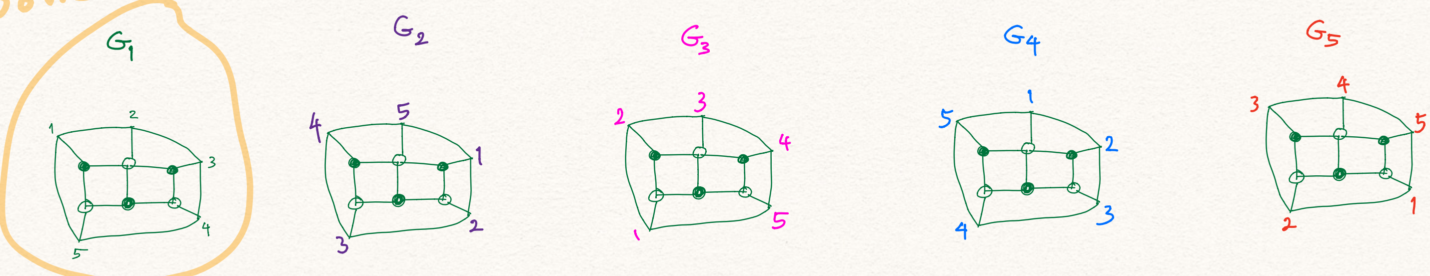


is the face labeling for G_1



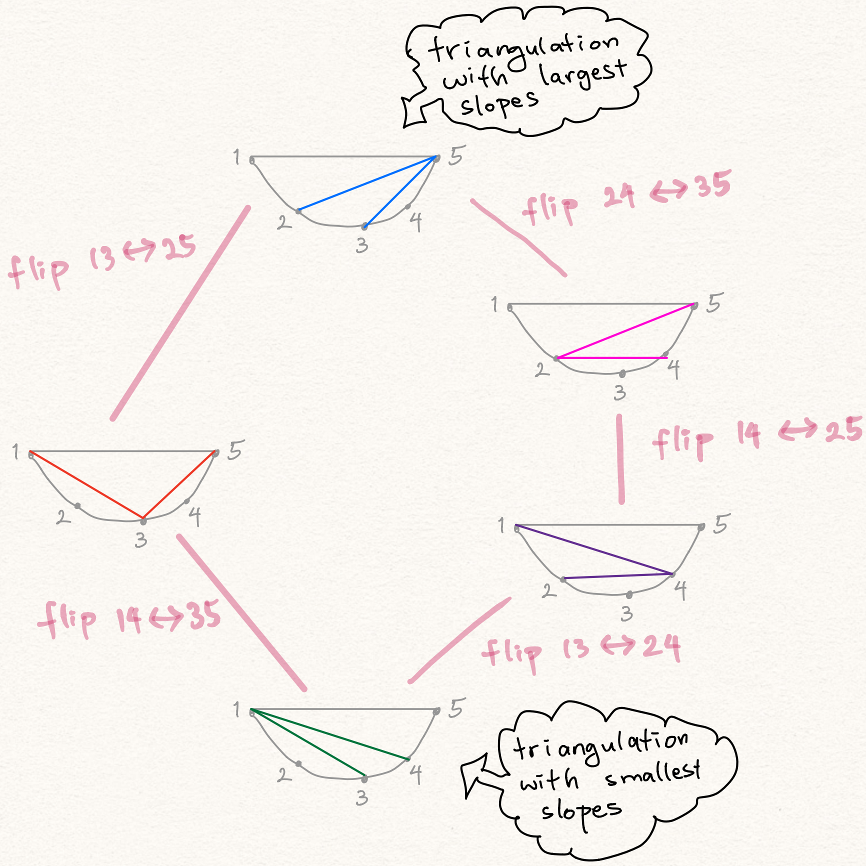
Example 3 Draw the face labeling of the following

Done above



Thm (Scott) The plabic graphs with trip permutation $3456\dots n12$ are Catalan objects.

HW 4 Below are the five triangulations of a 5-gon.



- Do you see a natural way to map the five plabic graphs (w/ trip permutation 34512) to these five triangulations? Explain the map.

the end

Ref

J. Scott "Grassmannians and Cluster Algebras"
A. Postnikov "Total Positivity, Grassmannians, and Networks"