The Pigeonhole Principle

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Introduction

Pigeonhole Principle: If we put m objects into n boxes and $m > n$ then at least two objects have to be put in the same box.

Introduction

- It is useful for proving existence theorems:
  - Increasing and decreasing sequences.
  - Periodicity of decimal expansions.
  - Good approximations of irrationals by rationals.
- It sometimes gives best possible results, but usually not.

If you can get away with an argument as simple as the pigeonhole principle, you have just been lucky.

E. W. Dijkstra

Increasing and Decreasing Sequences

Theorem (Erdős-Szekeres, 1935)

Any list of $n^2 + 1$ distinct real numbers contains an increasing or decreasing list of at least $n + 1$ numbers.

Example: In $\{4, 9, 8, 1, 6\}$ the list 9, 8, 6 is decreasing.

Proof.

Let the list be $\{a_1, \ldots, a_{n^2+1}\}$ and lengths of longest increasing and decreasing lists in it starting at a_k be i_k and d_k. If theorem is false then $1 \leq i_k, d_k \leq n$. The number of different pairs (i_k, d_k) is $\leq n^2$, while $1 \leq k \leq n^2 + 1$, so for some $k \neq \ell$, $(i_k, d_k) = (i_\ell, d_\ell)$. If $a_k < a_\ell$ then $i_k > i_\ell$; if $a_k > a_\ell$ then $d_k > d_\ell$. Contradiction! □

E-S showed any list of $mn + 1$ distinct numbers contains increasing list of length $m + 1$ or decreasing list of length $n + 1$. Some listing of $\{1, 2, \ldots, mn\}$ violates conclusion, so length $mn + 1$ is minimal.

Decimal Periodicity

$$\frac{93}{148} = .62837837837837\cdots = .62\overline{837}.$$

$$\begin{aligned} 10 \cdot 42 &= 148 \cdot \mathbf{2} + 124, \\ 10 \cdot 42 &= 148 \cdot \mathbf{2} + 124, \end{aligned}$$

Decimal Periodicity

For $0 < a < b$, the decimal expansion $a/b = .d_1d_2d_3\ldots$ comes from

$$\begin{aligned} 10a &= bd_1 + r_1, \\ 10r_1 &= bd_2 + r_2, \\ &\vdots \\ 10r_i &= bd_{i+1} + r_{i+1}, \\ &\vdots \end{aligned}$$

Setting $r_0 = a$, each remainder r_i is in $\{0, 1, \ldots, b-1\}$, so after at most $b+1$ steps there is a repetition: $r_i = r_j$ where $0 \leq i < j \leq b$. Then $d_{i+1} = d_{j+1}$, $r_{i+1} = r_{j+1}$, and a/b has decimal period $\leq b$.

What we don't learn: If $(10a, b) = 1$, the period of a/b is a factor of $\varphi(b)$. For instance, the period of reduced fractions with denominator 51 is not just ≤ 51 but divides 32 (in fact it's 16).

Lagrange's approximation theorem

Theorem (Lagrange)

If $\alpha \in \mathbf{R}$ is irrational then there are infinitely many rational numbers x/y, in reduced form with $y > 0$, such that

$$\left|\alpha - \frac{x}{y}\right| < \frac{1}{y^2}.$$

Lagrange's proof was constructive, using continued fractions. We will give a proof of Dirichlet (1842). It is useful to think about the inequality as

$$|y\alpha - x| < \frac{1}{y}.$$

The number $y\alpha$ is an integral multiple of α. We want to show some $y\alpha$ is close (in terms of y) to some integer.
For each $N \geq 1$ we will put the $N+1$ numbers $0, \alpha, 2\alpha, \ldots, N\alpha$ into N "boxes", so two of these must land in the same box.

Lagrange's approximation theorem

Pick $N \geq 1$ and for $k = 0, 1, 2, \ldots, N$ write $k\alpha = M_k + \varepsilon_k$, where $M_k \in \mathbf{Z}$ and $\varepsilon_k \in [0, 1)$. Each ε_k lies in

$$[0,1) = [0, 1/N) \cup [1/N, 2/N) \cup \cdots \cup [(N-1)/N, 1).$$

There are N intervals, so two of the $N+1$ numbers ε_k (objects) are in the same interval (box): closer than $1/N$ from each other. Let $|\varepsilon_k - \varepsilon_\ell| < 1/N$ where $k \neq \ell$ in $\{0, 1, \ldots, N\}$:

$$\begin{aligned} |\varepsilon_k - \varepsilon_\ell| < \frac{1}{N} &\implies |(k\alpha - M_k) - (\ell\alpha - M_\ell)| < \frac{1}{N} \\ &\implies |(k-\ell)\alpha - (M_k - M_\ell)| < \frac{1}{N}. \end{aligned}$$

Take $k > \ell$, so $0 < k - \ell \leq N \Rightarrow |(k-\ell)\alpha - (M_k - M_\ell)| < \dfrac{1}{k-\ell}$.
Set $x = M_k - M_\ell$ and $y = k - \ell$, so $0 < y \leq N$ and

$$|y\alpha - x| < \frac{1}{N} \implies \left|\alpha - \frac{x}{y}\right| < \frac{1}{Ny} \leq \frac{1}{y^2}.$$

Lagrange's approximation theorem

For each $N \geq 1$ we found x and y in $\mathbf{Z}$ such that $0 < y \leq N$ and

$$\left|\alpha - \frac{x}{y}\right| < \frac{1}{Ny} \Longrightarrow \left|\alpha - \frac{x}{y}\right| < \frac{1}{y^2} \text{ and } \left|\alpha - \frac{x}{y}\right| < \frac{1}{N}.$$

Maybe x/y isn't reduced. Let that be X/Y, so $0 < Y \leq y$. Then

$$\left|\alpha - \frac{X}{Y}\right| = \left|\alpha - \frac{x}{y}\right| < \frac{1}{y^2} \leq \frac{1}{Y^2}.$$

Therefore without loss of generality, x/y is in reduced form. Since α is irrational, $|\alpha - x/y| > 0$. We can choose $N' > 1$ so that

$$\frac{1}{N'} < \left|\alpha - \frac{x}{y}\right|.$$

There is a reduced fraction x'/y' with $y' > 0$ such that

$$\left|\alpha - \frac{x'}{y'}\right| < \frac{1}{y'^2} \text{ and } \left|\alpha - \frac{x'}{y'}\right| < \frac{1}{N'} < \left|\alpha - \frac{x}{y}\right|,$$

so $x'/y' \neq x/y$. Repeat *ad infinitum*, get $|\alpha - x_n/y_n| \to 0$.

Lagrange's approximation theorem

Example

To get $|\pi - x/y| < 1/y^2$ and $|\pi - x/y| < 1/1000$, set $N = 1000$. Look at 0, π, 2π, $\ldots$, 1000π and find two whose fractional parts are within $1/1000$ of each other. By a computer,

$$\pi = 3.1415926\ldots \text{ and } 114\pi = 358.1415625\ldots,$$

so (subtracting) $113\pi \approx 354.99996$. Thus $|113\pi - 355| \approx .00003$. Dividing by 113, $|\pi - 355/113| \approx .0000002 < 1/113^2$.

What we don't learn: we can find infinitely many solutions to

$$\left|\alpha - \frac{x}{y}\right| < \frac{1}{2y^2}.$$