CONTENU • CONTENTS

2) [Moral]

Obvious: mutation is based off the orientation relationships of adjacent arrows, which does not change when you swap the orientation of all arrows simultaneously.

3) [Moral]

mutation is a local process at the vertex. For two disconnected vertices (i.e. not sharing an arrow), mutation at both vertices impacts completely different sets of arrows.

CASS: 17th, Lecture 1 notes (Ralf Schiffler)

Cluster algebras

- introduced by Fomin-Zelevinsky 2002
- commutative algebra, subrational field of rational fcn's

Ground ring: $\mathbb{Z}\mathbb{P}$

$(\mathbb{P}, \cdot)$ a free abelian group on n generators

$y_1, \dots, y_n$

Define $\oplus$ addition by

$$\prod_i y_i^{a_i} \oplus \prod_i y_i^{b_i} = \prod_j y_j^{\min(a_j, b_j)}$$

ex:
- $y_1^3 y_2^{-1} y_3^4 \oplus y_1 y_5^{-1} = y_1 y_2^{-1} y_3^0 y_4^0 y_5^{-1} = y_1 y_2^{-1} y_5^{-1}$
- $y_1^4 y_2^{-3} y_3^4 \oplus 1 = y_2^{-3}$

$(\mathbb{P}, \oplus)$ is a semifield

$\mathbb{Z}\mathbb{P}$ = group ring of $(\mathbb{P}, \cdot)$ = ring of Laurent poly's in $y_1, \dots, y_n$ (w/ coeffs in $\mathbb{Z}$)

Ex: $\mathbb{P} = 1$, $\mathbb{Z}\mathbb{P} = \mathbb{Z}$ is the cluster alg. w/ trivial coeffs

$\mathbb{Q}\mathbb{P}$ field of fractions of $\mathbb{Z}\mathbb{P}$

$\mathcal{F} = \mathbb{Q}\mathbb{P}(x_1, \dots, x_n)$ field of rational fcns

Seeds and Mutation:

initial seed $(\mathbb{X}, \mathbb{Y}, Q)$, where Q is a quiver (directed graph) w/out loops ( ) or 2-cycles ( )

vertices arrows

$Q = (Q_0, Q_1)$, $|Q_0| = n$

AKA Principal coefficients

$\mathbb{Y} = (y_1, \dots, y_n)$ set of generators for free abelian group $\mathbb{P}$, called "initial coefficient tuple"

$\mathbb{X} = (x_1, \dots, x_n)$ n-tuple of variables in $\mathcal{F}$ called the "initial cluster"

$\mathbb{X}, \mathbb{X}'$ clusters, elements are cluster variables!

Seed mutation: μ_k, k is index $1, \dots, n$

$$\mu_k(\mathbb{X}, \mathbb{Y}, Q) = (\mathbb{X}', \mathbb{Y}', Q')$$

on clusters: $\mathbb{X}' = (\mathbb{X} \setminus \{x_k\}) \cup \{x'_k\}$,

$$x'_k = \frac{\frac{1}{y_k \oplus 1}\left(y_k \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i\right)}{x_k}$$

on coeffs: $\mathbb{Y}' = (y'_1, \dots, y'_n)$

$$y'_j = \begin{cases} y_k^{-1} & \text{if } k = j \\ y_j \prod_{k \to j} \frac{y_k}{y_k \oplus 1} \prod_{k \leftarrow j} (y_k \oplus 1) & \text{if } k \neq j \end{cases}$$

on quivers: Q' is obtained from Q in 3 steps

1. $\forall$ path $i \to k \to j$, add an arrow $i \to j$
2. Reverse all arrows at k
3. Remove 2-cycles

Ex:

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd} \xrightarrow{\mu_2} \begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}$$

$$\xrightarrow{\mu_2} \begin{tikzcd} 1 \arrow[r] \arrow[rr, bend left] & 2 \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd} = \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd}$$

Exercise: $\mu_k^2 = 1$, mutation is an involution

Ex

$$1 \Rrightarrow 2 \Rrightarrow 3 = 1 \xrightarrow{2} 2 \xrightarrow{3} 3$$

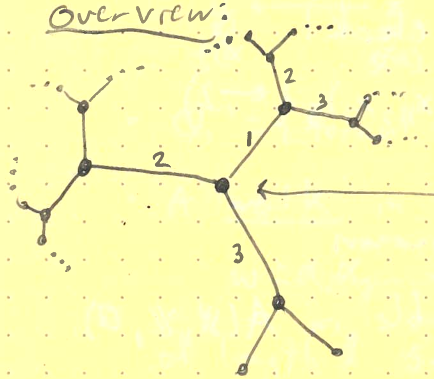
$$\Downarrow \mu_2$$

$$\begin{tikzcd} 1 \arrow[rr, bend left, "6"] & 2 \arrow[l, "2"'] & 3 \arrow[l, "3"'] \end{tikzcd}$$

$$\Downarrow \mu_3$$

$$\begin{tikzcd} 1 \arrow[r, shift left, "18"] & 2 \arrow[l, shift left, "2"] \arrow[r, "3"] & 3 \arrow[ll, bend right, "6"'] \end{tikzcd} = \begin{tikzcd} 1 \arrow[r, "16"] & 2 \arrow[r, "3"] & 3 \arrow[ll, bend right, "6"'] \end{tikzcd}$$

Overview:



initial seed

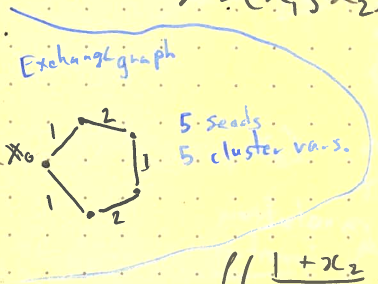
this is an n-regular graph!

Vertices = seeds
edges = mutations
$\mathcal{X}$ = all cluster variables obtained

Def^n: The cluster algebra $\mathcal{A}(\mathbb{X}, \mathbb{Y}, Q)$ is the $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}$ generated by $\mathcal{X}$

Ex Q: $1 \longrightarrow 2$, trivial coeffs. (y's $= 1$)

$\mathbf{x} = (x_1, x_2)$



$\Downarrow \mu_1$

$\left(\left(\frac{1+x_2}{x_1}, x_2\right), 1 \longleftarrow 2\right)$

$\Downarrow \mu_2$

$\left(\left(\frac{1+x_2}{x_1}, \frac{1+\frac{1+x_2}{x_1}}{x_2} = \frac{x_1+1+x_2}{x_1 x_2}\right), 1 \longrightarrow 2\right)$

$\Downarrow \mu_1$

$\left(\frac{1+\frac{x_1+1+x_2}{x_1 x_2}}{\frac{1+x_2}{x_1}} = \frac{x_1+1}{x_2}, \frac{x_2+1+x_1}{x_1 x_2}\right), 1 \longleftarrow 2\Big)$

$\Downarrow \mu_2$

$\left(\frac{x_1+1}{x_2}, \frac{\frac{x_1+1}{x_2}+1}{\frac{x_2+x_1+1}{x_1 x_2}} = x_1, 1 \longrightarrow 2\right)$

$\Downarrow \mu_1$ check!

$((x_2, x_1), 1 \longleftarrow 2)$

Thm: (FZ, '02) Laurent Phenomenon

Let u be any cluster variable in $\mathcal{A}(\mathbf{x}, \mathbf{y}, Q)$. Then

$$u = \frac{f(\mathbf{x})}{\prod_{i=1}^{n} x_i^{d_i}} \qquad f \in \mathbb{ZP}[\mathbf{x}]$$

Thm: (Lee-Schiffler, 15)

Positivity $f \in \mathbb{Z}_{\geq 0}\mathbb{P}[\mathbf{x}]$

CASS: 17th lecture (Emily Gunawan)

Maximal Almost-rigid (MAR) representations

- quiv. reps + geometric models

1. Quivers and strings

Def^n: Q quiver consists of

$Q_0 = \{\text{vertices}\}$ finite

$Q_1 = \{\text{arrows}\}$

maps $s: Q_1 \to Q_0$ arrow $\alpha \mapsto$ starting pt. of α

$t: Q_1 \to Q_0$ $\alpha \mapsto$ terminal pt. of α

if $s(\alpha) = t(\alpha)$, then α is a loop

- $\forall \alpha \in Q_1$, let α^{-1} denote the opposite arrow

$\underset{t(\alpha)}{y} \xleftarrow{\alpha} \underset{s(\alpha)}{x} \Longrightarrow \underset{s(\alpha^{-1})}{y} \xrightarrow{\alpha^{-1}} \underset{t(\alpha^{-1})}{x}$

- $Q_1^{-1} = \{\text{formal inverses } \alpha^{-1} \mid \alpha \in Q_1\}$
- A walk in Q is a finite sequence/word $w = \alpha_1 \alpha_2 \dots \alpha_\ell$ of length ℓ s.t. $\alpha_i \in Q_1 \cup Q_1^{-1}$ and $t(\alpha_i) = s(\alpha_{i+1})$
- A subwalk is contiguous subsequence of w
- A path in Q is a walk s.t. $\alpha_i \in Q_1$
- $\forall x \in Q_0$, associate a trivial path (or constant/lazy) e_x or 1_x w/ length $\ell = 0$

Ex:

$$Q = \begin{tikzcd} z \arrow[rr, "c"] \arrow[dr, "a"'] & & x \\ & y \arrow[ur, "b"'] & \end{tikzcd}$$

gentle when $R=\emptyset$, $R=\{ab\}$

$z \xrightarrow{a} x$ is locally gentle for $R=\emptyset$, gentle for $R=\{ca, bc\}$

paths $= \{e_x, e_y, e_z, a, b, c, ab\}$

infinite!

walks $=$ paths $\cup \{abc^{-1}a,\ cb^{-1}a^{-1}cb^{-1}, \dots\}$

Def^n: Assume a relation is a path of Q w/ length 2 or longer

quiver relations

(Q,R) is a gentle quiver w/ relations if:

(G0) if Q has an oriented cycle ($\rightleftarrows \to \cdots \to$), this cycle must contain a relation. (if $\exists$ a loop $\alpha \in Q_1$, $\alpha^m \in R$ for some m)

(G1) $\forall x \in Q_0$, indegree$(x) \le 2$ and outdegree$(x) \le 2$ (at most)

(G2) $\forall \alpha \in Q_1$ there is at most one arrow $\beta \in Q_1$ s.t. $\beta\alpha \notin R$ and at most one arrow $\gamma \in Q_1$ s.t. $\alpha\gamma \notin R$

(G3) R is a set of paths of length 2

(G4) $\forall a \in Q_1$, there is at most one arrow b' st. $b'a \in R$ and at most one arrow c' s.t. $ac' \in R$

Rem: · satisfying (G0–G2) yeilds a string algebra.

· satisfying (G1–G4) is a locally gentle quiver

Ex:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[r, bend left, "c"] & 3 \arrow[l, bend left, "d"] \end{tikzcd}$$

$R = \{ab, cd\}$ fails (G0) since $acdb$ is an oriented cycle not in R

$R = \{ab, dc, ba, cd\}$ gentle

Ex: $\begin{tikzcd} 1 \arrow[loop above, "a"] \end{tikzcd}$ $R = \{a^2\}$ gentle

Def^n: A string $w = \alpha_1\alpha_2\dots\alpha_\ell$ is a walk w/

- (no backtrack) no subwalk $\alpha\alpha^{-1}$ or $\alpha^{-1}\alpha$ for $\alpha \in Q_1$
- (no going through a relation) no subwalk v w/ $v \in R$ or $v^{-1} \in R$
- A trivial path e_x is a string w/ length $\ell = 0$
- the empty string or zero string will be denoted by $w = 0$
- Say strings v, w are equivalent if $v = w$ or $v = w^{-1}$, where

$$w^{-1} = \alpha_\ell^{-1} \dots \alpha_1^{-1}$$

Def^n: the diagram (or quiver) of w $\Gamma = \Gamma(w)$ is an orientation of type $A_{\ell+1}$ $(1 - 2 - \dots - \ell{+}1)$ s.t.

$i \to i+1$ in Γ if α_i is an arrow in Q_1

$i \leftarrow i+1$ in Γ if α_i is an inv. arrow in Q_1^{-1}

Ex: $\Gamma({}_y \overset{\alpha_1}{b}{}_x \overset{\alpha_2^{-1}}{c^{-1}}{}_z \overset{\alpha_3}{a}{}_y) =$ $\overset{y}{1} \xrightarrow{b=\alpha_1} \underset{x}{2} \xleftarrow{c=\alpha_2^{-1}} \overset{z}{3} \xrightarrow{a=\alpha_3} \underset{y}{4}$

$\Gamma(a^{-1}cb^{-1})$ is the same as an unlabelled directed graph

Quiver Representations:

K: an Algebraically closed field (usually $\mathbb{C}$)

Def^n: a representation $M=(M_x, \varphi_\alpha)_{x\in Q_0,\, \alpha\in Q_1}$ (vertices, arrows) of (Q,R) is given by

- A finite dimensional vector space M_x for each $x \in Q_0$
- A linear map $\varphi_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ for each $\alpha \in Q_1$

Such that the composition of maps along a path in R is the zero map, i.e.

if $\alpha\beta \in R$, then $\varphi_\beta \circ \varphi_\alpha$ = zero map

Ex:

$$\begin{tikzcd} & & 3 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & \\ & & 4 \end{tikzcd}$$

$ab = 0$

$$\begin{tikzcd} & & K \\ M = K^2 \arrow[r, "\varphi_a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}"'] & K^3 \arrow[ur, "\varphi_b = {[0\ 0\ 1]}"] \arrow[dr, "\varphi_c = {[1\ 2\ 3]}"'] & \\ & & K \end{tikzcd}$$

Note: $\varphi_b \circ \varphi_a = 0$ because ab is a relation

Def^n: a morphism $f: M \longrightarrow M'$, $M = (M_x, \varphi_\alpha)$, $M' = (M'_x, \varphi'_\alpha)$

is a collection $(f_x)_{x\in Q}$ of K-linear maps

$f_x : M_x \to M'_x$

s.t. $\forall\ i \xrightarrow{\alpha} j$ in Q_1,

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ M'_i \arrow[r, "\varphi'_\alpha"] & M'_j \end{tikzcd}$$

commutes

Def^n

1) if $f = (f_i): M \to N$ is a bijection, f is an isomorphism

2) The isoclass of M is the class of representations isomorphic to M)

3) $\text{rep}(Q,R)$ - category w/
   - objs = reprs of (Q,R)
   - morphisms = morphisms of repn.

Def^n:

$$M \oplus M' = \left(M_x \oplus M'_x,\ \left[\begin{array}{c|c} \varphi_\alpha & 0 \\ \hline 0 & \varphi_\alpha \end{array}\right] \right)$$

Ex:

$$\left(K^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} K^2 \xrightarrow{[1\ 0]} K \right) \oplus \left(K^2 \xrightarrow{[1\ 0]} K \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} K^2 \right)$$

$$= K^4 \xrightarrow{\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right]} K^3 \xrightarrow{\left[\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]} K^3$$

CASS: 17th Lecture 3 [Deping Weng]

Legendrian Links and Cluster Algebras (I)

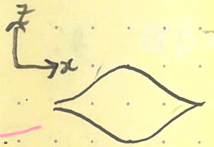
Def^{n}: The standard contact $\mathbb{R}^3_{xyz}$ is equipped w/ a contact 1-form

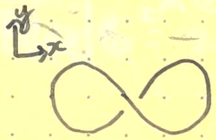
$$\alpha = dz - y\,dx$$

Def^{n}: A Legendrian Link is an oriented link $\Lambda \subseteq \mathbb{R}^3_{xyz}$ s.t. $\alpha|_\Lambda = 0$

Ex: $(x,y,z) = (-3\cos(t), \sin(2t), 2\sin^3(t))$ is a Legendrian unknot

In the xz-projection: , xy-projection





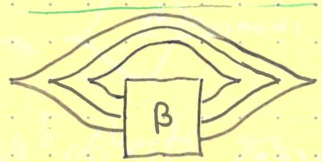
Since $\alpha|_\Lambda = 0$, $y = \frac{dz}{dx}$.

$\mathbb{R}^3_{xyz} \xrightarrow[\text{front projection}]{\pi_F} \mathbb{R}^2_{xz}$,

, no "round returns"

Ex: Fix some ≥ 0 integer n. Let β be a word (finite seq.) w/ letters from $\{1, \dots, n-1\}$ (sometimes called a positive braid-word), can draw a wiring diagram. e.g. $(1,1,2,3,2,2,1)$

The rainbow closure of β is obtained by



denoted Λ_β

Def^{n}: Two Legendrian links Λ_0 and Λ_1 are Leg. isotopic if $\exists$

$$H: S^1 \times [0,1] \to \mathbb{R}^3_{xyz}$$

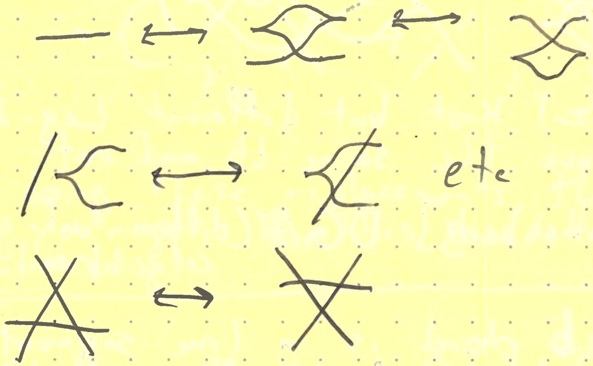
s.t. $H(t,0) \rightsquigarrow \Lambda_0$
$H(t,1) \rightsquigarrow \Lambda_1$
$H(t,s) \rightsquigarrow \Lambda_s$ leg. link

Legendrian isotopies can be discretized into Legendrian Reidemeister moves

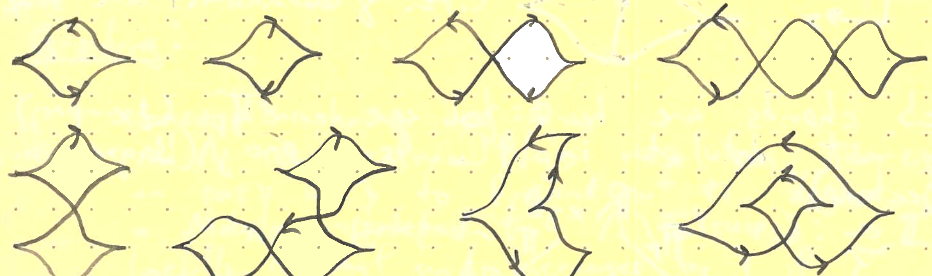
LR I.

LR II.

LR III



Exercise: determine which of the following unknots are legendrian isotopic



There are 2 numerical Leg. invariants

① Thurston-Bennequin number

$$tb(\Lambda) := \#(\times) + \#(\times) - \#(\times) - \#(\times) - \tfrac{1}{2}\#\text{cusps}$$

② rotation number

$$rot(\Lambda) = \tfrac{1}{2}\left(\#(\{) + \#(\}) - \#(\}) - \#(\{)\right)$$

Unfortunately, these are not complete invariants

Ex: (Chekanov Pair) google this

Same topological Knot but different Leg. Knots. But they have the same tb and rot

[Chekanov-Eliashberg] DGA (differential graded algebra)

Defn: A Reeb chord is a line segment connecting two points on Λ w/ the same x and y coordinates

Ex.

at these points, tangent lines (i.e. y coordinates) are same

Reeb chords are hard to see on Π_F (zx-proj) why not do it in the xy-proj (Lagrangian projection)

$$\begin{tikzcd} & \mathbb{R}^3_{xyz} \arrow[dl, "\Pi_F"'] \arrow[dr, "\Pi_L"] & \\ \mathbb{R}^2_{xz} & & \mathbb{R}^2_{xy} \end{tikzcd}$$

– Lagrangian projection

Reeb chords are intersections on the Lagrangian projection. However, Π_L is hard to draw typically.

we have the following Area condition

greens thm!

$$0 = \int_\Lambda dz = \int_\Lambda y\,dx = \text{Area enclosed by } \Pi_L(\Lambda),$$

ensuring this condition is met is hard in general

L. Ng came up with a way to "schematically" turn a front proj. to a Lag. proj:

After doing these replacements, then rescaling area, you get Π_L. This is called Ng's Resolution

Defn: Chekanov-Eliashberg DGA $\mathcal{A}(\Lambda)$

$\mathcal{A}(\Lambda)$ as an algebra is freely generated over $\mathbb{Z}/2$ by Reeb chords and Base points explained later!

Grading:

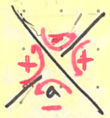
Denote Maslov potential by $m(\cdot)$

- base pts. have deg $= 0$
- if Λ comes from Ng's resolution, there is an easy way to figure out the degree. Assign a Maslov potential^m to $\Lambda \setminus \Pi_F^{-1}(\text{cusps}) \to \mathbb{Z}$ locally constant and increases by 1 if $\overset{m+1}{\underset{m}{\}}}$

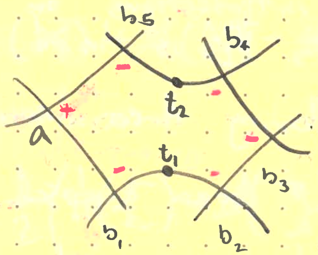
i.e. away from cusps

$\deg(\times) = m(\setminus) - m(/)$

Differential of deg 1



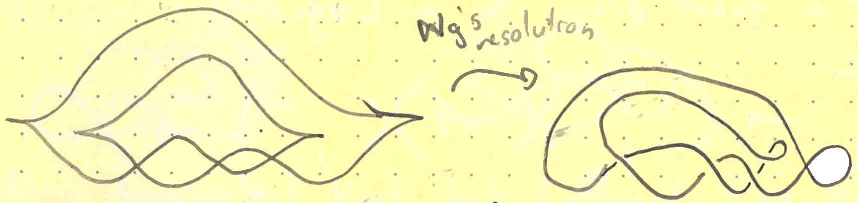
Look for immersed disks of form $\xrightarrow{\text{in } \Pi_L}$



where you start is the positive, the rest negative

$$\partial a = \cdots + b_1 t_1^{-1} b_2 b_3 b_4 t_2 b_5 + \cdots$$

trefoil



exercise compute ∂ of all the Reeb chords

Def^{n}: an augmentation is a DGA homomorphism
$$\varepsilon : \mathcal{A}(\Lambda) \to \mathbb{F}$$ "an alg. closed field if degree 2"

Def^{n} The augmentation variety is the moduli space of augmentation of $\mathcal{A}(\Lambda)$ $\mathrm{Aug}(\Lambda)$

$\mathrm{Aug}(\Lambda)$ is an affine variety generated by deg 0 generators modulo ∂(deg 1 gens)

Exercise: show that the $\mathcal{O}(\mathrm{Aug}(\Lambda_{(1,1,1)}))$ is an A_2-cluster algebra

coordinate ring, trefoil

CASS: 17th Lecture 4 (Khrystyna Serhiyenko)

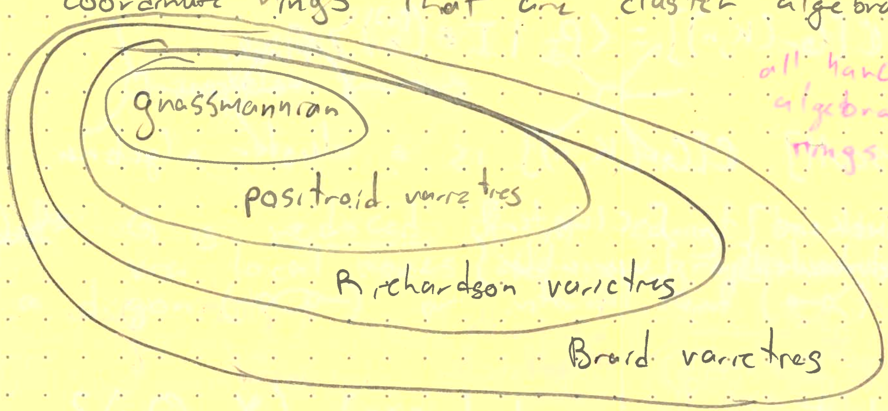
Cluster structure on Richardson Varieties and their categorification

Overview:

→ Cluster algebras introduced by FZ in 2001 and one motivation was to study total positivity
Def^{n}: $M \in \mathrm{Mat}_{k\times n}(\mathbb{R})$ is T.P. if all minors are positive

→ can we find an efficient way to test whether M is totally positive?

→ many important varieties from Lie theory have coordinate rings that are cluster algebras



all have cluster algebra coordinate rings

Goal: describe cluster algebra structure for these varieties, their combinatorics, and categorification

Grassmannian:

- $\mathrm{Gr}(k,n)$: set of k-dim linear subspaces of $\mathbb{C}^n$.
Elements of $\mathrm{Gr}(k,n)$ can think about as $k\times n$ matrices A

rowspan of A ⇝ k-dim linear subspaces

- $Gr(k,n) = GL_k \backslash Mat^*_{k\times n}$ ← full rank

  row ops

- $Gr(k,n)$ - projective variety
- $\mathbb{C}[Gr(k,n)]$ = homogeneous coordinate ring of the (affine cone) over $Gr(k,n)$
- P_I - Plücker coordinate

  $I \in \binom{[n]}{k}$ k-element subset of $[n] = 1, \dots, n$

- P_I - minor of A on columns indexed by I

<u>Thm</u>: $\mathbb{C}[Gr(k,n)] = \langle P_I \mid I \in \binom{[n]}{k} \rangle /$ Plücker relations

<u>Thm</u>: [Scott] $\mathbb{C}[Gr(k,n)]$ is a cluster algebra

$\left\{\begin{matrix}\text{Plucker} \\ \text{coordinates}\end{matrix}\right\} \subset \left\{\begin{matrix}\text{cluster} \\ \text{variables}\end{matrix}\right\}$

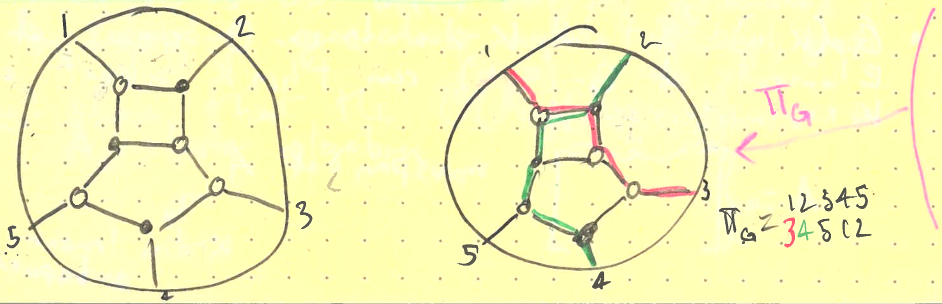
nice subset — generally infinite

cluster — Quiver

<u>Q?</u>: what is an initial seed $(\mathbf{x}, Q_{\mathbf{x}})$?

<u>Plabic Graph</u>: planar bi-coloured graph drawn inside a disk

<u>Ex</u>:



$\pi_G = \begin{matrix}1\,2\,3\,4\,5 \\ 3\,4\,5\,1\,2\end{matrix}$

<u>Local moves</u> on plabic graphs

- square move

- remove/add vertices of degree 2

- contract/add monochromatic edges

<u>Def^n</u>: G is <u>reduced</u> if it cannot be transformed via local moves to a graph containing a digon (—o○o—) or internal leaf (•—o)

any colours — opposite colours — not contractible

ⓟ ← lollipop, and are ok! assume no lollipops

* not an easy condition to check, but from now on we only consider reduced plabic graphs *

<u>Def^n</u> $G \rightsquigarrow \pi_G$ trip permutation

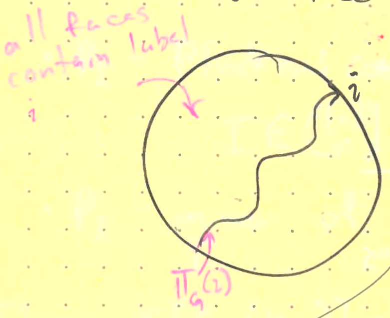
Starting w/ boundary vertex i, construct a path ending at $\pi_G(i)$ s.t.

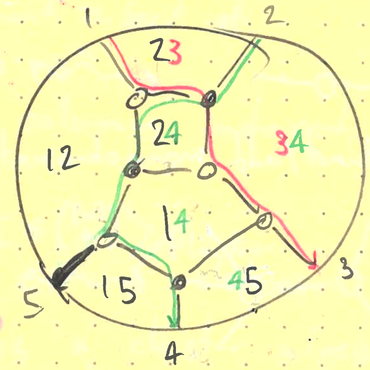
turn maximally left @ white vertices — turn maximally right @ black vertices

Thm: [Postnikov] reduced plabic graphs are move-equivalent iff they have the same trip permutation.

* label faces of G by subsets I s.t.

$i \in I$ if the face appears to the left of the trip ending at i

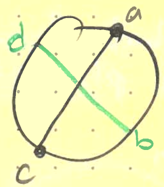




max weakly separated collections:
$\{12, 23, 34, 45, 51, 14, 24\}$

Thm: Face labels have constant cardinality (same # of labels)

Def^n: $I, J \in \binom{[n]}{k}$ are weakly separated if there does not exist $a, c \in I \setminus J$ $b, d \in J \setminus I$ s.t. a, b, c, d is cyclically ordered



not weakly separated

weakly separated

Thm: [Oh-Postnikov-Speyer]

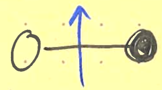
$\left\{\begin{matrix}\text{reduced} \\ \text{plabic graphs}\end{matrix}\right\} / \sim \longleftrightarrow \left\{\begin{matrix}\text{pairwise} \\ \text{weakly separated} \\ \text{labels}\end{matrix}\right\}$

move equivalent that preserve face labels

Def^n: $G \rightsquigarrow$ quiver $Q(G)$ as follows

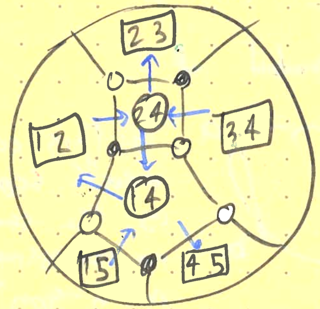
interior faces $\longleftrightarrow$ mutable vertices

boundary faces $\longleftrightarrow$ frozen vertices



$\longleftrightarrow$ arrows

Ex: of type $(2,5)$



Def^n: G is of type (k,n) if $\pi_G(i) = i + k$

Thm: $\left\{\begin{matrix}\text{reduced plabic} \\ \text{graphs } G \text{ of} \\ \text{type } (k,n)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{seeds in} \\ \mathbb{C}[Gr(k,n)] \text{ consisting} \\ \text{of Plücker} \\ \text{coordinates } (x_G, Q(G))\end{matrix}\right\}$

$\left\{\begin{matrix}\text{maximal} \\ \text{weakly} \\ \text{separated} \\ \text{collections}\end{matrix}\right\}$

faces labeled I $\longleftrightarrow$ Plücker coords. P_I

Type A

Defn: a cluster algebra $\mathcal{A}_Q$ is of type A_n if Q is mutation-equivalent to

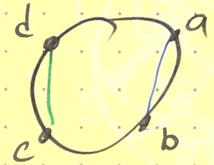
$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

Thm: $\mathbb{C}[Gr(2,n)]$ is a cluster algebra of type A_{n-3}. Moreover

$$\left\{\begin{matrix}\text{Plücker}\\ \text{coords}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{cluster}\\ \text{variables}\end{matrix}\right\}$$

$I, J \in \binom{[n]}{2}$ are weakly separated if

$I = \{a,b\}$
$J = \{c,d\}$ s.t.



in this context, weak separability is the same as non-crossing diagonals in n-gon

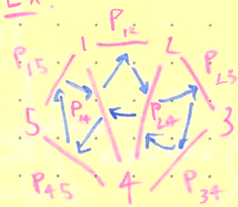
SO

Thm:

$$\left\{\begin{matrix}\text{reduced Plabic}\\ \text{graphs } G \text{ of}\\ \text{type } (2,n)\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{seeds in}\\ \mathbb{C}[Gr(k,n)]\\ (\mathbf{x}_G, Q(G))\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{triangulations}\\ \text{of } n\text{-gons}\end{matrix}\right\}$$

Ex!



Questions

① Can we understand cluster variables $\mathbb{C}[Gr(k,n)]$ that are not Plückers? (open)

② Do general plabic graphs give cluster structure for something? (A: Yes, positroids)

CASS: 17th questions

[Ralf Schiffler]

Q1) show mutation is an involution

Soln:

on clusters: 1 mutation @ k removes x_k and replaces it w/ x_k', where

$$x_k' = \frac{\frac{1}{y_k \oplus 1}\left(y_k \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i\right)}{x_k}$$

Since the 2nd mutation would remove x_k' and replace it w/ x_k'', it suffices to show that $x_k'' = x_k$. Notice that

$$\frac{\frac{1}{y_k \oplus 1}\left(y_k \prod_{i \to k} x_i + \prod_{i \leftarrow k} x_i\right)}{x_k} = \frac{(\text{numerator } y_k)\prod_{i \to k} x_i + (\text{denominator } y_k)\prod_{i \leftarrow k} x_i}{x_k}$$

maybe helpful? mutating once also flips y_k, by mutation on coefficients (since we're at k) and flips arrows, so the top term in the formula is the same after both mutations. i.e.

$$x_k'' = \frac{(\text{num. } y_k')\overbrace{\prod_{i \to k} x_i}^{x_k'} + (\text{denom. } y_k')\overbrace{\prod_{i \leftarrow k} x_i}^{x_k'}}{x_k'}$$

$$= \frac{\sim\sim\sim}{\left(\frac{(\text{num. } y_k)\overbrace{\prod_{i \to k} x_i}^{x_k} + (\text{denom. } y_k)\overbrace{\prod_{i \leftarrow k} x_i}^{x_k}}{x_k}\right)}$$

$$= x_k \left(\cancelto{0}{\frac{(\text{denom. } y_k)\prod_{i \leftarrow k} x_i + (\text{num. } y_k)\prod_{i \to k} x_i}{(\text{num. } y_k)\prod_{i \to k} x_i + (\text{denom. } y_k)\prod_{i \leftarrow k} x_i}}\right)$$

$$= x_k$$

on coefficients: for $k = j$ this is clear, assume $j \neq k$. Then

$$y_j' = y_j \prod_{k \to j} \frac{y_k}{y_k \oplus 1} \prod_{k \leftarrow j} (y_k \oplus 1)$$

$$= y_j \prod_{k \to j} (\text{numerator of } y_k) \prod_{k \leftarrow j} \frac{1}{(\text{denominator of } y_k)}$$

again, mutating flips arrows, so

$$y_j'' = y_j' \prod_{k \to j} (\text{num. } y_k) \prod_{k \leftarrow j} (\text{denom. } y_k)$$

$$= \Big(\underbrace{y_j \prod_{k \to j} (\text{num. } y_k) \prod_{k \leftarrow j} \frac{1}{(\text{denom. } y_k)}}_{y_j}\Big) \underbrace{\prod_{k \to j} (\text{num. } y_k) \prod_{k \leftarrow j} \frac{1}{(\text{denom. } y_k)}}_{y_j'}$$

$$= y_j \prod_{k \to j} \cancel{(\text{num. } y_k)} \prod_{k \leftarrow j} \cancel{\frac{1}{(\text{denom. } y_k)}} \prod_{k \leftarrow j} \cancel{(\text{denom. } y_k)} \prod_{k \to j} \cancel{\frac{1}{(\text{num. } y_k)}}$$

$$= y_j$$

on quivers: step 1) twice leaves directions unchanged. Arrows added in the first round of step 2) get deleted by the arrows added in the second round of step 2), via step 3 (since after the first mutation directions are reversed).

Q2) Compute the cluster variables for $1 \to 2 \leftarrow 3$

Q3) Show the following quivers are mutation-equivalent

$$\begin{tikzcd}[column sep=small] 1 \arrow[rr] & & 2 \arrow[rr] \arrow[dl] & & 3 \arrow[dl] \\ & 6 \arrow[ul] \arrow[rr] & & 4 \arrow[ul] \arrow[dl] & \\ & & 5 \arrow[ul] & & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] 1 \arrow[dr] & & 2 \arrow[ll] & & 3 \arrow[ll] \\ & 6 \arrow[dr] & & 4 \arrow[ur] & \\ & & 5 \arrow[ur] & & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & & & & 5 \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[ur] \arrow[dr] & \\ & & & & 6 \end{tikzcd}$$

Q3) Find all quivers mutation-equivalent to

$1 \to 2 \to 3 \to 4 \to 5$

(Q5) Consider the following recursion

$$x_0=1,\ x_1=1,\ x_k=\frac{x_{k-1}^3+1}{x_{k-2}}$$

Show that $x_k\in\mathbb{Z}$ for all k

[Emily Gunawan]

Q1)
$$Q=\begin{tikzcd} z \arrow[rr, "c"] \arrow[dr, "a"'] & & x \\ & y \arrow[ur, "b"'] & \end{tikzcd} \qquad ab=0$$

i) Count all non-zero strings w (up to inverse equivalence) and for some of them draw the diagrams $\Gamma(w)$.

ii) Convince yourself that any morphism $f=(f_x,f_y,f_z)$ from M to N is of the form

$$\begin{tikzcd} M: & K \arrow[rr, "Id"] \arrow[dr, "0"'] \arrow[dd, "\lambda"'] & & K \arrow[dd, "\lambda"] \\ & & 0 \arrow[ur, "0"'] \arrow[dd, "0"] & \\ N: & K \arrow[rr, "Id"] \arrow[dr, "0"'] & & K \\ & & K \arrow[ur, "Id"'] & \end{tikzcd}$$

, where $\lambda\in K$

iii) Find all morphisms from N to M

iv) Find all morphisms from

$$L:\ \begin{tikzcd} 0 \arrow[rr, "0"] \arrow[dr, "0"'] & & K \\ & K \arrow[ur, "Id"'] & \end{tikzcd}$$

to N.

Q2)
$$Q=\begin{tikzcd} z \arrow[rr, "c"] \arrow[dr, "a"'] & & x \arrow[r, "d"] & w \\ & y \arrow[ur, "b"'] & & \end{tikzcd} \qquad cd=0$$

i) Draw $\Gamma(w)$ for

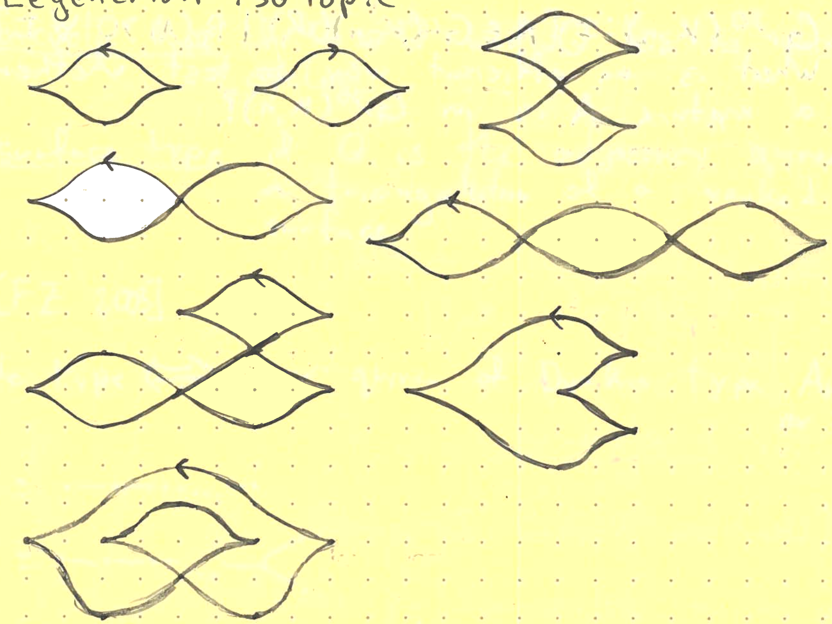
$w=abc^{-1}abd$

$w=bc^{-1}abc^{-1}abd$

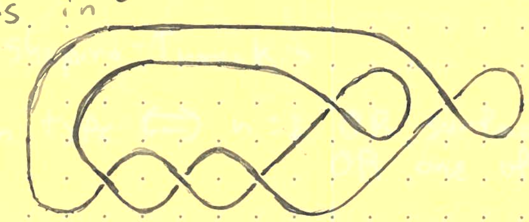
ii) How many strings are there?

[Daping Weng]

Q1) Determine whether the following unknots are Legendrian isotopic



Q2) Compute the grading and the differentials of Reeb chords in

$\Lambda_{(1,1,1)}=$ 

Q3) Show that $\mathcal{O}(Aug(\Lambda_{(1,1,1,1)}))$ is an A_2-cluster algebra by exhibiting the clusters and the quiver.

(Recall: $\mathcal{O}(Aug(\Lambda_{(1,1,1,1)}))$ is generated by the degree 0 Reeb chords modulo the differentials of the degree 1 Reeb chords.)

[Khrystyna Serhiyenko]

Q1) How do local moves on a plabic graph G affect the corresponding seed $(\mathbb{X}_G, Q(G))$?

Q2) What are the frozen variables for $\mathbb{C}[Gr(k,n)]$?

Q3) $Gr^{>0}(k,n) := \{A \in Gr(k,n)(\mathbb{R}) \mid P_I(A) > 0 \ \forall I\}$. What is an efficient way to test whether a matrix A is in $Gr^{>0}(k,n)$?

CASS: 18th (Ralf Schiffler)

- Google "quiver mutation Javascript". Downloadable script lets you add cluster variables and coefficients

Classifications

Def^n: A cluster algebra $\mathcal{A}(\mathbb{X}, \mathbb{Y}, Q)$ is said to be of

a) finite type if the number of cluster variables is finite

implies!

b) finite mutation type if the number of quivers $Q' \overset{\text{mut}}{\sim} Q$ is finite

mutation equiv.

c) acyclic type if $Q \underset{\text{mut}}{\sim} Q'$ and Q' has no oriented cycles

d) surface type if Q is the adjacency quiver of a triangulation of a marked surface

Thm: [FZ 2003]

Finite type $\Longleftrightarrow$ $Q \sim$ quiver of Dynkin type A_n, D_n or $E_{6/7/8}$

A_n =

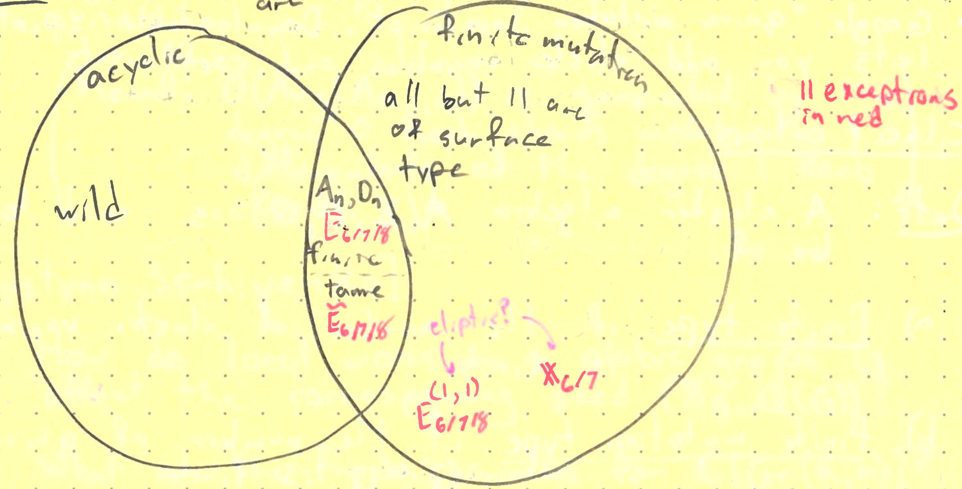
D_n =

E_{678} =

Thm: [Felikson-Shapiro-Tumarkin]

finite mutation type $\Longleftrightarrow$ $n=2$ OR surface type OR one of 11 exceptions

$n\geq 3$ most cluster algebras are wild and not acyclic



F-polynomial: $\mathcal{A}$ principal coeffs

initial seed

$\mathbb{X}_0$ $\qquad \mathbb{X}_t = (x_{1,t}, x_{2,t}, \dots, x_{n,t})$

$x_{i,t}$'s are cluster variables $\in \mathbb{Z}\mathbb{P}[x_1^{\pm}, \dots, x_n^{\pm}]$, $\mathbb{P}$ generated by $y_1, \dots, y_n$

$\big\downarrow x_i = 1$

$F_{i,t}$ F-polynomial

Thm [Derksen-Weyman-Zelevinsky]

$F_{i,t}$ is a polynomial (in $y_1, \dots, y_n$) w/ constant term 1 and there exists a maximal degree monomial and it has coefficient 1 and it is divisible by all the other monomials.

Consequences:

$x_{i,t}$ has a unique Laurent monomial that has no y's. The degree vector of this monomial is called the g-vector of $x_{i,t}$, denoted

$$g_{i,t} \in \mathbb{Z}^n$$

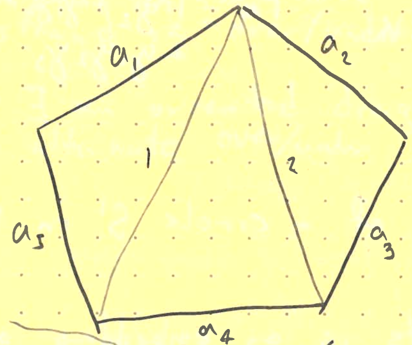
Thm: [FZ in Cluster Algebras IV] (for overview, start w/ CA IV)

$$x_{i,t} = \mathbb{X}^{g_{i,t}} F_{i,t}(\hat{y}_1, \dots, \hat{y}_n) \quad \text{where } \hat{y}_i = y_i \frac{\prod_{i\to j} x_j}{\prod_{j \to i} x_j}$$

(in Q, initial quiver!)

Coefficients as frozen variables

Ex



$\mathbb{P} = \text{Free Ab.}(a_1, \dots, a_5)$

$\mathbb{Y} = \left(\frac{a_5}{a_1 a_4}, \frac{a_2 a_4}{a_3}\right)$

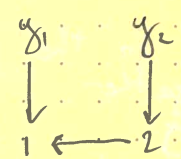
can mutate using formulas, or by mutating

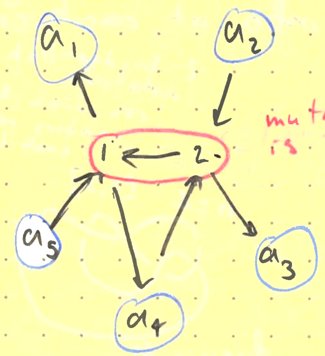
1, incident exceident in

2, incident exceident

$a_1, \dots, a_5$ frozen variables

Principal coeffs





mutable part is A_2

F-polynomials of quiver representations

Def^n: $M \in \operatorname{rep} Q$, $F_M = \sum_{e \in \mathbb{Z}^{|Q_0|}_{\geq 0}} \chi(Gr_e M)\, y^e$

euler-characteristic of the variety $Gr_e M$

quiver grassmannian $\{N \subset M \mid \dim N = e\}$ subrep

$\prod_{i \in Q_0} y_i^{e_i}$

Ex

$$1 \longrightarrow 2 \longleftarrow 3$$

$M: K \longrightarrow 0 \longleftarrow 0$ $\qquad F_M = 1 + y_1$

$M': K \xrightarrow{1} K \longleftarrow 0$ $\qquad F_{M'} = 1 + y_2 + y_1 y_2$

$M'': K \xrightarrow{1} K \xleftarrow{1} K$ $\qquad F_{M''} = 1 + y_2 + y_1 y_2 + y_2 y_3 + y_1 y_2 y_3$

2. Knot Theory

A Knot is an embedding of a circle S^1 in $\mathbb{R}^3$ (or S^3).

A link with n components is an embedding of n copies of S^1 in $\mathbb{R}^3$ (S^3)

A link diagram is a projection of a link in the plane (nice, no triple points, only 2-crossings) together with the over/under info at each crossing

Ex:

[2] $n=2$ ← # crossings

Hopf link

2-bridge-link notation: can draw X-ings on 2 levels, numbers record #X-ings on each level. (more interesting combinatorics to be found here)

$n=3$

[3]

trefoil

$n=4$ [2,2]

or [4]

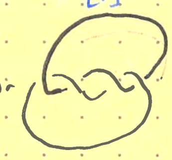
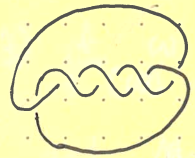


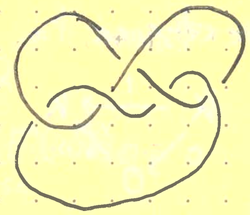
figure 8

$n=5$ [5]



[2,2]

[2,1,2]



Link

Borromean Rings

- not 2-bridge ← no blue label
- has no bigon

- examples so far have been alternating (following a strand is over/under/over/under...)

↳ $\exists$ an oriented diagram in which the crossings alternate over/under

CASS: 18^{th} (Emily Gunawan)

Day 1's exercise

M:
$$\begin{tikzcd} K \arrow[rr, "1"] \arrow[dr, "0"'] & & K \\ & 0 \arrow[ur, "0"'] & \end{tikzcd}$$

string $w = bc^{-1}a$ of
$$\begin{tikzcd} z \arrow[rr, "c"] \arrow[dr, "a"'] & & x \\ & y \arrow[ur, "b"'] & \end{tikzcd} \qquad ab = 0$$

N:
$$\begin{tikzcd} K \arrow[rr, "1"] \arrow[dr, "0"'] & & K \\ & K \arrow[ur, "1"'] & \end{tikzcd}$$

has diagram $\Gamma(w)$
$$\begin{tikzcd} \underset{y}{1} \arrow[r, "b"] & \underset{x}{2} & \underset{z}{3} \arrow[l, "c"'] \arrow[r, "a"] & \underset{y}{4} \end{tikzcd}$$

L:
$$\begin{tikzcd} 0 \arrow[rr, "0"] \arrow[dr, "0"'] & & K \\ & K \arrow[ur, "1"'] & \end{tikzcd}$$

MAR repr Day 2

Def^n: $\mathrm{Hom}(M, M') := \{\text{all morphisms } M \to M'\}$

↳ is a K-vector space

w/ add: addition of morphisms

scalar mult: scaling of morphisms

3. String repr

To each string w of (Q,R), define a

str rep $M(w) =: M = (M_x, \varphi_\alpha)_{x \in Q_0,\ \alpha \in Q_1}$

1) $w = 0$, $M(w)$ = zero repr

2) $w = 1_x$, $x \in Q_0$, $M_x = K$, and $M_y = 0$ if $y \neq x$, all φ_α are zero maps

Def^n: $M =: S(x)$ simple representation at x

3) if w visits every vertex at most once, $\Gamma(w)$ is a subquiver of Q

set $M_x = \begin{cases} K & \text{if } x \in (\Gamma(w))_0 \\ 0 & \text{otherwise} \end{cases}$

$\varphi_\alpha = \begin{cases} Id & \text{if } \alpha \in (\Gamma(w))_1 \\ 0 & \text{otherwise} \end{cases}$

Remark M, N, L of this form

4.) $w = \alpha_1 \alpha_2 \dots \alpha_\ell$ string

$\Gamma = \Gamma(w)$ diagram of w

4.a) Define a repr $N \in \mathrm{rep}\,\Gamma$

Ex: $\Gamma = $
$$\begin{tikzcd} 1 \arrow[r] & 2 & 3 \arrow[l] \arrow[r] & 4 \end{tikzcd}$$

1-dimensional

$K_i \cong K$

$$N = \begin{tikzcd} \overset{\{b_1\}}{K_1} \arrow[r, "Id"] & \underset{\{b_2\}}{K_2} & \overset{\{b_3\}}{K_3} \arrow[l, "Id"'] \arrow[r, "Id"] & \underset{\{b_4\}}{K_4} \end{tikzcd}$$

Rem: call both the diagram and repr Γ

4b) Define $M = M(w)$ of (Q,R) using Γ:

- Let $M_x := \bigoplus_{i \in I_x} K_i$ where $I_x = \{\text{vertices } i \text{ of } \Gamma \text{ corresp. to } x\}$

$$\begin{tikzcd} \underset{\{b_3\}}{K_3 = M_z} \arrow[rr] \arrow[dr] & & \underset{\{b_2\}}{M_x = K_2} \\ & M_y = K_1 \oplus K_4 \ \{b_1, b_4\} \arrow[ur] & \end{tikzcd}$$

Let b_i be a basis vector for K_i.

- For each arrow $\beta \in Q_1$: $s \xrightarrow{\beta} t$

define $\varphi_\beta : \underbrace{M_s}_{\oplus K_i} \to \underbrace{M_t}_{\oplus K_i}$

* if there is α_i s.t. $\alpha_i = \beta$, set $\varphi_\beta(b_i) = b_{i+1}$ $\left(i \xrightarrow{\alpha_i}_{\beta} i+1 \text{ in } \Gamma \right)$

* if there is α_i s.t. $\alpha_i = \beta^{-1}$ then set $\varphi_\beta(b_{i+1}) = b_i$ $\left(i \xleftarrow[\beta]{\alpha_i} i+1 \text{ in } \Gamma \right)$

* $\varphi_\beta(b_j) = 0$ if b_j is a basis vector for M_x (not sent to anywhere yet)

I think all of this so far has been how to solve the exercise? or comments relating to that!

Rem: From Γ, can construct repr $M(\omega)$ of (Q,R), so it's enough to work w/ Γ (type A diagram)

4. Indecomposables

Defn $M \in \mathrm{rep}(Q,R)$ is indecomposable if

$$M = N \oplus L \text{ implies } N = 0 \text{ or } L = 0$$

unique decomposition (thm?)

Every $M \cong M_1 \oplus \cdots \oplus M_t$ w/ M_i indec unique up to permutation of M_i

Defn: M is basic if no M_i is repeated

Thm: [Butler-Ringel]

every indec. rep of a string (Q,R) is a string rep or a band rep (diagram cyclic ex.)

Defn/Prop: For each vertex $x \in Q_0$ the indecomposable projective at x is

$$P(x) = M(\text{the string } \swarrow x \searrow \cdots) =: M$$

each vector space M_y will have bases {string paths from vertex x to y}

Ex

$$\begin{tikzcd} z \arrow[rr, "c"] \arrow[dr, "a"'] & & x \\ & y \arrow[ur, "b"'] & \end{tikzcd}$$

$P(x) = M(e_x)$ $0 \to k\{e_x\}$, $\to 0$

$P(y) = M(b)$ $0 \xrightarrow{0} k\{b\}$, $0 \to k\{e_y\}$ (Id)

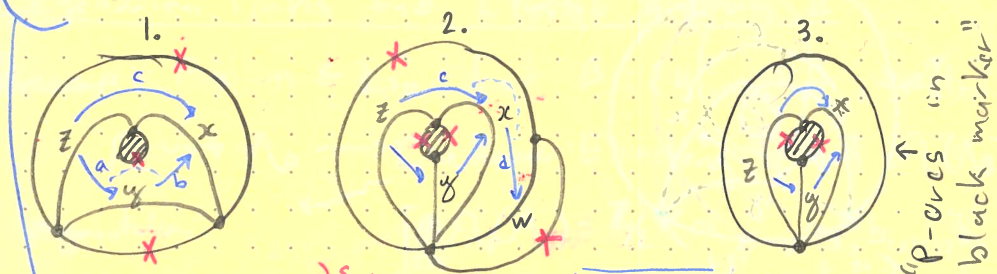
$P(z) = M(c\ a\ b)$ $\{e_z\}k \to k^2\{c, ab\}$, $k\{a\}$

5. Geometric mode for gentle (Q, R)

{gentle (Q,R)} ⊂ {string (Q,R)} ?

I think these authors work on this stuff!

[Opper-Plamondon-Schroll, Baur-Coelho Simões, 2018]



"generalization" of cl. alg from unpunctured surfaces [FST]

on ∂S or a puncture in interior of S

3. $Q = z \to x$, $\searrow y \nearrow$ $R = \emptyset$

2. $Q = z \to x \to w$, $\searrow y \nearrow$ $R = \{cd\}$

1. $Q = z \xrightarrow{c} x$, $a \searrow y \nearrow b$ $R = \{ab\}$

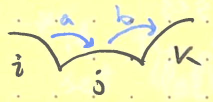
Rule (S, M, P, M) — tiling

oriented surface; points (•) on ∂S; P-arcs, partial triangulation (set of p-arcs that don't cross each other or itself)

From tiling to (Q_P, R_P)

P-angles: $[\, i \xrightarrow{a} j \longleftrightarrow i \xrightarrow{a} j \,]\ [\, i = j \longleftrightarrow i \circlearrowleft a \,]$

for R on next page

 in P $\longleftrightarrow$ ab is in R


not p-arc

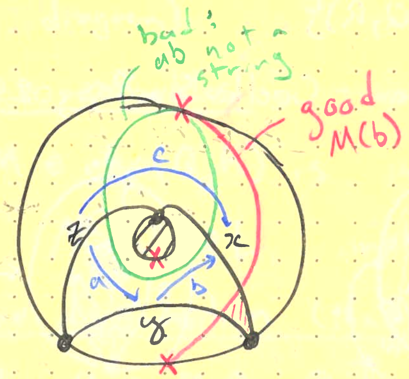
j can be a loop, or $i=k$ can be a loop, or both? i,j,k cannot all be equal?

Def^n A <u>permissible arc</u> in (S, M, M^*, P)

- endpoints are in M^*
- when a segment of γ crosses a "tile" Δ, they locally cut up a triangle 

Thm: permissible arcs $\longleftrightarrow$ string reps

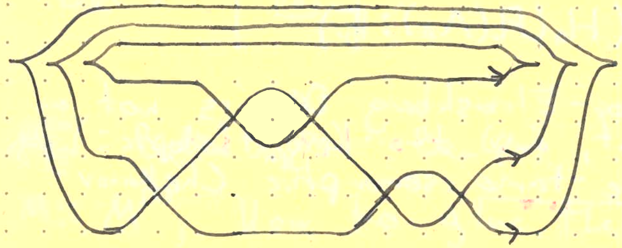
Ex



Did not follow. Notes are faithful representations of what was on the board, to the best of my ability

CASS: 18th [Daping Weng]

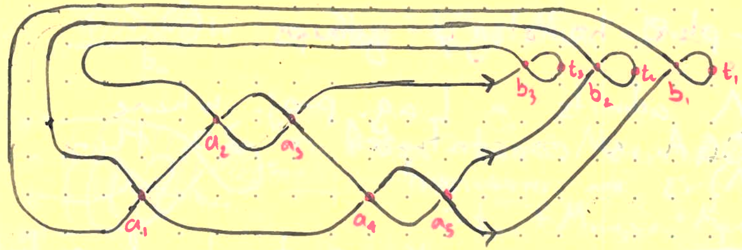
$n=3$ $\quad \beta=(1,2,2,1,1)$



$\Pi_F(\Lambda_\beta)$

Ng's resolution

$\Pi_L(\Lambda_\beta)$



Legendrian links and Cluster Algebras II

Recap: Fix $n \in \mathbb{Z}_{>0}$. Let $\beta = (k_1, \ldots, k_\ell)$, $k_i \in \{1, \ldots, n-1\}$

$\beta \rightsquigarrow \Lambda_\beta \rightsquigarrow$ Chekanov-Eliashberg DGA $\mathcal{A}(\Lambda_\beta)$

$\left[\bigoplus_{i\geq 0} \mathcal{A}_i(\Lambda_\beta) = \mathbb{Z}/2 \langle a_1, \ldots, a_\ell, b_1, \ldots, b_n, t_1^{\pm 1}, \ldots, t_n^{\pm 1} \rangle\right.$

augmentation: dga homomorphism

$\varepsilon: \mathcal{A}(\Lambda_\beta) \to \mathbb{F}$

$\mathbb{F}$ is an alg. closed field characteristic two 2

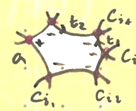
$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ \mathcal{A}_1 & \longrightarrow & 0 \\ \partial\downarrow & & \downarrow \\ \mathcal{A}_0 & \xrightarrow{\varepsilon} & \mathbb{F} \\ \downarrow & & \downarrow \\ \mathcal{A}_{-1}=0 & & 0 \end{array}$$

to define an augmentation, we need to assign an $\mathbb{F}$-value to each deg 0 generator modulo $\varepsilon\circ\partial = 0$

Grading: $|a\text{'s}| = |t\text{'s}| = 0$, $|b\text{'s}| = 1$

$\mathcal{A}(\Lambda_\beta)$ is concentrated in non-negative degrees

differential: $\partial: \mathcal{A}_i \to \mathcal{A}_{i-1}$



$\partial a = \cdots + c_{i_1} c_{i_2} c_{i_3} t_1 c_{i_4} t_2^{-1} + \cdots$

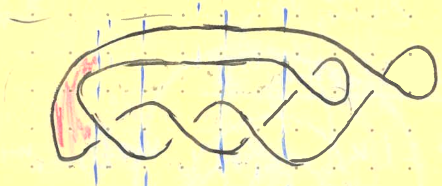
Def^n: The augmentation variety $Aug(\Lambda_\beta)$ is the moduli space of augmentation of $\mathcal{A}(\Lambda_\beta)$ when $\mathcal{A}(\Lambda_\beta)$ is concentrated in ≥ 0,

$$Aug(\Lambda_\beta) = Spec(H_0(\mathcal{A}(\Lambda_\beta);\mathbb{F}))$$

Rem: The Chekanov-Eliashberg DGA is not an invariant. But, any two Leg. isotopic Leg. links have stable-tame isomorphic Chekanov-Eliashberg DGA's

$\Rightarrow$ isomorphic homology groups

$\Rightarrow$ If Λ admits a Lag. proj. where the DGA is concentrated in ≥ 0, then $Aug(\Lambda)$ is an invariant

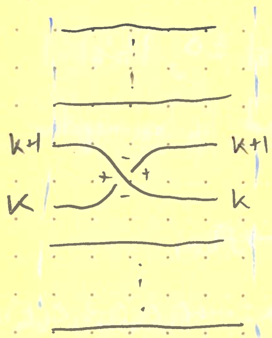


$M^{(0)}$ $M^{(1)}$ $M^{(2)}$ $M^{(3)}$

Keep track of partial disks

$M_{ij}^{(m)}$ = the partial dis between the i^{th} strand above and j^{th} strand below after the m^{th} X-ing

$M_{ij}^{(0)} = id$



$M_{ij}^{(m-1)}$ k_m $M_{ij}^{(m)}$

$$M_{ij}^{(m)} = \begin{cases} M_{ij}^{(m-1)} & \text{if } j \neq k, k+1 \\ M_{ik+1}^{(m-1)} & \text{if } j = k \\ M_{ik}^{(m-1)} + a_m M_{ik+1}^{(m-1)} & \text{if } j = k+1 \end{cases}$$

$$= M_{ij}^{(m-1)} \begin{array}{c|c} & \begin{bmatrix} 1 & & 0 & & 0 \\ & \ddots & & & \\ 0 & & 0 \ \ 1 & & 0 \\ & & 1 \ \ a_m & & \\ 0 & & 0 & \ddots & \\ & & & & 1 \end{bmatrix} \\ & \quad\quad k \ \ k{+}1 \end{array}$$

$\leftarrow$ Braid matrix

Def^n: The Braid matrix

$$B_k(a) := \begin{bmatrix} \ddots & & \\ & \boxed{\begin{matrix} 0 & 1 \\ 1 & a \end{matrix}} & \\ & & \ddots \end{bmatrix}$$

$$M_{ij}^{(m)} = B_{k_1}(a_1)B_{k_2}(a_2)\cdots B_{k_m}(a_m)$$

$M := M^{(\ell)}$, How to glue them? Look at b_n

$$\partial b_n = t_n^{-1} + M_{nn} = 0$$

$$\partial b_{n-1} = t_{n-1}^{-1} + M_{n-1,n-1} + M_{n-1,n} t_n M_{n,n-1}$$
$$= t_{n-1}^{-1} + M_{n-1,n-1} + M_{n-1,n} M_{nn}^{-1} M_{n,n-1}$$
$$= t_{n-1}^{-1} + t_n \underbrace{(M_{n-1,n-1}M_{n,n} + M_{n-1,n}M_{n,n-1})}_{\text{determinant, we have + instead of } - \text{ because characteristic 2.}}$$

$$= t_{n-1}^{-1} + t_n \det(M_{\geq n-1})$$

Iterating:

Product

$$\partial b_i = t_i^{-1} + t_{i+1} \times \cdots$$
$$\cdots \times t_n \det(M_{\geq i})$$
$$= 0$$

The eq^n to cut out the $Aug(\Lambda_\beta)$ are of the form

$$\det(M_{\geq i}) \neq 0$$

so

$$Spec\left(\mathbb{F}[a_1,\dots,a_\ell] \big/ \det(M_{\geq i}^{(\ell)}) \neq 0\right)$$

So, for

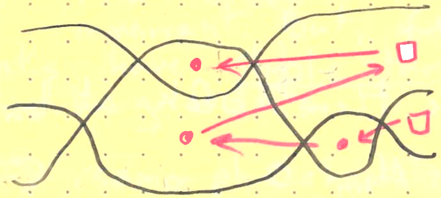
$$\begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & a_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & a_3 \end{bmatrix} = \begin{bmatrix} 1 & a_2 \\ a_1 & 1+a_1a_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & a_3 \end{bmatrix} = \begin{bmatrix} a_2 & 1+a_2a_3 \\ 1+a_1a_2 & a_1+a_3+a_1a_2a_3 \end{bmatrix}$$

yesterday: $\partial b_2 = 1 + a_1 + a_3 + a_1a_2a_3$, $\partial b_1 = 1 + (1+a_2a_3)(1+a_1a_2) + a_2$

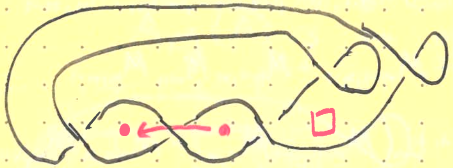
Thm: [Gao-Shen-Weng]

$\mathcal{O}(\mathrm{Aug}(\Lambda_\beta))$ is a cluster algebra with an initial seed.

Ex



Ex



w/ cluster variables

$$\det(M^{(m)}_{2k})$$

for any $m>0$ and $2\le k\le n$

Flags

Let V be a vector space $(\mathbb{F}^n, \mathbb{C}^n, \mathbb{R}^n)$.
A Flag is a nested sequence of subspaces

$$0\subset F^{(1)}\subset F^{(2)}\cdots\subset F^{(n)}=V$$

s.t.

$$\dim F^{(k)}=k$$

We can represent a flag with a matrix

$$\det\left(\begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}\right) = 0, \quad F=(0\subset \mathrm{span}(v_n)\subset\cdots\subset \mathrm{span}(v_1,\dots,v_n))$$

Exercise: Any two reps. of the same flag are related by a left multiple of B (upper triangular matrices)

Relative positions between flags: F and F' are said to be of relative position s_i if

$$\left.\begin{array}{ll} F^{(j)}=F'^{(j)} & \forall j\neq i \\ F^{(i)}\neq F'^{(i)} & \text{for } i \end{array}\right\} \; F \underset{s_i}{—} F'$$

Two flags F and F' are of relative position $w\in S_n$ if for a reduced word $s_{i_1}s_{i_2}\cdots s_{i_m}$ of w there exists a sequence of indeterminant flags

$$F \underset{s_{i_1}}{—} F_1 \underset{s_{i_2}}{—} F_2 \underset{s_{i_3}}{—} \cdots \underset{s_{i_m}}{—} F_m$$

Two flags F and F' are in general position if for every i,

$$F^{(i)}\cap F'^{(n-i)}=\{0\}$$

Exercise: Two flags are in general position iff they are w_0 (longest word in S_n) apart.

CASS: 18^{th} [Khrystyna Serhiyenko]

Categorification of cluster structure on $\mathbb{C}[Gr(k,n)]$

Recall: $\mathbb{C}[Gr(k,n)]$ is a cluster alg. whose combinatorics is described by plabic graphs

Idea for categorification: $\Longleftrightarrow$ in bijection

category $\mathcal{C}$	cluster character map	cluster algebra
• indecomposable (rigid) objects M	$\xrightarrow{\varphi}$	• cluster variables x_M
• cluster-tilting objects $M = \bigoplus_{i=1}^{n} M_i$	$\Longleftrightarrow$	• clusters $\mathbb{X}_M = (x_{M_1}, \dots, x_{M_n})$
• End(M)	$\Longleftrightarrow$	• quiver $\mathbb{X}_M$
• mutation	$\Longleftrightarrow$	• mutation

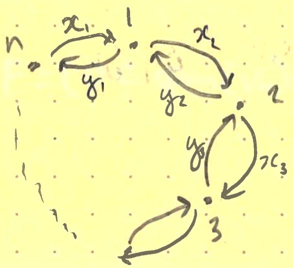
[Geiss-Leclerc-Schroer] - provide a categorification for $\mathbb{C}[Gr(k,n)]$ in terms of preprojective algebras, here $P_{1 \dots k} = 1$, not uniform, corresponding to the zero-module

[Jensen-King-Su] - give a different categorification of $\mathbb{C}[Gr(k,n)]$ in terms of "circle algebra", which is uniform but the modules are infinite dimensional

[JKS] categorification

Circle quiver

$Q_{k,n}$



not a c. alg quiver, 2-cycles ok.

$\mathbb{C}Q_{k,n}$ - path algebra of $Q_{k,n}$

bases correspond to paths in $Q_{k,n}$. multiplication corresponds to composition of paths

Ex: $x_1 x_2 x_3 y_3$, y_2

$x_1 x_2 x_3 y_3 \cdot y_2 = x_1 x_2 x_3 y_3 y_2$

$y_2 \cdot x_1 x_2 x_3 y_2 = 0$ ← does not make sense

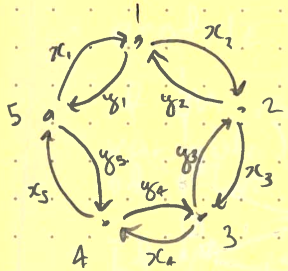
Defn:

$\widehat{\mathbb{C}Q_{k,n}}$ - completed path algebra ↳ "allow ∞-ly long paths" — might be infinite sum of finite paths? or something else for all subscripts

$B_{k,n} := \widehat{\mathbb{C}Q_{k,n}} / \langle xy - yx, x^k - y^{n-k} \rangle$ - still ∞-dimensional ⌟

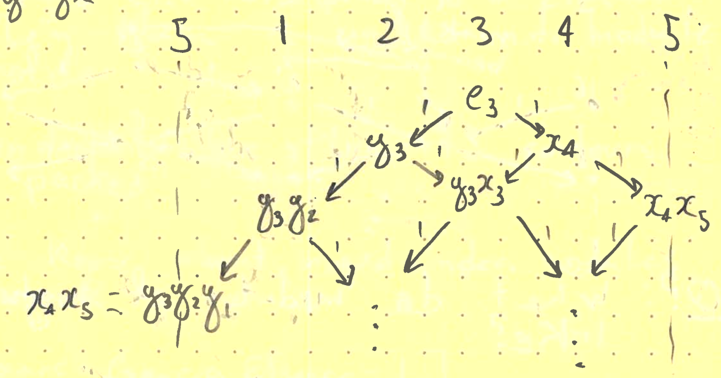
modules $\xleftrightarrow{\text{identified}}$ quiver representations

Ex: $k=2$, $n=5$



$x^2 = y^3$
$xy = yx$

compute $P(3)$.
↳ basis for $P(3) \leftrightarrow$ all paths starting at 3



• $CM(B_{k,n})$ - Cohen-Macaulay modules

$$CM(B_{k,n}) = \{ M \in \mathrm{Mod}\, B_{k,n} \mid \mathrm{Ext}^i(M, \mathrm{Proj}) = 0 \}$$

Projective $B_{k,n}$ modules become projective-injective objects in $CM(B_{k,n})$

[JKS] showed $CM(B_{k,n})$ provides categorification of $\mathbb{C}[Gr(k,n)]$ as follows

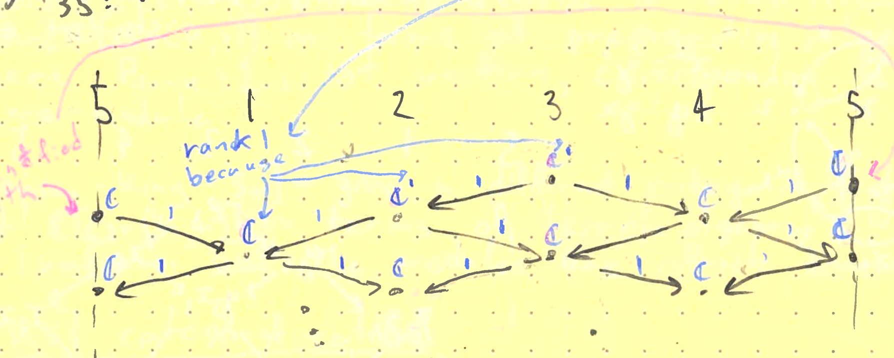
$CM(B_{k,n})$ $\mathrm{Ext}^1(M,M')=0$	$\mathbb{C}[Gr(k,n)]$
• ind. rigid modules M	• cluster variables x_M
• projective-injective objects $P(a)$	• Frozen variables Plückers $P_{a,a+1,\dots,a+k-1}$
• rank 1 modules M_I	• Plückers P_I

what do these rank 1 modules look like?

Ex $k=2$, $n=5$ $I=35$

M_{35}?:

1-dim copy of $\mathbb{C}$ @ each vertex



Q?: what do modules of higher rank look like?

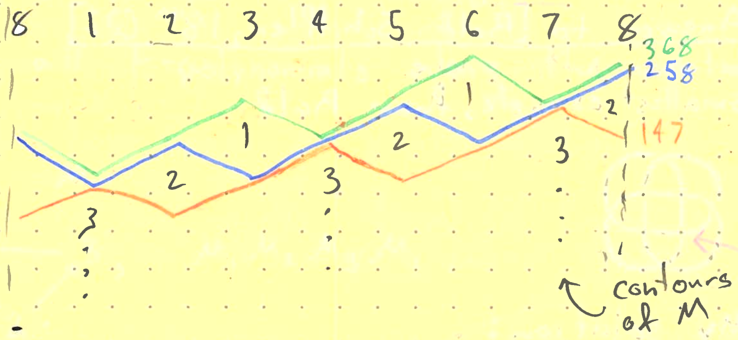
Prop: $M \in CM(B_{k,n})$ has a filtration by rank 1 modules

$$M_p \subset \cdots \subset M_2 \subset M_1 \subset M$$

$M_1/M_2 = M_{I_2}$ "rank 1 module" $M/M_1 = M_{I_1}$

M has profile $\dfrac{I_1}{\dfrac{I_2}{I_3}}$ ⋮

Ex M has profile $\dfrac{368}{\dfrac{258}{147}}$ in $Gr(3,8)$



Properties

• profile does not determine the module M

• if M is rigid (i.e. corresponds to a cluster variable) then there is unique ind. module of that profile "super-nice property"

• If M is indecomposable then its contours are "close packed" in above example, contours are close packed

Q?: What do we know about rigid indec. modules of higher rank?

Thm: [Baur - Bogdanic - Garcia Elsener - Li]

M is a rank 2 module w/ profile $\frac{I}{J}$. If the contours form exactly 3 "boxes", it is rigid

Conj: M is rigid iff the contours form 3 boxes

note: if ≤ 2 quasiboxes then $M_{\frac{I}{J}}$ is decomposible

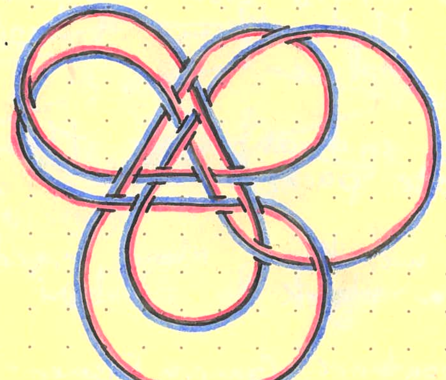
[Le-Yildirim] describe $P_{\frac{I}{J}}$ in terms of webs

Answer to [Ralf Schiffler, 18th Q2]

smallest example, from Ralf

My solution:

- alternating
- 9 3-chambers, 8 4-chambers, 1 5-chamber (exterior)
16 crossings



CASS: 18th questions

[Ralf Schiffler]

Q1) Compute the F-polynomials of the mutation sequence

$\cdot \xrightarrow{\mu_2} \cdot \xrightarrow{\mu_1} \cdot$ for $Q =$
$$\begin{tikzcd} 3 \arrow[r] \arrow[dr] & 1 \\ & 2 \arrow[u] \end{tikzcd}$$

check against Emily's Q1a)

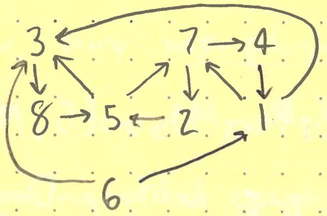
Q2) Find a knot diagram (not a link) without bigons, that is not the unknot

Q3) Compute all F-polynomials along the mutation sequence

a) $Q = 1$, μ_1

b) $Q =$
$$\begin{tikzcd} 1 \arrow[dr] & & 3 \arrow[ll] \\ & 2 \arrow[ur] & \end{tikzcd}$$
, $\mu_1 \mu_2 \mu_3 \mu_1$

Reinterpret your computation using a triangulated hexagon where mutation = flip of diagonal

c) $Q =$ 

Show that
i) Q is acyclic type
ii) Q is of surface type
iii) Mutate 4,8,2,6,1,5,3,7,6,2,8,4

[Emily Gunawan]

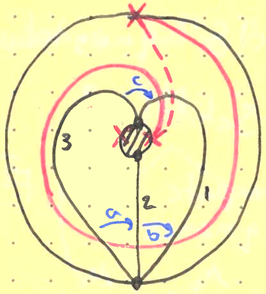
Rule: Tiling $\longleftrightarrow$ Quiver w/ relations

$(S, M, P, M^*) \longleftrightarrow (Q, R)$

P. angle  $\longleftrightarrow$ $i \xrightarrow{a} j$

 $\longleftrightarrow$ $ab \in R$

Q1)
$$Q = \begin{tikzcd} 3 \arrow[rr, "c"] \arrow[dr, "a"'] & & 1 \\ & 2 \arrow[ur, "b"'] & \end{tikzcd} \qquad R = \emptyset$$



$P(3) = M(c^{-1}ab)$

$P(1) = S(1)$

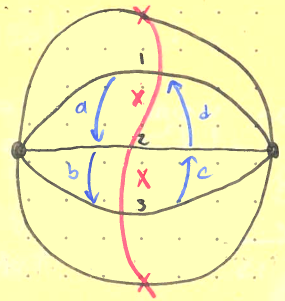
a) Find all iso classes of subrepresentations of the indecomposable projective $P(2)$. Compare with Ralph's Q1.

b) Put $P(1), P(2), P(3)$ as arcs of (S, M, P, M^*). Find 3 more arcs so that all 6 arcs have no crossings.

c) Find a string w whose arc crosses itself.

d) Find a longer string w' whose arc does not cross itself

Q2)



$$Q = \begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "d"] \arrow[r, bend left, "b"] & 3 \arrow[l, bend left, "c"] \end{tikzcd}$$

a) Find all relations R for this (S, M, P, M^*)

b) $P(2) = M(\overset{d}{\swarrow}\overset{b}{\searrow})$ has arc drawn. Draw all arcs for $P(1), P(2), P(3)$, and four other arcs that don't cross. Perform mutations (arc flips, see Ralph's Q3.b)) to get all possible sets T of noncrossing arcs

[Daping Weng]

Let

$$M = \begin{bmatrix} — V_1 — \\ — V_2 — \\ \vdots \\ — V_n — \end{bmatrix}$$

be an invertible matrix. This defines a flag

$$F_M := 0 \subset \operatorname{span}\{V_n\} \subset \operatorname{span}\{V_n, V_{n-1}\} \subset \cdots \subset \operatorname{span}\{V_n, \dots, V_1\} = V$$

Q1) Prove that if $F_M = F_{M'}$, then $M = UM'$ for some upper triangular matrix U

Q2) Prove that two flags are in general position iff they are w_0 apart

Q3) [Bonus] Prove that if

$$F_M \xrightarrow{s_i} F_{M'},$$

then $M' = M B_i(a)$ for some a

[Khrystyna Serhiyenko]

other than the trivial morphism

Q1) Show any morphism $M_I \to M_J$ is injective.

Q2) Show $\operatorname{Hom}_{\mathbb{C}}(M_I, M_J) \cong \mathbb{C}[t]$

Q3) Assuming the conjecture on rank 2 modules of [Baur-Bogdanic-Garcia Elsener-Li] holds, find rank 2 cluster variables for $Gr(2,n)$, $Gr(3,6)$, $Gr(3,7)$

CASS: 19^{th} [Ralf Schiffler]

Reidemeister Moves

R1

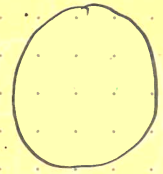
R2

R3

Alexander Polynomial

A Seifert Surface F of a link L is a connected compact oriented surface $\partial F = L$

Ex:

$L =$  $F =$ 

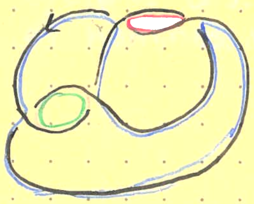
unknot Disk

Fact: $m = \#$ components of L

$H_1(F; \mathbb{Z}) = \bigoplus_{2g+m-1} \mathbb{Z}$, g = genus of F

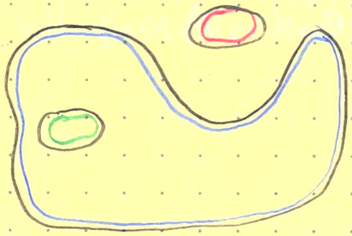
Ex

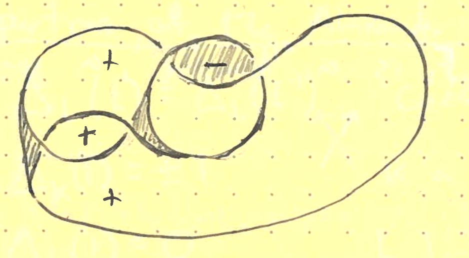
Figure 8 Knot



1. Smoothing: get seifert disks

Follow a strand. when at a crossing, swap strands and go with the flow





2. connect w/ twists according to crossings of original Knot

seifert surface

Note: the seifert surface of a knot is not unique. there are (potentially) many seifert surfaces. to get the canonical surface, start with a minimal diagram and use algorithm

Sign:

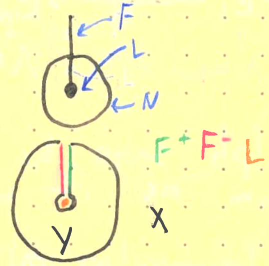
F seifert surface, $\partial F = L$

$L \subset N$ (N regular neighborhood)

$X = S^3 \setminus N$, $Y = X - F$

$X_\infty = \bigcup \{ Y_i \mid i \in \mathbb{Z} \}$, $Y_i = Y$,

Ralf was in Italy while making these notes



glue F^- of Y_i to F^+ of Y_{i+1}

$\exists\ t : X_\infty \to X_\infty$ homeomorphism mapping Y_i to Y_{i+1}

$\rightsquigarrow$ t-action on $H_1(X_\infty)$

$\rightsquigarrow$ $\Lambda = \mathbb{Z}[t, t^{-1}]$, $H_1(X_\infty)$ is a Λ-module called the Alexander module of L

Define

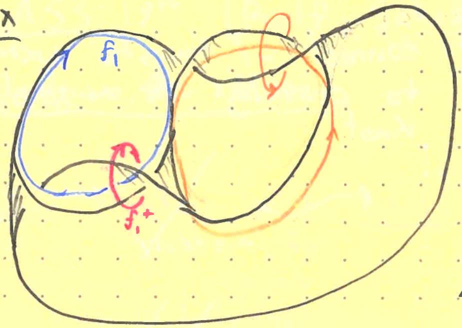
Seifert Matrix

$$A = \begin{pmatrix} lk(f_1, f_1^+) & lk(f_2, f_1^+) & \cdots \\ lk(f_1, f_2^+) & \ddots & \\ \vdots & & \end{pmatrix}$$

f_i: generators for $H_1(F)$

f_i^+: pushoff of f_i in + direction

Ex



F_1 is a "handle", a generator

F_2 is another, pushoff left as an exercise, though can maybe get it from

$$A = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$$

Thm: $tA - A^T$ is a presentation matrix for $H_1(X_\infty)$ as a Λ-module

exact sequence

$$\Lambda \otimes_{\mathbb{Z}} H_1(F) \hookrightarrow E_1 \xrightarrow{tA - A^T} E_0 \longrightarrow H_1(X_\infty) \longrightarrow 0$$

free Λ-modules $\qquad E_0 = \Lambda \otimes H_1(F)$

Ralph says he has details, but isn't providing them as it would take all week

$$tA - A^T = \begin{bmatrix} t-1 & 1 \\ -t & -t+1 \end{bmatrix}$$

The 1st Alexander ideal is generated by the maximal minor of $tA - A^T$.
The 2nd A. ideal is generated by the set of next smaller minors.
The 3rd, again the set of one smaller minors.
and so on

$$\Lambda \otimes \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} t-1 & 1 \\ -t & -t+1 \end{pmatrix}} \Lambda \otimes \mathbb{Z}^2 \longrightarrow H_1(X) \to 0$$

we want the image.

$$\mathrm{im}\begin{pmatrix} t-1 & 1 \\ -t & -t+1 \end{pmatrix} \overset{\text{row transform}}{=} \mathrm{im}\begin{pmatrix} 1 & t-2 \\ 0 & t^2-3t+1 \end{pmatrix} = \mathrm{im}\begin{pmatrix} 1 & 0 \\ 0 & t^2-3t+1 \end{pmatrix}$$

$$\Rightarrow H_1(X_\infty) \cong \Lambda / \langle t^2 - 3t + 1 \rangle$$

1st Alexander Ideal

Notated Δ

Alexander polynomial $\doteq \det(tA - A^T)$

up to multiplication by a unit in the ring ($\pm t^j$ for some $j \in \mathbb{Z}$)

Properties of the Alexander Polynomial

1. $\Delta_L(t) \doteq \Delta_L(t^{-1})$ Coeffs are palindromic
2. $\Delta_K(1) = \pm 1$ K a knot

   $\Delta_L(1) = 0$ L link w/ ≥ 2 components
3. K knot

   $\Delta_K(t) \doteq a_0 + a_1(t^{-1} + t) + a_2(t^{-2} + t^2) + \ldots$

Second defn: Skein relations

Δ normalised as in 3

The Alexander polynomial is the polynomial that satisfies

$$\Delta_0 = 1$$

$$\Delta_{L^+} - \Delta_{L^-} = (t^{-1/2} - t^{1/2}) \Delta_{L^0}$$

Ralf "I could have come up with this formulation, why did Jones get a Fields medal..."

Other invariants:

- Jones polynomial $\qquad V_0 = 1$

  $t^{-1} V_{L^+} - t V_{L^-} = (t^{1/2} - t^{-1/2}) V_{L^0}$

  people's names (acronym)
- HOMFLY-PT $\qquad P(0) = 1$

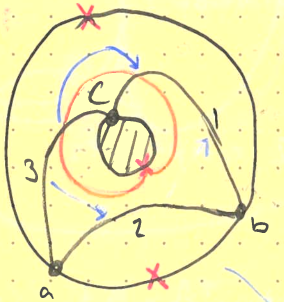
  $a P(L^+) - a^{-1} P(L^-) = z P(L^0)$

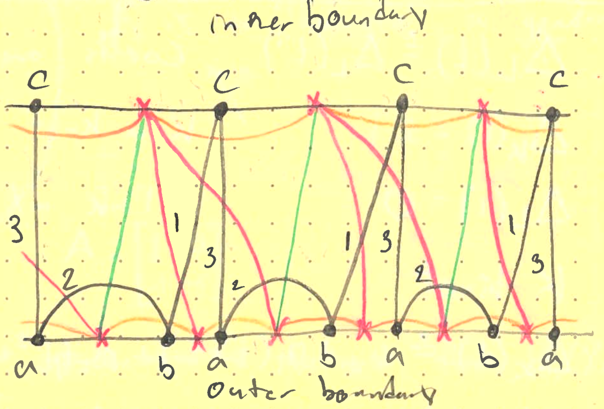
  Jones $\leftarrow a = t^{-1},\ z = t^{1/2} - t^{-1/2}$

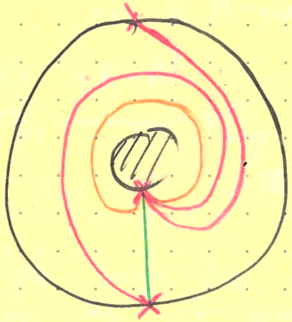
  Alexander $\leftarrow a = 1,\ z = t^{1/2} - t^{-1/2}$

CASS: 19th [Emily Gunawan]

ab=0







$(S, \cancel{M}, P, M^*)$
delete

outer bdry

$c \oplus a \oplus b \oplus e_2 \oplus e_1 \oplus c' \ldots$

$\parallel$

T

an example of a triangulation of (S, M^*)

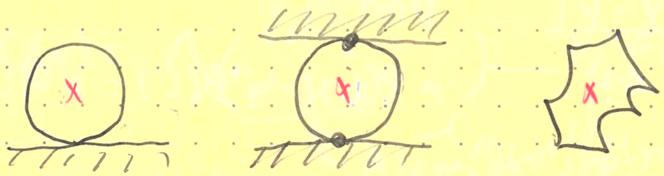
MAR reps Day 3: Cont. of 5. Geometric model

Remarks: $(S, \overset{(\bullet)}{M}, P, \overset{(x)}{M^*})$ is a tiled surface

P: collection of P-arcs s.t.:

- P is connected
- P cuts (S, M, P, M^*) into n-gons Δ called tiles of type (i) or (ii):

(i) Δ is internal n-gon ($n \geq 1$) containing a unique red point $p^* \in M^*$ in interior



n=1 n=2 n>2

(ii) Δ is n-gon ($n \geq 2$) w/ exactly 1 boundary edge (and 1 red point x on this boundary edge).
no red points in interior of Δ

n=2 n>2

Rem

$M^* \longleftrightarrow$ tiles of (S, M, P, M^*)

$M \longleftrightarrow$ certain strings (ex)

MAR reps

Defn / Fact:

$L, M, N \in \text{rep}(Q, R)$

- An extension of N by L is a short exact sequence (s.e.s.)

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$$

$$\text{im}(f) = \ker(g)$$

- A s.e.s. is split if $M \cong L \oplus N$

Ex $Q = 1 \longrightarrow 2$

a non-split s.e.s.

$$\begin{tikzcd}
L \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & K \arrow[d] \\
M \arrow[d, "g"] & K \arrow[r, "\text{id}"] \arrow[d, "\text{Id}"] & K \arrow[d, "0"] \\
N & K \arrow[r] & 0
\end{tikzcd}$$

f injective

$\ker g = \text{im} f$

g surjective

a split s.e.s.

$$\begin{tikzcd}
L \arrow[d, "f'"] & 0 \arrow[r, "0"] \arrow[d] & K \arrow[d] \\
M \arrow[d, "g'"] & K \arrow[r, "0"] \arrow[d] & K \arrow[d] \\
N & K \arrow[r, "0"] & 0
\end{tikzcd}$$

$\cong L \oplus N$

$\text{im} f' = \ker g'$

Def^n: $T \in rep(Q,R)$ is called rigid if $Ext^1(T,T)=0$

Def^n: A basic (no repeated summands) $T \in rep(Q,R)$ is almost rigid if

(1) T is a sum of string modules

(2) $\forall$ pairs M,N of indecomposable summands of T, if

$$0 \longrightarrow N \longrightarrow E \longrightarrow M \longrightarrow 0$$

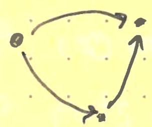
is a s.e.s, then it either splits or the middle term E is indecomposable

Def^n: an almost rigid T is maximal almost rigid (MAR) if T is maximal with respect to (2), i.e.

(3) for every non-zero $L \in rep(Q,R)$, $T \oplus L$ is not almost rigid.

Prop: we can construct a MAR rep for every gentle (Q,R)

Rem



has 6 summands ($3+3$)

has 7 summands ($3+4$)

Prop: Condition (3) can be replaced with

$$\#\text{summands of } T = \underbrace{|Q_0|}_{\text{vertices}} + \underbrace{|Q_1|}_{\text{arrows}}$$

MAR Thm 1

$$\{\text{MAR reps}\} \longleftrightarrow \left\{\begin{matrix}\text{(permissible)} \\ \text{triangulations} \\ \text{of } (S,M^*)\end{matrix}\right\}$$

arcs correspond to strings

7. Auslander-Reiten quiver for $rep(Q,R)$

Def^n/Prop: $\omega = \alpha_1 \cdots \alpha_\ell$ string of (Q,R)

$\omega' = \alpha_i \cdots \alpha_j$ subwalk of ω

1) $\Gamma(\omega')$ is a down-set (ideal for poset people) of $\Gamma(\omega)$ if

(i) α_{i-1} is an arrow β or $i=1$

(ii) α_{j+1} is an inverse arrow γ^{-1} or $j=\ell$

Picture.

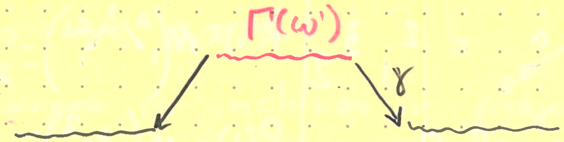
$\Gamma(\omega) =$

Then there is an injective morphism

$$M(\omega) \hookrightarrow M(\omega')$$

2) $\Gamma(\omega')$ is an up-set of $\Gamma(\omega)$ if

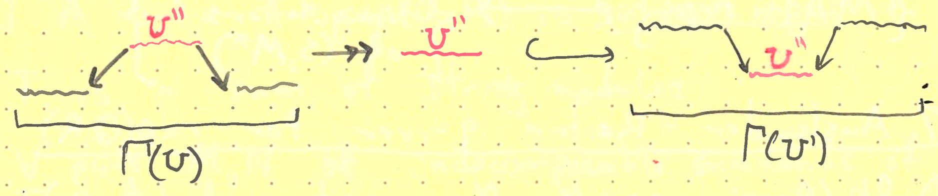
$\Gamma(\omega) =$



Then there exists a surjective morphism

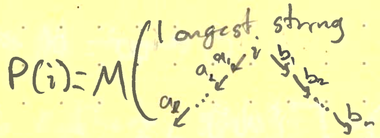
$$M(\omega) \twoheadrightarrow M(\omega')$$

3) Basis of Hom(M(U), M(U')) is

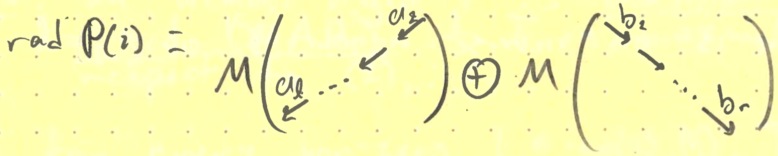


Def^n/Facts:

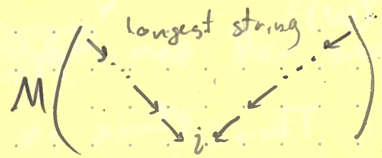
- indec. projective at i,



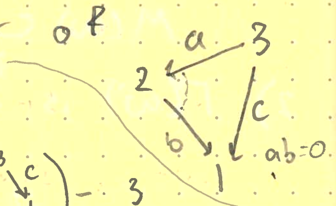
- The radical of $P(i)$,



- Injection rad $P(i) \hookrightarrow P(i)$
- Indec. injective at i,

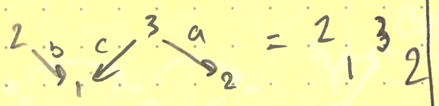


The Auslander-Reiten (AR) quiver of

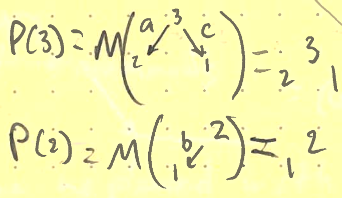


shorthand of diagram $\Gamma(w)$

Ex



white vertices of Q corresponding to dim(M(w))



$P(1) = S(1) = 1$

rad $P(3) = 2 \oplus 1$

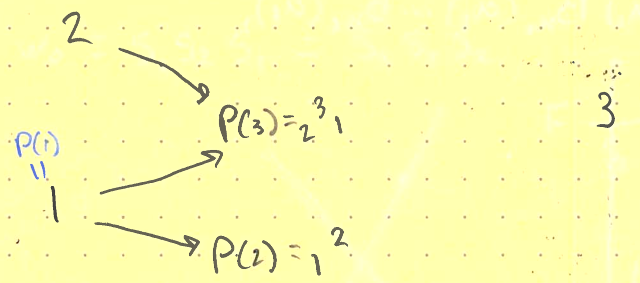
rad $P(2) = 1$

rad $P(1)$ = zero rep

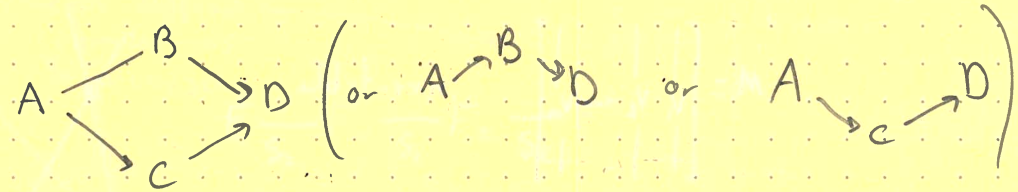
Algorithm doesn't always work but does sometimes

- Start w/ all $S(i)$ Step 1
- If $S(i)$ is a summand of rad $P(j)$, then draw $S(i) \to P(j)$
- If $P(j)$ is a summand of rad $P(k)$, then draw $P(j) \to P(k)$

Ex

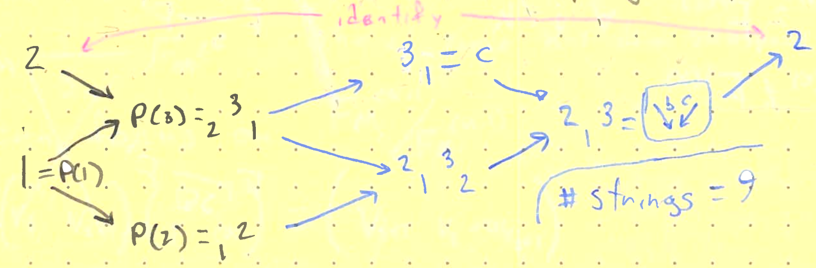


Step 2 Knitting Rule



- dim D + dim A = dim B + dim C
- Pictures are s.e.s.
- each arrow is inclusion or projection

Ex



CASS: 19th [Daping Weng]

Quick Recap

$$\beta \longrightarrow \Lambda_\beta \longrightarrow \mathcal{A}(\Lambda_\beta) \longrightarrow \mathrm{Aug}(\Lambda_\beta)$$

If $\beta = (k_1, \dots, k_\ell)$ with $1 \le k_i \le n$, then

$$\mathcal{O}(\mathrm{Aug}(\Lambda_\beta)) = \mathbb{F}[a_1, \dots, a_\ell] / (\det(M_{\ge i}) \ne 0)$$

where $M = B_{k_1}(a_1) B_{k_2}(a_2) \cdots B_{k_\ell}(a_\ell)$,

$$B_k(a) = \begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & a & \\ 0 & & & & \ddots \\ & & & & 1 \end{bmatrix}$$

Flags, suppose

$$M = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$ invertible

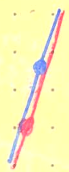
swapped rows for columns because of

Define $F_M = (0 \subset \mathrm{span}(v_n) \subset \cdots \subset \mathrm{span}(v_n, \dots, v_1) = \mathbb{F}^n)$

Two flags F and F' are in relative position s_k if

$F^{(i)} = F'^{(i)} \quad \forall\, i \ne k$

$F^{(k)} \ne F'^{(k)}$



$F \underset{s_1}{—} F'$

$F \underset{s_2}{—} F'$

In general, we say that two flags are w apart

$F \underset{w}{—} F'$ for some $w \in S_n$

if for any reduced word of $w = (k_1, k_2, \dots, k_m)$ there exists a sequence of intermediate flags

$$F \underset{s_{k_1}}{—} F_1 \underset{s_{k_2}}{—} F_2 — \cdots \underset{s_{k_m}}{—} F_m = F'$$

This sequence is unique for each reduced word

Ex w_0 longest element in S_n. In S_3,

$w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$

note: these are projective flags

$F \underset{s_1}{—} F_1 \underset{s_2}{—} F_2 \underset{s_1}{—} F'$

$F \underset{s_2}{—} F_1 \underset{s_1}{—} F_2 \underset{s_2}{—} F'$

Suppose we fix F_M. What do flags that are of s_k relative position to F_M look like?

$F_M \underset{s_k}{—} F_{M'}$ share same $k-1$ dim subspace

we can focus on $F_M^{(k+1)} / F_M^{(k-1)}$

$$\begin{pmatrix} v_i & v_{i+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} = \begin{pmatrix} v_{i+1} & v_i + a v_{i+1} \end{pmatrix}$$

matrix multiplication convention got mixed up, fixed

Claim: If $F_{M'}$ is a flag s.t. $F_M \xrightarrow{s_K} F_{M'}$, then $M' = M B_K(a)$ for some $a \in \mathbb{F}$

So

$$F_{id} \underset{s_{k_1}}{\text{———}} F_{B_{k_1}(a_1)} \underset{s_{k_2}}{\text{———}} F_{B_{k_2}(a_2)} \text{———} \cdots \underset{s_{k_\ell}}{\text{———}} F_{B_{k_\ell}(a_\ell)}$$

From a positive braid word $\beta = (k_1, \dots, k_\ell)$ and a collection $a_i \in \mathbb{F}^\ell$, we get a sequence of flags

$$F_{id} \underset{s_{k_1}}{\text{———}} F_{M^{(1)}} \underset{s_{k_2}}{\text{———}} F_{M^{(2)}} \text{———} \cdots \underset{s_{k_\ell}}{\text{———}} F_{M^{(\ell)}}$$

$$F_{id} \underset{w_0}{\text{———}} F_{w_0} \underset{w_0}{\text{———}} F_{M^{(\ell)}}$$

$$w_0 = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \qquad e_1 \wedge e_2 \wedge \cdots e_i$$

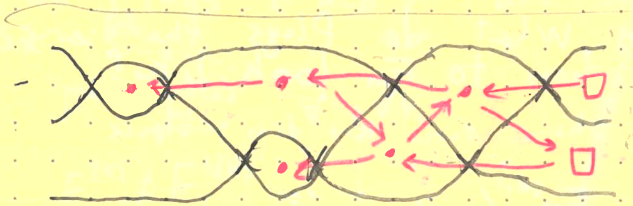
id

things got messed up due to a convention mistake, should be mostly good

Def^n:

$$\mathrm{Aug}(\Lambda_\beta) = \left\{ (a_1, \dots, a_\ell) \in \mathbb{F}^\ell \;\middle|\; F_{id} \underset{s_{k_1}}{\text{—}} F_{M^{(1)}} \underset{s_{k_2}}{\text{—}} \cdots \underset{s_{k_\ell}}{\text{—}} F_{M^{(\ell)}},\; F_{id} \underset{w_0}{\text{—}} F_{w_0} \underset{w_0}{\text{—}} F_{M^{(\ell)}} \right\}$$

$\beta = (k_1, \dots, k_\ell)$



$\det(M^{(m)}_{\geq k}) \neq 0$

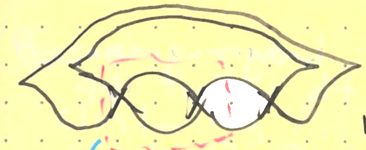
for the initial seed, $\rightsquigarrow$ an algebraic torus inside $\mathrm{Aug}(\Lambda_\beta)$, cut out by setting cluster var. $= 0$

$\cong (\mathbb{F}^\times)^{\cdots}$

$$F_{id} \text{—} F_1 \text{—} F_2 \text{—} \cdots \text{—} F_\ell, \quad F_{id}, F_1, F_2, F_\ell \overset{w_0}{\text{—}} F_{w_0}$$

EX Positive trefoil

$S_2 = \{e, s\}$, $s = w_0$



$n = 2$

$$F_{id} \text{—} F_1 \text{—} F_2 \text{—} F_3, \quad F_{w_0}$$

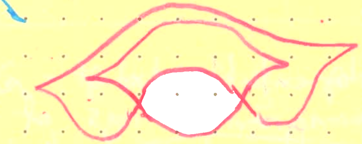
This is a pentagon. initial seed is a triangulation

replaced 2 X-ings with 1

mutate

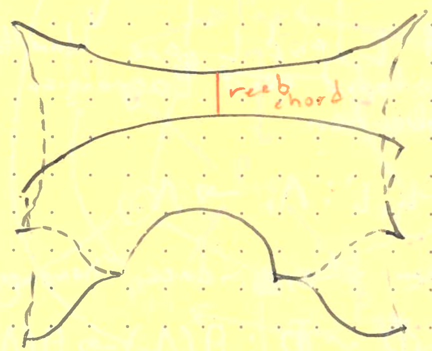
$$F_{id} \text{—} F_1 \text{—} F_2 \text{—} F_3, \quad F_{w_0}$$

initial seed for Hopf-link



can do this replacement w/ cobordism (but not by shrinking the reeb chord between the 2 X-ings we want to replace

Saddle cobordism



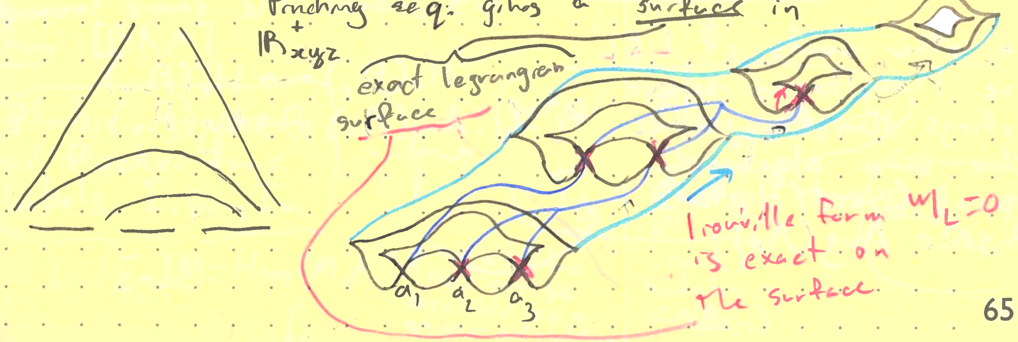
each vertical slice is a legendrian link

Exercise:

realize  to  by Reidemeister moves and the saddle cobordism

Pinching seq. gives a surface in $\mathbb{R}^4_{xyzt}$

exact legrangian surface

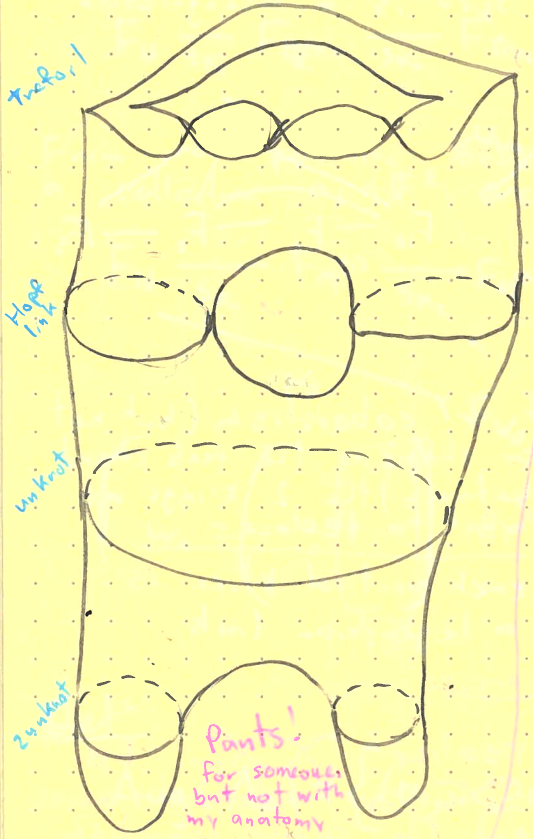


liouville form $w|_L = 0$ is exact on the surface

Thm:

There exists a unique exact Lagrangian surface that bounds the max tb unknot up to Hamiltonian isotopy

thurston-bennequin



Defn: an exact Lagrangian filling of Λ is an exact Lagrangian surface L s.t. $\partial \Lambda$

Thm:

All orientable exact Lagrangian fillings of the same Legendrian link have the same genus

Fact/thm:

If L and L' are Hamiltonian isotopic exact Lagrangian cobordism

$L, L' : \Lambda_+ \to \Lambda_-$

then they induce homomorphism

$\Phi_L \sim \Phi_{L'} : \mathcal{A}(\Lambda_+) \to \mathcal{A}(\Lambda_-)$

$\rightsquigarrow$ induce the same homomorphism

$H_0(\mathcal{A}_+) \to H_0(\mathcal{A}_-)$

$\varphi_L^* = \varphi_{L'} : \mathrm{Aug}(\mathcal{A}_-) \to \mathrm{Aug}(\mathcal{A}_+)$

in other words if L & L' are Hamiltonian isotopic fillings then they induce the same algebraic torus (seed/cluster) inside $\mathrm{Aug}(\Lambda_\beta)$

CASS: 19th [Khrystyna Serhiyenko]

Recap

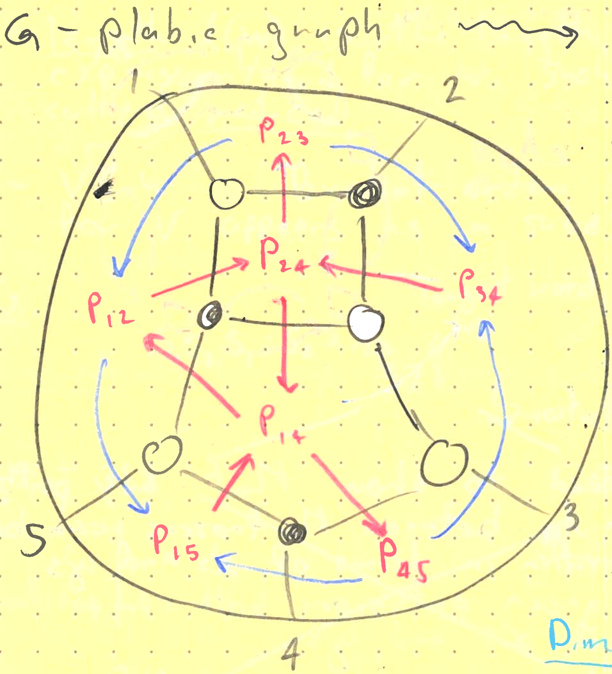
cluster alg $\mathbb{C}[Gr(k,n)]$

Plabic graphs

categorification $CM(B_{k,n})$

? [Baur-King-Marsh]

G - plabic graph $\rightsquigarrow$ quiver $Q(G)$



add arrows between neighboring frozen variables i.e. boundary faces s.t. get oriented cycles

$\mathbb{C}(Q(G))$ - path alg

$I_G = \langle p - q \mid$ for every internal arrow α $p \overset{\alpha}{\underset{q}{\rightleftarrows}}$ $\rangle$

(equivalent paths p and q)

$A_G = \mathbb{C}(Q(G)) / I_G$

Dimer algebra

Thm: [BKM]

Plabic graph G w/ Plücker face labels $I_1, \dots, I_m$ $\rightsquigarrow$ cluster tilting object $T_G = \bigoplus_{j=1}^{m} M_{I_j}$ $\to$ $\mathrm{End}_{B_{k,n}} T_G$

rank 1 modules in $CM(B_{k,n})$

$\mathrm{End}\, M = \mathrm{Hom}(M, M)$ is an algebra that can be described by a quiver w/ relations

vertices $\longleftrightarrow$ indec summands of M
arrows $\longleftrightarrow$ irreducible morphisms between the summands
for $\mathrm{End}\,M = \mathrm{Hom}(M,M)$

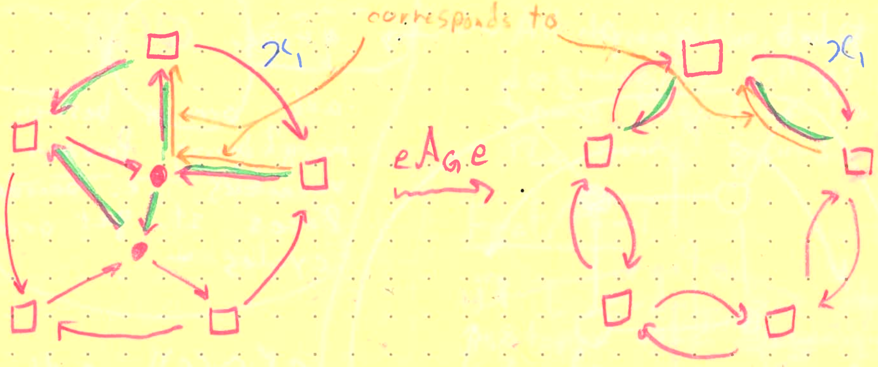
then

① $\mathrm{End}_{B_{k,n}} T_G \simeq A_G$

② $\underbrace{e A_G e}$ $\cong B_{k,n}$ ← Boundary algebra

$e = \sum e_i$ — frozen vertices; e_i — trivial paths e_i

only consider paths that start and end at the frozen vertices



*need to check relations!

$x^2 = y^3$

$xy = yx$

need to show that any path in A_G between the frozens can be written in terms of x's and y's

green paths are both compositions of x's and y's

Note different plabic graphs of type (k,n) yield the same algebra

$eA_Ge \cong B_{k,n}$

Richardson Varieties

Symmetric Group: S_n

$s_i = (i, i+1)$ — simple transposition

$$S_n = \langle s_i \mid i \in [n-1] \rangle \Big/ \begin{array}{l} s_i^2 = 1 \\ s_i s_j = s_j s_i,\ |i-j|>1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \end{array}$$

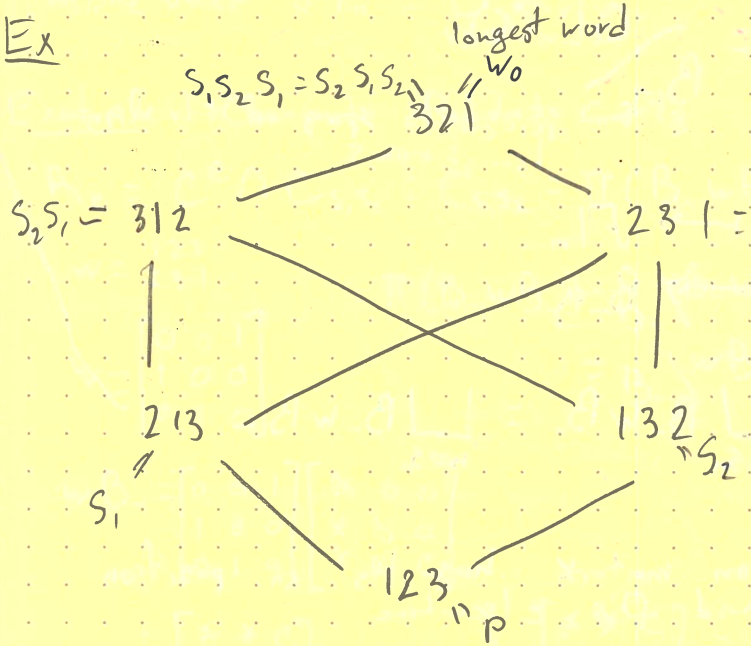
$w \in S_n$ can be expressed as a product of s_i's

Def^{n}.

- length $\ell(w)$ is the length of the shortest expression/word for w. Such an expression is called reduced

- $v \leq w$ in Bruhat order if a reduced expression for v appears as a subexpression for w

Ex



Richardson Variety

$Gl_n(\mathbb{C})$

B_+, B_- — upper/lower Δ matrices in $GL_n(\mathbb{C})$

Fl_n - (complete) flag variety

$$= \{ V_0 \subset V_1 \subset \cdots \subset V_n \mid \dim V_i = i \}$$

$g \in Gl_n(K)$,

$$g = \begin{bmatrix} \text{---}\; g_1 \;\text{---} \\ \text{---}\; g_2 \;\text{---} \\ \vdots \\ \text{---}\; g_n \;\text{---} \end{bmatrix} \rightsquigarrow \text{flag} \qquad V_i = \text{Span}\{g_1, \dots, g_i\}$$

So

$$Fl_n = B_- \backslash GL_n(\mathbb{C})$$

← stabilizes span of the first $1, \dots, i$ rows

$$\pi : GL_n \longmapsto Fl_n$$
$$g \longmapsto B_- g$$

$$GL_n = \bigsqcup_{w \in S_n} B_- w B_- = \bigsqcup_{w \in S_n} B_- w B_+$$

w = permutation matrix has 1's in position $(w(i), i)$ and 0's elsewhere

$$\pi \to Fl_n = \bigsqcup_{w \in S_n} \pi(B_- w B_-) = \bigsqcup_{w \in S_n} \pi(B_- w B_+)$$

Schubert cell $C_w \cong \mathbb{C}^{\ell(w)}$ opp. Schubert cell $C^w \cong \mathbb{C}^{\ell(w_0) - \ell(w)}$

Ex $C_e = \pi(B_- e B_-) = \pi(B_-) = B_- \backslash B_- = \text{point}$

$C^e = \pi(B_- e B_+) = \pi(GL_n^\circ) = B_- \backslash GL_n(\mathbb{C}) = Fl_n^\circ$

Defn: $C_w = \pi(B_- w B_-)$
$C^w = \pi(B_- w B_+)$

Def^n

$R_{v,w} := C^v \cap C_w$ — (open) Richardson Variety

Note: $R_{v,w} \neq 0$ iff $v \leq w$

$\dim = \ell(w) - \ell(v)$

moreover $Fl_n = \bigsqcup_{v \leq w} R_{v,w}$

Example: Compute $R_{e, s_1 s_2} \subset Fl_3$

$$R_e = C^e \cap C_{s_1 s_2} = C_{s_1 s_2} = \pi(B_- w B_-) \cong \mathbb{C}^2$$

$w = \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix}$

$$\pi(B_- w B_-) = B_- \backslash B_- w B_-$$

$$w = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad = B_- \backslash w B_-$$

sus. but yeah whatever

$$w B_- = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ * & b & 0 \\ * & * & c \end{bmatrix} = \begin{bmatrix} * & * & c \\ a & 0 & 0 \\ * & b & 0 \end{bmatrix}$$

$$B_- \backslash \begin{bmatrix} * & * & c \\ a & 0 & 0 \\ * & b & 0 \end{bmatrix} = \begin{bmatrix} * & * & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cong \mathbb{C}^2 = \mathbb{C}^{\ell(w)} \qquad \dim = 2$$

consider projection

$$\pi_k : Fl_n \longrightarrow Gr(k,n)$$
$$\{V_0 \subset \cdots \subset V_n\} \longmapsto V_k$$

So

$$Gr(k,n) = \bigsqcup_{v \le w} \pi_k(R_{v,w})$$

w is k-grassmann permutation

- w is k-gr. perm if each reduced expression for w ends in s_k
- $\pi_k(R_{v,w}) \cong R_{v,w} :=$ (Open) positroid variety $\Pi_{v,w}$

Special case

- if $v=e$ then $\Pi_{e,w}$ is a Schubert variety
- if $v=e$, w is longest k-grassmann perm then

$$\Pi_{e,w} = Gr^o(k,n)$$ ← where none of the frozen Plückers vanish

[Leclerc], building on the work of [GLS], provides a cluster structure on

$$\mathbb{C}[R_{v,w}]$$

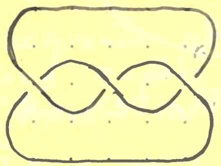
using representation theory of preprojective algebras

CASS: 19th Questions

[Ralf Schiffler]

Q1) Let K be a link diagram with n crossing points. Prove that K has 2n segments and n+2 regions (including unbounded regions)

Q2) Construct a seifert surface for the Trefoil. Compute the Seifert matrix and the Alexander polynomial.

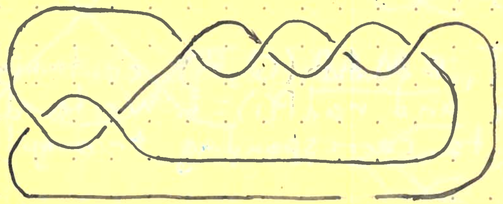


– Trefoil

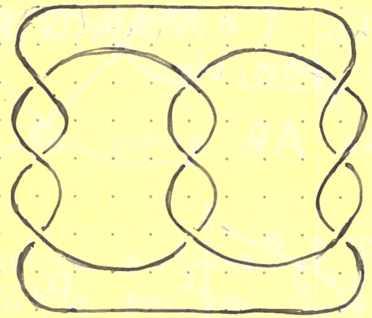
Q3) Compute the Alexander polynomial of the Trefoil using skein relations

Q4) Let

K =



P =



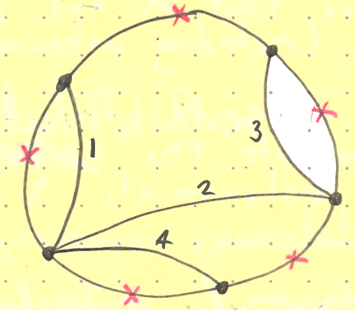
Note: P is not alternating!

Show that both knots have the same Alexander polynomial but different second Alexander ideals.

Q5) Prove that $\Delta(t) \doteq \Delta(t^{-1})$

[Emily Gunawan]

Q1) a) Find (Q,R) for the tiled surface



b) Finish the Auslander-Reiten quiver of rep(Q,R)

$$\begin{tikzcd}
P(3)=3 \arrow[dr] & & ? \arrow[r] & ? \\
 & P(2)=\begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix} \arrow[ur] \arrow[dr] & \cdots & \\
P(4)=4 \arrow[ur] & & ? &
\end{tikzcd}$$

Note: google "string applet"

c) Let $T_1 \in MAR(Q,R)$ containing summands $P(2)=\begin{smallmatrix} & 2 & \\ 3 & & 4 \end{smallmatrix}$ and rad $P(1)=\begin{smallmatrix} 2 \\ 4 \end{smallmatrix}$. Write down T_1 and draw its corresponding triangulation T_1 of (S, M^*)

d) Find all other $T \in MAR(Q,R)$ by flipping arcs when possible.

Q2) a) Finish the AR quiver of

$$Q = \begin{tikzcd}
 & 3 \arrow[dl, "a"'] \arrow[dd, "c"] \\
2 \arrow[dr, "b"'] & \\
 & 1
\end{tikzcd} \qquad R=\{ab\}$$

Note: google "string applet".

b) Starting from

$$T_1 = \underset{\substack{\uparrow \\ \text{inner boundary}}}{c} \oplus \underset{\substack{\uparrow \\ \text{outer boundary}}}{a} \oplus b \oplus \underbrace{e_2 \oplus e_1}_{\text{rad } P(3)} \oplus \underset{\substack{\uparrow \\ P(3)}}{c'a} \in MAR(Q,R)$$

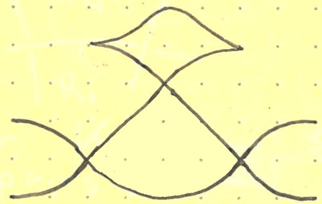
find all other $T \in MAR(Q,R)$

Q3) Conjecture a bijection between points $(\cdot)$ in M on ∂S and "certain strings"

[Daping Weng]

Q1) Find a sequence of Legendrian Reidemeister moves and saddle cobordisms to realize the pinching move:

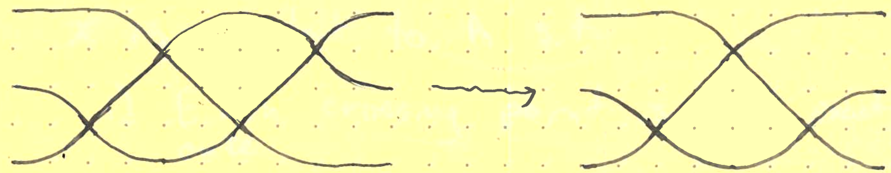
Hint:



Bonus 2:

Bonus 3:

Bonus:



Q2) Draw projective pictures of flags in Fl_3 that are of relative position $s_1 s_2$ and $s_2 s_1$

Q3) By using a reduced word, we can realize a permutation w as a positive braid. Prove that $Aug(\Lambda_w) \cong R_{e,w}$ ← open Richardson

[Khrystyna Serhiyenko]

Q1) Compute $R_{213,231}$

Q2) a) Show w is a k-grassmann perm iff

$$w(1) < w(2) < \dots < w(k)$$

$$w(k+1) < w(k+2) < \dots < w(n)$$

(use fact $\ell(w) = \#$ of inversions of w
i.e. $i<j$ and $w(i) > w(j)$

b) Show that Schubert varieties in $Gr(k,n)$ are in bijection with Young diagrams in $k \times (n-k)$ rectangles

Q3) Show that in our running example G, there is an arrow $M_{24} \to M_{14}$ in $End_{B_{k,n}} T_G$

CASS: 20th [Ralf Schiffler]

Kauffman States

(2)

K a link diagram (oriented) without curls

n = # crossing points

$= |K_0|$ thinking about K as a simplicial complex

K_1 = segments

K_2 = regions (including unbounded region)

$|K_1| = 2n \qquad |K_2| = n+2$

choose a segment $i \in K_1$ and label the adjacent regions R_i, R_i'

Define a Kauffman state is a set of pairs (markers)

$$(x, R) \in K_0 \times K_2 \qquad \text{s.t.}$$

x is incident to R s.t.

i) Each crossing point is used exactly once

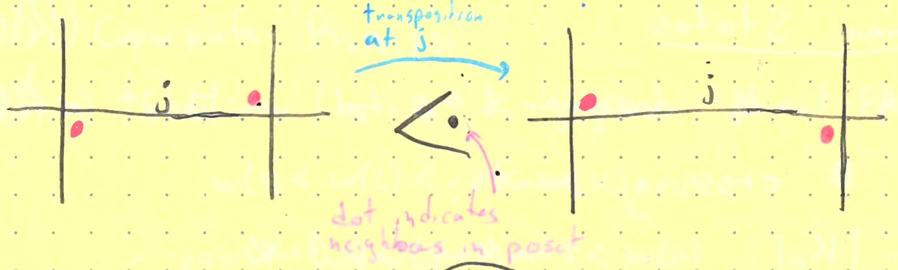
ii) Each region, except R_i, R_i', is used exactly once

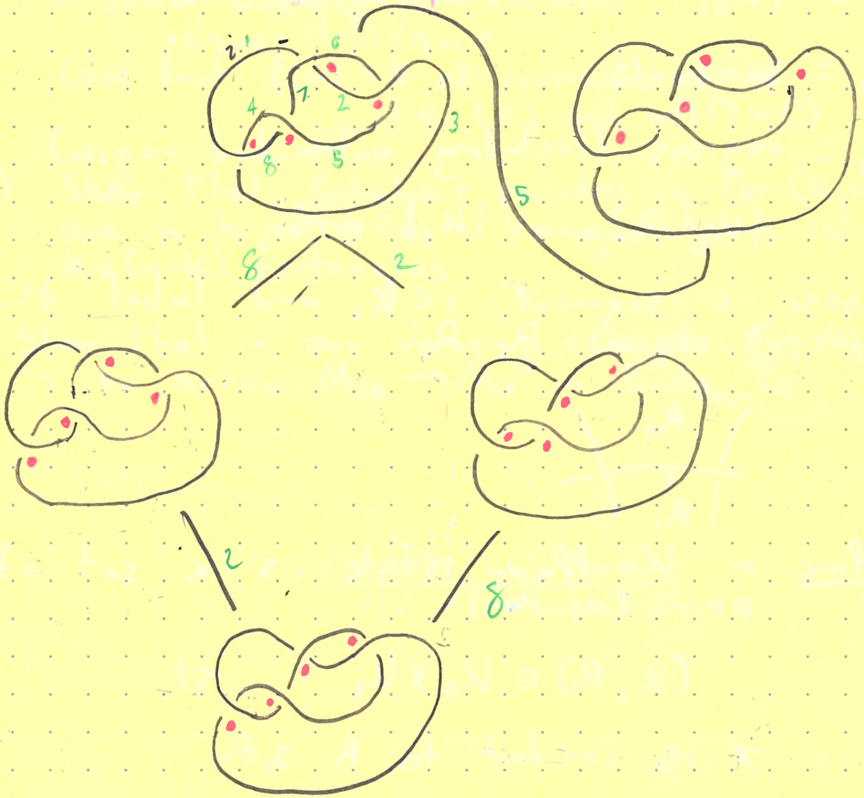
R_i
R_i'
*

Kauffman state in red

Order and poset structure

marker moves ccw around the crossing

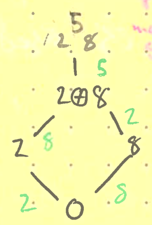




Observation: $2 \leftarrow 5 \rightarrow 8 = Q$

$K \leftarrow K \rightarrow K = M$

2	$K \leftarrow 0 \rightarrow 0$	submodules
8	$0 \leftarrow 0 \rightarrow K$	
$2 \oplus 8$	$K \leftarrow 0 \rightarrow K$	
$^{5}_{2\,8}$	$K \leftarrow K \rightarrow K$	
0	$0 \leftarrow 0 \rightarrow 0$	



Perfect matching of snake graph

$\boxed{2|5|8}$

Kauffman: associated a weight function $w(S)$ to the states

$$\Delta = \sum_{S \text{ state}} \pm w(S)$$

[Schiffler - Bazier - Matte] - Alexander polynomial

[Lee - Schiffler] - Jones polynomial of 2-bridge knots

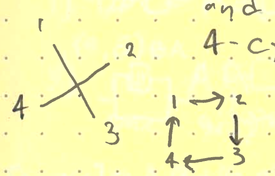
[Schiffler - Whiting] - tilting modules for small 2-bridge knots

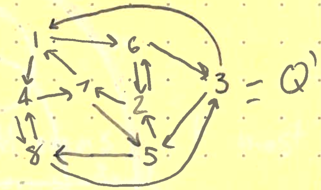
Cluster algebra $\longleftrightarrow$ Knot

K link diagram, $n = |K_0|$

$Q' =$ quiver, $Q_0 = K_1$

arrows: every crossing point gives 4-cycle and oriented clockwise





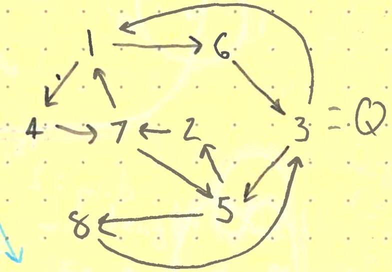
Jacobian algebra

$B = Jac(Q', W)$ Potential

$W = \sum_{\text{cycles}}^{cw} - \sum_{\text{cycles}}^{ccw}$

$R = \{\partial_\alpha W, \mid \alpha \in Q_1\}$

To get Q, remove 2-cycles



Then $A(Q)$ is a cluster algebra (principal coefficients)

Now, $\forall i \in K_1$, construct

$T(i) \in rep(Q, R)$
$\in \mod B$

by taking successive boundaries of $K \setminus \{i\}$

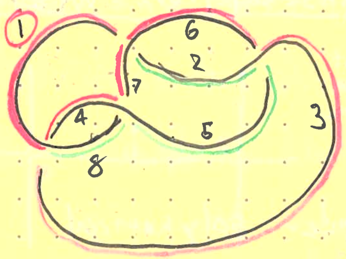
partial derivative: $\alpha = 4 \to 7$

$\alpha: 4 \to 7 \quad W = 4 \xrightarrow{\alpha} 7 \to 5 \to 8$

$- 4 \xrightarrow{\alpha} 7 \to 1 + \ldots$

$\partial W = 7 \to 5 \to 8 \to 4$

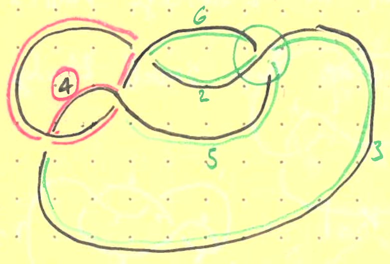
$- 7 \to 1 \to 4$

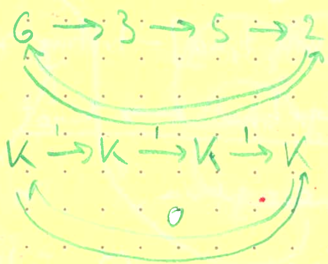


$\dim T(1)_j = 0$

$\dim T(1)_j = 1$

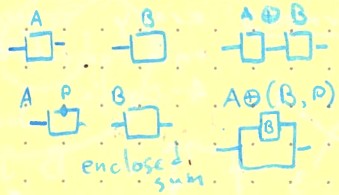
T(4)



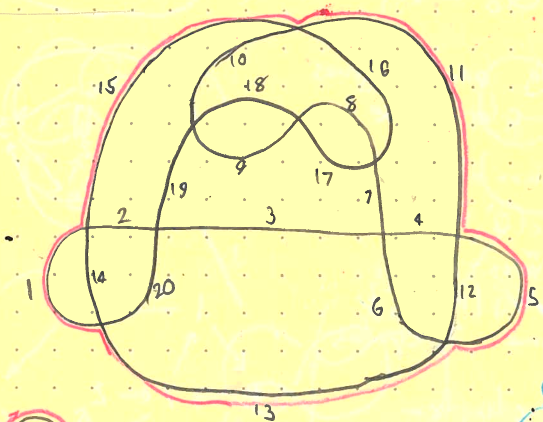


T(4)

How to compute boundaries

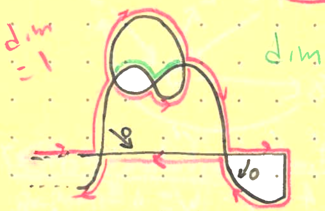


$\partial(A\oplus(B,P)) = \partial A \oplus [\partial B, P]$



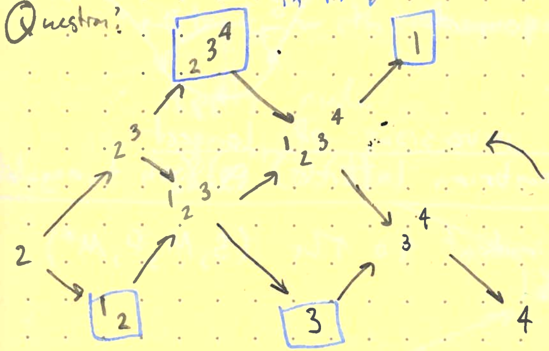
dim = 1

dim = 2



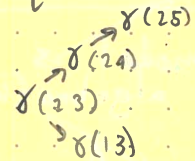
CASS: 20th [Emily Gunawan] (Type A_n)

Question? in Req(Q,R)



AR quiver of rep(Q)

Q:  R = ∅



8. Required summands

$\mathrm{Req}(Q,R) = \{$ indecomposable M s.t. every $T \in \mathrm{max}(Q,R)$ has M as a summand $\}$

Prop TFAE:

1) $M \in \mathrm{Req}(Q,R)$

2) in AR quiver for rep(Q,R), M has at most one arrow going in and one going out

At worst $\searrow M \nearrow$ or $\nearrow M \searrow$

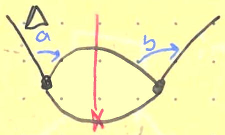
3) $M = M(w)$ is s.t.

boundary segments

- w is a max direct string w/ length $\ell \geq 1$
  Ex Q $3 \xrightarrow{c} 1$, $R = \{ab\}$ a, b, 2 — c, a, b   Ex if $R = \emptyset$, c, ab
- $w = 1_i$ w/ degree$(i) = 1$ $\to i$ $i \leftarrow$
  OR
- $w = 1_i$ w/ $\deg(i) = 2$ where ab is a string
  OR $\xrightarrow{a} i \xrightarrow{b}$
- $w = 1_i$ w/ $\deg(i) = 2$ where $ab \in R$ $\xrightarrow{a} i \xrightarrow{b}$ (*)

boundary segments ⟷ points (∘) in tiled surface

(*) ⟷ "digon/bigon" connector



9. Type A model using inversions of longest permutation w_0 / Cambrian lattice: polygon triangulations

Rem: This model is equivalent to the (S, M, P, M^*) gentle algebra model

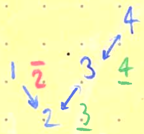
Rule: Type A_n quiver

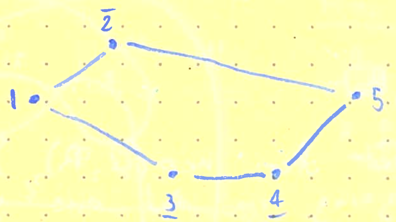
$1, \dots, n+1$ left to right

1 • ← on the horizontal → • n+1

other # above or below based on Rule:
$j-1 \to j$ in Q means $\bar{j}$ goes above
otherwise $\underline{j}$ goes below

Ex





Equivalence F: inversions/line segments $\gamma(i,j), 1 \leq i < j \leq n+1$ ⟶ string representations

$$(i,j) \longmapsto M(i, \dots, j-1)$$

Def^n Let c be a coxeter element

$$c = s_2 s_3 s_1 s_4 = (23)(34)(12)(45) = (1\,3\,4\,5\,\bar{2})$$

Ex

to create AR quiver where vertices are $(i<j)$, do:

- start with any pair $(i<j)$
- Apply rule

$$\begin{tikzcd}
\gamma(c^{-1}(i)<j) \arrow[dr] & & \gamma(i<c(j)) \\
& \gamma(i<j) \arrow[ur] \arrow[dr] & \\
\gamma(i<c^{-1}(j)) \arrow[ur] & & \gamma(c(i)<j)
\end{tikzcd}$$

Ex:

$$\begin{tikzcd}
& & \gamma(25) \arrow[dr] & & \gamma(12) & \\
& \gamma(24) \arrow[ur] \arrow[dr] & & \gamma(15) \arrow[ur] \arrow[dr] & & \\
\gamma(23) \arrow[ur] \arrow[dr] & & \gamma(14) \arrow[ur] \arrow[dr] & & \gamma(35) \arrow[dr] & \\
& \gamma(13) \arrow[ur] & & \gamma(34) \arrow[ur] & & \gamma(45)
\end{tikzcd}$$

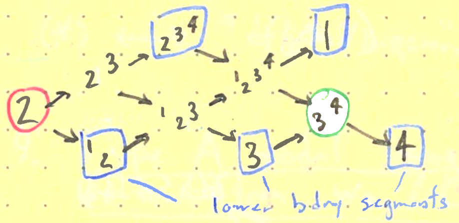
Rem: • Coxeter element description AR quiver
"charmed roots & Kronecker compliment" [D-F-I-SA-T-W]

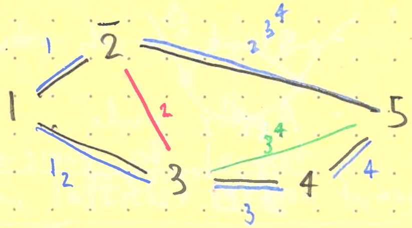
• Polygon model is used to study MAR (type A)
"Cambrian combinatorics on quiver reprs" [B-G-M-S]

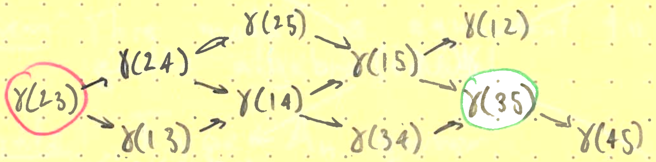
Same MAR Thm holds

{Triangulations (including boundary segments)}
⟷
mar(Q)

Ex







Prop

① In type A_n, mar(Q) forms a semidistributive lattice which is isomorphic to

② the oriented exchange graph of a seed of the A_{n-2} cluster algebra;

③ also isomorphic to c-Cambrian lattice (ex. Tamari lattice)

④ also the lattice of support τ-tilting of a type A quiver

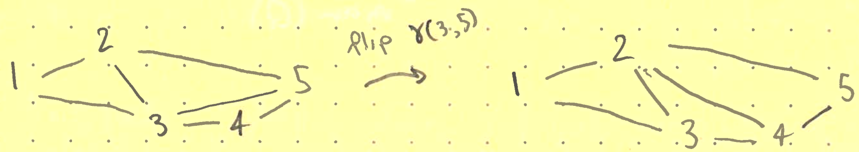
⑤ and many more

Triangulations $T_1 < T_2$ if the difference between T_1, T is two line segments

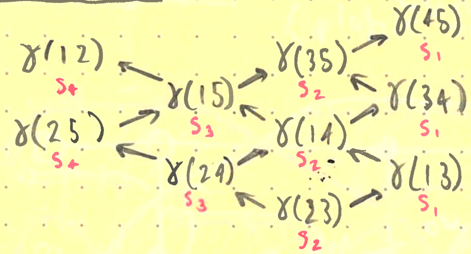
γ_1, γ_2

s.t. slope of γ_1 is smaller than slope of γ_2

Ex



Inversions



H corresponds to a set of reduced expressions of w_0

commutation class

allowed $s_i s_j \leftrightarrow s_j s_i$

not $s_i s_{i+1} s_i \leftrightarrow s_{i+1} s_i s_{i+1}$

labels come from projectives

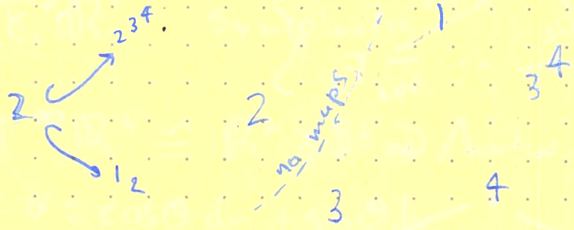
as a poset

Ex $s_2 s_1 s_3 s_4 \dots = w_0$

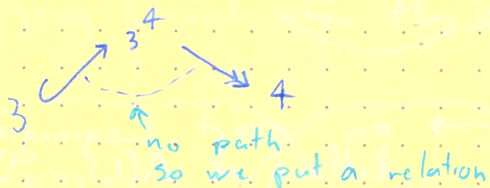
23 24 13

$12345 \xrightarrow{\text{right multiply } (s_2)} 13245 \xrightarrow{\text{multiply by } (s_1)} 31245$

Ex



End(T) = Hom(T,T)

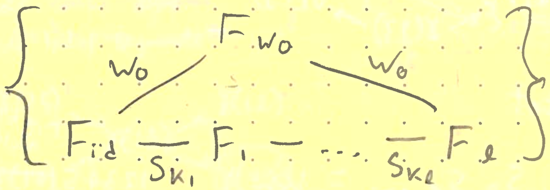


CASS: 20th [Daping Weng]

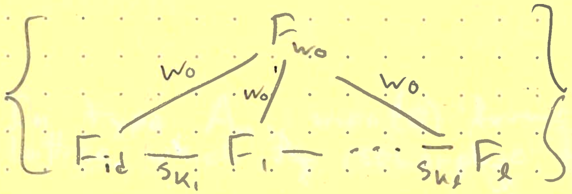
Recap

$n \geq 0$. $\beta = (k_1, \dots, k_\ell)$

$\mathrm{Aug}(\Lambda_\beta) =$

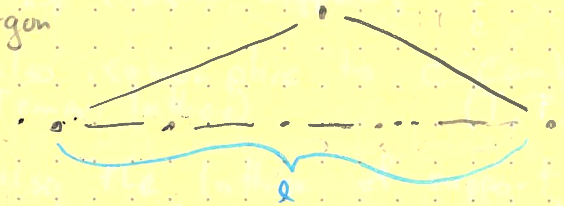


with initial seed



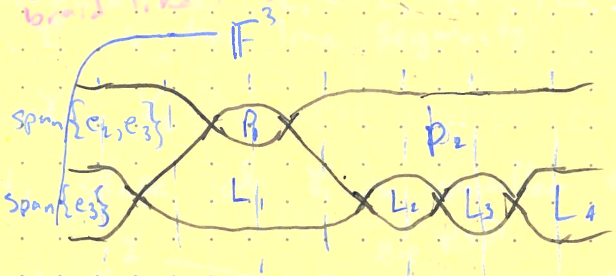
In particular, when $n=2$,

$(\ell+2)$-gon

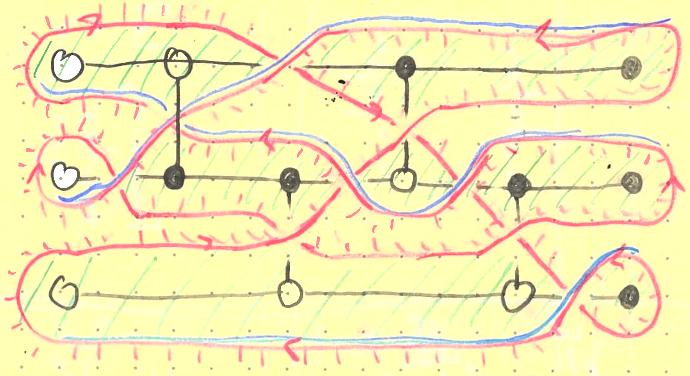


a triangulation $\longrightarrow$ an exact Legrangian filling of $\Lambda(\underbrace{1,1,\dots,1}_{\ell})$

apparently we get flags from a braid like this?



Plabic graphs (do not require reducedness) (plabic fences)



The "hairs" are called co-normals

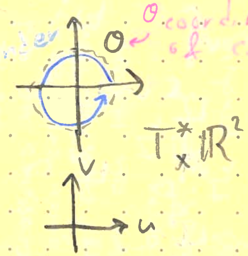
notice Hairs "point" to black vertex regions

Daping says: "if I've made a mistake w/ over/under fix it yourself"

$T^*\mathbb{R}^2$ symplectic

shows over/under

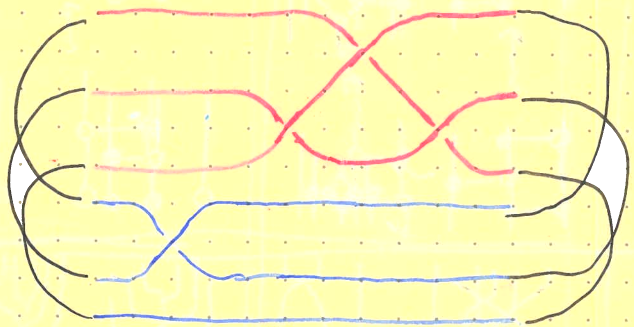
θ coordinates of circle



$T^\infty \mathbb{R}^2 \cong \mathbb{R}^2 \times S^1 \supset \Lambda$

contact 1-form $\rightarrow$ $\alpha = \cos\theta \, du + \sin\theta \, dv$

Left going strand always on top of right going strands when they cross.

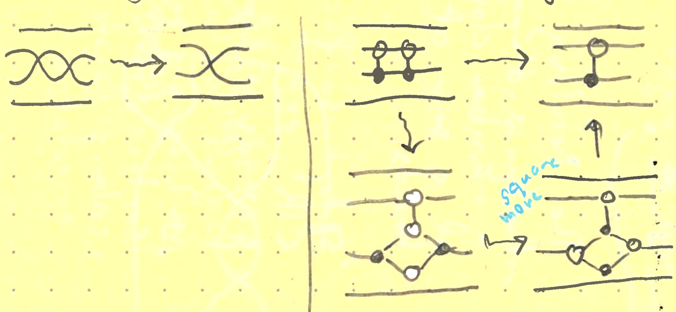
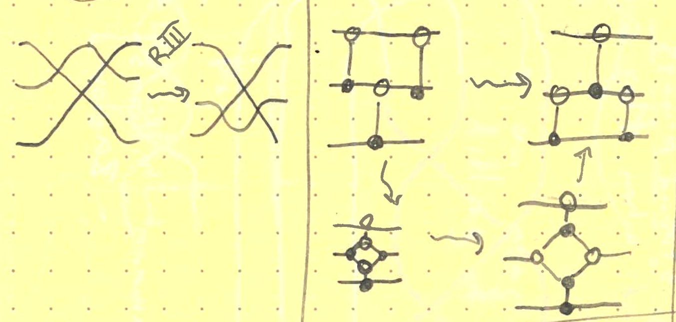
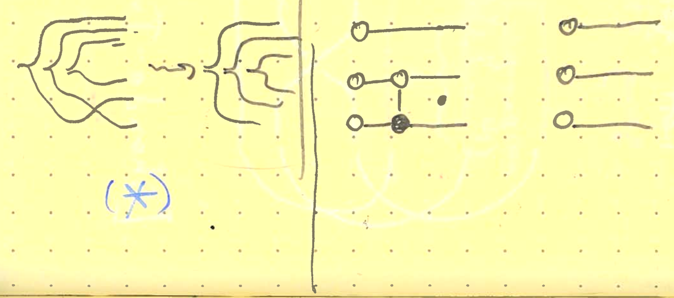
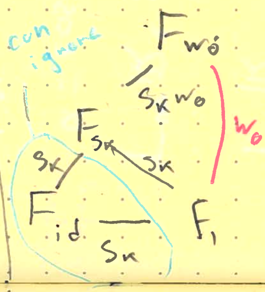
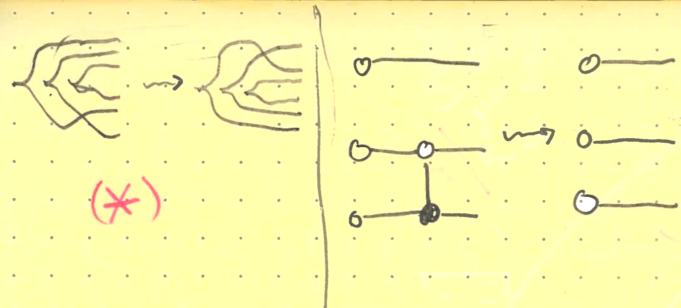
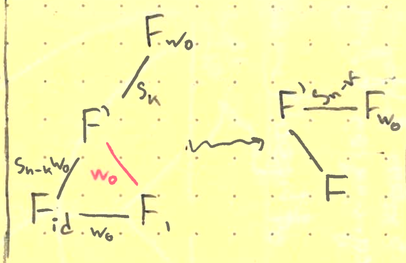
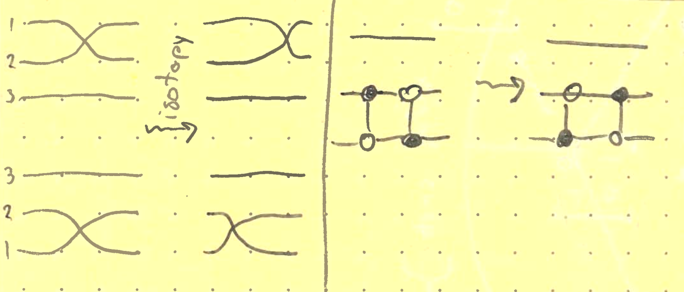
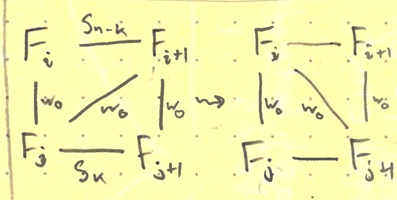


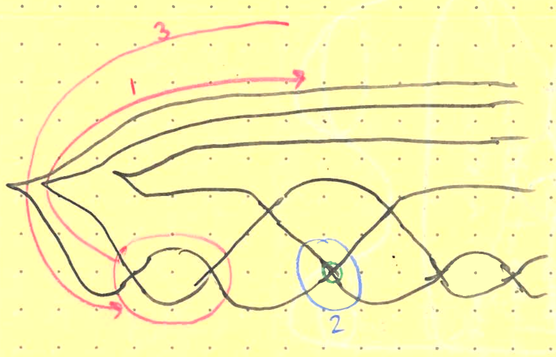
The resulting Leg link is always a rainbow closure of a pair of braids

[Shende-Treumann-Williams-Zaslow]

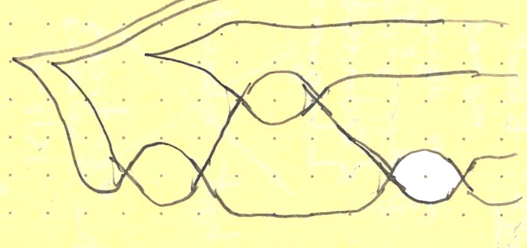
Seifert surface $\subset \mathbb{R}^3$ exact Legrangian filling $\subset \mathbb{R}^4$ regions containing vertices are part of surface, vertex colours denote orientation

Dictionary.

leg. link / plabic graph	Flags	cluster Alg.
	$F_{m-1} \xrightarrow{s_k} F_m \xrightarrow{s_k} F_{m+1}$ ⇝ $F_{m-1} \xrightarrow{s_k} F_{m+1}$	mutation + Freezing
	$F_m \xrightarrow{s_{k+1}} F_{m+1} \xrightarrow{s_k} F_{m+2} \xrightarrow{s_{k+1}} F_{m+3}$ $F_m \xrightarrow{s_k} F'_{m+1} \xrightarrow{s_{k+1}} F'_{m+2} \xrightarrow{s_k} F_{m+3}$	mutation
		Freezing leftmost vertex
		Nothing
		mutation



using (*) to move the circled crossings up and over, the green crossing is the leftmost crossing. Then, using (*), we can remove ∅, then bring the circled crossings back. Thus we can access the pinching manouvre?



Defn: Exact lagrangian fillings built from these moves are called admissible

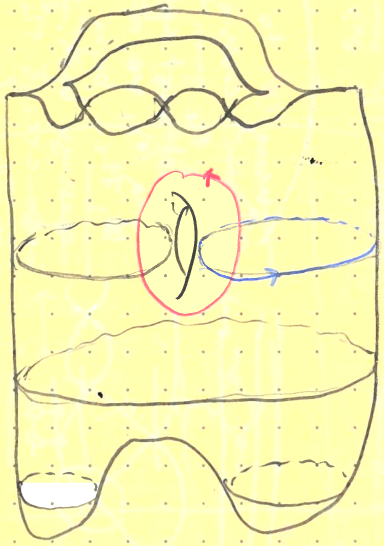
$$\left\{\begin{matrix}\text{admissible Exact} \\ \text{Legrangian fillings}\end{matrix}\right\} \rightsquigarrow \left\{\begin{matrix}\text{clusters} \\ \text{in } Aug(\Lambda_\beta)\end{matrix}\right\}$$

What about the quiver?

vertices $\longleftrightarrow$ certain 1-cycles on the (oriented) exact Legrangian filling

Arrows $\longleftrightarrow$ pairings between these 1-cycles

which 1-cycles?

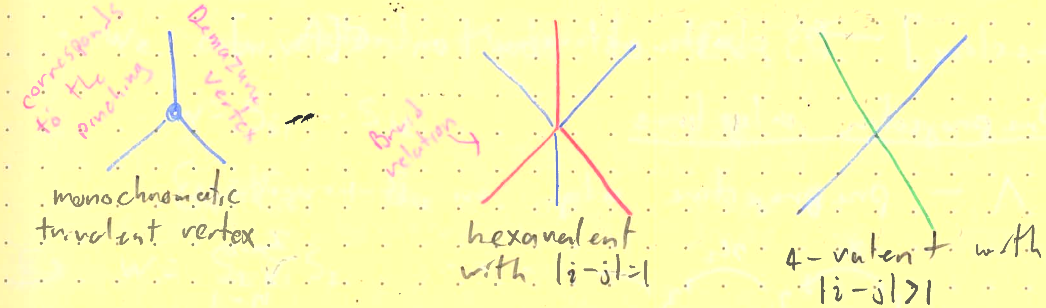


$\bullet \longrightarrow \bullet$

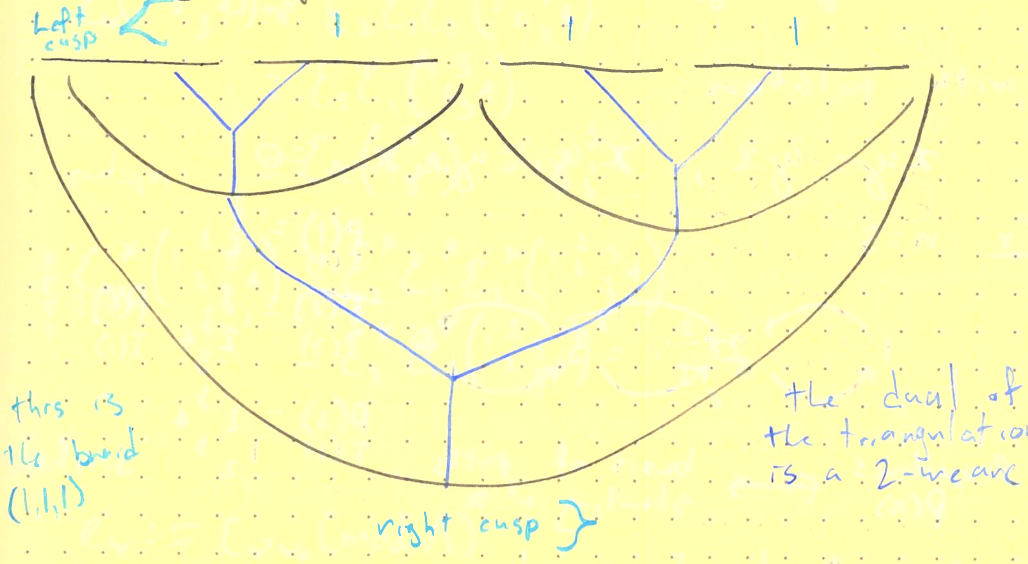
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Legendrian Weaves

Defn: an n-weave is a planar graph on a disk with coloured edges $\in \{1, 2, \ldots, n-1\}$ and the following types of vertices:



Something to play with triangulation



CASS - 20th [Khrystyna Serhiyenko]

Last time:

$v \le w \in S_n$ Richardson variety $\boxed{R_{v,w}}$

[Leclerc] $\rightsquigarrow$ cluster structure on $\mathbb{C}[R_{v,w}]$

Preprojective algebras

Λ – preprojective alg. on $n-1$ vertices

with relations

$$xy = yx, \quad x_1 y_1 = y_{n-2} x_{n-2} = 0$$

injectives

Ex $n = 5$

$P(1) = \begin{smallmatrix} 1 \\ 2 \\ 3 \\ 4 \end{smallmatrix}$ $I(4)$

$P(2) = \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}$ $I(3)$

$P(3) = \begin{smallmatrix} & & 3 & \\ & 2 & & 4 \\ 1 & & 3 & \\ & 2 & & \end{smallmatrix}$ $I(2)$

$P(4) = \begin{smallmatrix} 4 \\ 3 \\ 2 \\ 1 \end{smallmatrix}$ $I(1)$

Projectives $P(x)$ $\longleftrightarrow$ basis of paths starting at x

Λ is finite dimensional

Leclerc's Construction:

define two functors

$$\mathcal{E}_i, \mathcal{E}_i^* : \text{mod}\,\Lambda \longrightarrow \text{mod}\,\Lambda$$

- $\mathcal{E}_i(M)$ is obtained from M by removing all $S(i)$'s (simple module at vertex i) from the top of M
- $\mathcal{E}_i^*(M)$ is obtained from M by removing all $S(i)$'s from the bottom of M
- We can extend these to $\mathcal{E}_w, \mathcal{E}_w^*$

$$w = s_{i_1} \cdots s_{i_\ell}$$

$$\mathcal{E}_w = \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_\ell}$$

Ex $w = s_2 s_1 s_2$

$$\mathcal{E}_w\left(\begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}\right) = \mathcal{E}_2\mathcal{E}_1\mathcal{E}_2\left(\begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}\right)$$
$$= \mathcal{E}_2\mathcal{E}_1\left(\begin{smallmatrix} 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}\right)$$
$$= \mathcal{E}_2\left(\begin{smallmatrix} & 3 & \\ 2 & & 4 \\ & 3 & \end{smallmatrix}\right) = \begin{smallmatrix} & 3 & \\ 2 & & 4 \\ & 3 & \end{smallmatrix}$$

$$\mathcal{E}_w^*\left(\begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}\right) = \mathcal{E}_2^*\mathcal{E}_1^*\mathcal{E}_2^*\left(\begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}\right)$$
$$= \mathcal{E}_2^*\mathcal{E}_1^*\left(\begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}\right) = \begin{smallmatrix} & 2 & & \\ 1 & & 3 & \\ & 2 & & 4 \\ & & 3 & \end{smallmatrix}$$

define

$$e_w := \mathcal{E}_{w w_0}(\text{mod}\,\Lambda)$$

$$e^w := \mathcal{E}^*_{w^{-1}}(\text{mod}\,\Lambda)$$

Thm:

has projective/injective objects

$$e_{v,w} := e^v \cap e_w \qquad \text{Frobenius cluster category}$$

cluster structure

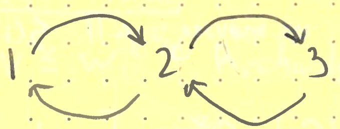
moreover $\exists$ cluster character map

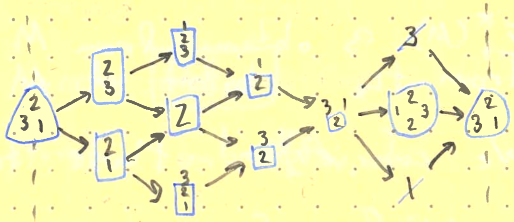
$$\varphi : e_{v,w} \longrightarrow \mathbb{C}[R_{v,w}]$$
$$M \longmapsto \varphi_M$$

induces cluster (sub)algebra in $\mathbb{C}[R_{v,w}]$

Ex $n=4$

AR quiver of Λ





compute $e_{e,s_2s_3s_1s_2}$ ⇝ cat. of $\mathbb{C}[Gr(2,4)]$

note: $w = \begin{smallmatrix}1&2&3&4\\3&4&1&2\end{smallmatrix}$ longest 2-grassmann perm

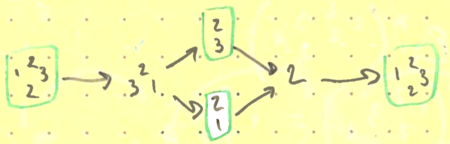
One Mutable vertex

$$e^e = \mathcal{E}^*_{e^{-1}}(\mathrm{mod}\,\Lambda) = \mathrm{mod}\,\Lambda$$

$$e_w = \mathcal{E}_{w^{-1}w_0} = \mathcal{E}_{s_3s_1}$$

$$\mathcal{E}_{s_3s_1}(\mathrm{mod}\,\Lambda)$$

yeilds



$= e_{v,w}$

Projective-Injective objects $\square$

one s.e.s. $0 \to {}^{\;2}_{3\;1} \to {}^2_3 \oplus {}^2_1 \to 2 \to 0$

${}^{\;2}_{3\;1}$, 2 have an extension

clusters ⟷ c.t.o. (cluster tilting object) i.e. max objects w/o extensions

c.t.o. c.t.o.

"nice" seeds in $\mathbb{C}[R_{v,w}]$

$\underline{w} = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ fixed reduced expression for w

Find an expression for v in $\underline{w}$ that is right most in $\underline{w}$ — somehow relates to Soc?

Ex: $\underline{w} = s_2\, s_1\, s_4\, s_3\, s_2\, s_5\, s_4\, s_3$ (① ② ③ ④ ⑤ ⑥ ⑦ ⑧) $\quad v = s_4 s_2$

$v = s_{i_1}^v s_{i_2}^v \cdots s_{i_\ell}^v \qquad s_{i_j}^v \in \{e, s_{i_j}\}$

$w(j) = s_{i_\ell}\cdots s_{j+1}\, s_{i_j} \qquad v(j) = s^v_{i_\ell}\cdots s^v_{i_j}$

Ex $v = e\,e\,e\,e\,s_2\,e\,s_4\,e \qquad w(6) = s_3 s_4 s_5 \qquad v(6) = s_4$

Defn

$$T_{v,\underline{w}} = \bigoplus_{j=1}^{\ell} T_j$$

$$T_j = \mathcal{E}^*_{v(j)}\left(\mathrm{Soc}_{w(j)}(I(i_j))\right)$$

injective at i_j

removes simples from the bottom according to $v(j)$

submodule of $I(i_j)$ built according to $w(j)$

Ex $T_6 = \mathcal{E}^*_4\left(\mathrm{Soc}_{3,4,5}\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}\right)$

$i_j = 5 = \mathcal{E}^*_4\begin{pmatrix}3\\4\\5\end{pmatrix} = \begin{smallmatrix}3\\4\\5\end{smallmatrix}$

ex $T_3 = \mathcal{E}^*_{v(3)}\left(\mathrm{Soc}\left(\begin{smallmatrix}1&&&&\\&2&&4&\\&&3&&5\\&&&4&\end{smallmatrix}\right)\right) = \mathcal{E}^*_{v(3)}\left(\begin{smallmatrix}&3&&\\2&&4&\\&3&&5\\&&4&\end{smallmatrix}\right) = \begin{smallmatrix}&3&\\2&&4\\&3&&5\end{smallmatrix}$

$\boxed{\mathrm{Soc}_{413454} = \begin{pmatrix}1&&&&\\&2&&4&\\&&3&&5\\&&&4&\end{pmatrix} = \begin{smallmatrix}&4&\\3&&5\\&4&\end{smallmatrix}}$

Thm $T_{v,w}$ is a cluster-tilting object in $\mathcal{C}_{v,w}$ which induces a seed in $\mathbb{C}[R_{v,w}]$ as follows

① $T_{v,w}$ has $\ell(w)-\ell(v)$ distinct ind. summands which yeilds cluster variables s.t. $\varphi_{T_j} = \Delta_{v(j)[i_j],\, w(j)[i_j]}$ minor on rows, columns $[i_j] = \{1,\dots,i_j\}$

② projective-injective objects are summands of $\bigoplus_{i=1}^{n-1} \mathcal{E}^*_{v^{-1}}(\mathrm{Soc}_{w^{-1}}(I(i)))$ which corresponds to frozen variables

③ the quiver $Q_{v,w}$ is End $T^{\text{basic}}_{v,w}$

Difficulties

① T_j's may be decomposable

$\rightsquigarrow \Delta_{v(i)[i;j],\, w(i)[i;j]}$ are reducible

② $T_i = T_j$ for $i \neq j$ but $\varphi_{T_i} = \varphi_{T_j}$

③ compute the quiver?

Aside: Computing Soc (according to me)

Say we've got $\mathrm{Soc}_w(\text{"stuff"})$, where w is some list of numbers, $w_n w_{n-1} \cdots w_2 w_1$. Starting from w_1, "build" w_i if its grounded (at the bottom) in "stuff", or has full support (has built numbers to either side below it)

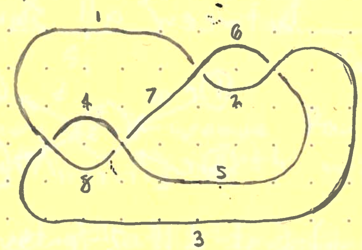
Ex: From before, $\mathrm{Soc}_{413454}\begin{pmatrix} 1\,2\,3 \\ 2\,3\,4\,5 \\ 3\,4 \end{pmatrix} = \begin{smallmatrix} & 4 & \\ 3 & & 5 \\ & 4 & \end{smallmatrix}$

- w_1 is 4, and 4 is grounded in the argument of Soc, so we put a 4-brick down
- w_2 is 5, and our 4-brick provides support to 5, since the other direction down from 5 is empty
- w_3 is 4; we are missing support from where 3 is
- w_4 is 3, and for the same reason as the 5-brick, we can lay down a 3-brick
- w_5 is 1, no support, no 1-brick
- w_6 is 4, and we now have full support, so lay the 4-brick on top

CASS: 20th Questions

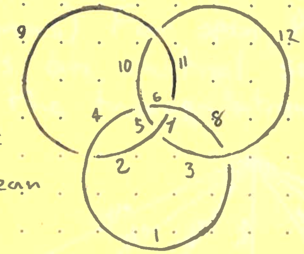
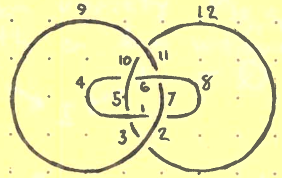
[Ralf Schiffler]

Q1) Compute the lattice of Kauffmann states of



relative to the segment 2 and show it is isomorphic to the lattice of submodules of $T(2)$.

Q2) Let K be a knot diagram without curls and bigons. Prove that K has at least 8 triangular regions (see Q3 for an example).

Q3) $K =$  $K' =$ 

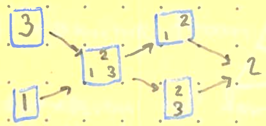
Borromean Rings

Compute the quivers Q, Q' of K, K' and show that Q' is obtained from Q by the mutation sequence at $1,2,3,1$ or $1,2,3,1,2,3,2,3,2$.

[Emily Gunawan]

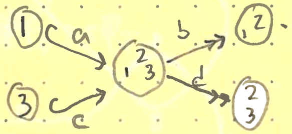
Example

$Q = 1 \leftarrow 2 \rightarrow 3$, AR quiver



Q1) Draw the poset model of $\mathrm{mar}(Q)$ using polygon model

Q2) To compute $\mathrm{End}(T)$: i) write all nonzero, non-isomorphic maps between all summands:



ii) List all 2-paths corresponding to compositions of maps which equal zero:

ad and cb

example: see boxed summands

Q3) Do the same for

$$Q = 1 \rightarrow 2 \leftarrow 3 \leftarrow 4$$

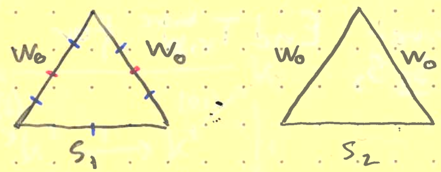
Q4) Do the same for

$$Q = 1 \xrightarrow{a} 2 \begin{smallmatrix} \xrightarrow{b} 3 \\ \xrightarrow{c} 4 \end{smallmatrix} \qquad R = \{ab\}$$

[Daping Weng]

Q1) Recall that for $n=2$, we can get a 2-weave from a triangulation by

Now for $n=3$, try to draw a 3-weave with a single trivalent vertex inside the following triangles: (But other vertices are allowed)



colour conventions: s_1 s_2 s_3 s_4

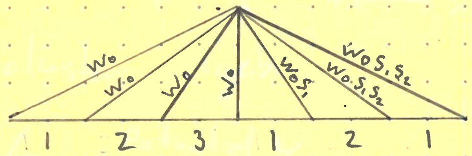
Q2) Draw a 3-weave for the initial seed for $\mathrm{Aug}(\Lambda_{1122121})$ with $n=3$

Q3) Prove that the conormal lift of a curve in $\mathbb{R}^2_{u,v}$ is a Legendrian in $T^\infty \mathbb{R}^2 (\cong \mathbb{R}^2_{u,v} \times S^1_\theta)$ with contact form $\alpha = \cos\theta\, du + \sin\theta\, dv$

Hint: assume the curve is $\gamma(t) = (u(t), v(t))$ with $\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial v}{\partial t}\right)^2 = 1$

Q4) Add a weave column to the dictionary table

Q5) (Bonus) For open Richardson variety $R_{s_1 s_2, s_1 s_2 s_3 s_1 s_2 s_1}$ has an initial seed with a 4-weave associated to the triangulation



Draw it. (Hint: at most 1 trivalent vertex per triangle)

[Khrystyna Serhiyenko]

Q1) a) Compute $T_{v,\underline{w}}$ for $v=e$ $w = s_2 s_3 s_1 s_2$

b) Find cluster variables coming from this seed

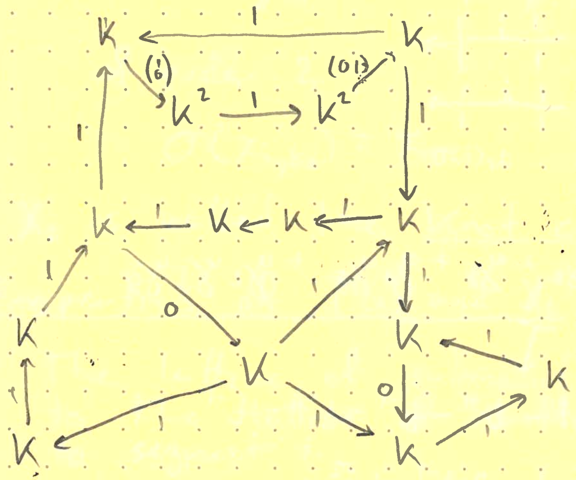
c) Find another reduced expression $\underline{w}'$ for w such that $T_{v,\underline{w}'}$ and $T_{v,\underline{w}}$ give distinct seeds in $\mathcal{C}_{v,w}$

Q2) Find an example of $v, \underline{w}$ such that a summand T_j of $T_{v,\underline{w}}$ is decomposable

Q3)* Compute the quiver $\operatorname{End} T^{\text{basic}}_{v,\underline{w}}$ for $\underline{w} = s_2 s_1 s_4 s_3 \underline{s_2} s_5 \underline{s_4} s_3$

<u>CASS</u>: 21st [Ralf Schiffler]

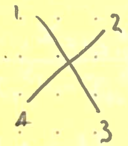
<u>Ex</u> $T(1)$



<u>Recall</u>

K (oriented) link diagram

Q quiver



$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3 \arrow[l] \end{tikzcd}$$

$\mathcal{A}(Q)$ cluster algebra

$B = kQ/I$ ← Potential W

$\forall$ segment $i \in K$, $T(i) \in \operatorname{rep}(Q, I)$

$F_{T(i)}$ ~ F-polynomial

<u>Thm 1</u>: [Bazier Matte - Schiffler]

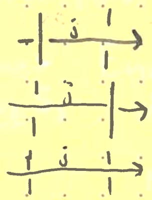
$$F_{T(i)} = \sum_{\substack{L \subset T(i) \\ \text{submodules}}} \prod_{i \in Q_0} y_i^{\dim L_i}$$

(i.e. each submodule of $T(i)$ is determined by its dimension vector

(i.e. $Gr_e(T(i))$ is a point or $\emptyset$)

$F_{T(i)}$ specializes to the Alexander polynomial under

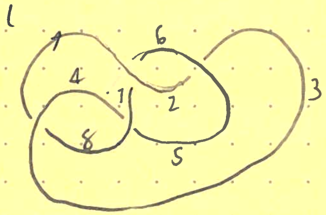
$$y_j = \begin{cases} -t & \text{if} \\ -t^{-1} & \text{if} \\ -1 & \end{cases}$$



Ex: figure 8

$T(1) = {}_{2}{}^{5}{}_{8}$ $\quad F_{T(1)} = 1 + y_2 + y_8 + y_2 y_8 + y_2 y_5 y_8$

$$\downarrow$$



$$\Delta = 1 - t^{-1} - t^{-1} - t^{-2} - t^{-1}$$
$$= 1 - 3t^{-1} + t^{-2}$$

$T(2) = \begin{smallmatrix} 8 \\ 3 \\ 1 \\ 4 \end{smallmatrix}$ $\quad F_{T(2)} = 1 + y_4 + y_4 y_1 + y_4 y_1 y_3 + y_4 y_1 y_3 y_8$

$$\downarrow$$

$$\Delta = 1 - t^{-1} + t^{2} + 1 - t + 1$$
$$= 3 - t - t^{-1}$$

Thm 2 [BM-S]

The cluster algebra $\mathcal{A}(Q)$ contains a seed $(\mathbb{X}_t, \mathbb{Y}_t, Q_t)$ s.t. $\quad (n = |K_0|,\ 2n = |Q_0|)$

1) every cluster variable in $\mathbb{X}_t$ specializes to the Alexander polynomial under $x_i \mapsto 1$ and (Thm1)

2) The permutation $\sigma \in S_{2n}$ that induces an isomorphism

$$\sigma : Q \longrightarrow Q_t^{op}$$

3) σ induces a cluster automorphism (period 2)

$$\sigma : \mathcal{A}(Q)_{\mathbb{Y}=1} \longrightarrow \mathcal{A}(Q)_{\mathbb{Y}=1}$$ trivial coefficients

of order 2:

$$\sigma(x_{i,t_0}) = x_{\sigma(i),t}$$

$\mathbb{X}_t$ is called the Knot cluster

Properties of $T(i)$ and $\mathbb{X}_t$

- The lattice of submodules of $T(i)$ is isomorphic to the lattice of Kauffman states relative to segment i.
- Symmetry of dimension:

$$\dim T(i)_j = \dim T(j)_i$$

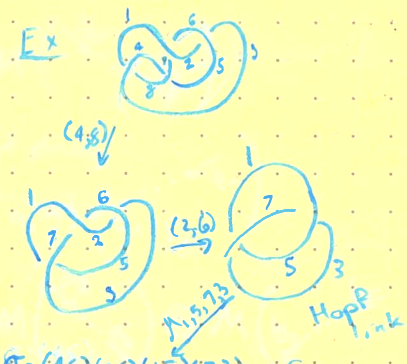
- $x_{i,t} = \dfrac{f(x_1,\dots,x_{2n}, y_1,\dots,y_{2n})}{x_1^{d_1}\cdots x_{2n}^{d_{2n}}}$ Laurent polynomial by Laurent phenomenon

then $d_j = 0$ or $d_j = 1$ $\forall j$ "grin" - grassmannian thin

There exists an explicit mutation sequence

$$\mu : t_0 \longrightarrow t$$

Ex

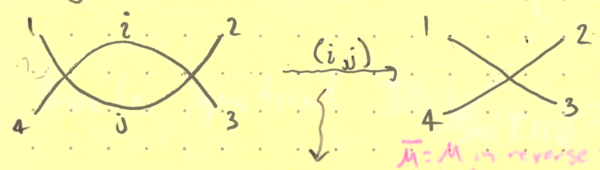


$\sigma = (48)(26)(15)(73) \in S_{2n}$

$\mu = \underbrace{4,8,2,6}_{\mu_{red}},\underbrace{1,5,7,3}_{\mu_{Hopf}},\underbrace{2,6,4,8}_{\mu_{red}}$

2 cases:

1. Bigon Reductions

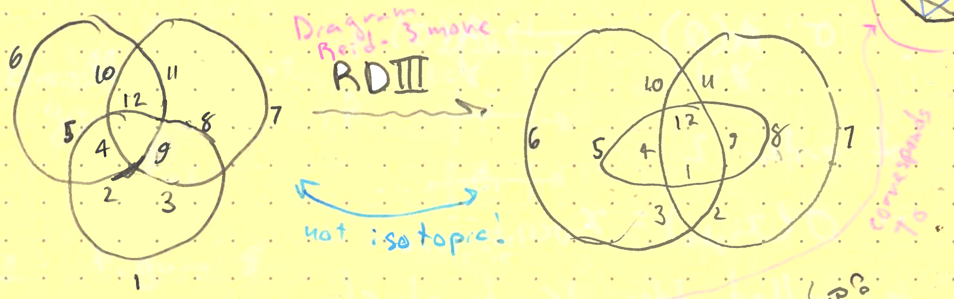


$\bar{\mu} = \mu$ in reverse order

$\mu_i^\sigma = \mu_{\sigma(i)}$

$$t_0 \xrightarrow{\mu_{red}} t_1 \xrightarrow{\mu_{Hopf}} t_2 \xrightarrow{\overline{\mu^\sigma_{red}}} t_3 = t$$

2. what if there is no bigon



$$\underbrace{Q \xrightarrow{M_{123123232}} Q'}_{M_{red}}$$

complete Bigon reduction

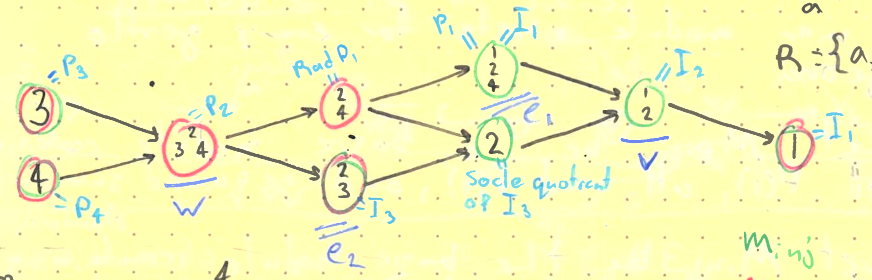
Thm: for every diagram K there exists a sequence of RDIII moves that produces a Bigon (performed at triangular regions)

2 general case: $M_{red} \, M_{Hopf} \, \overline{M^{\sigma}_{red}}$

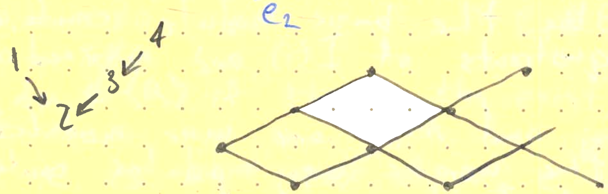
CASS: 21st [Emily Gunawan]

$Q = 1 \xrightarrow{a} 2 \xrightarrow{b} 3$, $2 \xrightarrow{c} 4$

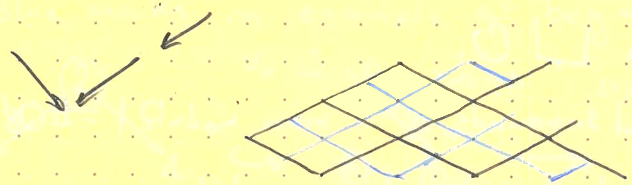
$R = \{a, b\}$



Q: $1 \to 2 \leftarrow 3 \leftarrow 4$



Q':



10. nur modules exist

Def/facts

- indecomposable projective $P(i) = M\left(\text{longest string } a_\ell \cdots \xleftarrow{a_1} i \xrightarrow{b_1} \cdots b_r\right)$
- $\text{rad}\, P(i) = M(a_\ell \cdots a_2) \oplus M(b_2 \cdots b_\ell)$
- indec. injective $I(i) = M\left(\text{longest string } a_\ell \to \cdots \xrightarrow{a_1} i \xleftarrow{b_1} \cdots \leftarrow b_r\right)$
- Socle quotient $I(i)/\text{Soc}(I(i)) = M(a_\ell \cdots a_2) \oplus M(b_2 \cdots b_r)$

Prop

a) An max module exists for every gentle algebra

- Let M_{proj} be the basic module containing all $P(i)$, all rad $P(i)$, and required summands $Z_{i,1}$
- Let M_{inj} be the basic module containing all socle quotients of $I(i)$ and required summands
- Then M_{proj} and M_{inj} are max modules and they are the unique max modules containing the $P(i)$ and $I(i)$, respectively

b) $Z \overset{bij}{\longleftrightarrow} \underset{\text{vertices}}{Q_0} \sqcup \underset{\text{arrows}}{Q_1}$

↳ thus |summands of a max module| $= \#Q_0 + \#\text{arrows}$

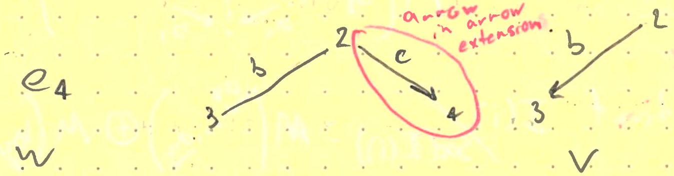
II. Extensions between string modules

Defn: An arrow extension (of string modules) of $M(v)$ by $M(w)$ is an s.e.s:
(possibly same)

$$0 \longrightarrow M(w) \hookrightarrow M(vaw) \twoheadrightarrow M(v) \longrightarrow 0$$

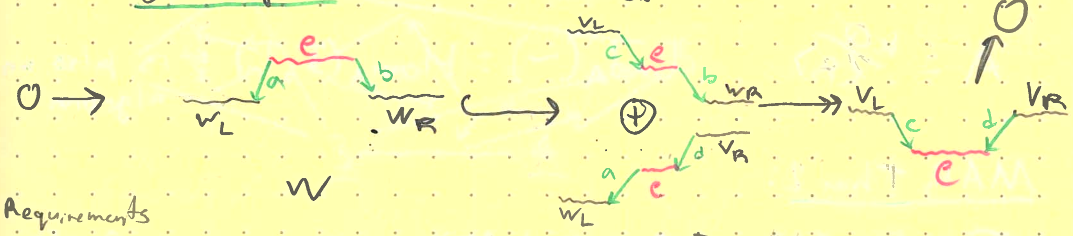
where a is an arrow in Q_1.

Ex:



$e_4 \hookrightarrow b^{-1}ce_4 = b^{-1}c \twoheadrightarrow b^{-1}$

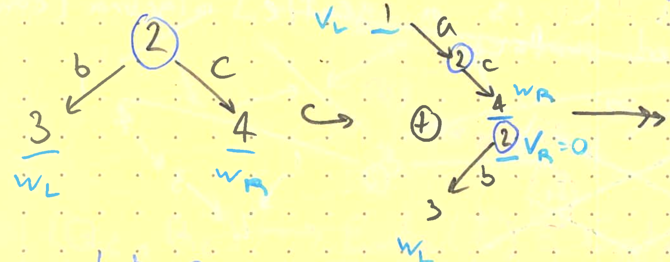
Defn: An overlap extension of $M(v)$ by $M(w)$ with overlap e is a s.e.s.



Requirements
- $a \neq 0$ or $c \neq 0$
- $b \neq 0$ or $c \neq 0$
- $cb, da \notin \langle R \rangle$ if these paths exist

Rem: e is an upset of w, e is a downset of v

Ex Blue marks in example at beginning



overlap e_2

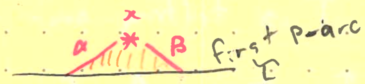
Prop (seeing s.e.s in permissible (red)-triangulations (*))

① crossing of *-arcs ⟷ overlap extension

② suppose *-arcs α and β share an endpoint x

Case II:

Case I:



∃ morphism $M(\alpha) \to M(\beta)$

case III:

case II and III contributes to an arrow extension

12. Endomorphism algebra of a mar module

$A = kQ/\langle R \rangle$ $\quad \mathrm{End}_A(-) = \mathrm{Hom}_A(T,T)$ ← is also an algebra

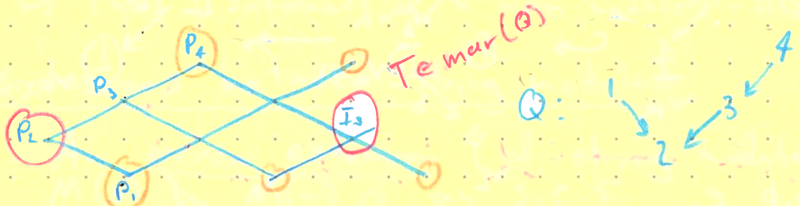
MAR Thm 2:

$T \in \mathrm{mar}(A)$ $\quad C := \mathrm{End}_A(T)$ then

i) $C \cong \mathrm{End}_{\bar{A}}(\bar{T})$ where $\bar{A} = k\bar{Q}/\langle \bar{R} \rangle$ is a bigger gentle algebra, and $\bar{Q}$ has $n = \#Q_0 + \#Q_1$ and $\bar{T}$ is a tilting module of $\bar{A}$
\# summands $= n$

ii) This shows C is a gentle algebra (can have a tiled surface

Ex:



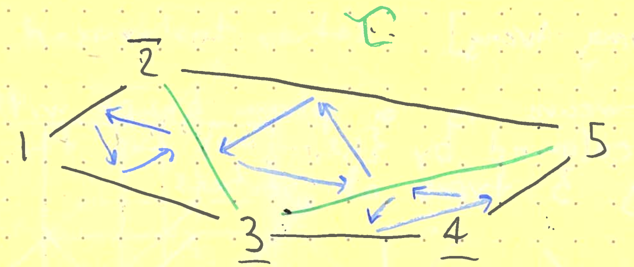
$\bar{T}$ is tilting over $\bar{Q}$

Rem: In type A, C is a tilting algebra (End. alg. of a tilting module of a finite dimensional path algebra)

iii) The bounded quiver (Q_C, R_C) can be read from $*$-triangulation τ corresponding to mar module T.

 · How? 1) Draw adjacency quiver of τ (bdry lines too)

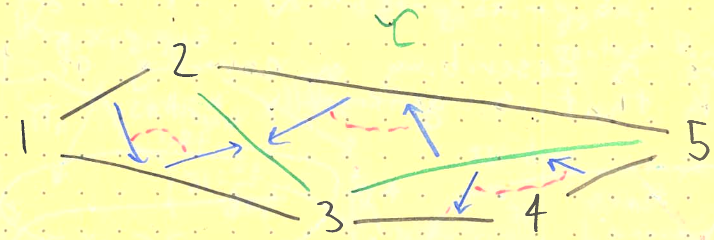
Ex



2) · Remove arrow between $\gamma(i,j)$ and $\gamma(j,k)$ $i<j<k$

· The remaining path in that is inserted as a relation in R_C

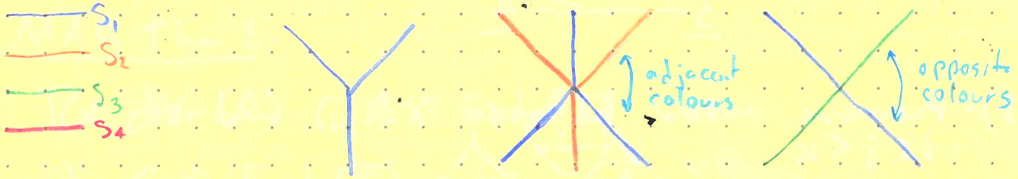
Ex



$(Q_C, R_C) \rightsquigarrow \mathrm{End}_A(T) \cong \mathrm{End}_{\bar{A}}(\bar{T})$

CASS: 21^{st} [Deping Weng]

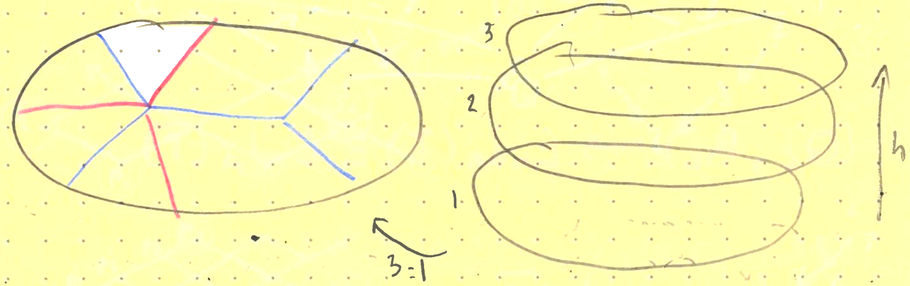
Recap An n-weave is a planar graph with edges coloured by $\{1, 2, \dots, n-1\}$ and the following 3 types of vertices



$(s_i s_j)^3 = id$ $\quad$ $(s_i s_j)^2 = id$

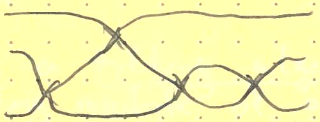
$|i-j| = 1$ $\quad$ $|i-j| > 1$

Each weave is describing a covering of the base disk that is generically $n:1$

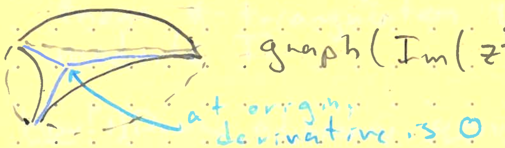


Edges are the singular locus

Ex 1211

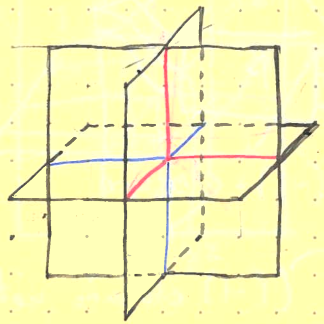


At a vertex



graph$(\mathrm{Im}(z^{\frac{3}{2}}) : \mathbb{C} \to \mathbb{R})$

At hexavalent vertex



weave $w \longrightarrow \Sigma_w$ an immersed surface in $\mathbb{R}^3_{x_1 x_2 x_3}$

leg. surface

Λ_w $\quad$ $\mathbb{R}^5_{x_1 x_2 y_1 y_2 z}$ $\qquad \alpha = dz - y_1 dx_1 - y_2 dx_2$

contact form

π_F $\quad$ cotangent lift $\quad y_i = \frac{\partial z}{\partial x_i}$ $\quad \pi_L$

k+1-dim $\quad$ 2k-dim

$\mathbb{R}^3_{x_1 x_2 x_3} \supset \Sigma_w$ $\qquad \mathbb{R}^4_{x_1 x_2 y_1 y_2}$ exact Lagrangian

Def^n: A weave w is called a free weave if Λ_w does not have any Reeb chords

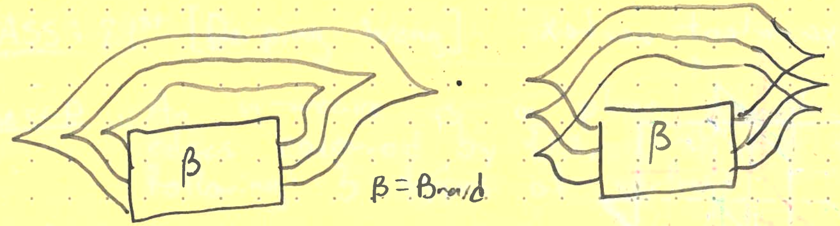
Exercise: Show that a 2-weave is free iff it is a tree (Hint: Mean Value Theorem)

What topological information can we get from a weave?

Euler characteristic

$\chi(\Lambda_w) = n - \#$ trivalent vertices (Exercise: Prove it)

∂L_w is a Legendrian link, and it is given by the (-1)-closure of the boundary braid

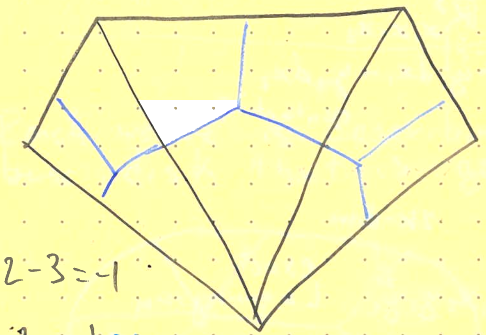


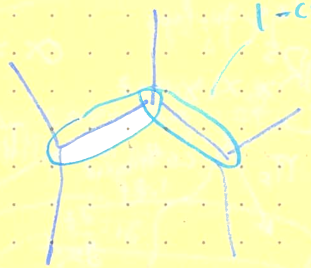
rainbow closure

(-1)-closure

Fact/Exercise: $\Lambda_\beta = \Lambda^{(-1)}_{w_0 \beta w_0}$ ((-1) closure notation)

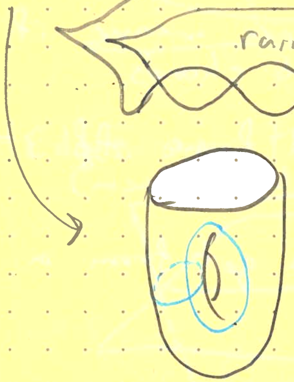
Ex





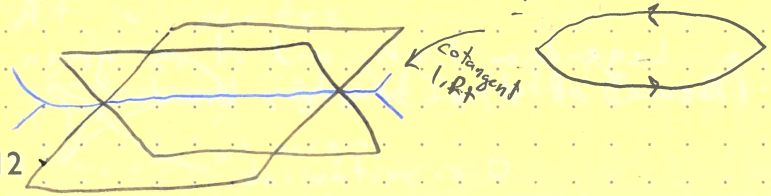
$\chi = 2-3 = -1$
$\quad \overset{\shortparallel}{2} - 2g - b = 1$

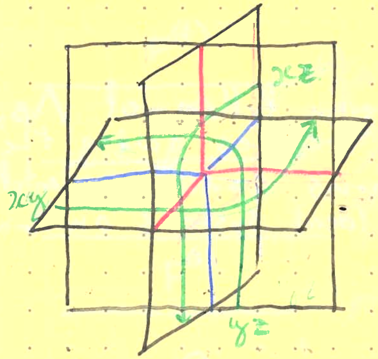
$s_1\ s_1\ s_1\ s_1\ s_1 \quad (1,1,1,1,1,1)$



1-cycles correspond to generators of homology

We can see some 1-cycles on the weave





"this is somehow describing the generators of the homology"

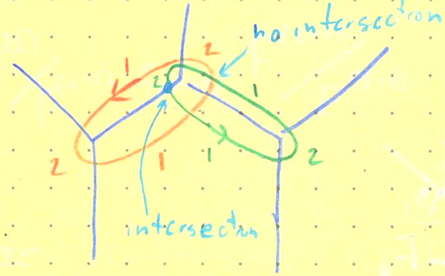
In nice cases (Λ_β rainbow closure), we get a basis of $H_1(L_w)$ using on-weave representation.

In general (Richardson that are not rainbow closure; plabic fence) we get a linearly independent subset of $H_1(L_w)$ using on-weave cycles.

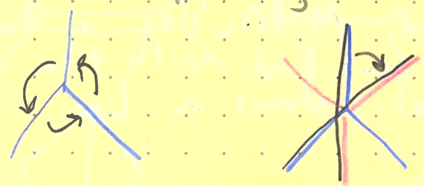
Such lin. indep. subset gives the mutable vertices of the quiver

arrows are given by the intersection pairing between these cycles

Ex



Exercise prove this is the correct intersection pairing



Thm [STWZ]

If L is an exact Lagrangian filling of Λ_β, and we have a collection of 1-cycles giving the quiver, Then the cluster torus can be identified with the space of rank 1 local systems on L_w s.t.

$$y_i = \text{monodromy along } \gamma_i$$

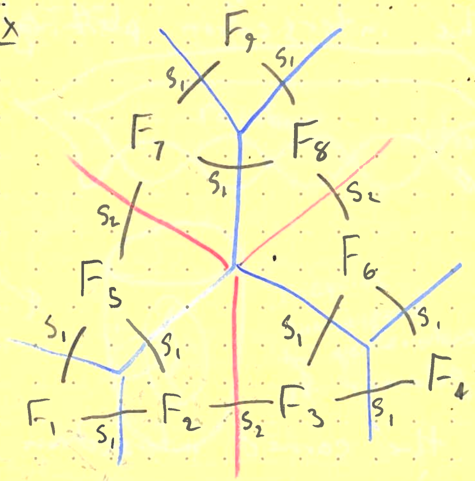
"local system is vector bundle with a flat connection"

$M_1(\Lambda_w)$ flag moduli space

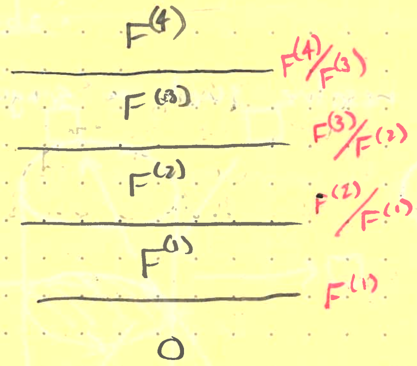
$\parallel$

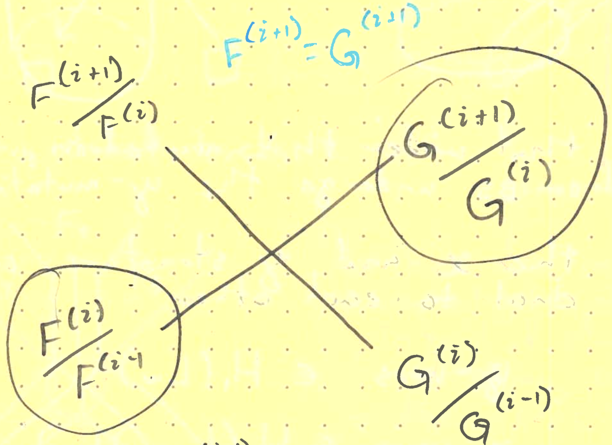
$\left[\begin{array}{l}\text{assign a flag to each face of the weave,}\\ \text{satisfying the relative position condition inspired}\\ \text{by } w\end{array}\right] / GL_n$

Ex



Fact/Thm: if w is free, $M_1(\Lambda_w)$ is a torus and can be identified with the space of local systems on L_w



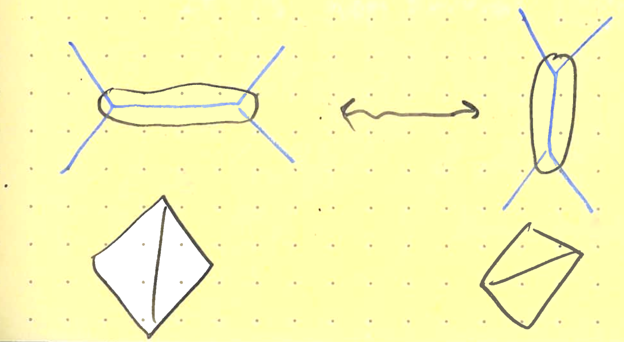


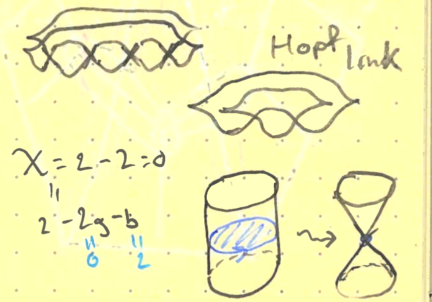
$$F^{(i+1)} = G^{(i+1)}$$

$$F^{(i)}/F^{(i-1)} \hookrightarrow F^{(i+1)}/F^{(i-1)} = G^{(i+1)}/G^{(i-1)} \twoheadrightarrow$$

$$\cong$$

Mutation





$\chi = 2-2 = 0$

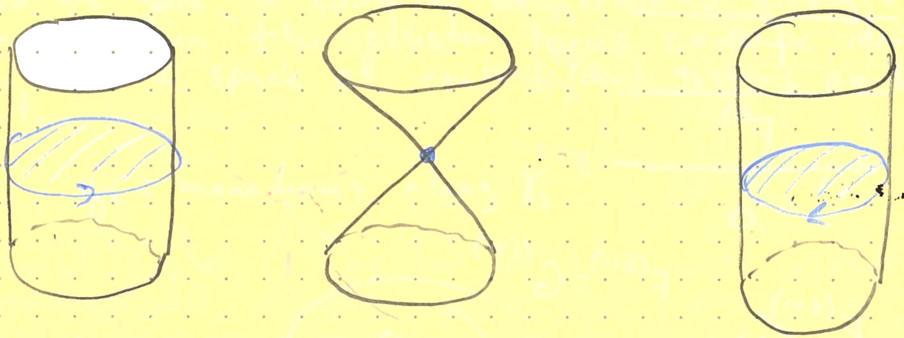
$\quad\parallel$

$2-2g-b$

$\quad\; 0 \quad\;\; 2$

Thm (Polterovich)

There are two ways to resolve such a singularity



Exercise: verify that under this mutation, the monodromies undergo the y-mutation

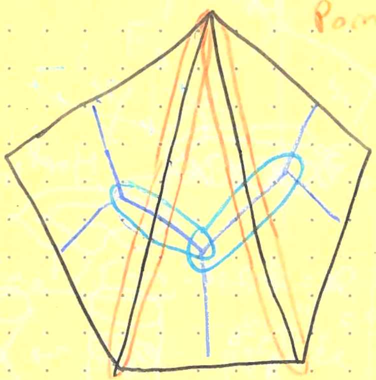
Cluster Duality: the x and y structure are dual to each other

Poincaré Duality

y-vars $\in H_1(L_w)$

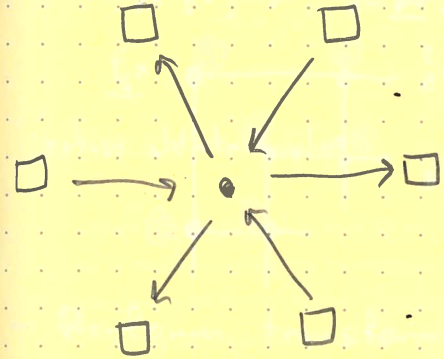
x-vars $\in H_1(L_w, \partial L_w)$

Ex for trefoil

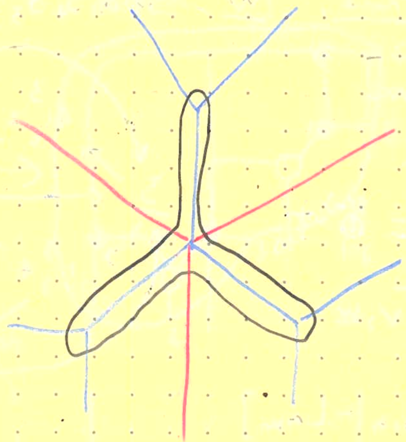


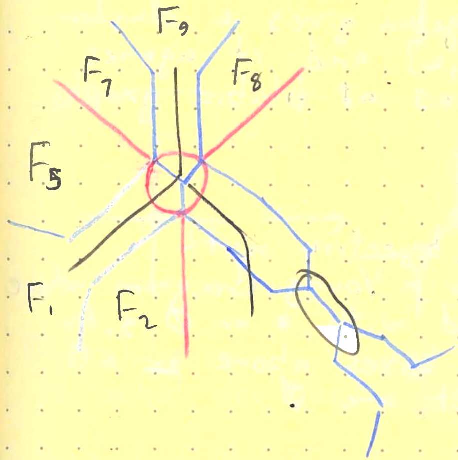
Poincaré duals

Aside:



train track trick





no idea what this means, what else is new. Something about "if stuck with the mutations, can mutate using the weave"

Note: Huge issue w/ signs, hence why this work was done over characteristic 2. If over characteristic 0, there is a fix, but it is nontrivial

CASS: 21st [Khrystyna Serhiyenko]

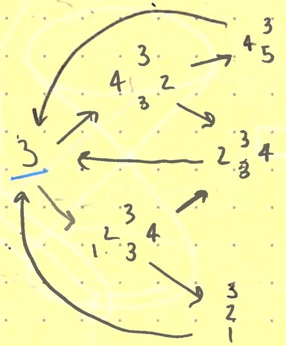
Last time: $\underline{w} = s_2 s_1 s_4 s_3 \underline{s_2} s_5 \underline{s_4} s_3$ $v = s_2 s_4$

$\mathbb{C}[R_{v,w}]$

$v \leq w$

$T_{v,w} = \bigoplus_{j=1}^{\ell(w)} T_j$

$\mathrm{End}\, T_{v,w}$



only mutable vertex

[Galashin - Lam]

$\Pi_{v,w}$ Leclerc's cluster algebra gives a cluster structure on $\mathbb{C}[\Pi_{v,w}]$ and it agrees with the combinatorics of plabic graphs

w = k-grassmann perm.

Le-diagram

[Postnikov] Positroids are in bijection with Le-diagrams i.e. a Young diagram where every box is filled with • or ∅ s.t. for every ∅ all boxes above or all boxes to the left are ∅

Ex

•	∅	∅	∅	•
•	∅	•	∅	•
∅	•	•	∅	•
•	•	•		
∅	•			

Ex

$n-k$, $k=3$

•	∅	•
∅	•	•
•	•	

$\overset{v,w}{\rightsquigarrow}$

s_3	s_4	s_5
s_2	s_3	s_4
s_1	s_2	

1 2 3 4 5 6 7 8

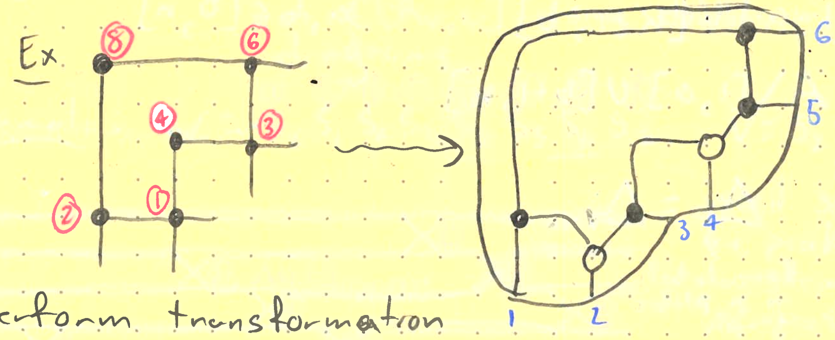
$w = s_2 s_1 s_4 s_3 \underline{s_2} s_5 \underline{s_4} s_3$

$v = s_2 s_4$

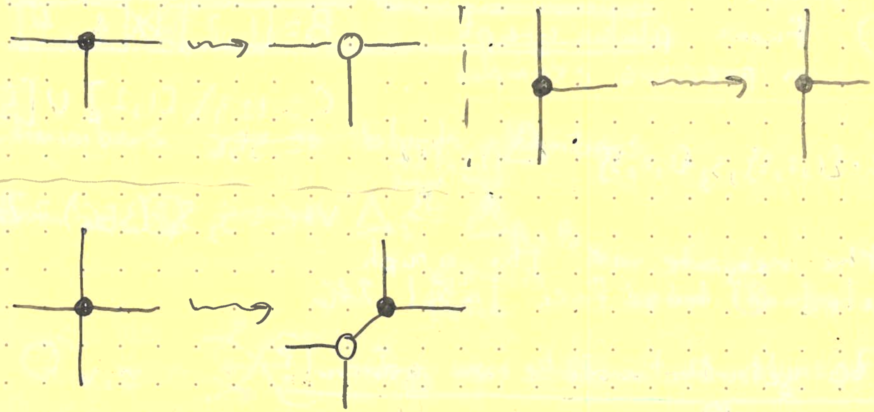
Le-diagrams ⟶ Plabic graphs

- For every • draw a hook

Ex

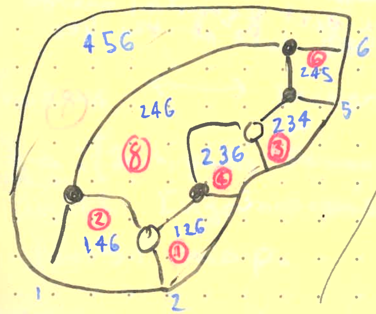


- Perform transformation



Conventions

- label vertices counterclockwise
- label faces to the right of the trip by the source of the trip
- turn right at •
  left at ○



Le-diagram ↙ ↘ Plabic graph $\overset{?}{\longleftrightarrow}$ $v \leq w$, $T_{v,w}$

face labels ⟷ $\Pi_{v,w}$

Suppose $\Delta_{A,B}$ Leclere's generalized minor, how to convert it to a Plücker?

$B = [1,a] \cup [K+1, b]$ $a, b \in [0,n]$

$C := A \setminus [1,a] \cup [b+1, n]$

claim: $\Delta_F = \Delta_{v \cdot C}$

face label in a plabic graph

Ex ⑧ from plabic graph in previous examples

$\Delta_{e\cdot\{1,2,3\}, s_3\{1,2,3\}} = \Delta_{\underbrace{123}_{A}, \underbrace{124}_{B}}$

$B = [1,2] \cup [4,4]$

$C = 123 \setminus [1,2] \cup [5,6]$
$= 356$

$v \cdot C = s_2 s_4 \cdot (356) = 246$

so the region in the graph labeled ⑧ has face label 246

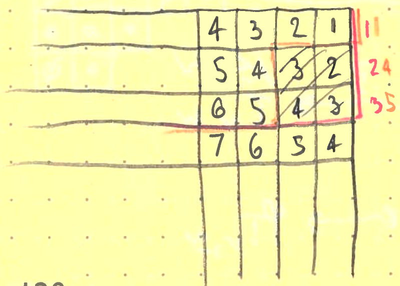
How to get the module in general?

$\underline{w} = s_2 s_1 s_4 \overset{④}{s_3} \underline{s_2} s_5 \underline{s_4} s_3$ Example

$\Delta_{v(4)[123], w(4)[123]}$

$v(4)$ means all elements of v left of ④

$\Delta_{123, 145}$



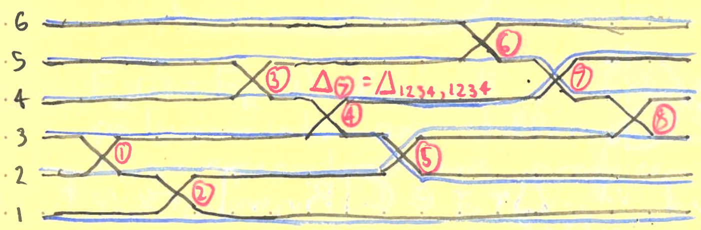
draw line down, keeping the next sequential number to the left of the line

$\Rightarrow$ 2 3 4 3

How to understand $R_{v,\underline{w}}$ seed in general

[Ingermansron] constructs upper cluster algebra in $\mathbb{C}[R_{v,w}]$ using double wiring diagrams

Example: $\underline{w} = s_2 s_1 s_4 s_3 \underline{s_2} s_5 \underline{s_4} s_3$ $v = s_2 s_4$



need $\underline{w}$ to be unipeak: w-strings have at most one peak

chambers $\rightsquigarrow$ block chambers

chamber ⓘ $\rightsquigarrow$ $\Delta_i = \Delta_{A,B}$

v strands below   w strands below chamber

$Q'_{v,\underline{w}}$

Ing uses different conventions $Fl_n = GL_n / B^+$

quiver of the seed $Q_{v,\underline{w}}$ vertices $\leftrightsquigarrow$ irreducible factors of $\Delta_{ⓘ}$

arrows $\longleftrightarrow$ $\Delta_{ⓘ} \to \Delta_{ⓙ}$ in $Q'_{v,\underline{w}}$

get arrows $\Delta \to \Delta'$ for every Δ, Δ' factors of $\Delta_{ⓘ}, \Delta_{ⓙ}$ respectively

Frozens $\longleftrightarrow$ minors in open left chambers

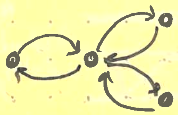
Thm: [Serhiyenko-Sherman-Bennett]. Leclere's seed agrees with Ingermanson's after applying the twist map.

Open Problems

* Describe higher rank cluster variables in $\mathbb{C}[Gr(k,n)]$?

* [Leclerc] works for types A, D, E, can we find a combinatorial interpretation in the other types?

ex



* For $\mathbb{C}[R_{v,w}]$ if $\Delta_{I,J} \in \mathbb{C}[R_{v,w}]$ is it a cluster variable?

* Do we get a correspondence between Leclerc's seeds and seeds consisting of minors?

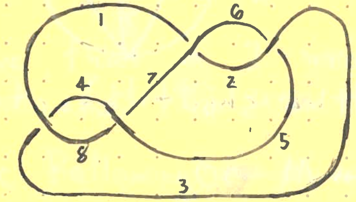
* [JKS] categorification for $\mathbb{C}[R_{v,w}]$?

* One can consider plabic graphs on general surfaces, get a cluster algebra and categorification, but do they come from coordinate ring of other varieties?

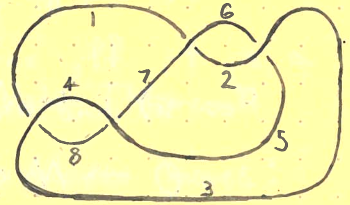
CASS: 21st Questions

[Ralf Schiffler]

Q1) $K =$



$K' =$



$F_K(i) = F_{K'}(i)$ for all i, but the Alexander polynomials are different. Show that the two specializations produce the correct $\Delta_K, \Delta_{K'}$.

Q2) Compute the F-polynomial of $T(3)$ for the diagram in ① and find a specialization for the y_i that produces the <u>Jones</u> polynomial.

Q3) Can you do the same for $T(2)$?

Q4) Take your favourite knot diagram K and associate a plabic graph G to it such that the quivers Q_K, Q_G are the same.

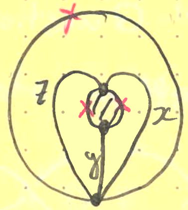
[Emily Gunawan] (Infinite rep type)

Q1) a) Match the two sides

Strings	attributes
• $w_1 = bc^{-1}$	i) no self extensions $(0 \to M(w) \hookrightarrow E \twoheadrightarrow M(w) \to 0)$
• $w_2 = abc^{-1}ab$	ii) has self-arrow extension $(0 \to M(w) \hookrightarrow M(waw) \twoheadrightarrow M(w) \to 0)$
• $w_3 = bc^{-1}abc^{-1}b$	iii) has an overlap self-extension $(0 \to M(w) \hookrightarrow E_1 \oplus E_2 \twoheadrightarrow M(w) \to 0)$

b)

$$Q = \begin{tikzcd} z \arrow[rr, "c"] \arrow[dr, "a"'] & & x \\ & y \arrow[ur, "b"'] & \end{tikzcd} \qquad R = \{\emptyset\}$$



For ii) and iii), write the short exact sequence

$$0 \to M \xhookrightarrow{f} E \xrightarrow{g} M \to 0$$

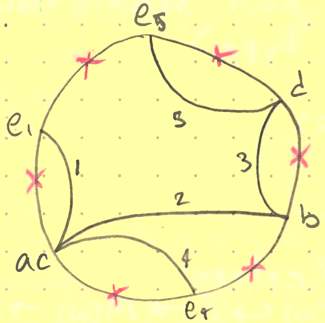
including the morphisms $f=(f_x, f_y, f_z)$, $g=(g_x, g_y, g_z)$ and maps $\varphi_a, \varphi_b, \varphi_c$ for the reps M and E

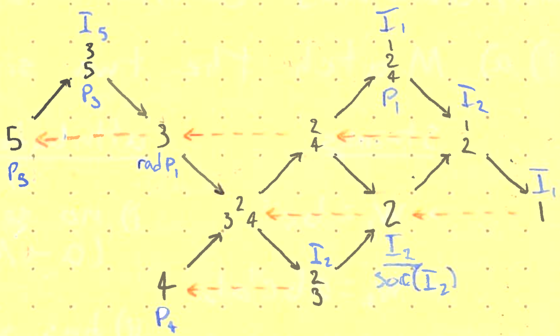
c) Pick a $T \in \mathrm{mar}(Q)$. Compute $\overline{Q}$, the tilting module $\overline{T} \in \mathrm{rep}(\overline{Q})$, and $\mathrm{End}_{kQ}(T) \cong \mathrm{End}_{k\overline{Q}}(\overline{T})$ using any method

Q2) (Finite rep type .

Compute $\mathrm{End}_A(T)$ for all three $T \in \mathrm{mar}(K)$ for

$$A = \frac{kQ}{\langle R \rangle}, \quad Q = \begin{tikzcd} & & 3 \arrow[r, "d"] & 5 \\ 1 \arrow[r, "a"] & 2 \arrow[ur, "b"] \arrow[dr, "c"'] & & \\ & & 4 & \end{tikzcd} \qquad R = \{ab, bd\}$$





[Daping Weng]

Q1) Prove that $\chi(\Lambda_w) = n - \#$ trivalent vertices (Hint: make Λ_w into a CW complex)

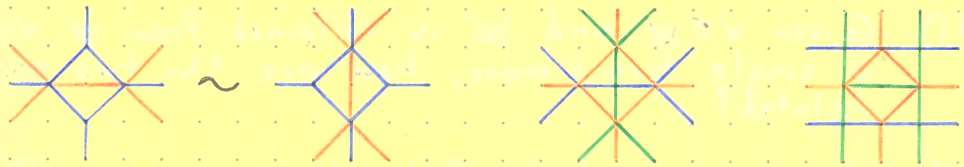
Q2) Prove that a 2-weave is free iff it is a Tree (Hint: use the Mean Value Theorem)

Q3) The following are weave equivalence moves:

Candy Twist:

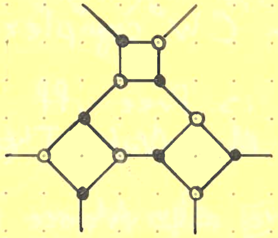
Push through

Flop



Prove that

Q4) Find the weave that gives the same quiver as the reduced plabic graph



and mutate the weave at the center vertex.

Q5) (Open Problem) Prove that if L_w and $L_{w'}$ are Hamiltonian isotopic, then w and w' are related by weave equivalences.

Q6) (Open Problem) Find a weave with a non-tree cluster cycle. Prove that it can always turn into a tree.

[Khrystyna Serhiyenko]

Q1) Given $v \leq \underline{w}$ and $\underline{w}'$ is obtained from $\underline{w}$ via a single braid move, how are the two seeds related?

Q2) How can you read of Leclerc's minors φ_{T_j} easily from the wiring diagram?

Q3) Compute and compare Ingermonson's and Leclerc's seeds, find quivers for

$$\underline{w} = s_2 s_1 \underline{s_4} s_3 s_2 \underline{s_3} s_4$$