Cluster Algebras from Triangulations of Surfaces

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Slides available at `umn.edu/home/egunawan`

Overview

Cluster Algebras (Fomin - Zelevinsky, 2001)

* Commutative rings with rich combinatorial structure.
* Appear in discrete dynamical systems, quiver representations, geometry, string theorey, etc.

Cluster algebras from surfaces (Fock - Goncharov, Gekhtman - Shapiro - Vainshtein, and Fomin - Shapiro - Thurston, 2003–2006)

Almost all cluster algebras with finitely many matrices arise from surfaces (Felikson, Shapiro, Tumarkin 2008).

T-path formulas (Schiffler - Thomas, Dupont - Thomas, 2007 - 2011)

Generalize to surfaces with punctures (G. and Musiker)

Atomic Basis

Combinatorial proof for type D cluster algebras (G. and Musiker)

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An integer sequence

Consider the recurrence $x_n = \frac{x_{n-1}^2+1}{x_{n-2}}$

Set $x_1 = x_2 = 1$

- $x_3 = \frac{x_2^2+1}{x_1} = (1^2 + 1)/1 = 2$

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- $x_4 = \frac{x_3^2+1}{x_2} = (2^2 + 1)/1 = 5$

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- $x_5 = \frac{x_4^2+1}{x_3} = (5^2 + 1)/2 = 26/2 = 13$

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- $x_8 = \frac{x_7^2+1}{x_6} = (89^2+1)/34 = 7922/34 = 233$
- $x_9 = \frac{x_8^2+1}{x_7} = (233^2+1)/89 = 54290/89 = 610$

Integer sequence $1, 1, 2, 5, 34, 89, 233, 610, \ldots$

Do you recognize this sequence?

A Laurent polynomial sequence

The same recurrence $x_n = \frac{x_{n-1}^2+1}{x_{n-2}}$

- $x_3 = \frac{x_2^2+1}{x_1}$

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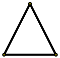
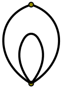
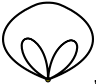
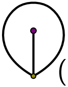
Laurent polynomials in $x_1,\ x_2$, with positive coefficients!

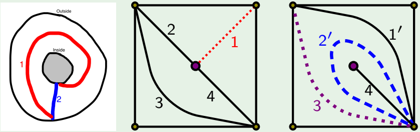
Cluster algebras from marked surfaces (Fomin-Shapiro-Thurston 2006) [1]

- Fix an oriented Riemann surface S + marked points.
- Points are either on the boundary of S or in the interior (called **punctures**).
- An **ideal triangle** is one the following:

 ,  ,  , or  (a self-folded "triangle").

- An **ideal triangulation** T^o cuts S into ideal triangles.

Example



[1] based on Fock-Goncharov and Gekhtman-Shapiro-Vainshtein, 2003.

Seeds

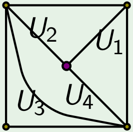
Definition

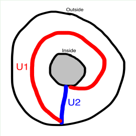
A **seed** $(\{U_1, U_2, \ldots, U_n\}, T)$ is a triangulation T together with a transcendence basis

$$\{U_1, U_2, \ldots, U_n\} \subset \mathbb{Q}(x_1, x_2, \ldots, x_n) \text{ (a } \textbf{cluster}\text{)}$$

identified with the internal diagonals (**arcs**) of T.

Example



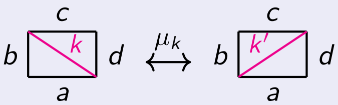


Seed mutations with Ptolemy rule

- In practice, start with an initial seed $(\{x_1, \ldots, x_n\}, T_{\text{initial}})$.
- To produce all seeds, repeatedly perform a **mutation** μ_k in each of the n positions:

Definition (Mutations)

- Replace diagonal k with k'



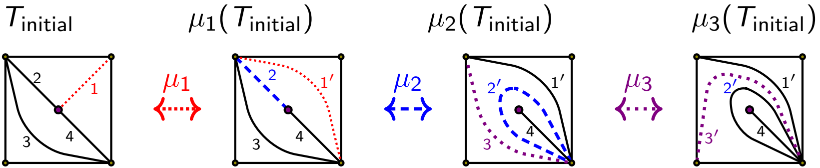
- Replace $\{U_1, \ldots, U_n\}$ with $\{U_1, \ldots, U_n\} \setminus \{U_k\} \cup \{U'_k\}$, where

$$U_k \, U'_k = U_a \, U_c + U_b \, U_d .$$

Set weight of a boundary edge to 1.

- Remark: μ_k is an involution.

Example: once-punctured disk



$\{x_1, \ldots, x_4\}$	$\{x_1', x_2, x_3, x_4\}$	$\{x_1', x_2', x_3, x_4\}$	$\{x_1', x_2', x_3', x_4\}$
Initial cluster	$x_1' = \frac{x_2+x_4}{x_1}$	$x_2' = \frac{x_1'x_4+x_3x_4}{x_2}$	$x_3' = \frac{x_1'+x_2'}{x_3}$

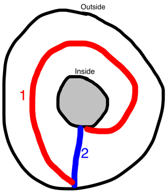
Remark

Arc 4 in the two right-most triangulations looks like it cannot be flipped, but there is a way to mutate at 4.

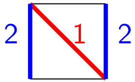
Example: Annulus with 2 marked points

Initial seed

$(T_{\text{initial}}, \{x_1, x_2\})$



inside

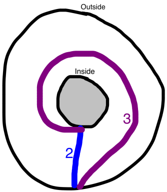


outside

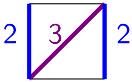
$\{x_1, x_2\}$

mutate at 1

$(\mu_1(T_{\text{initial}}), \{x_3, x_2\})$



inside

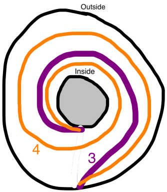


outside

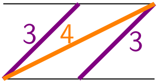
$$x_3 = \frac{x_2^2+1}{x_1}$$

mutate at 1, then 2

$(\mu_2\mu_1(T_{\text{initial}}), \{x_3, x_4\})$



inside inside



outside outside

$$x_4 = \frac{x_3^2+1}{x_2}$$

$$= \frac{x_2^4+2x_2^2+x_1^2+1}{x_2x_1^2}$$

There is exactly one triangulation up to relabeling

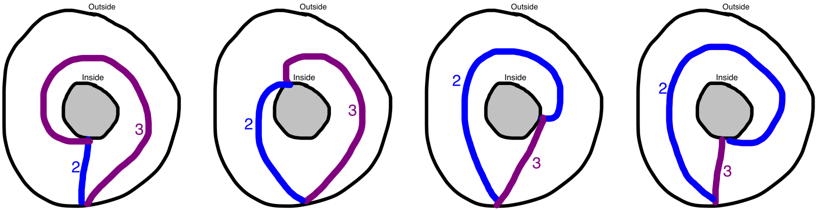
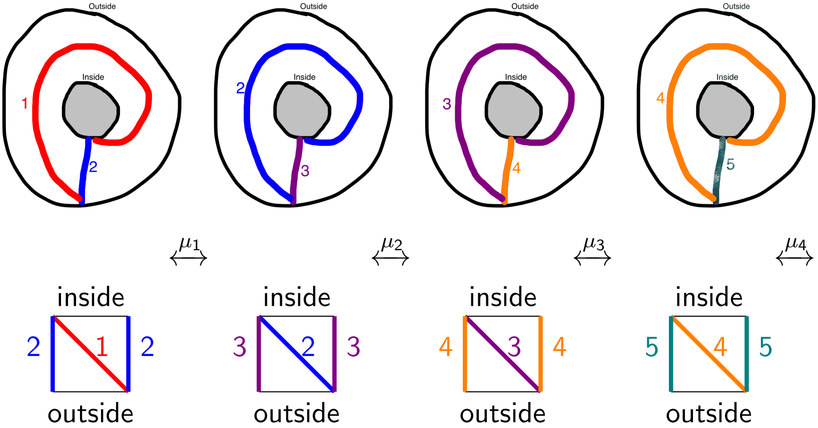


Figure: Rotating the inside boundary clockwise $\mu_1(T_{\text{initial}})$

Remark

Mutating a triangulation results in the same triangulation, but there are infinitely many diagonals.

Mutations correspond to the recurrence $x_n x_{n-2} = x_{n-1}^2 + 1$



$x_1\, x_3 = x_2^2 + 1 \qquad x_2\, x_4 = x_3^2 + 1 \qquad x_3\, x_5 = x_4^2 + 1 \qquad \ldots$

Definition

- Two seeds are **mutation equivalent** if they are connected by a sequence of mutations.
- The set of **cluster variables** is the union of the U_is in all mutation-equivalent seeds.

Definition (Fomin-Zelevinsky 2001)

The **cluster algebra** is the subring of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all cluster variables.

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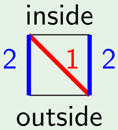
Theorem

- **Laurent Phenomenon**: *each cluster variable can be expressed as a Laurent polynomial in* $\{x_1, ..., x_n\}$.
- **Positivity**: *this Laurent polynomial has positive coefficients (Lee - Schiffler, Gross - Hacking - Keel - Kontsevich, 2014, and special cases by others).*

Seeds from skew-symmetric matrices and quivers

A **seed** is $(\{U_1, \ldots, U_n\}, B)$ where B is a skew-symmetric matrix.

Example



corresponds to the quiver $2 \Rightarrow 1$ and the matrix $B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$.

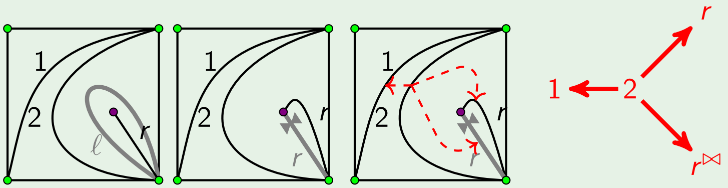
- $\mathcal{A}$ is said to be of **finite mutation type** it there are finitely many quivers/B-matrices.
- Because there are finitely many triangulations of (S, M), any $\mathcal{A}(S, M)$ is of finite mutation type.

Finite type classification

$\mathcal{A}$ is of **finite type** if there are finitely many seeds.

- The finite type cluster algebras are classified by the Dynkin diagrams.
- Type A and D are modeled by marked surfaces.

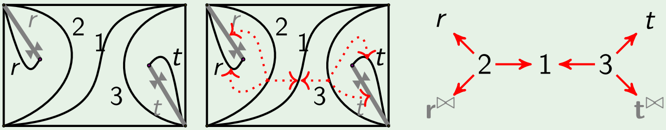
Example (Type D_4)



- Type B and C are modeled by orbifolds.

Surfaces with punctures

Example (Twice-punctured $(n-3)$-gon, $n = 7$, type $\widetilde{D}_6$)



- When the marked surface has punctures, each endpoint of the internal diagonals is assigned a choice of two tags.
- This introduces new technical difficulties.

Goal

Generalize results to surfaces with punctures.

T-paths

Result 1

We extend T-path definitions and formulas [Schiffler - Thomas, Dupont - Thomas, 2008-2011] to surfaces with punctures.

Definition ((T, γ)-path)

Let T be a triangulation and let γ be an arc that crosses T. A (T, γ)-**path** $\alpha = (\alpha_1, \ldots, \alpha_{2d+1})$ is a concatenation of edges of T such that:

(T1) Each even step α_{2k} is the k-th arc that γ crosses.

(T2) The path α is homotopic to γ, and satisfies stronger local homotopy condition.

T-path formulas

Definition (Laurent monomial from a T-path α)

If $\alpha = (\alpha_1, \ldots, \alpha_{2d+1})$ is a T-path,

$$x(\alpha) := \left(\prod_{i \text{ odd}} x_{\alpha_i} \right) \left(\prod_{i \text{ even}} {x_{\alpha_i}}^{-1} \right).$$

Theorem (T-path formula for plain arcs)

The cluster variable x_γ expressed in the variables of T is

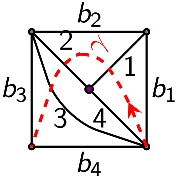
$$x_\gamma = \sum_\alpha x(\alpha)$$

over all (T^o, γ)-paths α of γ.

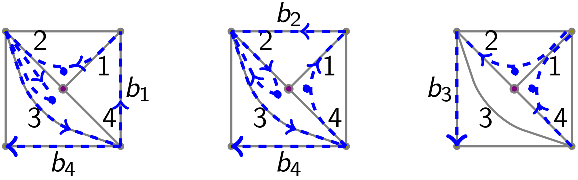
We have T-path formulas for arcs with decorations on each endpoint (tagged arcs) and for closed curves in the interior of S.

Example of T-paths

Let T and γ be:



Three of the five (T^o, γ)-paths:



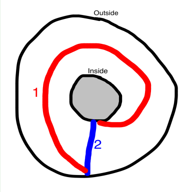
$w = (b_1, \mathbf{1}, 2, \mathbf{2}, 2, \mathbf{3}, b_4)$ $w = (4, \mathbf{1}, b_2, \mathbf{2}, 2, \mathbf{3}, b_4)$ $w = (4, \mathbf{1}, 1, \mathbf{2}, 3, \mathbf{3}, b_3)$

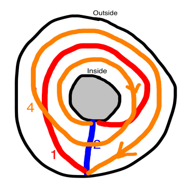
$x(w) = \frac{b_1\, x_2\, x_2\, b_4}{x_1\, x_2\, x_3}$ $x(w) = \frac{x_4\, b_2\, x_2\, b_4}{x_1\, x_2\, x_3}$ $x(w) = \frac{x_4\, x_1\, x_3\, b_3}{x_1\, x_2\, x_3}$

Direct formula vs. recursive definition

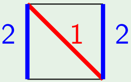
It's not clear which sequence of mutations will lead us to a triangulation containing an arc γ. Use the T-path formula to compute any cluster variable directly.

Example (Compute arc 4 recursively)





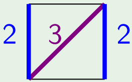
inside



outside

$\overset{\mu_1}{\longleftrightarrow}$

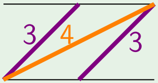
inside



outside

$\leftrightarrow \mu_2$

inside inside



outside outside

$\{x_1, x_2\}$

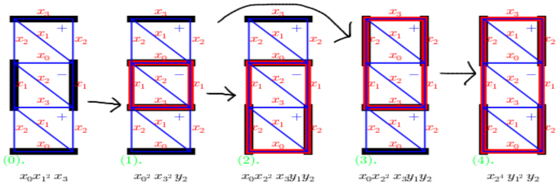
$x_3 = \frac{x_2^2+1}{x_1}$

$x_4 = \frac{x_3^2+1}{x_2} = \frac{x_2^4+2x_2^2+x_1^2+1}{x_2 x_1^2}$

Snake graphs (Musiker - Schiffler - Williams, 2009)

Proposition (G. and Musiker)

Generalized to surface with punctures the (natural) bijection between T-paths are in natural bijection to snake graphs.



Note: Figures drawn and calculated using SageMath

- Grading: the number of boxes enclosed in a red cycle.
- This data is related to a certain lamination on the surface:
- r boxes enclosed in a red cycle $\leftrightarrow$ r even steps in the T-path have orientations that agree with the lamination.

Atomic bases

Introduced by Sherman - Zelevinsky in 2003

Definition

Let $\mathcal{A}$ be a cluster algebra.

- $y \in \mathcal{A}$ is a **positive element** if:
  y is a positive Laurent polynomial with respect to **any** cluster, *i.e*, $y \in \mathbb{Z}_{\geq 0}[U_1^{\pm 1}, \ldots, U_n^{\pm 1}]$ for *any* cluster $\{U_1, \ldots, U_n\}$.
  For example, cluster variables are positive elements.

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  For example, cluster variables are positive elements.
- An **indecomposable positive element** cannot be written as a sum of two positive elements.
- The **atomic basis** is a basis which is precisely the set of all indecomposable positive elements.

Application of T-paths: atomic basis proof

Definition

A **cluster monomial** is a product of cluster variables all coming from the same cluster, e.g. a^5be^2 is a cluster monomial if $\{a, b, c, d, e\}$ is a cluster.

Theorem (Cerulli Irelli, 2011)

For a cluster algebra of type D, the basis of cluster monomials is atomic.

- Cerulli Irelli's proof is by representation theory and also work for type A and E cluster algebras.

Result 2 (G. and Musiker)

A combinatorial proof (using the T-path formula) for above.

- Type D cluster algebra is modeled by once-punctured disk.

Question: which cluster algebras have atomic bases?

Remark

Atomic bases don't always exist. For some cluster algebras, the set of indecomposable positive elements fail to be linearly independent.

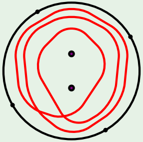
For example, the cluster algebra corresponding to $\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix}$ with $r \geq 3$ (Lee, Li, Zelevinsky, 2012).

Atomic bases for other surfaces

Conjecture (Fomin - Shapiro - Thurston)

The atomic basis is the **bracelets collection**, *the cluster monomials + bracelets.*

Example



A bracelet

- True for:
  - disks with ≤ 1 puncture and annuli (types A_n, D_n, $\widetilde{A}_{p,q}$) [Cerulli Irelli, Dupont - Thomas]
  - certain orbifolds (type B_n, C_n, and $C_n^{(1)}$) [Felikson - Tumarkin].
- The bracelets collection forms a basis for surfaces [Musiker - Schiffler - Williams, 2011] and orbifolds [Felikson - Tumarkin, 2015] without punctures.

Thank you

Slides available at `umn.edu/home/egunawan`

Proof for atomic bases for type A and D

Definition

- A **proper Laurent monomial in a cluster** $\mathbf{x} = \{x_{i_1}, \ldots, x_{i_n}\}$ is a product $x_{i_1}^{c_1} \ldots x_{i_n}^{c_n}$ where one of the c_k is negative.
- $\mathcal{A}$ has the **proper Laurent monomial property** if for any cluster $\mathbf{x} = \{x_{i_1}, \ldots, x_{i_n}\}$ and any cluster monomial Σ, the cluster monomial Σ is a proper Laurent polynomial in the variables of $\mathbf{x}$ (unless all factors of Σ come from $\mathbf{x}$).

Theorem (Cerulli Irelli, Keller, Labardini, and Plamondon, 2012)

All cluster algebras arising from directed graphs (note: these include type D) have the proper Laurent monomial property.

Corollary

For type A and D, the basis of cluster monomials is atomic.