Involutions in Coxeter Groups

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Abstract

A Coxeter group is a group which is generated by involutions. They are often used to study geometrical symmetries. Two finite Coxeter groups were studied, those of type B_n and those of type D_n. The research group, led by Professor Ruth Haas, was primarily interested in the conjugacy classes of the involutions of each group and the relationships between these conjugacy classes. The elements of each involution conjugacy class of B_n were explicitly determined. Formulae were found to count the order of each involution conjugacy class of B_n and to count the number of involution conjugacy classes in B_n. A relationship between the involution conjugacy classes of B_n was determined. The involution conjugacy classes in the subgroup D_n were studied.

The Coxeter Group B_n

For any $n \in \mathbb{N}$, consider the set $\{1, 2, \ldots, n, -1, -2, \ldots, -n\}$. The Coxeter group B_n is the group generated by the permutations $s_1, s_2, \ldots, s_n$ where

$$\begin{aligned} s_1 &= (1\,2)(-1\ -2) \\ s_2 &= (2\,3)(-2\ -3) \\ &\vdots \\ s_{n-1} &= ((n-1)\,n)(-(n-1)\ -n) \\ s_n &= (1\ -1) \end{aligned}$$

Properties of B_n :

- Note that each generator s_i is an involution.
- A defining characteristic of B_n is that for any element σ of B_n and any $i, j \in \{1, \ldots, n, -1, \ldots, -n\}$, if $\sigma(i) = j$, then $\sigma(-i) = -j$.

Properties of Involutions

Lemma 1: If w is an involution and s is any element in a group G, then sws^{-1} is an involution. *Proof.* Then $(sws^{-1})(sws^{-1}) = sws^{-1}sws^{-1} = swws^{-1} = ss^{-1} = e$.

Lemma 2: Suppose w and s are involutions in a group G. If $sws^{-1} = w$, then sw is an involution. *Proof.* Then $swsw = (sws^{-1})w = ww = e$.

Theorem: For a group G with generating set S, all involutions of G can be generated by repeated application of Lemma 1 and Lemma 2 using S, starting with the identity.

Involution Posets of B_n

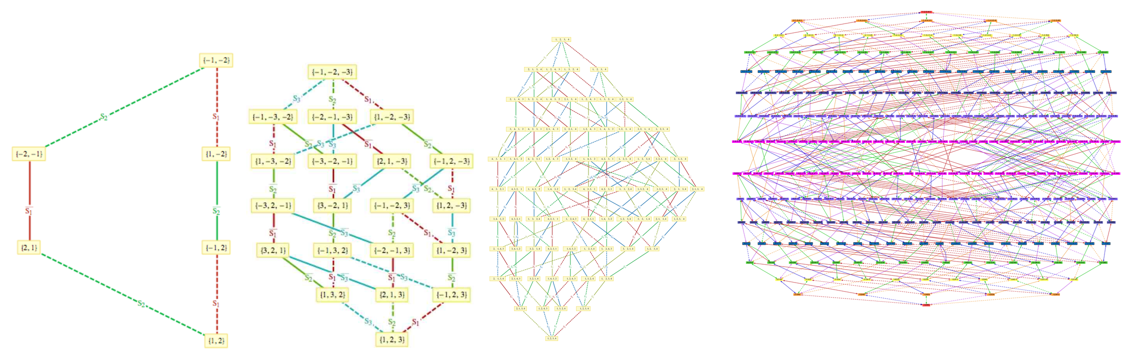

Figure 1: (From left to right) The involution posets for B_2, B_3, B_4, and B_5.

The Conjugacy Classes of B_n

Definition: Elements of B_n of the form $(a\ -a)$ will be called α-cycles, and elements of the form $(a\,b)(-a\ -b)$ (where $b \neq \pm a$) will be called β-cycles. We say that two involutions σ and τ of B_n have the same *cycle type* if they consist of the same number of α-cycles and the same number of β- cycles.

Theorem 1: Let σ and τ be involutions in B_n. Then σ is conjugate to τ if and only if σ and τ have the same cycle type.

Notation: Let C be a conjugacy class in B_n. Then every element of C is composed of s α-cycles and t β-cycles, and we can denote the class C by $[s, t]$.

Theorem 2: In B_n, the number of elements in each conjugacy class $[s, t]$ is given by

$$\frac{n!}{(n-2t)!t!}\binom{n-t}{s}.$$

Theorem 3: For each n, the number of conjugacy classes in B_n is given by

$$\sum_{k=0}^{n}\left(\left\lfloor\frac{k}{2}\right\rfloor + 1\right).$$

B_n Supergraphs

Define the *Supergraph* of B_n to be the graph where each conjugacy class $[s, t]$ of B_n is a vertex, and an edge between two vertices, $[s, t]$ and $[s', t']$, indicates that there exists some $\sigma \in [s, t]$ and a generator s_i of B_n such that $s_i \sigma s_i^{-1} = \sigma$ and $s_i\sigma \in [s', t']$.

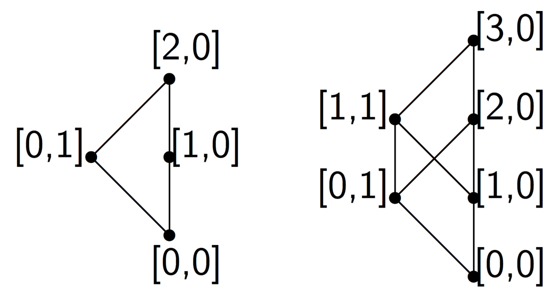

Figure 2: The Supergraphs of for B_2 and B_3.

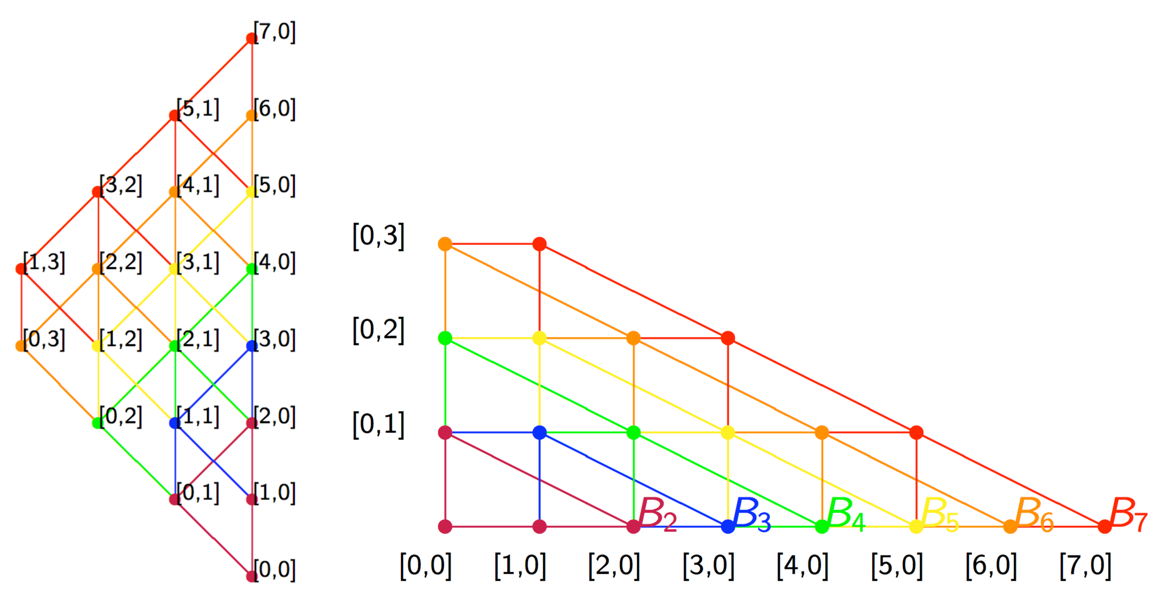

Figure 3: The Supergraph of for B_7, also shown mapped onto the Cartesian plane.

Properties of the B_n Supergraphs

Notice that the Supergraph of B_n always contains the Supergraphs of B_k for all $k \leq n$ as sub-graphs.

Theorem 4: The conjugacy class $[s, t]$ is adjacent to the conjugacy class $[s', t']$ in the Supergraph of B_n iff one of the following hold:

1. $s + 1 = s'$ and $t = t'$ (or $s = s' + 1$ and $t = t'$)
2. $s = s'$ and $t + 1 = t'$ (or $s = s'$ and $t = t' + 1$)
3. $s + 2 = s'$ and $t - 1 = t'$ (or $s = s' + 2$ and $t = t' - 1$)

The Coxeter Group D_n

Coxeter Groups of type D_n are generated by the elements

$$\begin{aligned} s_1 &= (1\,2)(-1\ -2) \\ s_2 &= (2\,3)(-2\ -3) \\ &\vdots \\ s_{n-1} &= ((n-1)\,n)(-(n-1)\ -n) \\ s_n &= (1\ -2)(-1\,2) \end{aligned}$$

Properties of D_n:

- D_n is a subgroup of B_n
- Every element of D_n, when written in bottom row notation, must have an even amount of negative signs. For example $[3\ -4\,2\ -1\,5]$ is an element of D_5 but $[1\ -2\,3\,4\,5]$ is not.

Involution Posets for D_n

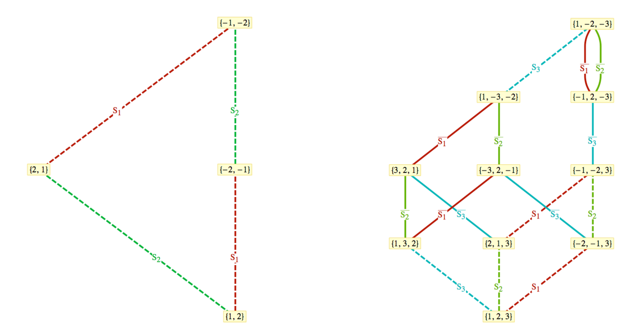

Figure 4: The involution poset for D_2 (left) and the involution poset for D_3 (right).

Future Work

In the future, we plan to study the conjugacy classes of D_n by:

- Examining the involution posets for D_n
- Determining the Supergraphs for D_n
- Determining which properties of the conjugacy classes of B_n hold in D_n

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