# Atomic Bases and 7-path Formula for Cluster Algebras of Type D

Emily Gunawan\* and Gregg Musiker - University of Minnesota, Minneapolis, USA

## What is a cluster algebra? (Fomin and Zelevinsky, 2000)

- A (coefficient-free) cluster algebra of rank n is a  $\mathbb{Z}$ -subalgebra of  $\mathbb{Q}(x_1,\ldots,x_n)$  generated by elements called cluster variables:
  - Start with an initial seed: a cluster  $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$  and a skew-symmetrizable exchange matrix  $B = (b_{ij})$ .
  - For each  $k = 1, \ldots, n$ , we can *mutate* in the k-th direction  $(\{x_1, \ldots, x_k, \ldots, x_n\}, B) \xrightarrow{\mu_k} (\{x_1, \ldots, x_k', \ldots, x_n\}, \mu_k(B))$  to obtain a new seed where

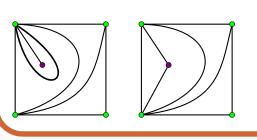
$$x'_{k} = \frac{1}{x_{k}} \left( \prod_{b_{ik} > 0} x_{i}^{b_{ik}} + \prod_{b_{ik} < 0} x_{i}^{-b_{ik}} \right) \text{ and } \mu_{k}(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j \\ b_{ij} + \frac{1}{2} \left( b_{ik} |b_{kj}| + b_{ik} |b_{kj}| \right) & \text{otherwise} \end{cases}$$

- Apply all possible sequences of mutations to produce all cluster variables (usually infinitely many).
- Laurent Phenomenon: each cluster variable can be expressed as a Laurent polynomial in  $\{x_1, \ldots, x_n\}$ .
- **Positivity:** this Laurent polynomial has positive coefficients (2014, [Lee and Schiffler], [Gross, Hacking, Keel, and Kontsevich], and special cases by others).

## Cluster algebras from surfaces

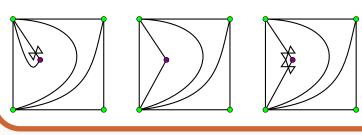
#### Definition: ordinary arcs

- An ordinary arc  $\gamma$  is a non-contractible curve between marked points such that  $\gamma$  does not cross itself or the boundary, and  $\gamma$  is not homotopic to a boundary edge.
- An *ideal triangulation* is a maximum collection of distinct arcs that pairwise do not cross.



#### Definition: tagged arcs

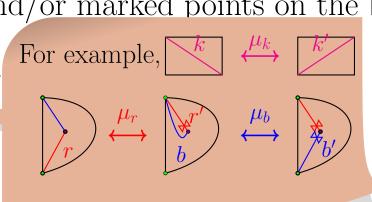
- A tagged arc is an ordinary arc (which does not cut out a monogon with 1 puncture  $\ell \longrightarrow$  ) decorated (plain or with a  $\bowtie$ ) at each endpoint.
- A tagged triangulation is a maximum collection of distinct tagged arcs that are pairwise "compatible".



#### Theorem (Fomin, Shapiro, and Thurston, 2006)

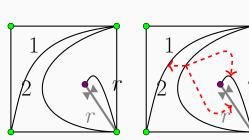
One can define a cluster algebra from a Riemann surface + interior points (called *punctures*) and/or marked points on the boundary:

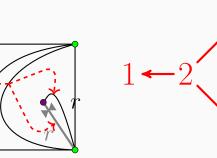
seed  $(\mathbf{x}_T, B_T) \longleftrightarrow$  tagged triangulation  $T = \{\tau_1, ... \text{ For example}, \mathbb{Z} \}$ cluster variable  $x_{\gamma} \longleftrightarrow \text{tagged arc } \gamma$ cluster mutation ←→ "flipping diagonals"

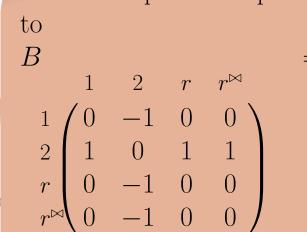


### Example: once-punctured 4-gon

A tagged triangulation of a once-punctured square  $\longleftrightarrow$  A quiver that is mutation equivalent to an orientation of a type  $D_4$  This example corresponds







# Example: once-punctured 3-gon

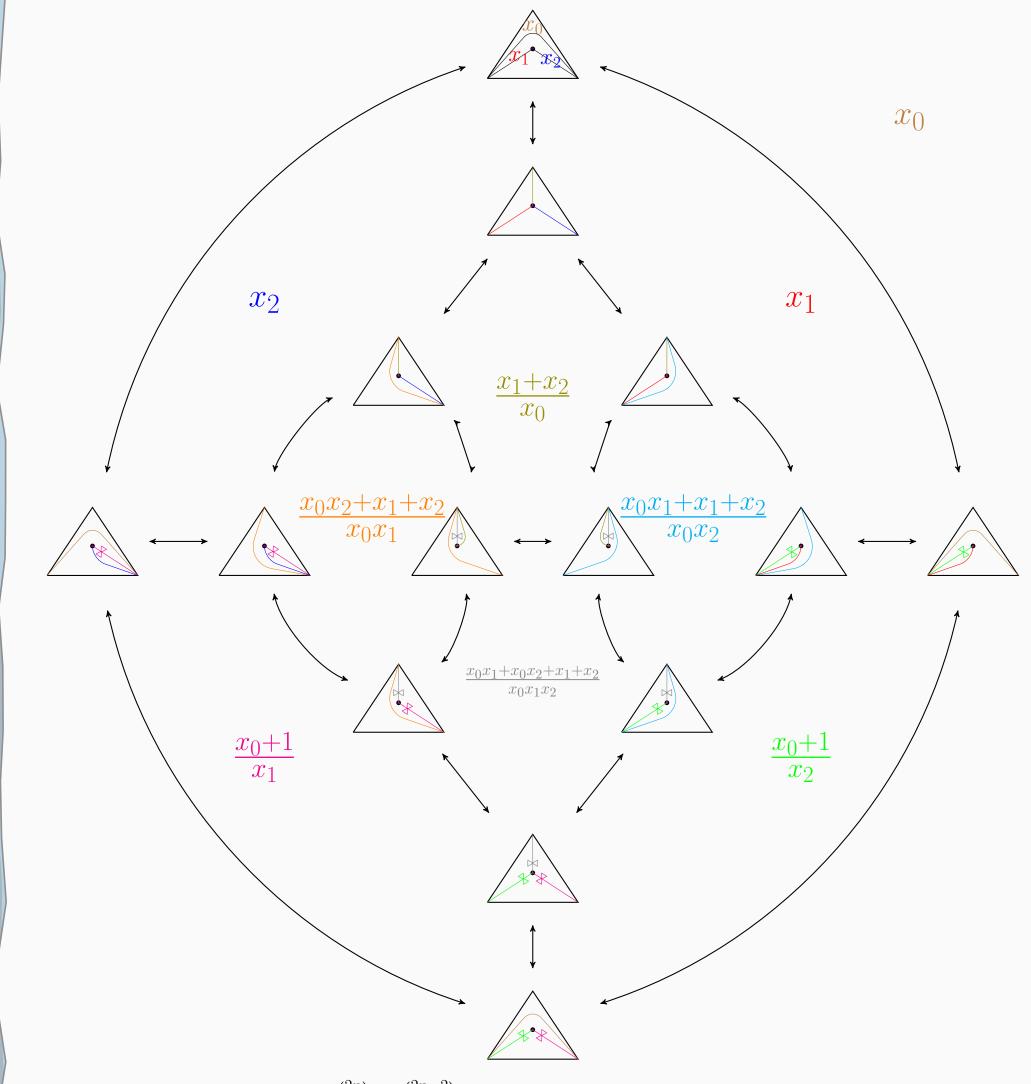


Fig. 1: There are  $\binom{2n}{n} - \binom{2n-2}{n-1}$  tagged triangulations of a once-punctured *n*-gon.

## Result 1: T-path formula for cluster variables (type D)

We extend Schiffler and Thomas' T-path definition and formula for unpunctured surfaces (2009). Let  $T^o$  be an ideal triangulation (of an unpunctured surface or a 1-punctured disk) and  $\gamma$  an arc that crosses  $T^o$ . Let  $\Delta_k$  denote the k-th ideal triangle crossed by  $\gamma$ .

#### Definition: quasi-arc

If  $\tau$  is an ordinary arc, let an associated quasi-arc  $\tau'$  be a curve (not passing through the puncture P) which agrees with  $\tau$  outside of a small radius- $\epsilon$  disk  $D_{\epsilon}$  around P.

#### Definition: T-path

A (complete)  $(T^o, \gamma)$ -path  $w = (w_1, \dots, w_{2d+1})$  is a concatenation of quasi-arcs and boundary edges such that:

- (T1) Each even step  $w_{2k}$  (k = 1, ..., d) is the k-th arc that  $\gamma$  crosses.
- (T2) The path w is homotopic to  $\gamma$ , and satisfies the following: Let  $p_1, \ldots, p_d$  be the intersection points of  $\gamma$  and  $T^o$ . Let  $\gamma_k$  be the segment along  $\gamma$  between  $p_k$  and  $p_{k+1}$ . Then the segment  $\gamma_k$  is homotopic to the segment from  $p_k$  following  $w_{2k}$ , following  $w_{2k+1}$ , following  $w_{2k+2}$  until  $p_{k+1}$ .
- (T3) The step  $w_{2k+1}$  traverses a side of the triangle  $\Delta_k$ , and starts and finishes in the interior of  $\Delta_k$  or at a boundary marked point.

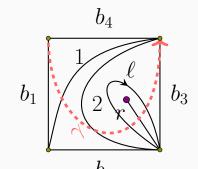
#### Theorem (T-path formula)

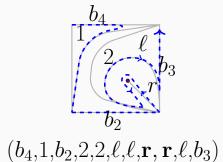
We expect to generalize this to other punctured surfaces.

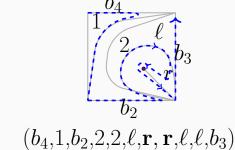
The cluster variable  $x_{\gamma}$  expressed in the variables corresponding to  $T^{o}$  is

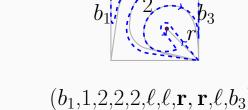
$$x_{\gamma} = \sum_{w} x(w)$$

over all  $(T^o, \gamma)$ -paths  $w = (w_1, \dots, w_{2d+1})$ , where  $x(w) := \left(\prod_{i \text{ odd}} x_{w_i}\right) \left(\prod_{i \text{ even}} x_{w_i}^{-1}\right)$ .









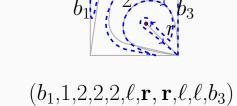
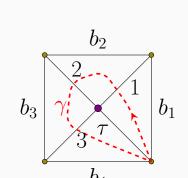
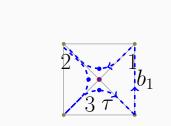
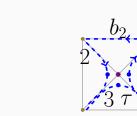
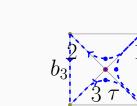


Fig. 2: Four of the nine  $(T^o, \gamma)$ -paths from the first figure. All backtracks (2, 2) and  $(\ell, \ell)$  have been omitted.









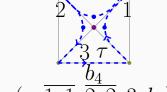
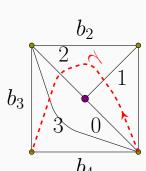


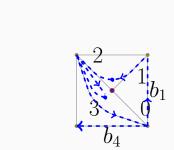
Fig. 3: The four  $(T^o, \gamma)$ -paths of the ideal triangulation  $T^o$  and  $\gamma$  of the first figure.

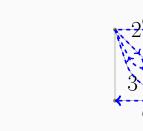
natural bijection with Musiker, Schiffler, and Williams'

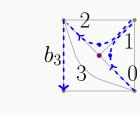
snake graph

matchings.

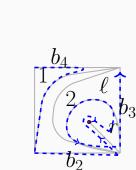




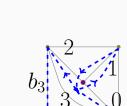




ig. 4: Three of the five  $(T^o, \gamma)$ -paths of the ideal triangulation  $T^o$  and the arc  $\gamma$  of the first figure.







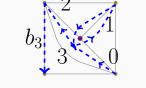


Fig. 5: Examples of  $non-(T^o, \gamma)$ -paths for the situations in Figures 2, 3, and 4. Either (T2) or (T3) is not satisfied.

#### Atomic bases

#### Definition: atomic bases (Sherman and Zelevinsky, 2003)

Let  $\mathcal{A}$  be a (coefficient-free) cluster algebra.

- Let the positive cone of  $\mathcal{A}$  be  $\mathcal{A}^+ := \{\text{positive elements}\} = \{\text{elements that are positive }\}$ Laurent polynomials with respect to every cluster.
- The subset  $\mathcal{B}$  of all indecomposable positive elements (i.e., those that cannot be written as a sum of two positive elements) is called the *atomic basis* if it forms a  $\mathbb{Z}$ -basis of  $\mathcal{A}$ .

The existence of this atomic basis is not known in general. The cluster algebra with the exchange matrix  $\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$  has no atomic basis if  $bc \geq 5$ .

### Result 2: T-path proof for type D cluster algebras

#### Definition: cluster monomial

a cluster monomial is a product of cluster variables all coming from the same cluster, e.g.  $a^5be^2$  is a cluster monomial if  $\{a, b, c, d, e\}$  is a cluster.

#### Theorem (atomic basis)

For a cluster algebra of type A, D, or E, the basis of cluster monomials is atomic.

- Representation theory proof by [Cerulli Irelli, 2011] and [Cerulli Irelli, 2011]

- We give a combinatorial proof (using the T-path formula) and Thomas, 2011].

A cluster monomial corresponds to a multi-tagged dissection (i.e. a partial tagged triangulation allowing multiple copies of tagged arcs).

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#### Atomic bases for other surfaces

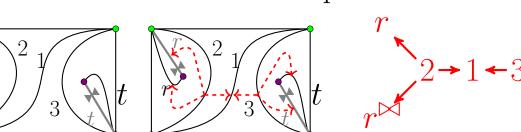
Conjecture (Fomin, Shapiro, and Thurston, 2008, unpublished, based on Fock and Goncharov, 2006)

A candidate for atomic bases: the "bracelets collection" consisting of all cluster monomials + a class of elements. A bracelet is a closed loop in the interior of the surface which wraps around itself once or multiple times ar marked points.

- True for annuli, type  $\widetilde{A}$  (Dupont and Thomas, 201

Further directions

Type  $D_{n-1}$  cluster algebras ((n-3)-gons with 2 punctures), e.g. type  $D_6$  cluster algebra comes from a twice-punctured disk with 4 marked points on the boundary.



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- Email: egunawan@umn.edu

- Home page: umn.edu/home/egunawan

- The bracelets collection forms a basis for unpunctured surfaces (Musiker, Schiffler, and Williams, 2011).