Atomic Bases and τ-path Formula for Cluster Algebras of Type D

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What is a cluster algebra? (Fomin and Zelevinsky, 2000)

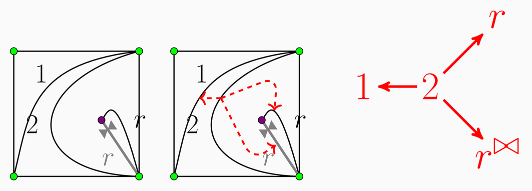
- A (coefficient-free) *cluster algebra* of rank n is a $\mathbb{Z}$-subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by elements called *cluster variables*:
  - Start with an *initial seed*: a *cluster* $\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$ and a skew-symmetrizable *exchange matrix* $B = (b_{ij})$.
  - For each $k = 1, \ldots, n$, we can *mutate* in the k-th direction $(\{x_1, \ldots, x_k, \ldots, x_n\}, B) \xrightarrow{\mu_k} (\{x_1, \ldots, x_k', \ldots, x_n\}, \mu_k(B))$ to obtain a new seed where

$$x_k' = \frac{1}{x_k}\left(\prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}\right) \quad \text{and} \quad \mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j \\ b_{ij} + \frac{1}{2}\left(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|\right) & \text{otherwise} \end{cases}$$

  - Apply all possible sequences of mutations to produce all cluster variables (usually infinitely many).
- **Laurent Phenomenon:** each cluster variable can be expressed as a Laurent polynomial in $\{x_1, \ldots, x_n\}$.
- **Positivity:** this Laurent polynomial has positive coefficients (2014, [Lee and Schiffler], [Gross, Hacking, Keel, and Kontsevich], and special cases by others).

Example: once-punctured 4-gon

A tagged triangulation of a once-punctured square $\longleftrightarrow$ A quiver that is mutation equivalent to an orientation of a type D_4 Dynkin diagram.



This example corresponds to

$$B = \begin{array}{c|cccc} & 1 & 2 & r & r^{\bowtie} \\ \hline 1 & 0 & -1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ r & 0 & -1 & 0 & 0 \\ r^{\bowtie} & 0 & -1 & 0 & 0 \end{array}$$

Example: once-punctured 3-gon

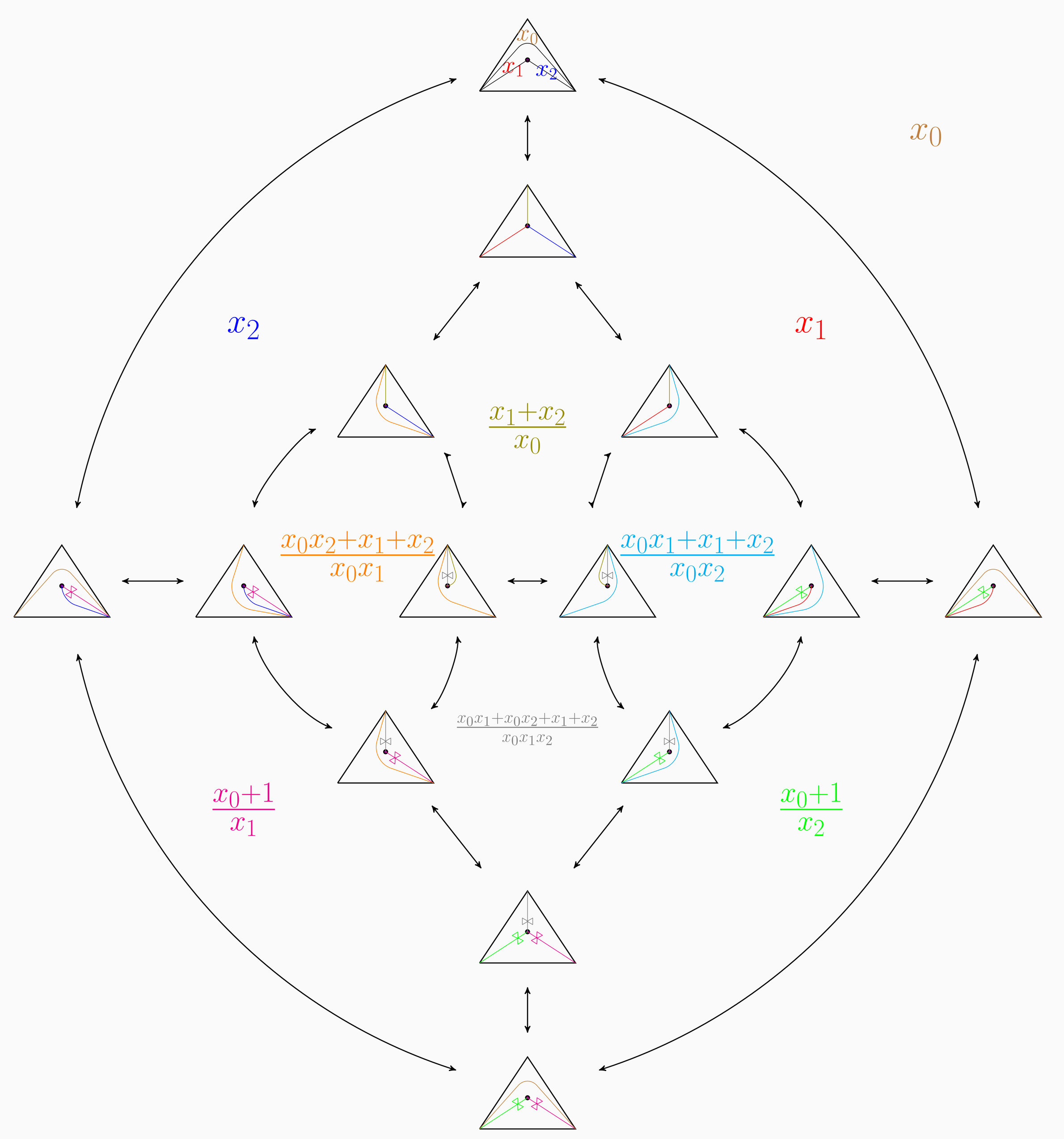
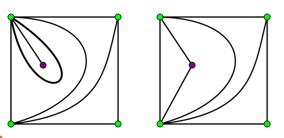


Fig. 1: There are $\binom{2n}{n} - \binom{2n-2}{n-1}$ tagged triangulations of a once-punctured n-gon.

Cluster algebras from surfaces

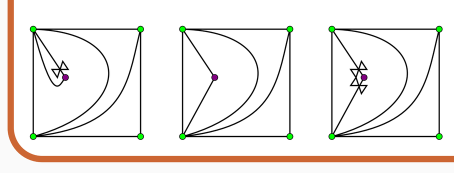
Definition: ordinary arcs

- An *ordinary arc* γ is a non-contractible curve between marked points such that γ does not cross itself or the boundary, and γ is not homotopic to a boundary edge.
- An *ideal triangulation* is a maximum collection of distinct arcs that pairwise do not cross.



Definition: tagged arcs

- A *tagged arc* is an ordinary arc (which does not cut out a monogon with 1 puncture ℓ) decorated (plain or with a $\bowtie$) at each endpoint.
- A *tagged triangulation* is a maximum collection of distinct tagged arcs that are pairwise "compatible".

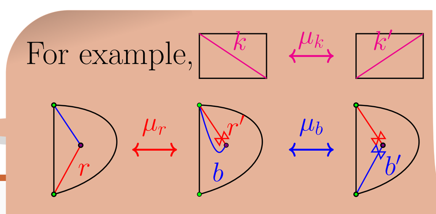


Theorem (Fomin, Shapiro, and Thurston, 2006)

One can define a cluster algebra from a Riemann surface + interior points (called *punctures*) and/or marked points on the boundary:

seed $(\mathbf{x}_T, B_T) \longleftrightarrow$ tagged triangulation $T = \{\tau_1, \ldots, \tau_n\}$
cluster variable $x_\gamma \longleftrightarrow$ tagged arc γ
cluster mutation $\longleftrightarrow$ "flipping diagonals"

For example,



Result 1: τ-path formula for cluster variables (type D)

We extend Schiffler and Thomas' T-path definition and formula for unpunctured surfaces (2009). Let T^o be an ideal triangulation (of an unpunctured surface or a once-punctured disk) and γ an ordinary arc that crosses T^o. Let $\triangle_k$ denote the k-th ideal triangle crossed by γ.

Definition: quasi-arc

If τ is an ordinary arc, let an associated *quasi-arc* τ' be a curve (not passing through the puncture P) which agrees with τ outside of a small radius-ϵ disk D_ϵ around P.

Definition: T-path

A *(complete)* (T^o, γ)*-path* $w = (w_1, \ldots, w_{2d+1})$ is a concatenation of quasi-arcs and boundary edges such that:

(T1) Each even step w_{2k} $(k = 1, \ldots, d)$ is the k-th arc that γ crosses.

(T2) The path w is homotopic to γ, and satisfies the following:
Let $p_1, \ldots, p_d$ be the intersection points of γ and T^o. Let γ_k be the segment along γ between p_k and p_{k+1}. Then the segment γ_k is homotopic to the segment from p_k following w_{2k}, following w_{2k+1}, following w_{2k+2} until p_{k+1}.

(T3) The step w_{2k+1} traverses a side of the triangle $\triangle_k$, and starts and finishes in the interior of $\triangle_k$ or at a boundary marked point.

new

Theorem (T-path formula)

The cluster variable x_γ expressed in the variables corresponding to T^o is

$$x_\gamma = \sum_w x(w)$$

over all (T^o, γ)-paths $w = (w_1, \ldots, w_{2d+1})$, where $x(w) := \left(\prod_{i \text{ odd}} x_{w_i}\right)\left(\prod_{i \text{ even}} x_{w_i}^{-1}\right)$.

We expect to generalize this to other punctured surfaces.

The T-paths are in natural bijection with Musiker, Schiffler, and Williams' snake graph matchings.

Also, a *(complete)* T-path is uniquely determined by its sequence of labels.

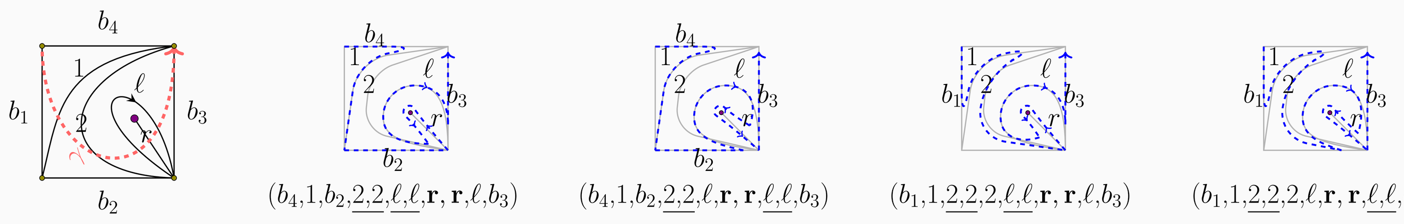


Fig. 2: Four of the nine (T^o, γ)-paths from the first figure. All *backtracks* $(\underline{2,2})$ and $(\underline{\ell,\ell})$ have been omitted.

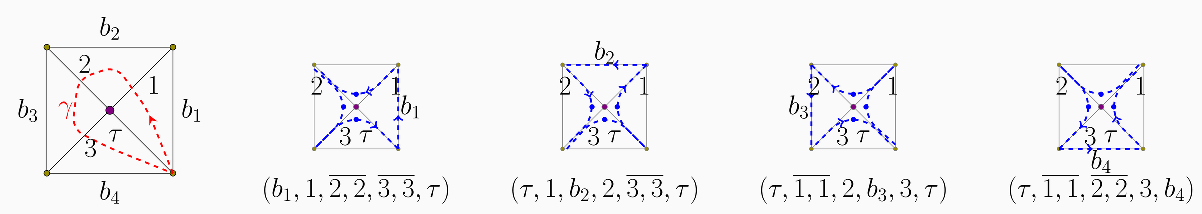


Fig. 3: The four (T^o, γ)-paths of the ideal triangulation T^o and the arc γ of the first figure.

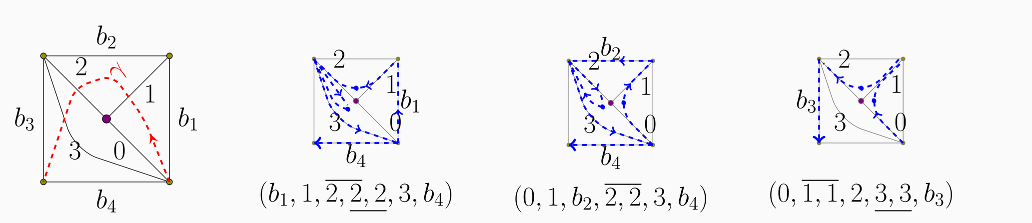


Fig. 4: Three of the five (T^o, γ)-paths of the ideal triangulation T^o and the arc γ of the first figure.

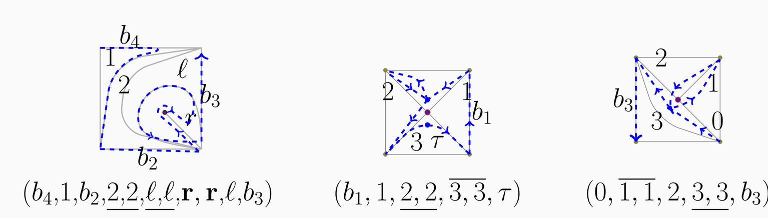


Fig. 5: Examples of *non*-(T^o, γ)-paths for the situations in Figures 2, 3, and 4. Either (T2) or (T3) is not satisfied.

Atomic bases

Definition: atomic bases (Sherman and Zelevinsky, 2003)

Let $\mathcal{A}$ be a (coefficient-free) cluster algebra.

- Let the *positive cone* of $\mathcal{A}$ be $\mathcal{A}^+ := \{$positive elements$\} = \{$elements that are positive Laurent polynomials with respect to every cluster$\}$.
- The subset $\mathcal{B}$ of all indecomposable positive elements (*i.e.*, those that cannot be written as a sum of two positive elements) is called the *atomic basis* if it forms a $\mathbb{Z}$-basis of $\mathcal{A}$.

The existence of this atomic basis is not known in general. The cluster algebra with the exchange matrix $\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$ has no atomic basis if $bc \geq 5$.

Result 2: τ-path proof for type D cluster algebras

Definition: cluster monomial

a *cluster monomial* is a product of cluster variables all coming from the same cluster, e.g. a^5be^2 is a cluster monomial if $\{a, b, c, d, e\}$ is a cluster.

Theorem (atomic basis)

For a cluster algebra of type A, D, or E, the basis of cluster monomials is atomic.

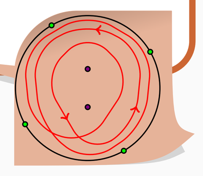
A cluster monomial corresponds to a multi-tagged dissection (*i.e.* a partial tagged triangulation allowing multiple copies of tagged arcs).

- Representation theory proof by [Cerulli Irelli, 2011] and [Cerulli Irelli, Keller, Labardini-Fragoso, and Plamondon, 2012].
- We give a combinatorial proof (using the T-path formula) for type D, inspired by work on types A and $\widetilde{A}$ by [Dupont and Thomas, 2011].

Atomic bases for other surfaces

Conjecture (Fomin, Shapiro, and Thurston, 2008, unpublished, based on Fock and Goncharov, 2006)

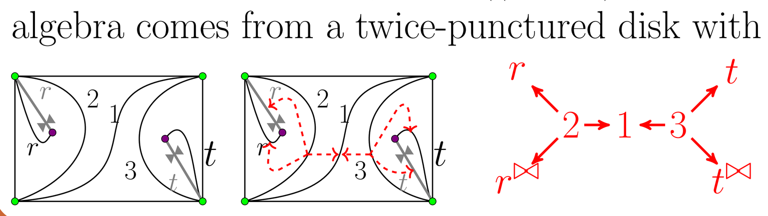
A candidate for atomic bases: the *"bracelets collection"* consisting of all cluster monomials + a class of elements. A *bracelet* is a closed loop in the interior of the surface which wraps around itself once or multiple times and avoids marked points.



- True for annuli, type $\widetilde{A}$ (Dupont and Thomas, 2011).
- The bracelets collection forms a basis for unpunctured surfaces (Musiker, Schiffler, and Williams, 2011).

Further directions

Type $\widetilde{D}_{n-1}$ cluster algebras ($(n-3)$-gons with 2 punctures), *e.g.* type $\widetilde{D}_6$ cluster algebra comes from a twice-punctured disk with 4 marked points on the boundary.



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