

ATOMIC BASES AND T -PATH FORMULA FOR CLUSTER ALGEBRAS OF TYPE D

Emily Gunawan* and Gregg Musiker

University of Minnesota, School of Mathematics, Minneapolis, USA

What is a cluster algebra? (Fomin and Zelevinsky, 2000)

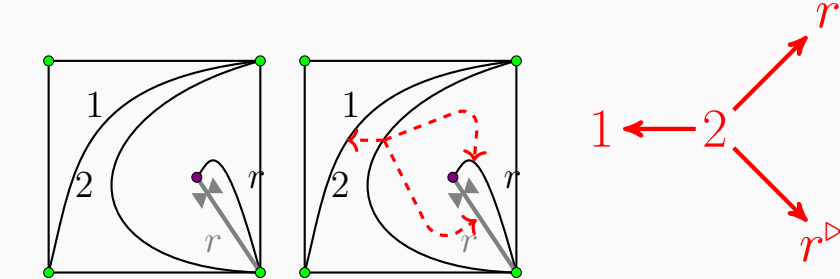
- A (coefficient-free) *cluster algebra* of rank n is a \mathbb{Z} -subalgebra of $\mathbb{Q}(x_1, \dots, x_n)$ generated by elements called *cluster variables*:
 - Start with an *initial seed*: a *cluster* $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ and a skew-symmetrizable *exchange matrix* $B = (b_{ij})$.
 - For each $k = 1, \dots, n$, we can *mutate* in the k -th direction ($\{x_1, \dots, x_k, \dots, x_n\}, B$) $\xrightarrow{\mu_k} (\{x_1, \dots, x'_k, \dots, x_n\}, \mu_k(B))$ to obtain a new seed where

$$x'_k = \frac{1}{x_k} \left(\prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \right) \quad \text{and} \quad \mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

- Apply all possible sequences of mutations to produce all cluster variables (usually infinitely many).
- Laurent Phenomenon:** each cluster variable can be expressed as a Laurent polynomial in $\{x_1, \dots, x_n\}$.
- Positivity:** this Laurent polynomial has positive coefficients (2014, [Lee and Schiffler], [Gross, Hacking, Keel, and Kontsevich], and special cases by others).

Example: once-punctured 4-gon

A tagged triangulation of a once-punctured square \longleftrightarrow A quiver that is mutation equivalent to an orientation of a type D_4 Dynkin diagram.



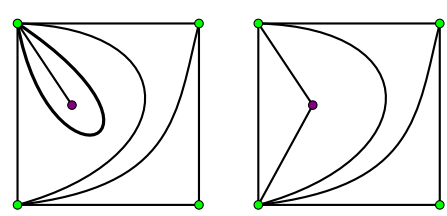
This example corresponds to

$$B = \begin{pmatrix} 1 & 2 & r & r^\infty \\ 0 & -1 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ r & 0 & -1 & 0 \\ r^\infty & 0 & -1 & 0 \end{pmatrix}$$

Cluster algebras from surfaces

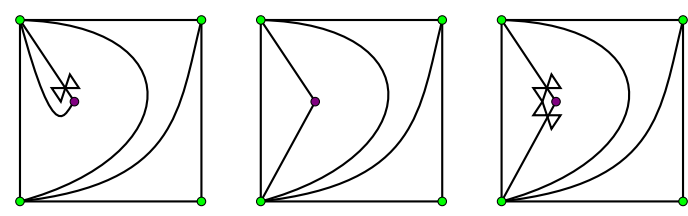
Definition: ordinary arcs

- An *ordinary arc* γ is a non-contractible curve between marked points such that γ does not cross itself or the boundary, and γ is not homotopic to a boundary edge.
- An *ideal triangulation* is a maximum collection of distinct arcs that pairwise do not cross.



Definition: tagged arcs

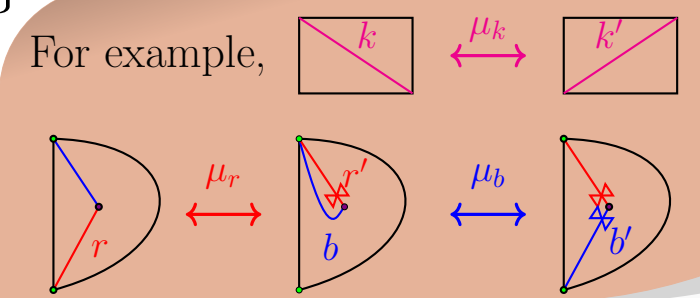
- A *tagged arc* is an ordinary arc (which does not cut out a monogon with 1 puncture $\ell \circlearrowright$) decorated (plain or with a ∞) at each endpoint.
- A *tagged triangulation* is a maximum collection of distinct tagged arcs that are pairwise “compatible”.



Theorem (Fomin, Shapiro, and Thurston, 2006)

One can define a cluster algebra from a Riemann surface + interior points (called *punctures*) and/or marked points on the boundary:

seed $(\mathbf{x}_T, B_T) \longleftrightarrow$ tagged triangulation $T = \{\tau_1, \dots, \tau_n\}$
 cluster variable $x_\gamma \longleftrightarrow$ tagged arc γ
 cluster mutation \longleftrightarrow “flipping diagonals”



Example: once-punctured 3-gon

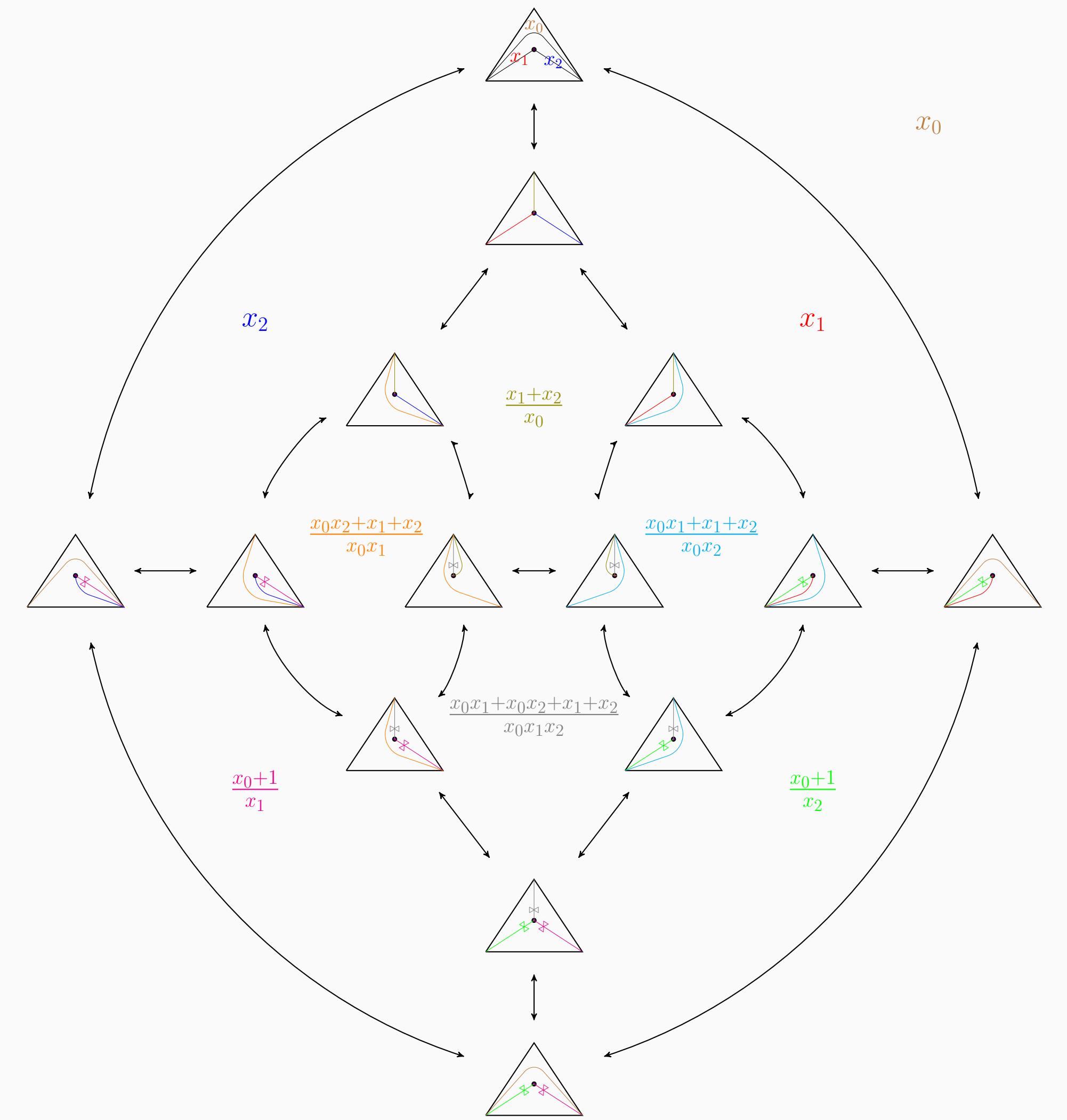


Fig. 1: There are $\binom{2n}{n} - \binom{2n-2}{n-1}$ tagged triangulations of a once-punctured n -gon.

Result 1: T -path formula for cluster variables for type D cluster algebras

We extend Schiffler - Thomas' T -path definition and formula for unpunctured surfaces (2009). Let T° be an ideal triangulation (of an unpunctured surface or a once-punctured disk) and let γ be an arc that crosses T° . Let Δ_k denote the k -th ideal triangle crossed by γ .

Definition: quasi-arc

If τ is an ordinary arc, let an associated *quasi-arc* τ' be a curve (not passing through the puncture p) which agrees with τ outside of a small radius- ϵ disk D_ϵ around p . If τ is not adjacent to the puncture, let the associated quasi-arc be τ itself.

Definition: T -path

A (*complete*) (T°, γ) -path $w = (w_1, \dots, w_{2d+1})$ is a concatenation of quasi-arcs and boundary edges such that:

- (T1) Each even step w_{2k} ($k = 1, \dots, d$) is the k -th arc that γ crosses.
- (T2) The path w is homotopic to γ , and satisfies the following:
 Let p_1, \dots, p_d be the intersection points of γ and T° . Let γ_k be the segment along γ between p_k and p_{k+1} . Then the segment γ_k is homotopic to the segment from p_k following w_{2k} , following w_{2k+2} until p_{k+1} .
- (T3) The step w_{2k+1} traverses a side of the triangle Δ_k , and starts and finishes in the interior of Δ_k or at a boundary marked point.

Theorem (T -path formula)

The cluster variable x_γ expressed in the variables corresponding to T° is

$$x_\gamma = \sum_w x(w)$$

over all (T°, γ) -paths $w = (w_1, \dots, w_{2d+1})$, where

$$x(w) := \left(\prod_{i \text{ odd}} x_{w_i} \right) \left(\prod_{i \text{ even}} x_{w_i}^{-1} \right).$$

We expect to generalize this to other punctured surfaces.

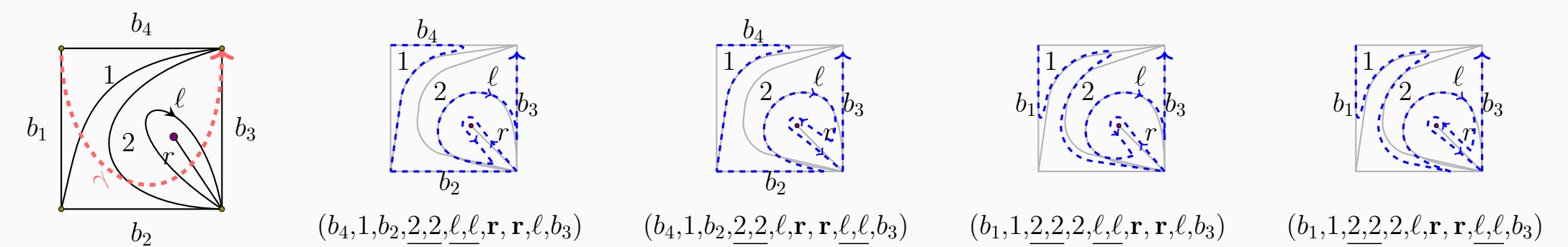


Fig. 2: Four of the nine (T°, γ) -paths from the first figure. All *backtracks* $(\underline{2}, \underline{2})$ and $(\underline{\ell}, \underline{\ell})$ have been omitted.

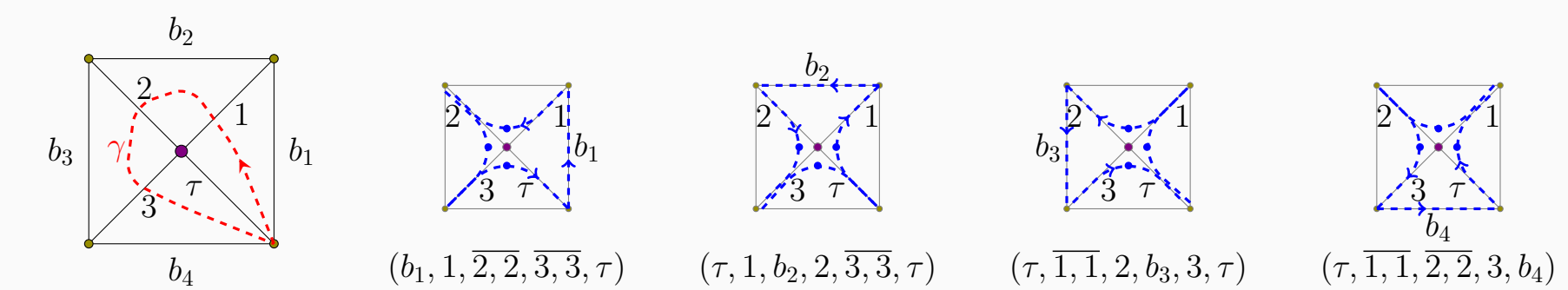


Fig. 3: The four (T°, γ) -paths of the ideal triangulation T° and the ℓ -loop γ of the first figure.

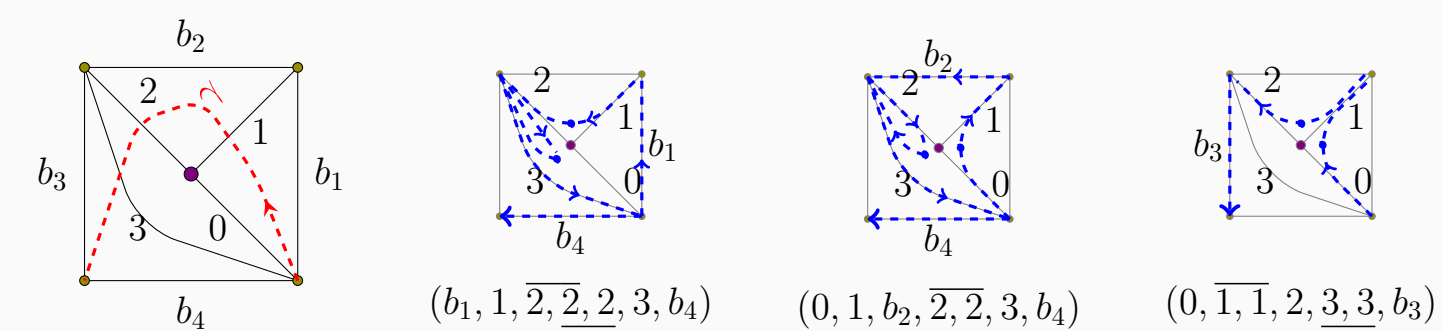


Fig. 4: Three of the five (T°, γ) -paths of the ideal triangulation T° and the arc γ of the first figure.

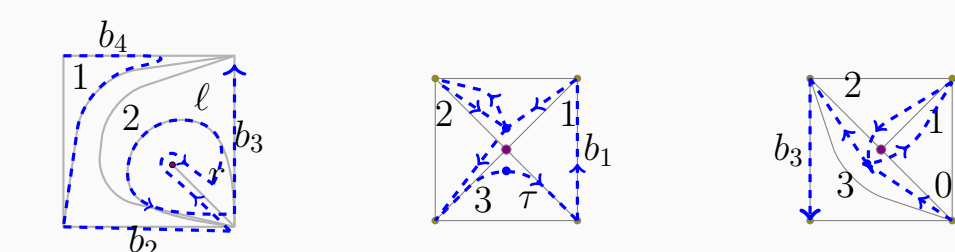


Fig. 5: Examples of *non*-(T°, γ)-paths for the situations in Figures 2, 3, and 4. Each path is homotopic to γ and satisfies (T1) but fails (T2) or (T3).

Atomic bases

Definition: atomic bases (Sherman and Zelevinsky, 2003)

Let \mathcal{A} be a (coefficient-free) cluster algebra.

- Let the *positive cone* of \mathcal{A} be $\mathcal{A}^+ := \{\text{positive elements}\} = \{\text{elements that are positive Laurent polynomials with respect to every cluster}\}$.
- The subset \mathcal{B} of all indecomposable positive elements (*i.e.*, those that cannot be written as a sum of two positive elements) is called the *atomic basis* if it forms a \mathbb{Z} -basis of \mathcal{A} .

The existence of this atomic basis is not known in general. The cluster algebra with the exchange matrix $\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix}$ has no atomic basis if $bc \geq 5$.

Result 2: T -path proof for type D cluster algebras

Definition: cluster monomial

a *cluster monomial* is a product of cluster variables all coming from the same cluster, e.g. $a^5 b e^2$ is a cluster monomial if $\{a, b, c, d, e\}$ is a cluster.

Theorem (atomic basis)

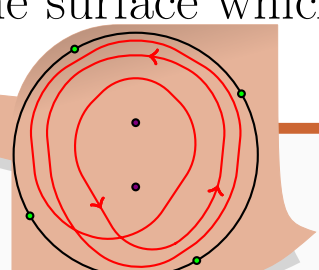
For a cluster algebra of type A , D , or E , the basis of cluster monomials is atomic.

- Representation theory proof by [Cerulli Irelli, 2011] and [Cerulli Irelli, Keller, Leardini, Posner, and Plamondon, 2019].
- We give a combinatorial proof (using the T -path formula) for type D . A cluster monomial corresponds to a multi-tagged dissection (*i.e.*, a partial tagged triangulation allowing multiple copies of tagged arcs). [Gunawan, 2011].

Atomic bases for other surfaces

Conjecture (Fomin, Shapiro, and Thurston, 2008, unpublished, based on Fock and Goncharov, 2006)

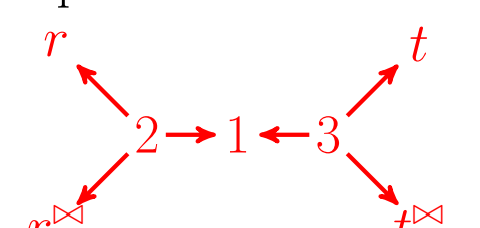
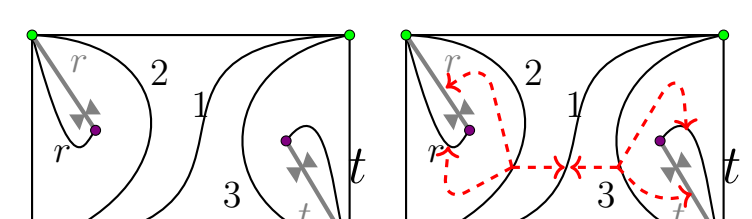
A candidate for atomic bases: the “*bracelets collection*” consisting of all cluster monomials + a class of elements. A *bracelet* is a closed loop in the interior of the surface which wraps around itself once or multiple times and avoids marked points.



- True for annuli, type \tilde{A} (Dupont and Thomas, 2011).
- The bracelets collection forms a basis for unpunctured surfaces (Musiker, Schiffler, and Williams, 2011).

Further directions

Type \tilde{D}_{n-1} cluster algebras ($(n-3)$ -gons with 2 punctures), *e.g.* type \tilde{D}_6 cluster algebra comes from a twice-punctured disk with 4 marked points on the boundary.



- The authors were supported by NSF Grants DMS-1067183 and DMS-1148634, and would like to thank P. Pylyavskyy, V. Reiner, H. Thomas, and P. Webb.
- Preprint arXiv:1409.3610 (2014) to appear in SIGMA.
- Email: egunawan@umn.edu
- Home page: umn.edu/home/egunawan