Atomic Bases and τ-path Formula for Cluster Algebras of Type D

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What is a cluster algebra? (Fomin and Zelevinsky, 2000)

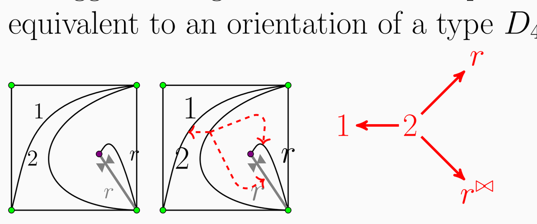
- A (coefficient-free) *cluster algebra* of rank n is a $\mathbb{Z}$-subalgebra of $\mathbb{Q}(x_1,\dots,x_n)$ generated by elements called *cluster variables*:
  - Start with an *initial seed*: a *cluster* $\mathbf{x}=\{x_1,x_2,\dots,x_n\}$ and a skew-symmetrizable matrix $B=(b_{ij})$.
  - For each $k=1,\dots,n$, we can *mutate* in the k-th direction $(\{x_1,\dots,x_k,\dots,x_n\},B)\xrightarrow{\mu_k}(\{x_1,\dots,x_k',\dots,x_n\},\mu_k(B))$ to obtain a new seed where

$$x_k'=\frac{1}{x_k}\left(\prod_{b_{ik}>0}x_i^{b_{ik}}+\prod_{b_{ik}<0}x_i^{-b_{ik}}\right)\quad\text{and}\quad \mu_k(B)_{ij}=\begin{cases}-b_{ij} & \text{if } k=i \text{ or } k=j\\ b_{ij}+b_{ik}b_{kj} & \text{if } b_{ik}>0 \text{ and } b_{kj}>0\\ b_{ij}-b_{ik}b_{kj} & \text{if } b_{ik}<0 \text{ and } b_{kj}<0\\ b_{ij} & \text{otherwise}\end{cases}$$

  - Apply all possible sequences of mutations to produce all cluster variables (usually inifinite).
- **Laurent Phenomenon:** each cluster variable can be expressed as a Laurent polynomial in $\{x_1,\dots,x_n\}$.
- **Positivity:** this Laurent polynomial has positive coefficients (2014, [Lee and Schiffler], [Gross, Hacking, Keel, and Kontsevich], and others).

Example: once-punctured 4-gon

A tagged triangulation of a once-punctured square $\longleftrightarrow$ A quiver that is mutation equivalent to an orientation of a type D_4 Dynkin diagram.



This example corresponds to

$$B=\begin{array}{c|cccc} & 1 & 2 & r & r^{\bowtie}\\ \hline 1 & 0 & -1 & 0 & 0\\ 2 & 1 & 0 & 1 & 1\\ r & 0 & -1 & 0 & 0\\ r^{\bowtie} & 0 & -1 & 0 & 0\end{array}$$

Example: once-punctured 3-gon

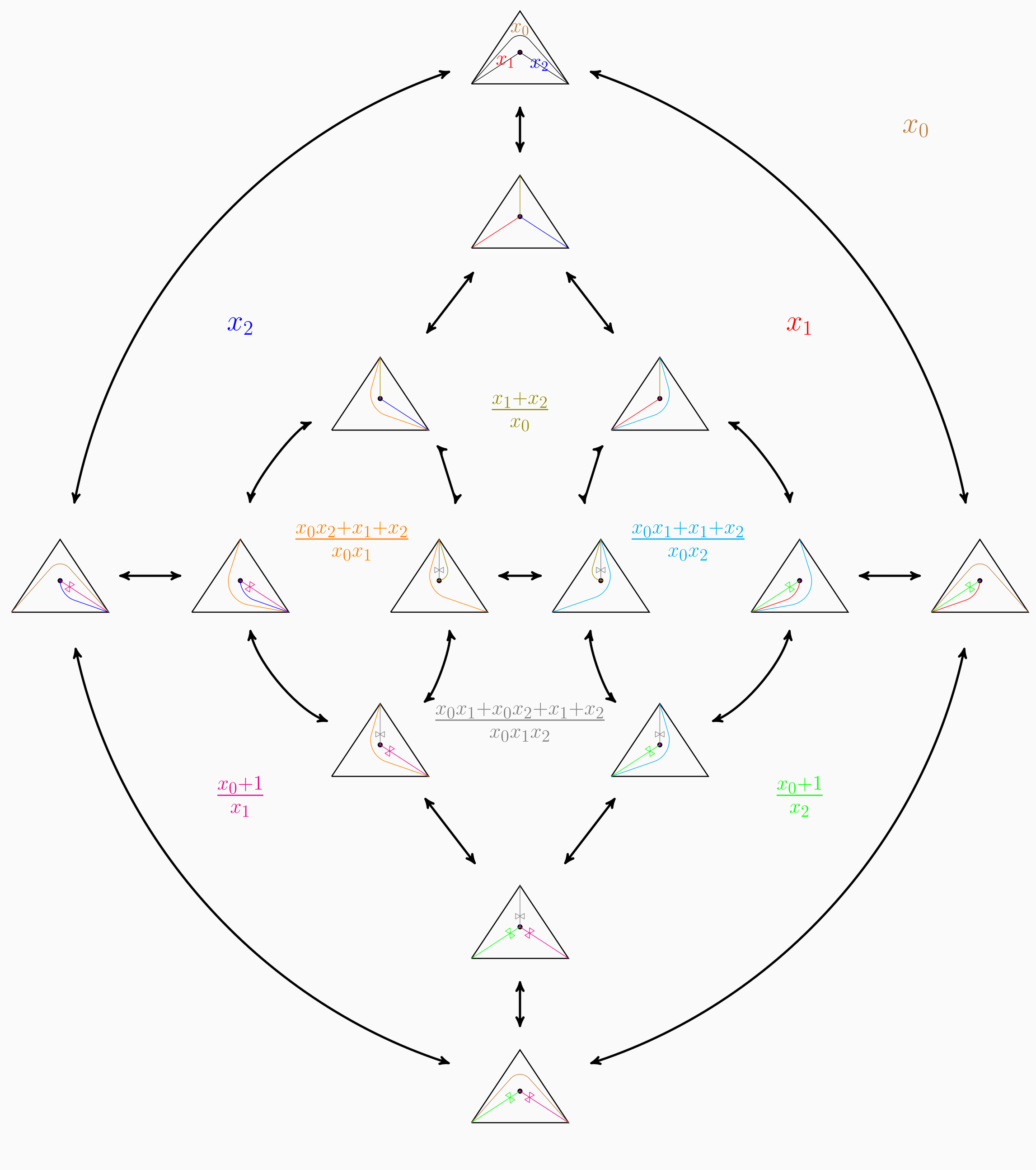
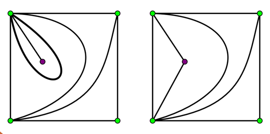


Fig. 2: There are $\binom{2n}{n}-\binom{2n-2}{n-1}$ tagged triangulations of a once-punctured n-gon.

Cluster Algebras from Surfaces

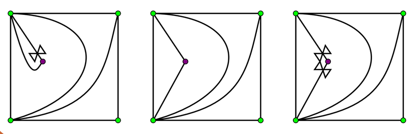
Definition: ordinary arcs

- An *ordinary arc* γ is a non-contractible curve between marked points such that γ does not cross itself or the boundary, and γ is not homotopic to a boundary edge.
- A loop ℓ that cuts out a monogon with 1 puncture is called a *noose*.
- An *ideal triangulation* is a maximum collection of distinct arcs that pairwise do not cross.



Definition: tagged arcs

- A *tagged arc* is an ordinary arc (not a noose) decorated (plain or with a $\bowtie$) at each endpoint.
- A *tagged triangulation* is a maximum collection of distinct tagged arcs that are pairwise "compatible".

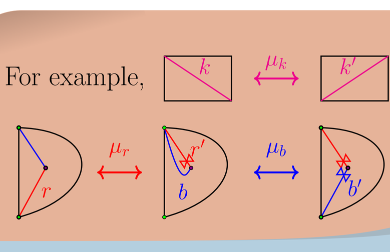


Theorem (Fomin, Shapiro, and Thurston, 2006)

One can define a cluster algebra from a Riemann surface + interior points (called *punctures*) and/or marked points on the boundary:

seed $(\mathbf{x}_T,B_T)\longleftrightarrow$ tagged triangulation $T=\{\tau_1,\dots,\tau_n\}$
cluster variable $x_\gamma\longleftrightarrow$ tagged arc γ
cluster mutation $\longleftrightarrow$ "flipping diagonals"

For example,



Result 1: T-path formula for cluster variables for type D cluster algebras

We extend Schiffler - Thomas' T-path definition and formula for unpunctured surfaces (2009). Let T^o be an ideal triangulation (of an unpunctured surface or a once-punctured disk) and an arc γ that crosses T^o.

Definition: quasi-arc

If τ is an ordinary arc, let an associated *quasi-arc* τ' be a curve (not passing through the puncture P) which agrees with τ outside of a small radius-ϵ disk D_ϵ around P. If τ is not adjacent to the puncture, let the associated quasi-arc be τ itself.

Definition: T-path

A *complete* (T^o,γ)*-path* (or T^o*-path* for short) $w=(w_1,\dots,w_{2d+1})$ is a concatenation of quasi-arcs and boundary edges such that:

(T1) Each even step w_{2k} $(k=1,\dots,d)$ is the k-th arc that γ crosses.

(T2) The path w is homotopic to γ, and satisfies the following:
Let $p_1,\dots,p_d$ be the intersection points of γ and T^o. Let γ_k be the segment along γ between p_k and p_{k+1}. Then the segment γ_k is homotopic to the segment from p_k following w_{2k+1}, following w_{2k+2} until p_{k+1}.

(T3) The step w_{2k+1} starts and finishes in the interior of $\triangle_k$ or at a boundary marked point.

new

Theorem (T-path formula)

The cluster variable x_γ expressed in the variables corresponding to T^o is

$$x_\gamma=\sum_w x(w)$$

over all (T^o,γ)-paths $w=(w_1,\dots,w_{2d+1})$, where

$$x(w):=\left(\prod_{i\text{ odd}}x_{w_i}\right)\left(\prod_{i\text{ even}}x_{w_i}^{-1}\right).$$

We expect to generalize this to other punctured surfaces.

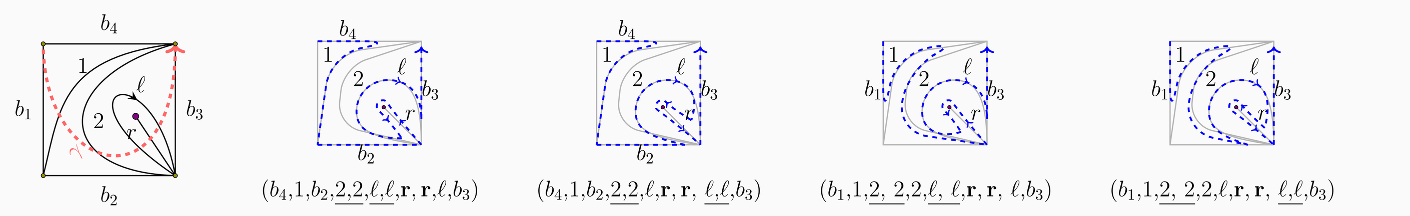


Fig. 3: Four of nine (T^o,γ)-paths. All backtracks $(\underline{2,2})$ and $(\underline{\ell,\ell})$ have been omitted.

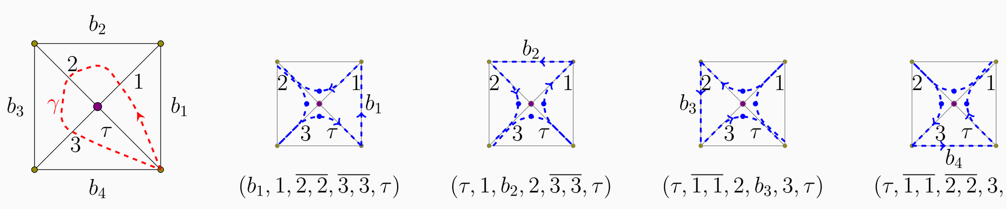


Fig. 4: The four (T^o,γ)-paths of the ideal triangulation T^o and the ℓ-loop γ of the first figure.

The T-paths are in natural bijection with Musiker, Schiffler, and Williams' snake graph matchings.

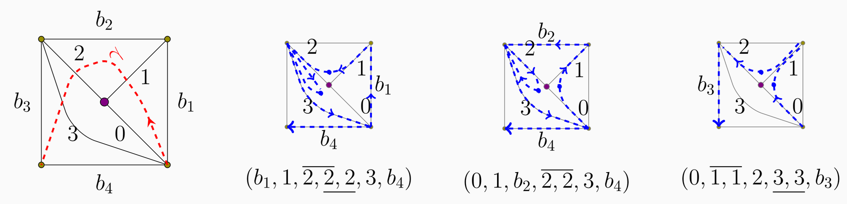


Fig. 5: Three of the five (T^o,γ)-paths of the ideal triangulation T^o and the arc γ of the first figure

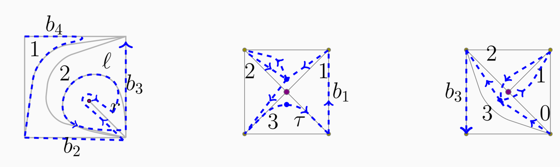


Fig. 6: Examples of *non*-(T^o,γ)-paths for the situations in Figures 3, 4, and 5. Each path is homotopic to γ and satisfies (T1) but fails (T2) or (T3).

Definition: Atomic Bases and Cluster Monomials

Let $\mathcal{A}$ be a (coefficient-free) cluster algebra.

- The *positive cone* of $\mathcal{A}$ is {positive elements} = {elements that are positive Laurent polynomials with respect to every cluster}.
- The subset $\mathcal{B}$ of all indecomposable positive elements (*i.e.*, those that cannot be written as a sum of two positive elements) is called the *atomic basis* if it forms a $\mathbb{Z}$-basis of $\mathcal{A}$.
- A *cluster monomial* is a product of cluster variables all coming from the same cluster, e.g. a^5be^2 is a cluster monomial if $\{a,b,c,d,e\}$ is a cluster.

A cluster monomial corresponds to a multi-tagged dissection D (*i.e.* a partial tagged triangulation allowing multiple copies of tagged arcs).

Result 2: T-path proof for type D cluster algebras

Theorem (atomic basis)

For a cluster algebra of type A, D, or E, the basis of cluster monomials is atomic.

- Representation theory proof by [Cerulli Irelli, 2011] and [Cerulli Irelli, Keller, Labardini-Fragoso, and Plamondon, 2012]
- We give a combinatorial proof (using the T-path formula) for type D, inspired by work on types A and $\widetilde{A}$ by [Dupont and Thomas, 2011].

Atomic bases for other surfaces (with 2 or more marked points)

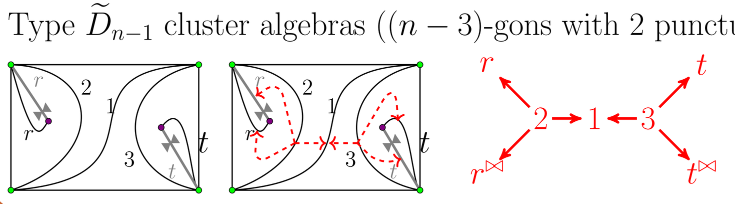
Conjecture (Fomin, Shapiro, and Thurston, 2006)

A candidate for atomic bases: the *"bracelets collection"* consisting of all cluster monomials + a class of elements. A *bracelet* is a closed loop in the interior of the surface which avoids marked points.

- True for annuli, type $\widetilde{A}$ (Dupont and Thomas, 2011).
- The bracelets collection forms a basis for unpunctured surfaces (Musiker, Schiffler, and Williams, 2011).

Current work

Type $\widetilde{D}_{n-1}$ cluster algebras ($(n-3)$-gons with 2 punctures), *e.g.* type $\widetilde{D}_6$ cluster algebra comes from a square with 2 punctures.



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