

# ATOMIC BASES AND $T$ -PATH FORMULA FOR CLUSTER ALGEBRAS OF TYPE $D$

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## What is a cluster algebra? (Fomin and Zelevinsky, 2000)

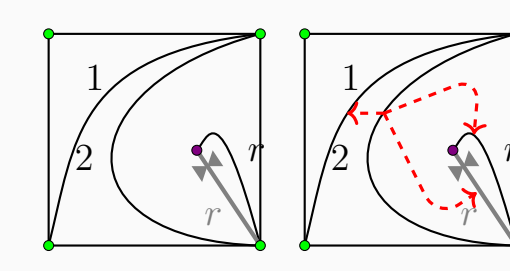
- A (coefficient-free) *cluster algebra* of rank  $n$  is a  $\mathbb{Z}$ -subalgebra of  $\mathbb{Q}(x_1, \dots, x_n)$  generated by elements called *cluster variables*:
  - Start with an *initial seed*: a *cluster*  $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$  and a skew-symmetrizable matrix  $B = (b_{ij})$ .
  - For each  $k = 1, \dots, n$ , we can *mutate* in the  $k$ -th direction  $(\{x_1, \dots, x_k, \dots, x_n\}, B) \xrightarrow{\mu_k} (\{x_1, \dots, x'_k, \dots, x_n\}, \mu_k(B))$  to obtain a new seed where

$$x'_k = \frac{1}{x_k} \left( \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}} \right) \quad \text{and} \quad \mu_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0 \\ b_{ij} & \text{otherwise} \end{cases}$$

- Apply all possible sequences of mutations to produce all cluster variables (usually infinitely many).
- Laurent Phenomenon**: each cluster variable can be expressed as a Laurent polynomial in  $\{x_1, \dots, x_n\}$ .
- Positivity**: this Laurent polynomial has positive coefficients (2014, [Lee and Schiffler], [Gross, Hacking, Keel, and Kontsevich], and special cases by others).

## Example: once-punctured 4-gon

A tagged triangulation of a once-punctured square  $\longleftrightarrow$  A quiver that is mutation equivalent to an orientation of a type  $D_4$  Dynkin diagram.



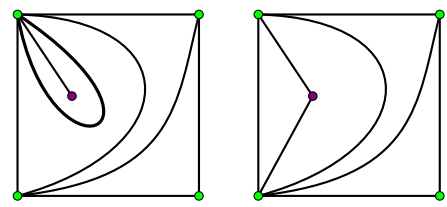
This example corresponds to

$$B = \begin{pmatrix} 1 & 2 & r & r^\infty \\ 0 & -1 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ r & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

## Cluster Algebras from Surfaces

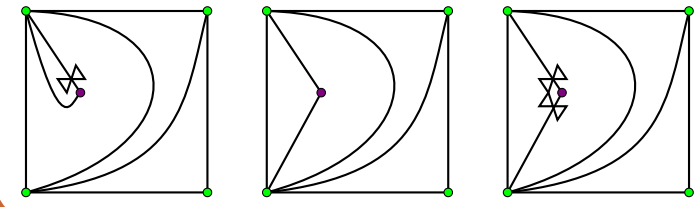
### Definition: ordinary arcs

- An *ordinary arc*  $\gamma$  is a non-contractible curve between marked points such that  $\gamma$  does not cross itself or the boundary, and  $\gamma$  is not homotopic to a boundary edge.
- An *ideal triangulation* is a maximum collection of distinct arcs that pairwise do not cross.



### Definition: tagged arcs

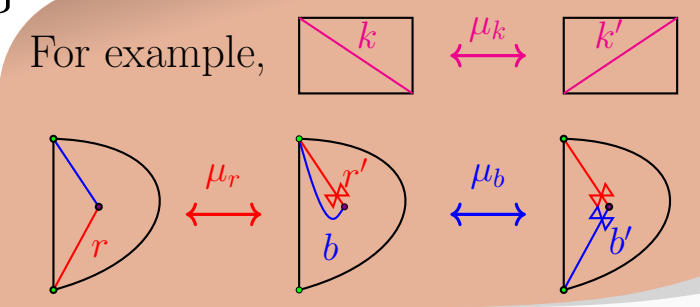
- A *tagged arc* is an ordinary arc (which does not cut out a monogon with 1 puncture  $\ell \circlearrowleft$ ) decorated (plain or with a  $\infty$ ) at each endpoint.
- A *tagged triangulation* is a maximum collection of distinct tagged arcs that are pairwise “compatible”.



### Theorem (Fomin, Shapiro, and Thurston, 2006)

One can define a cluster algebra from a Riemann surface + interior points (called *punctures*) and/or marked points on the boundary:

seed  $(\mathbf{x}_T, B_T) \longleftrightarrow$  tagged triangulation  $T = \{\tau_1, \dots, \tau_n\}$   
 cluster variable  $x_\gamma \longleftrightarrow$  tagged arc  $\gamma$   
 cluster mutation  $\longleftrightarrow$  “flipping diagonals”



## Example: once-punctured 3-gon

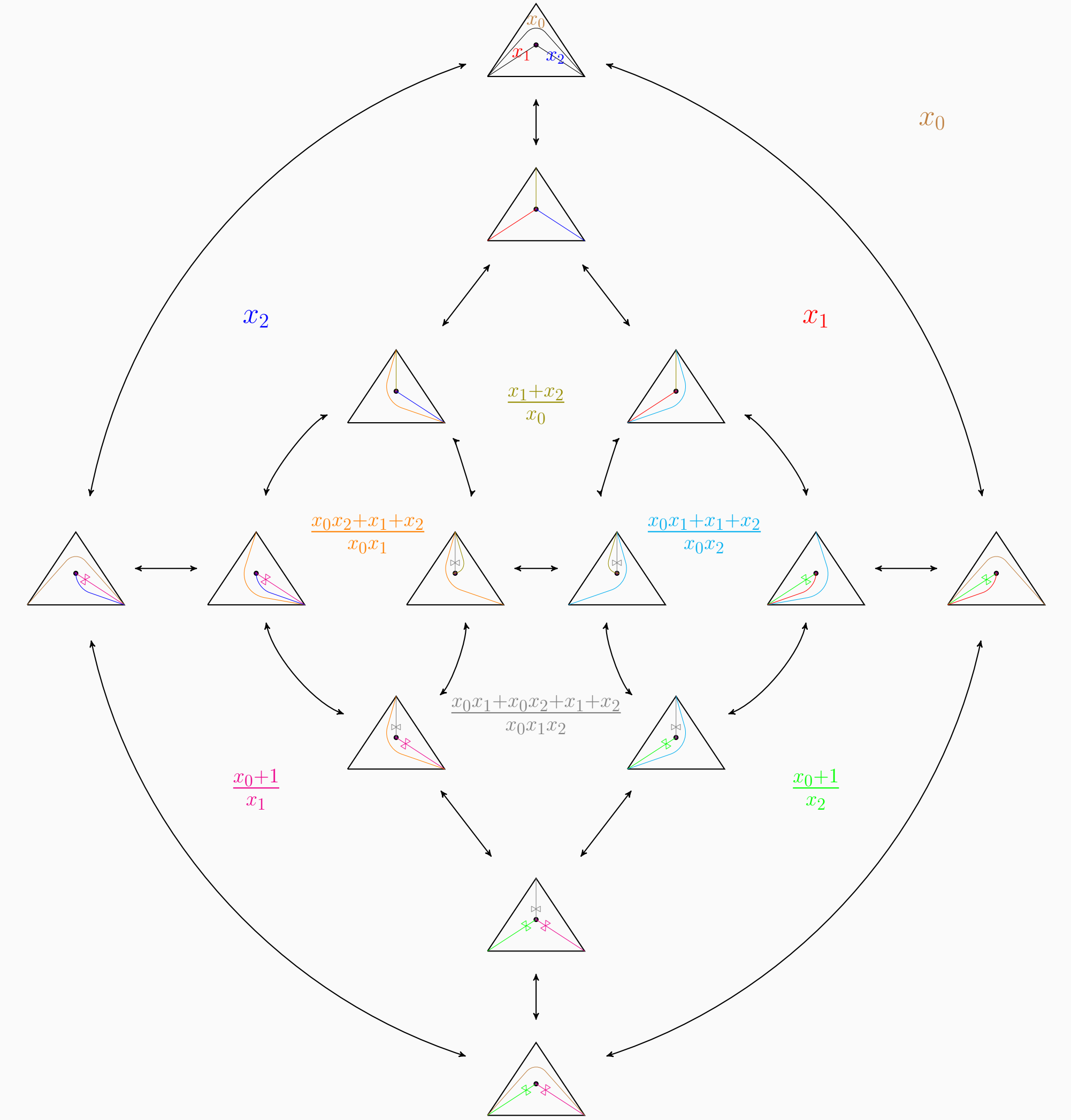


Fig. 1: There are  $\binom{2n}{n} - \binom{2n-2}{n-1}$  tagged triangulations of a once-punctured  $n$ -gon.

## Result 1: $T$ -path formula for cluster variables for type $D$ cluster algebras

We extend Schiffler - Thomas'  $T$ -path definition and formula for unpunctured surfaces (2009). Let  $T^\circ$  be an ideal triangulation (of an unpunctured surface or a once-punctured disk) and an arc  $\gamma$  that crosses  $T^\circ$ .

### Definition: quasi-arc

If  $\tau$  is an ordinary arc, let an associated *quasi-arc*  $\tau'$  be a curve (not passing through the puncture  $p$ ) which agrees with  $\tau$  outside of a small radius- $\epsilon$  disk  $D_\epsilon$  around  $p$ . If  $\tau$  is not adjacent to the puncture, let the associated quasi-arc be  $\tau$  itself.

### Definition: $T$ -path

A *complete*  $(T^\circ, \gamma)$ -*path* (or  $T^\circ$ -*path* for short)  $w = (w_1, \dots, w_{2d+1})$  is a concatenation of quasi-arcs and boundary edges such that:

- (T1) Each even step  $w_{2k}$  ( $k = 1, \dots, d$ ) is the  $k$ -th arc that  $\gamma$  crosses.
- (T2) The path  $w$  is homotopic to  $\gamma$ , and satisfies the following:  
 Let  $p_1, \dots, p_d$  be the intersection points of  $\gamma$  and  $T^\circ$ . Let  $\gamma_k$  be the segment along  $\gamma$  between  $p_k$  and  $p_{k+1}$ . Then the segment  $\gamma_k$  is homotopic to the segment from  $p_k$  following  $w_{2k}$ , following  $w_{2k+1}$ , following  $w_{2k+2}$  until  $p_{k+1}$ .
- (T3) The step  $w_{2k+1}$  traverses a side of the triangle  $\Delta_k$ , and starts and finishes in the interior of  $\Delta_k$  or at a boundary marked point.

### Theorem ( $T$ -path formula)

The cluster variable  $x_\gamma$  expressed in the variables corresponding to  $T^\circ$  is

$$x_\gamma = \sum_w x(w)$$

over all  $(T^\circ, \gamma)$ -paths  $w = (w_1, \dots, w_{2d+1})$ , where

$$x(w) := \left( \prod_{i \text{ odd}} x_{w_i} \right) \left( \prod_{i \text{ even}} x_{w_i}^{-1} \right).$$

We expect to generalize this to other punctured surfaces.

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The  $T$ -paths are in natural bijection with Musiker, Schiffler, and Williams' snake graph matchings.

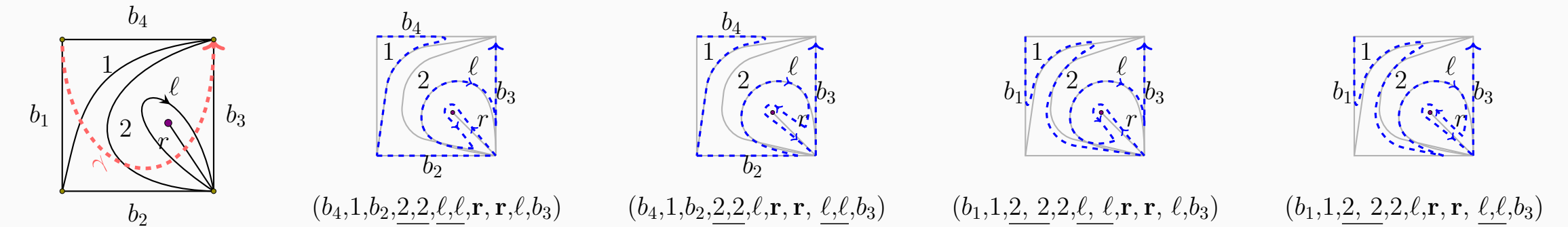


Fig. 2: Four of nine  $(T^\circ, \gamma)$ -paths. All backtracks  $(\underline{2}, \underline{2})$  and  $(\underline{\ell}, \underline{\ell})$  have been omitted.

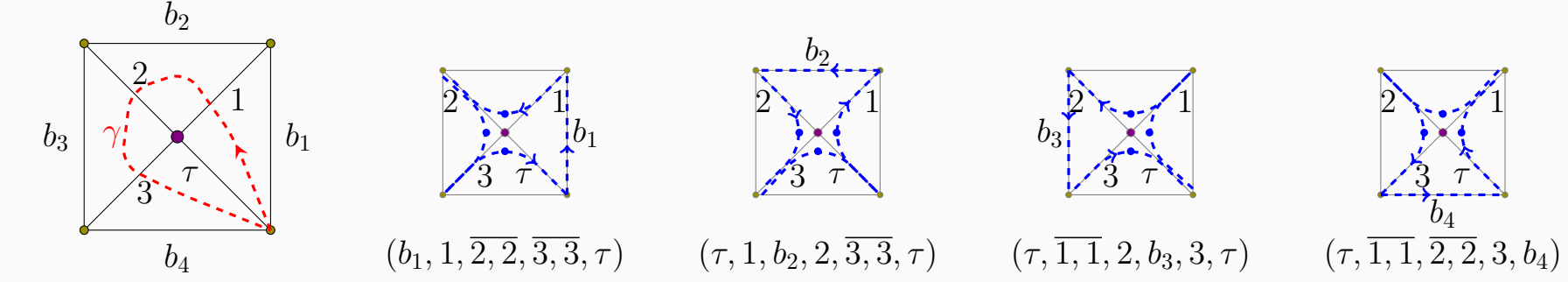


Fig. 3: The four  $(T^\circ, \gamma)$ -paths of the ideal triangulation  $T^\circ$  and the  $\ell$ -loop  $\gamma$  of the first figure.

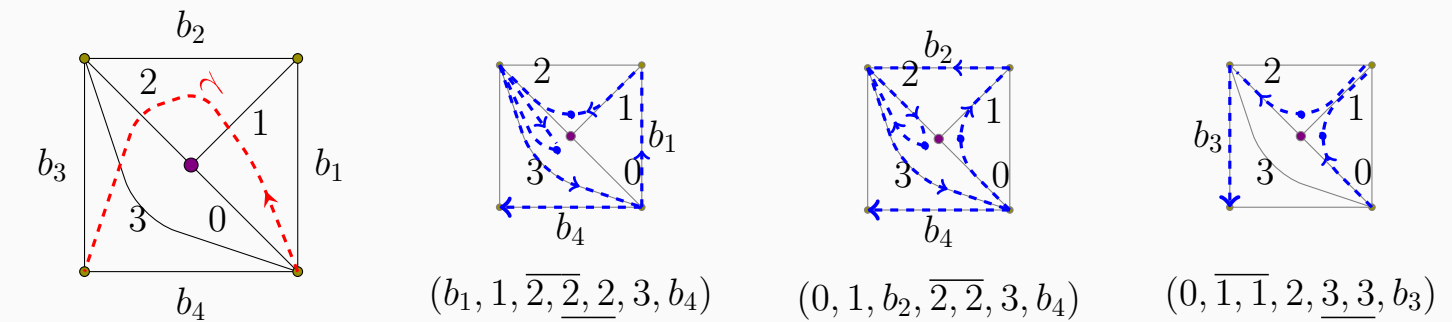


Fig. 4: Three of the five  $(T^\circ, \gamma)$ -paths of the ideal triangulation  $T^\circ$  and the arc  $\gamma$  of the first figure

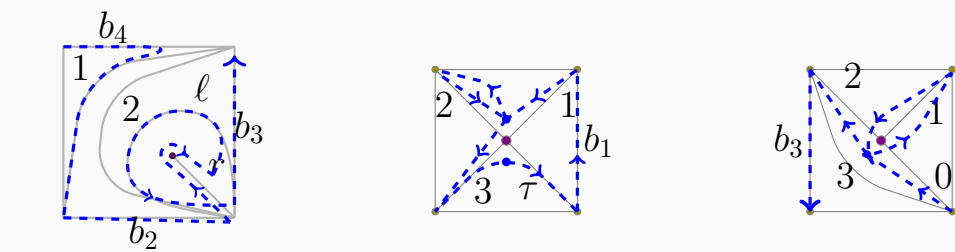


Fig. 5: Examples of *non*-( $T^\circ, \gamma$ )-paths for the situations in Figures 2, 3, and 4. Each path is homotopic to  $\gamma$  and satisfies (T1) but fails (T2) or (T3).

## Definition: Atomic Bases and Cluster Monomials

Let  $\mathcal{A}$  be a (coefficient-free) cluster algebra.

- The *positive cone* of  $\mathcal{A}$  is  $\{\text{positive elements}\} = \{\text{elements that are positive Laurent polynomials with respect to every cluster}\}$ .
- The subset  $\mathcal{B}$  of all indecomposable positive elements (*i.e.*, those that cannot be written as a sum of two positive elements) is called the *atomic basis* if it forms a  $\mathbb{Z}$ -basis of  $\mathcal{A}$ .
- A *cluster monomial* is a product of cluster variables all coming from the same cluster, e.g.  $a^5be^2$  is a cluster monomial if  $\{a, b, c, d, e\}$  is a cluster.

A cluster monomial corresponds to a multi-tagged dissection  $D$  (*i.e.*, a partial tagged triangulation allowing multiple copies of tagged arcs).

## Result 2: $T$ -path proof for type $D$ cluster algebras

### Theorem (atomic basis)

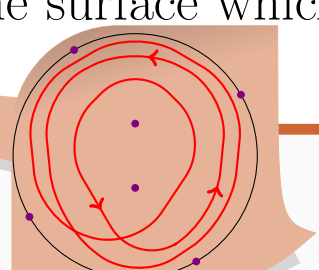
For a cluster algebra of type  $A$ ,  $D$ , or  $E$ , the basis of cluster monomials is atomic.

- Representation theory proof by [Cerulli Irelli, 2011] and [Cerulli Irelli, Keller, Labardini-Fragoso, and Plamondon, 2012]
- We give a combinatorial proof (using the  $T$ -path formula) for type  $D$ , inspired by work on types  $A$  and  $\tilde{A}$  by [Dupont and Thomas, 2011].

## Atomic bases for other surfaces

**Conjecture** (Fomin, Shapiro, and Thurston, 2008, unpublished, based on Fock and Goncharov, 2006)

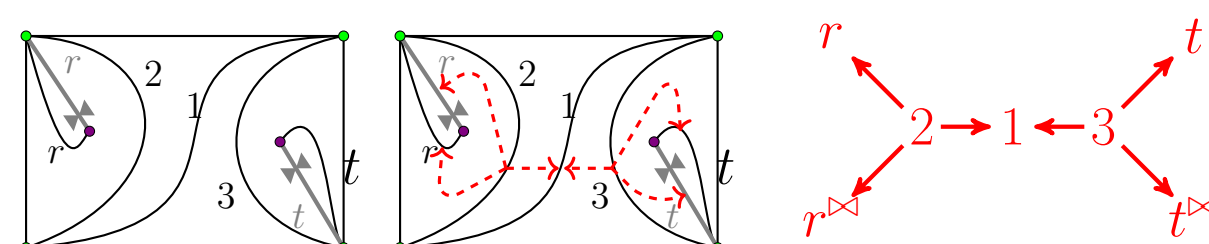
A candidate for atomic bases: the “*bracelets collection*” consisting of all cluster monomials + a class of elements. A *bracelet* is a closed loop in the interior of the surface which wraps around itself once or multiple times and avoids marked points.



- True for annuli, type  $\tilde{A}$  (Dupont and Thomas, 2011).
- The bracelets collection forms a basis for unpunctured surfaces (Musiker, Schiffler, and Williams, 2011).

### Further directions

Type  $\tilde{D}_{n-1}$  cluster algebras  $((n-3)$ -gons with 2 punctures), *e.g.* type  $\tilde{D}_6$  cluster algebra comes from a twice-punctured disk with 4 marked points on the boundary.



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