The Box-Ball System: Soliton Decomposition and Robinson-Schensted algorithm (University of Connecticut Math REU 2020)

2020 MRC workshop: Combinatorial Applications of Computational Geometry and Algebraic Topology

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Ben Drucker

Eli Garcia

Aubrey Rumbolt

Rose Silver

Motivation: Soliton waves

- At time $-\infty$, soliton waves are traveling through space at different speeds, not minding each other.
- At some time, they begin to collide with one another, causing interference, and for a while you have a mess.
- But eventually by time $+\infty$ the interference sorts itself out, and the solitons continue on their way as if it hadn't happened.

Box-Ball System: Example—$\pi = 452361$

Start with an initial configuration $\pi = \pi_1\pi_2\pi_3 \ldots \pi_k$, where π is a permutation.

$\boxed{\textit{Step 1:}}$ Write the permutation on a strip of infinite boxes:

$t = 0$

4	5	2	3	6	1							…

$\boxed{\textit{Step 2:}}$ To complete a box-ball move, let each number (or "ball") jump to the next available spot (or "box") to the right. First move 1, then move 2, and so on.

$t = 0$

4	5	2	3	6	1							…
4	5	2	3	6		1						…

Box-Ball System: Example—$\pi = 452361$

$\boxed{\textit{Step 3:}}$ Continue moving numbers from smallest to largest to their nearest available spots until every number in the permutation has been moved.

$t = 0$	4	5	2	3	6	1							…
	4	5	2	3	6		1						…
	4	5		3	6	2	1						…
	4	5			6	2	1	3					…
		5	4		6	2	1	3					…
			4	5	6	2	1	3					…
$t = 1$			4	5		2	1	3	6				…

We are now at the $t = 1$ state and we have completed one BBS move.

Box-Ball System: Example—$\pi = 452361$

$\boxed{\textit{Step 4:}}$ Continue making BBS moves.

(Here, 4 moves are shown).

$t = 0$	4	5	2	3	6	1							…							
$t = 1$			4	5		2	1	3	6				…							
$t = 2$					4	5	2			1	3	6	…							
$t = 3$							4	2	5				1	3	6			…		
$t = 4$								4		2	5					1	3	6		…

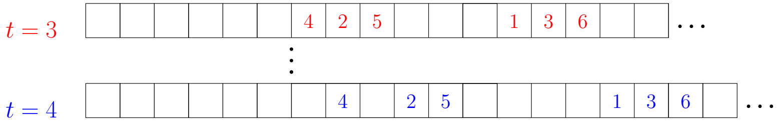
Note. (Backwards BBS)

- Move balls from **largest-to-smallest** to their nearest available spaces to the **left**.
- The time-values after each backwards box-ball move now **decrease**.

Box-Ball System: Soliton Decomposition

At a certain point, the system reaches a *steady state* where:

- blocks of increasing sequences (or *solitons*) move together at a speed equal to their length.
- the sizes of the solitons are weakly increasing from left to right
- order of the solitons remain unchanged



Box-Ball System: Example—$\pi = 452361$

$\boxed{\textit{Step 5:}}$ After reaching steady state, create a *soliton decomposition* diagram $\mathrm{SD}(\pi)$ by stacking solitons from right to left.

$t = 4$

							4		2	5					1	3	6		$\cdots$

The shape of the diagram always forms a partition (weakly decreasing sequence of positive integers):

$$\text{Soliton decomposition } \mathrm{SD}(\pi) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \text{ with shape } (3, 2, 1).$$

REU Questions.

When does a permutation reach its steady-state?

How many permutations in S_n first reach its steady-state by a given time t?

Tableaux

Definition. (Young Tableaux)

- A *tableau* is an arrangement of numbers $\{1, 2, ..., n\}$ in Young diagram (sequence of weakly decreasing rows).
- A tableau is *standard* if the rows and columns are increasing sequences.
- The *reading word* of a standard Young tableau is the permutation formed by concatenating the rows of the tableau from bottom to top.

Example. (Standard Young Tableau)

- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$ is a standard tableau. Its reading word is 53412.
- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$ is a nonstandard tableau.

REU Questions.

When is a soliton decomposition standard?

Robinson-Schensted insertion algorithm

$$\pi = 452361 \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = P$$

- The *Robinson-Schensted (RS) insertion algorithm* is a famous bijection from permutations to pairs of standard tableaux.
- Given a permutation $\pi = \pi_1 \cdots \pi_n$, the first tableau in the pair, denoted $P(\pi)$, is called the insertion tableau or P-tableau of π.

REU Questions.

For what permutations π do we have $SD(\pi) = P(\pi)$?

Soliton decomposition vs P-tableau

Theorem

The following are equivalent:

1. $\mathrm{SD}(\pi) = \mathrm{P}(\pi)$.
2. $\mathrm{SD}(\pi)$ is a standard tableau.

Conjecture

The following is equivalent to (1) and (2).

3. $\operatorname{sh}\mathrm{SD}(\pi) = \operatorname{sh}\mathrm{P}(\pi)$.

Knuth Relations

Definition

Suppose π, $\sigma \in S_n$ and $x < y < z$.

- π and σ differ by a *Knuth relation of the* ***first kind*** (K_1) if
$$\pi = x_1 ... yxz ... x_n \text{ and } \sigma = x_1 ... yzx ... x_n$$
- π and σ differ by a *Knuth relation of the* ***second kind*** (K_2) if
$$\pi = x_1 ... xzy ... x_n \text{ and } \sigma = x_1 ... zxy ... x_n$$
- π and σ differ by *Knuth relations of* ***both kinds*** (K_B) if
$$\pi = x_1 ... y_1 xzy_2 ... x_n \text{ and } \sigma = x_1 ... y_1 zxy_2 ... x_n$$
for $x < y_1, y_2 < z$

Soliton Decomposition and Knuth moves

- Let r denote the reading word of $P(\pi)$. The RSK theory tells us there is a path of Knuth moves from π to r.

Theorems: Soliton decomposition and Knuth moves

- If there is a path from π to r such that no move along the path is K_B, then $\operatorname{sh}\mathrm{SD}(\pi) = \operatorname{sh} P(\pi)$.
- If there exists a path from π to r containing an odd number of K_B moves, then $\mathrm{SD}(\pi) \neq P(\pi)$.

Steady-State Times

Let $a_{n,t}$ be the number of permutations in S_n which first reach their soliton decompositions at time t. Let $F_n(x) = a_{n,0} + a_{n,1}x + a_{n,2}x^2 + a_{n,3}x^3 + \dots$ be the generating function of the sequence $\{a_{t,n}\}_{t>0}$.

Theorem: classification of permutations with steady-state value of $t = 0$

- A permutation r reaches its soliton decomposition at $t = 0$ if and only if r is the reading word of a standard tableau.
- In particular, the constant value of $F_n(x)$ is the number of standard tableaux with n boxes.

Theorem: a class of permutations with steady-state value of $t = 1$

If a permutation π is related to a reading word of a standard tableau by one K_1 or K_2 move (but not K_B), then π first reaches its soliton decomposition at $t = 1$.

$n - 3$ conjecture

The generating function of $a_n(t)$ is a polynomial of degree at most $n - 3$.

Insertion algorithm for soliton decomposition

"Carrier" Algorithm

- Given a BBS state at time t, compute the state at time $t + 1$

Theorem: "M-carrier" algorithm and insertion algorithm

- Given a BBS state at time t, compute the state at time $t + M$

Work in progress: RSK-like insertion algorithm for soliton decomposition

- Define an "unlimited-carrier" algorithm
- Compute the soliton decomposition using insertion/bumping similar to Robinson-Schensted (RS) insertion algorithm.
- When $SD(\pi) = P(\pi)$, the "unlimited-carrier" algorithm is equivalent to the usual RS insertion algorithm

Thank You!