

The Box-Ball System: Soliton Decomposition and Robinson-Schensted algorithm (University of Connecticut Math REU 2020)

2020 MRC workshop: Combinatorial Applications of Computational Geometry and Algebraic Topology

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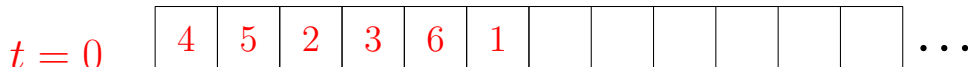
Motivation: Soliton waves

- ▶ At time $-\infty$, soliton waves are traveling through space at different speeds, not minding each other.
- ▶ At some time, they begin to collide with one another, causing interference, and for a while you have a mess.
- ▶ But eventually by time $+\infty$ the interference sorts itself out, and the solitons continue on their way as if it hadn't happened.

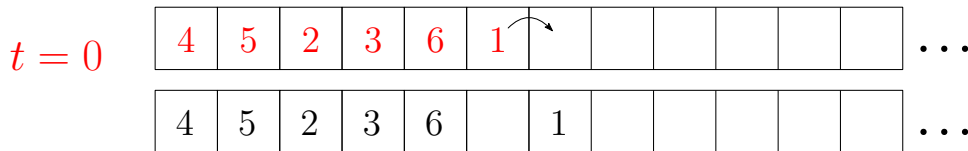
Box-Ball System: Example— $\pi = 452361$

Start with an initial configuration $\pi = \pi_1\pi_2\pi_3...\pi_k$, where π is a permutation.

Step 1: Write the permutation on a strip of infinite boxes:

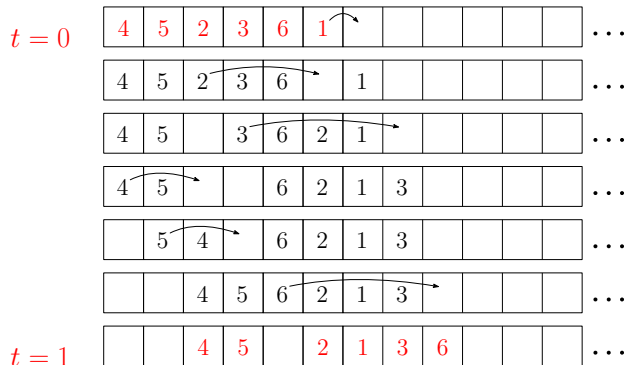


Step 2: To complete a box-ball move, let each number (or “ball”) jump to the next available spot (or “box”) to the right. First move 1, then move 2, and so on.



Box-Ball System: Example— $\pi = 452361$

Step 3: Continue moving numbers from smallest to largest to their nearest available spots until every number in the permutation has been moved.

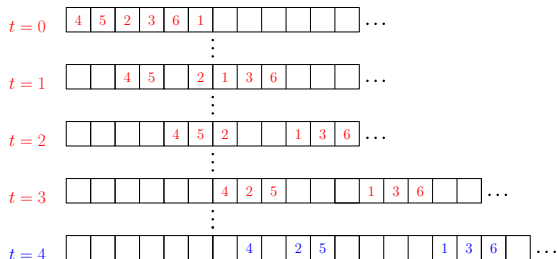


We are now at the $t = 1$ state and we have completed one BBS move.

Box-Ball System: Example— $\pi = 452361$

Step 4: Continue making BBS moves.

(Here, 4 moves are shown).



Note. (Backwards BBS)

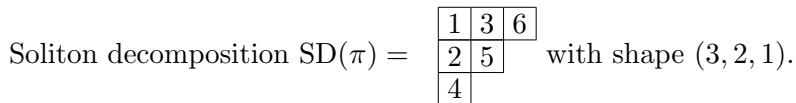
- Move balls from **largest-to-smallest** to their nearest available spaces to the **left**.
- The time-values after each backwards box-ball move now **decrease**.

Box-Ball System: Example— $\pi = 452361$

Step 5: After reaching steady state, create a *soliton decomposition* diagram $\text{SD}(\pi)$ by stacking solitons from right to left.



The shape of the diagram always forms a partition (weakly decreasing sequence of positive integers):



REU Questions.

When does a permutation reach its steady-state?

How many permutations in S_n first reach its steady-state by a given time t ?

Tableaux

Definition. (Young Tableaux)

- ▶ A *tableau* is an arrangement of numbers $\{1, 2, \dots, n\}$ in Young diagram (sequence of weakly decreasing rows).
- ▶ A tableau is *standard* if the rows and columns are increasing sequences.
- ▶ The *reading word* of a standard Young tableau is the permutation formed by concatenating the rows of the tableau from bottom to top.

Example. (Standard Young Tableau)

- ▶

1	2
3	4
5	

 is a standard tableau. Its reading word is 53412.
- ▶

1	2
4	
5	
3	

 is a nonstandard tableau.

REU Questions.

When is a soliton decomposition standard?

Robinson-Schensted insertion algorithm

$$\pi = 452361 \rightarrow \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \hline 4 & & \\ \hline \end{array} = P$$

- ▶ The *Robinson-Schensted (RS) insertion algorithm* is a famous bijection from permutations to pairs of standard tableaux.
- ▶ Given a permutation $\pi = \pi_1 \cdots \pi_n$, the first tableau in the pair, denoted $P(\pi)$, is called the insertion tableau or P -tableau of π .

REU Questions.

For what permutations π do we have $SD(\pi) = P(\pi)$?

Soliton decomposition vs P-tableau

Theorem

The following are equivalent:

1. $SD(\pi) = P(\pi)$.
2. $SD(\pi)$ is a standard tableau.

Conjecture

The following is equivalent to (1) and (2).

3. $sh\ SD(\pi) = sh\ P(\pi)$.

Knuth Relations

Definition

Suppose $\pi, \sigma \in S_n$ and $x < y < z$.

- ▶ π and σ differ by a Knuth relation of the **first kind** (K_1) if

$$\pi = x_1 \dots yxz \dots x_n \text{ and } \sigma = x_1 \dots yzx \dots x_n$$

- ▶ π and σ differ by a Knuth relation of the **second kind** (K_2) if

$$\pi = x_1 \dots xzy \dots x_n \text{ and } \sigma = x_1 \dots zxy \dots x_n$$

- ▶ π and σ differ by Knuth relations of **both kinds** (K_B) if

$$\pi = x_1 \dots y_1 xz y_2 \dots x_n \text{ and } \sigma = x_1 \dots y_1 zxy_2 \dots x_n$$

for $x < y_1, y_2 < z$

Soliton Decomposition and Knuth moves

- ▶ Let r denote the reading word of $P(\pi)$. The RSK theory tells us there is a path of Knuth moves from π to r .

Theorems

- ▶ If there exists a path from π to r such that no move along the path is K_B , then $\text{sh SD}(\pi) = \text{sh } P(\pi)$.
- ▶ If there exists a path from π to r containing an odd number of K_B moves, then $\text{SD}(\pi) \neq P(\pi)$.

Results Involving Steady-State Times

Let $a_{n,t}$ be the number of permutations in S_n which first reach their soliton decompositions at time t . Let $F_n(x) = a_{n,0} + a_{n,1}x + a_{n,2}x^2 + a_{n,3}x^3 + \dots$ be the generating function of the sequence $\{a_{t,n}\}_{t \geq 0}$.

Theorem: classification of permutations with steady-state value of $t = 0$

- ▶ A permutation r reaches its soliton decomposition at $t = 0$ if and only if r is the reading word of a standard tableau.
- ▶ In particular, the constant value of $F_n(x)$ is the number of standard tableaux with n boxes.

Theorem: a class of permutations with steady-state value of $t = 1$

If a permutation π is related to a reading word of a standard tableau by one K_1 or K_1 move (but not K_B), then π first reaches its soliton decomposition at $t = 1$.

$n - 3$ conjecture

The generating function $F_n(x)$ is a polynomial of degree at most $n - 3$.

Insertion algorithm for soliton decomposition

“Carrier” Algorithm

- ▶ Given a BBS state at time t , compute the state at time $t + 1$

Theorem: “ M -carrier” algorithm and insertion algorithm

- ▶ Given a BBS state at time t , compute the state at time $t + M$

Work in progress: RSK-like insertion algorithm for soliton decomposition

- ▶ Define an “unlimited-carrier” algorithm
- ▶ Compute the soliton decomposition using insertion/bumping similar to Robinson-Schensted (RS) insertion algorithm.
- ▶ When $SD(\pi) = P(\pi)$, the “unlimited-carrier” algorithm is equivalent to the usual RS insertion algorithm

Thank You!