# The Box-Ball System

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# Overview

1. Introduction: permutations, tableaux, Knuth relations, and the RSK algorithm
2. The box-ball system
3. Results and conjectures: steady state times and soliton decompositions

# Introduction

## Definition. (Permutation)

A permutation $\pi$ of $n$ numbers is a bijective function from the set of natural numbers $\{1, 2, ..., n\}$ to itself. Let $S_n$ be the set of all possible permutations of $n$ numbers.

## Example. (Permutation)

In $S_6$, a permutation $\sigma$ may consist of the following mappings:

$$
\begin{aligned}
1 &\mapsto 3 \\
2 &\mapsto 4 \\
3 &\mapsto 2 \\
4 &\mapsto 1 \\
5 &\mapsto 5 \\
6 &\mapsto 6
\end{aligned}
$$

# Permutations: Notation

## Example. (2-line Notation)

We may write permutations compactly using 2-line notation. For example, in 2-line notation,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 1 & 5 & 6 \end{pmatrix} \begin{matrix} \leftarrow \text{(1 through n)} \\ \leftarrow \text{(mappings)} \end{matrix}$$

## Example. (1-line Notation)

We may write permutations even more compactly using 1-line notation. For example, in 1-line notation, $\sigma = 342156 \leftarrow$ (just mappings).

## Note.

From now on, we will use the 1-line notation whenever permutations are used.

## Definition

Suppose $\pi$, $\sigma \in S_n$ and $x < y < z$.

- $\pi$ and $\sigma$ differ by a Knuth relation of the **first kind** $(K_1)$ if
$$\pi = x_1 ... yxz ... x_n \text{ and } \sigma = x_1 ... yzx ... x_n$$
- $\pi$ and $\sigma$ differ by a Knuth relation of the **second kind** $(K_2)$ if
$$\pi = x_1 ... xzy ... x_n \text{ and } \sigma = x_1 ... zxy ... x_n$$
- $\pi$ and $\sigma$ differ by Knuth relations of **both kinds** $(K_B)$ if for $x < y_1 < z$ and $x < y_2 < z$
$$\pi = x_1 ... y_1xzy_2 ... x_n \text{ and } \sigma = x_1 ... y_1zxy_2 ... x_n$$

# Tableaux

## Definition. (Young Tableaux)

- A Young tableau (pl. tableaux) is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
- The Young tableau is *standard* if each row is an increasing sequence (going from left to right) and each column is an increasing sequence (going from top to bottom).
- The *reading word* of a standard Young tableau is the permutation formed by concatenating the rows of the tableau from bottom to top.

## Example. (Standard Young Tableau)

- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1} \end{array}$ is a standard Young tableau. Its reading word is 53412.
- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & \\ \cline{1-1} 5 & \\ \cline{1-1} 3 & \\ \cline{1-1} \end{array}$ is a nonstandard Young tableau.

$$\pi = 452361 \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} = P$$

Let $P$ be the empty tableau. Given a permutation $\pi = \pi_1 \cdots \pi_n$, the *Robinson-Schensted-Knuth (RSK) insertion algorithm* is a method of insertion of each element of $\pi$ into $P$. The RSK insertion algorithm ensures that the result—after all elements of $\pi$ have been inserted—will be a standard Young tableau.

# The Box-Ball System

Consider an initial configuration $\pi = \pi_1\pi_2\pi_3...\pi_k$, where $\pi$ is a permutation in 1-line notation.

To complete a box-ball move, we let each number (or “ball”) jump to the next available spot (or “box”) to the right. We first move 1, then we move 2, and so on.

# Box-Ball System: Example—$\pi = 452361$

We will use the example permutation $\pi = 452361$.

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*Step 1:* Write the permutation on a strip of infinite boxes as shown below.
This configuration corresponds $t = 0$.

$t = 0$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 2 | 3 | 6 | 1 | | | | | | | ⋯ |

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| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 2 | 3 | 6 | 1 | | | | | | | $\dots$ |

*Step 2:* Going from smallest to largest, move each number to the next empty box to the right.

$t = 0$

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 2 | 3 | 6 | 1 | | | | | | | $\dots$ |
| 4 | 5 | 2 | 3 | 6 | | 1 | | | | | | $\dots$ |

$\boxed{\textit{Step 3:}}$ Continue moving numbers from smallest to largest to their nearest available spots until every number in the permutation has been moved. We are now at the $t = 1$ state and we have completed one BBS move.

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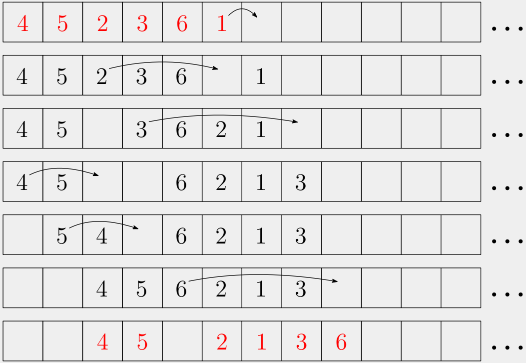



$t = 1$

$\boxed{\textit{Step 4:}}$ Continue making BBS moves.

(Here, 4 moves are shown).

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t = 0$ | 4 | 5 | 2 | 3 | 6 | 1 | | | | | | | … | | | | | | | |
| $t = 1$ | | | 4 | 5 | | 2 | 1 | 3 | 6 | | | | … | | | | | | | |
| $t = 2$ | | | | | 4 | 5 | 2 | | | 1 | 3 | 6 | … | | | | | | | |
| $t = 3$ | | | | | | | 4 | 2 | 5 | | | | 1 | 3 | 6 | | | … | | |
| $t = 4$ | | | | | | | | 4 | | 2 | 5 | | | | | 1 | 3 | 6 | | … |

$\boxed{\textit{Step 4:}}$ Continue making BBS moves.

(Here, 4 moves are shown).

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t = 0$ | 4 | 5 | 2 | 3 | 6 | 1 | | | | | | | … | | | | | | | |
| $t = 1$ | | | 4 | 5 | | 2 | 1 | 3 | 6 | | | | … | | | | | | | |
| $t = 2$ | | | | | 4 | 5 | 2 | | | 1 | 3 | 6 | … | | | | | | | |
| $t = 3$ | | | | | | | 4 | 2 | 5 | | | | 1 | 3 | 6 | | | … | | |
| $t = 4$ | | | | | | | | 4 | | 2 | 5 | | | | | 1 | 3 | 6 | | … |

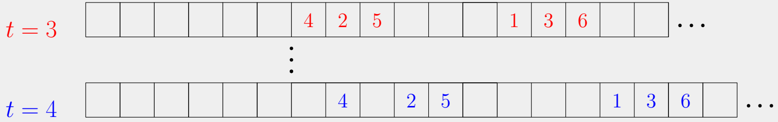
**Note.** (Backwards BBS)

- We now move balls from **largest-to-smallest** to their nearest available spaces to the **left**.
- The time-values after each backwards box-ball move now **decrease**.

# Box-Ball System: Soliton Decomposition

At a certain point, the system reaches a *steady state* where:

- blocks of increasing sequences (or *solitons*) move together at a speed equal to their length.
- the sizes of the solitons are weakly increasing from left to right
- order of the solitons remain unchanged



# Box-Ball System: Example—$\pi = 452361$

$\boxed{\textit{Step 5:}}$ After reaching steady state, create a diagram by stacking solitons from right to left. The shape of the diagram forms a (not necessarily standard) Young tableau:

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t = 4$ | | | | | | | | 4 | | 2 | 5 | | | | | 1 | 3 | 6 | | … |

Soliton decomposition $\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$ with shape $(3, 2, 1)$.

## Central Questions.

When is soliton decomposition standard? When is soliton decomposition equal to the original permutation's $P$-tableau? How long does it take for a given permutation to reach its steady-state?

# Results and Conjectures

- The remainder of the presentation will focus on results and conjectures about the *soliton decomposition* of a permutation, which we denote $\mathrm{SD}(\pi)$.

**Big question (from earlier)**

For what permutations $\pi$ do we have $\mathrm{SD}(\pi) = \mathrm{P}(\pi)$?

# Reformulating the question

- A way to simplify (and answer) the big question is by formulating equivalent statements.

**Theorem**

The following are equivalent:

1. $\mathrm{SD}(\pi) = \mathrm{P}(\pi)$.
2. $\mathrm{SD}(\pi)$ is a standard tableau.

**Conjecture**

The following is equivalent to (1) and (2).

3. $\operatorname{sh}\mathrm{SD}(\pi) = \operatorname{sh}\mathrm{P}(\pi)$.

- We have shown (1) $\iff$ (2) and (1) $\implies$ (3). It remains to prove (3) $\implies$ (1).

# Knuth move perspective

- Yet another way of attacking this problem is from the perspective of Knuth moves[1]
- Let $r$ denote the reading word of $P(\pi)$. The RSK theory tells us there is a path of Knuth moves from $\pi$ to $r$.

## Theorems

- If there exists a path from $\pi$ to $r$ such that no move along the path is $K_B$, then $\mathrm{SD}(\pi) = P(\pi)$.
- If there exists a path from $\pi$ to $r$ containing an odd number of $K_B$ moves, then $\mathrm{SD}(\pi) \neq P(\pi)$.

[1] The theorem on this slide is due in part to a result in [LLPS19].

# Results Involving Steady-State Times

**Theorem: Classification of permutations with steady-state value of $t = 0$**

A permutation $r$ has a box-ball steady-state value of $t = 0$ if and only if $r$ is the reading word of a standard tableau.

**Theorem: Some permutations with steady-state value of $t = 1$**

Let $r$ be the reading word of a standard tableau $P$. If $K$ represents the action of a $K_1$ or $K_2$ move (but not $K_B$), then $r' = K(r)$ has a box-ball steady-state value of $t = 1$.

**$n - 3$ conjecture**

The steady-state time for a permutation in $S_n$ is no more than $n - 3$.

## The $M$-carrier Algorithm

- The "carrier algorithm"
  - Method of computing $t = k + 1$ state given $t = k$
- The "$M$-carrier algorithm"
  - (Possible) "closed form" method of computing solition decomposition given the original permutation

Thank You!
Questions?

# References

[LLPS19] Joel Lewis, Hanbaek Lyu, Pavlo Pylyavskyy, and Arnab Sen. **Scaling limit of soliton lengths in a multicolor box-ball system.** Preprint `arXiv:1911.04458`, 2019.