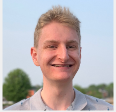
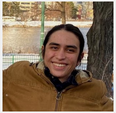
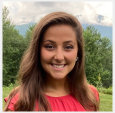
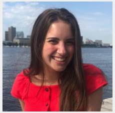
The Box-Ball System

UConn REU

July 24, 2020

Ben Drucker Eli Garcia Aubrey Rumbolt Rose Silver

Overview

1. Introduction: permutations, tableaux, Knuth relations, and the RSK algorithm
2. The box-ball system
3. Results and conjectures: steady state times and soliton decompositions

Introduction

Definition. (Permutation)

A permutation π of n numbers is a bijective function from the set of natural numbers $\{1, 2, ..., n\}$ to itself. Let S_n be the set of all possible permutations of n numbers.

Example. (Permutation)

In S_6, a permutation σ may consist of the following mappings:

$$\begin{gathered} 1 \mapsto 3 \\ 2 \mapsto 4 \\ 3 \mapsto 2 \\ 4 \mapsto 1 \\ 5 \mapsto 5 \\ 6 \mapsto 6 \end{gathered}$$

Permutations: Notation

Example. (2-line Notation)

We may write permutations compactly using 2-line notation. For example, in 2-line notation,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 2 & 1 & 5 & 6 \end{pmatrix} \begin{matrix} \leftarrow \text{(1 through n)} \\ \leftarrow \text{(mappings)} \end{matrix}$$

Example. (1-line Notation)

We may write permutations even more compactly using 1-line notation. For example, in 1-line notation, $\sigma = 342156 \leftarrow$ (just mappings).

Note.

From now on, we will use the 1-line notation whenever permutations are used.

Knuth Relations

Definition

Suppose π, $\sigma \in S_n$ and $x < y < z$.

- π and σ differ by a Knuth relation of the **first kind** (K_1) if
$$\pi = x_1 ... yxz ... x_n \text{ and } \sigma = x_1 ... yzx ... x_n$$
- π and σ differ by a Knuth relation of the **second kind** (K_2) if
$$\pi = x_1 ... xzy ... x_n \text{ and } \sigma = x_1 ... zxy ... x_n$$
- π and σ differ by Knuth relations of **both kinds** (K_B) if for $x < y_1 < z$ and $x < y_2 < z$
$$\pi = x_1 ... y_1 xzy_2 ... x_n \text{ and } \sigma = x_1 ... y_1 zxy_2 ... x_n$$

Tableaux

Definition. (Young Tableaux)

- A Young tableau (pl. tableaux) is an arrangement of numbers $\{1, 2, \ldots, n\}$ into rows whose lengths are weakly decreasing.
- The Young tableau is *standard* if each row is an increasing sequence (going from left to right) and each column is an increasing sequence (going from top to bottom).
- The *reading word* of a standard Young tableau is the permutation formed by concatenating the rows of the tableau from bottom to top.

Example. (Standard Young Tableau)

- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1}\end{array}$ is a standard Young tableau. Its reading word is 53412.
- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ is a nonstandard Young tableau.

$$\pi = 452361 \longrightarrow \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & \\ \cline{1-2} 4 & & \\ \cline{1-1}\end{array} = P$$

Let P be the empty tableau. Given a permutation $\pi = \pi_1 \cdots \pi_n$, the *Robinson-Schensted-Knuth (RSK) insertion algorithm* is a method of insertion of each element of π into P. The RSK insertion algorithm ensures that the result—after all elements of π have been inserted—will be a standard Young tableau.

The Box-Ball System

Consider an initial configuration $\pi = \pi_1\pi_2\pi_3...\pi_k$, where π is a permutation in 1-line notation.

To complete a box-ball move, we let each number (or "ball") jump to the next available spot (or "box") to the right. We first move 1, then we move 2, and so on.

Box-Ball System: Example—$\pi = 452361$

We will use the example permutation $\pi = 452361$.

Box-Ball System: Example—$\pi = 452361$

We will use the example permutation $\pi = 452361$.

$\boxed{\textit{Step 1:}}$ Write the permutation on a strip of infinite boxes as shown below.
This configuration corresponds $t = 0$.

$t = 0$	4	5	2	3	6	1							$\cdots$

Box-Ball System: Example—$\pi = 452361$

We will use the example permutation $\pi = 452361$.

Step 1: Write the permutation on a strip of infinite boxes as shown below. This configuration corresponds $t = 0$.

$t = 0$

4	5	2	3	6	1							$\cdots$

Step 2: Going from smallest to largest, move each number to the next empty box to the right.

$t = 0$

4	5	2	3	6	1							$\cdots$
4	5	2	3	6		1						$\cdots$

Box-Ball System: Example—$\pi = 452361$

$\boxed{\textit{Step 3:}}$ Continue moving numbers from smallest to largest to their nearest available spots until every number in the permutation has been moved. We are now at the $t = 1$ state and we have completed one BBS move.

Box-Ball System: Example—$\pi = 452361$

Step 3: Continue moving numbers from smallest to largest to their nearest available spots until every number in the permutation has been moved. We are now at the $t = 1$ state and we have completed one BBS move.

$t = 0$	4	5	2	3	6	1							…
	4	5	2	3	6		1						…
	4	5		3	6	2	1						…
	4	5			6	2	1	3					…
		5	4		6	2	1	3					…
			4	5	6	2	1	3					…
$t = 1$			4	5		2	1	3	6				…

Step 4: Continue making BBS moves.

(Here, 4 moves are shown).

$t = 0$	4	5	2	3	6	1							…							
$t = 1$			4	5		2	1	3	6				…							
$t = 2$					4	5	2			1	3	6	…							
$t = 3$							4	2	5				1	3	6			…		
$t = 4$								4		2	5					1	3	6		…

Box-Ball System: Example—$\pi = 452361$

Step 4: Continue making BBS moves.

(Here, 4 moves are shown).

$t = 0$	4	5	2	3	6	1							…							
$t = 1$			4	5		2	1	3	6				…							
$t = 2$					4	5	2			1	3	6	…							
$t = 3$							4	2	5				1	3	6			…		
$t = 4$								4		2	5					1	3	6		…

Note. (Backwards BBS)

- We now move balls from **largest-to-smallest** to their nearest available spaces to the **left**.
- The time-values after each backwards box-ball move now **decrease**.

At a certain point, the system reaches a *steady state* where:

- blocks of increasing sequences (or *solitons*) move together at a speed equal to their length.
- the sizes of the solitons are weakly increasing from left to right
- order of the solitons remain unchanged

$t = 3$							4	2	5				1	3	6			...		
⋮																				
$t = 4$								4		2	5					1	3	6		...

Box-Ball System: Example—$\pi = 452361$

$\boxed{\textit{Step 5:}}$ After reaching steady state, create a diagram by stacking solitons from right to left. The shape of the diagram forms a (not necessarily standard) Young tableau:

$t = 4$

							4		2	5					1	3	6		…

Soliton decomposition $\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 4 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$ with shape $(3, 2, 1)$.

Central Questions.

When is soliton decomposition standard? When is soliton decomposition equal to the original permutation's P-tableau? How long does it take for a given permutation to reach its steady-state?

Results and Conjectures

- The remainder of the presentation will focus on results and conjectures about the *soliton decomposition* of a permutation, which we denote $\mathrm{SD}(\pi)$.

Big question (from earlier)

For what permutations π do we have $\mathrm{SD}(\pi) = \mathrm{P}(\pi)$?

Reformulating the question

- A way to simplify (and answer) the big question is by formulating equivalent statements.

Theorem

The following are equivalent:

1. $\mathrm{SD}(\pi) = \mathrm{P}(\pi)$.
2. $\mathrm{SD}(\pi)$ is a standard tableau.

Conjecture

The following is equivalent to (1) and (2).

3. $\operatorname{sh}\mathrm{SD}(\pi) = \operatorname{sh}\mathrm{P}(\pi)$.

- We have shown (1) $\iff$ (2) and (1) $\implies$ (3). It remains to prove (3) $\implies$ (1).

- Yet another way of attacking this problem is from the perspective of Knuth moves[1]
- Let r denote the reading word of $P(\pi)$. The RSK theory tells us there is a path of Knuth moves from π to r.

Theorems

- If there exists a path from π to r such that no move along the path is K_B, then $\operatorname{sh}\mathrm{SD}(\pi) = \operatorname{sh}P(\pi)$.
- If there exists a path from π to r containing an odd number of K_B moves, then $\mathrm{SD}(\pi) \neq P(\pi)$.

[1] The theorem on this slide is due in part to a result in [LLPS19].

Results Involving Steady-State Times

Theorem: Classification of permutations with steady-state value of $t = 0$

A permutation r has a box-ball steady-state value of $t = 0$ if and only if r is the reading word of a standard tableau.

Theorem: Some permutations with steady-state value of $t = 1$

Let r be the reading word of a standard tableau P. If K represents the action of a K_1 or K_2 move (but not K_B), then $r' = K(r)$ has a box-ball steady-state value of $t = 1$.

$n - 3$ conjecture

The steady-state time for a permutation in S_n is no more than $n - 3$.

The M-carrier Algorithm

- The "carrier algorithm"
  - Method of computing $t = k + 1$ state given $t = k$
- The "M-carrier algorithm"
  - (Possible) "closed form" method of computing solition decomposition given the original permutation

Thank You!
Questions?

References

[LLPS19] Joel Lewis, Hanbaek Lyu, Pavlo Pylyavskyy, and Arnab Sen. **Scaling limit of soliton lengths in a multicolor box-ball system.** Preprint `arXiv:1911.04458`, 2019.