3283W Functions Cheat Sheet

The setup is as follows: $f\colon A \to B$, with the inclusions $C \subseteq A$, $D \subseteq B$.

Definition (Injective)**.** The function f is *injective* if

$$\textit{for all } x, x' \in A,\ f(x) = f(x') \implies x = x'$$

Equivalently,

$$\textit{for all } x, x' \in A,\ x \neq x' \implies f(x) \neq f(x'),$$

which is the contrapositive.

Definition (Surjective)**.** The function f is *surjective* if

$$\textit{for all } y \in B,\ \textit{there exists } x \in A \textit{ such that } f(x) = y.$$

Equivalently,

$$f(A) = B.$$

Definition (Image)**.** An element $y \in B$ in the codomain is in the *image* of a set C if $f(x) = y$ for some $x \in C$. *In other words,*

$$y \in f(C) \iff \text{there exists } x \in C \text{ s.t. } f(x) = y.$$

Definition (Preimage)**.** An element $x \in A$ in the domain is in the *preimage* of a set D if $f(x) \in D$. *In other words,*

$$x \in f^{-1}(D) \iff f(x) \in D.$$

Exercise (Injectivity and composition)*.* Let $f\colon A \to B$ and $g\colon B \to C$. Prove that if $g \circ f$ is injective, then f is injective.[1]

Exercise (Surjectivity and composition)*.* Let $f\colon A \to B$ and $g\colon B \to C$. Prove that if $g \circ f$ is surjective, then g is surjective.[2]

Exercise (Injections, surjections in finite sets)*.* Let A, B be finite sets and $f\colon A \to B$. Show that if $|A| = |B|$, then f injective iff f bijective iff f surjective. (*Count elements.*)

Remark (Final note)*.* It is true that

$$x \in C \implies f(x) \in f(C).$$

However, $f(x) \in f(C)$ *does not imply* $x \in C$. *If you assume that f is injective*, then

$$f(x) \in f(C) \implies x \in C.$$

[1] *Note*: The contrapositive says that f failing to be injective implies $g \circ f$ fails to be injective. *You should try to prove the statement (i) directly and (ii) by proving the contrapositive.* Lastly, *draw a (simple) picture* of sets A, B, C and maps f and g showing what is happening, for the proof via contrapositive.

[2] See last footnote.