Unitary friezes and frieze vectors

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Frieze

In architecture, a *frieze* is an image that repeats itself along one direction.

Conway and Coxeter, 1970s

Definition

A **Conway − Coxeter frieze pattern** is an array of positive integers such that:

1. it is bounded above and below by a row of 1s
2. every diagram

$$
\begin{array}{ccc}
 & b & \\
a & & d \\
 & c & \\
\end{array}
$$

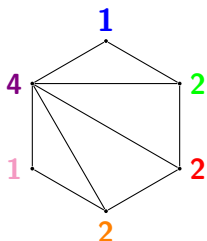satisfies the diamond rule $ad - bc = 1$.

```
1   1   1   1   1   1   1   1   1   1   1   1
  1   4   1   2   2   2   1   4   1   2   2   2
    3   3   1   3   3   1   3   3   1   3   3   1
      2   2   1   4   1   2   2   2   1   4   1   2
        1   1   1   1   1   1   1   1   1   1   1   1
```

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Conway and Coxeter, 1970s

Theorem

A Conway – Coxeter frieze pattern with n nontrivial rows ⟷ a triangulation of an $(n + 3)$-gon.

```
1    1    1    1    1    1    1    1    1    1    1    1
  1    4    1    2    2    2    1    4    1    2    2    2
    3    3    1    3    3    1    3    3    1    3    3    1
      2    2    1    4    1    2    2    2    1    4    1    2
        1    1    1    1    1    1    1    1    1    1    1    1
```

Fomin and Zelevinsky, 2001

Start with a quiver (directed graph) Q on n vertices with no loops and no 2-cycles.
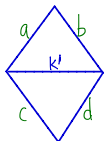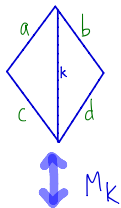
Example: type $\widetilde{\mathbb{A}}_{p,q}$

An acyclic quiver Q is of type $\widetilde{\mathbb{A}}_{p,q}$ if and only if

- its underlying graph is a circular graph with $n = p + q$ vertices,
- the quiver Q has p counterclockwise arrows and q clockwise arrows
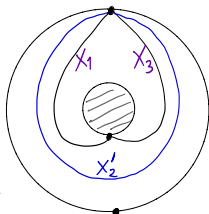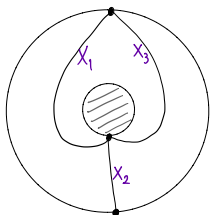
For example, this is a quiver of type $\widetilde{\mathbb{A}}_{1,2}$ $\rightarrow$
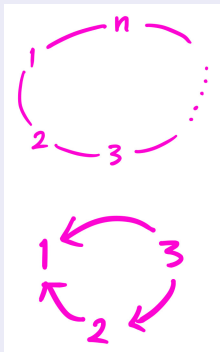
Annulus with $p+q$ marked points on the boundary
(Fomin - Shapiro - Thurston, 2006)

- An <u>arc</u> is an internal curve between marked points

- A <u>triangulation</u> is a maximal collection of non-crossing arcs
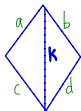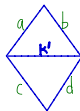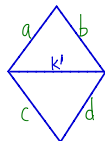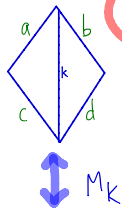
- A <u>flip</u> M_k replaces

with

Ptolemy Rule
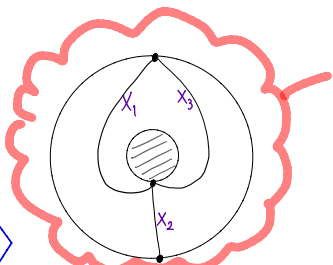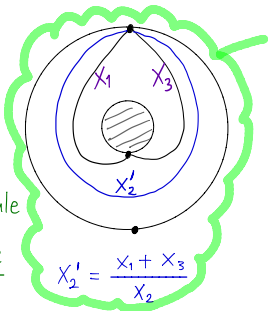
$$k' = \frac{ad+bc}{k}$$

$$x_2' = \frac{x_1 + x_3}{x_2}$$

a cluster $\{x_1, x_2, x_3\}$

mutation at x_2

M_2

a new cluster $\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$
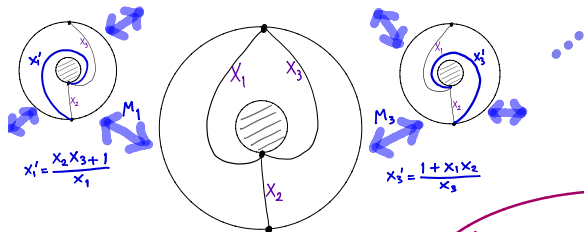
M_k

Ptolemy Rule

$k' = \dfrac{ad+bc}{k}$

$x'_2 = \dfrac{x_1+x_3}{x_2}$

$$x_1' = \frac{x_2 x_3 + 1}{x_1}$$

$$x_3' = \frac{1 + x_1 x_2}{x_3}$$

X_1 X_3

X_2

M_1 M_3

M_2

Repeat this mutation process to produce all clusters

$$x_1'' = \frac{x_2 + x_3^2}{x_1}$$

$$x_3'' = \frac{x_1' + x_1^2}{x_3}$$

X_1 X_3

X_2'

M_1 M_3

$$X_2' = \frac{x_1 + x_3}{x_2}$$

Repeat this mutation process to produce all clusters
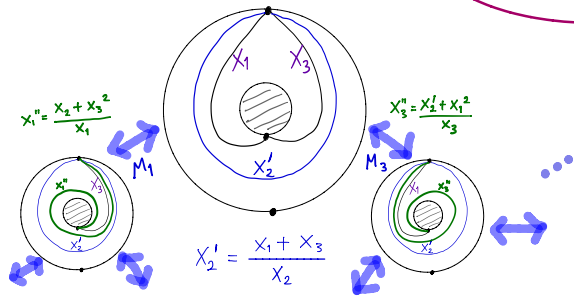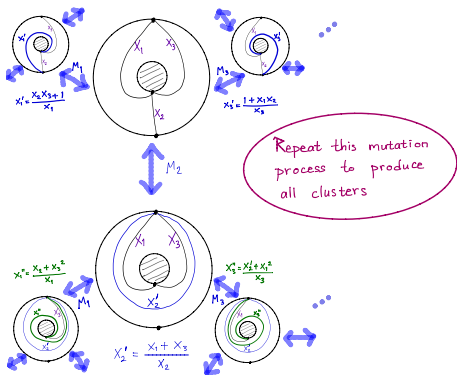
$$x_1' = \frac{x_2 x_3 + 1}{x_1}$$

$$x_3' = \frac{1 + x_1 x_2}{x_3}$$

$$x_1'' = \frac{x_4 + x_3^2}{x_1}$$

$$x_3'' = \frac{x_4^2 + x_1}{x_3}$$

$$x_2' = \frac{x_1 + x_3}{x_2}$$

a b
k
c d

M_k

a b
k'
c d

Ptolemy Rule

$$k' = \frac{ad+bc}{k}$$

$X_1 \leftarrow X_3$
X_2

M_2
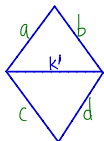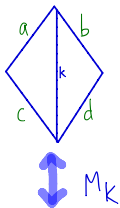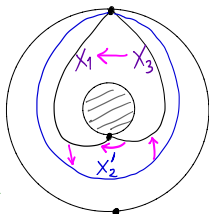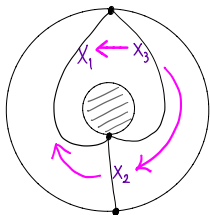
$X_1 \leftarrow X_3$
X_2'

$$X_2' = \frac{X_1 + X_3}{X_2}$$

If j follows i
counterclockwise
along a
triangle

$j \triangle i$ then

draw $j \leftarrow i$

1   3
2

1 $\leftarrow$ 3
2

Friezes

Definition

Let Q be a quiver and $\mathcal{A}(Q)$ the cluster algebra from Q.

- A **frieze** of type Q is a ring homomorphism $\mathcal{F} : \mathcal{A}(Q) \to R$
- We say that $\mathcal{F}$ is **positive integral** if $R = \mathbb{Z}$ and $\mathcal{F}$ maps every cluster variable to a positive integer

Examples:

- The identity frieze $Id : \mathcal{A}(Q) \to \mathcal{A}(Q)$.
- A frieze $\mathcal{F} : \mathcal{A}(Q) \to \mathbb{Z}$ defined by fixing a cluster $\mathbf{x}$ and sending each cluster variable in $\mathbf{x}$ to 1.

Friezes examples

1		1		1		1	
	x_3		$\frac{x_1 x_3 + 1 + x_2}{x_2 x_3}$		$\frac{x_2+1}{x_1}$		x_1
x_2		$\frac{x_1 x_3 + 1}{x_2}$		$\frac{x_2^2 + 2x_2 + 1 + x_1 x_3}{x_1 x_2 x_3}$		x_2	
	x_1		$\frac{x_1 x_3 + 1 + x_2}{x_1 x_2}$		$\frac{x_2+1}{x_3}$		x_3
1		1		1		1	

Figure: The identity frieze $Id : \mathcal{A}(Q) \to \mathcal{A}(Q)$ for the type $\mathbb{A}_3$ quiver $Q = 1 \to 2 \leftarrow 3$.

1		1		1		1	
	1		3		2		1
1		2		5		1	
	1		3		2		1
1		1		1		1	

Figure: Setting $x_1 = x_2 = x_3 = 1$ produces a Conway – Coxeter frieze pattern.

Unitary friezes

Definition

We say that a frieze $\mathcal{F}$ is **unitary** if there exists a cluster $\mathbf{x}$ in $\mathcal{A}(Q)$ such that $\mathcal{F}$ maps every cluster variable in $\mathbf{x}$ to 1.

Proposition 1 (G – Schiffler)

Let $\mathcal{F}$ be a positive unitary integral frieze, i.e., there is a cluster $\mathbf{x}$ such that $\mathcal{F}(u) = 1$ for all $u \in \mathbf{x}$. Then $\mathbf{x}$ is unique.

Sketch of Proof: If u is a cluster variable not in a cluster $\mathbf{x}$, then the Laurent expansion of u in $\mathbf{x}$ has two or more terms.

Remark

All positive integral friezes of type $\mathbb{A}$ are unitary (due to Conway and Coxeter), but there are non-unitary positive integral friezes of type $\mathbb{D}$, $\widetilde{\mathbb{D}}$, $\mathbb{E}$, and $\widetilde{\mathbb{E}}$ (due to Fontaine and Plamondon).

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Theorem 2 (G – Schiffler)

All positive integral friezes of type $\widetilde{\mathbb{A}}_{p,q}$ are unitary.

Example: There are the two friezes of type $\widetilde{\mathbb{A}}_{1,2}$, up to translation.
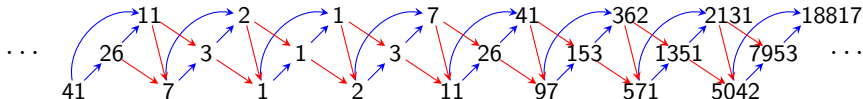

Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.
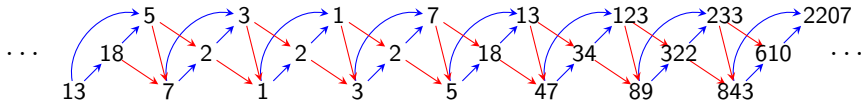

Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Algorithm for finding the cluster where each cluster variable has frieze value 1:

Let $\mathcal{F}$ be a positive integral frieze of type $\widetilde{\mathbb{A}}_{p,q}$. Pick any acyclic cluster $\mathbf{x_0} := \{x_1, \ldots, x_n\}$. If not all cluster variables of $\mathbf{x_0}$ have weight 1, we mutate $\mathbf{x_0}$ at x_k with maximal frieze value. Let $x_k' := \mu_k(x_k)$ Then:

- $\mathcal{F}(x_k') < \mathcal{F}(x_k)$
- Furthermore, if the vertex k is not a sink/source, then $\mathcal{F}(x_k') = 1$

If not every cluster variable in $\mathbf{x_1} := \{x_k'\} \cup \mathbf{x_0} \setminus \{x_k\}$ has weight 1, repeat this procedure, and so on. Since $\mathcal{F}$ is positive integral, this process must stop.

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$
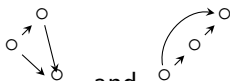
Every acyclic shape, for example, and tells us the frieze values of a cluster.

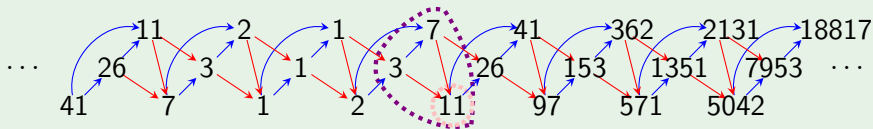## Example (A possible step in the algorithm)

Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

Mutating at the position with frieze value 11 produces a new frieze value $\frac{3 \times 7 + 1}{11} = 2 < 11$.

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Every acyclic shape, for example, and tells us the frieze values of a cluster.

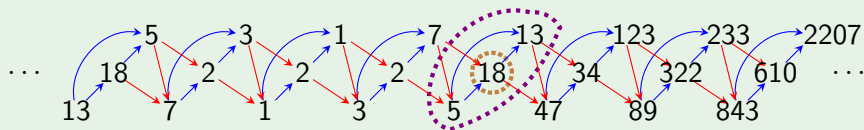## Example (A possible step in the algorithm)

Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

Mutating at the position with frieze value 18 (which is not a sink/source) produces a new frieze value $\frac{5+13}{18} = 1$.

Frieze vectors

Definition

Fix a cluster $\mathbf{x} = (x_1, \ldots, x_n)$.

- A vector $(a_1, \ldots, a_n) \in \mathbb{Z}_{>0}^n$ can be used to define a frieze $\mathcal{F} : \mathcal{A}(Q) \to \mathbb{Q}$ by defining $\mathcal{F}(x_i) = a_i$ for all $i = 1, \ldots, n$.
- We say that $(a_1, \ldots, a_n)$ is a **positive integral frieze vector** relative to $\mathbf{x}$ if $\mathcal{F}$ maps every cluster variable to a positive integer.
- If $(a_1, \ldots, a_n)$ determines a unitary frieze, we say that $(a_1, \ldots, a_n)$ is a **unitary** frieze vector.
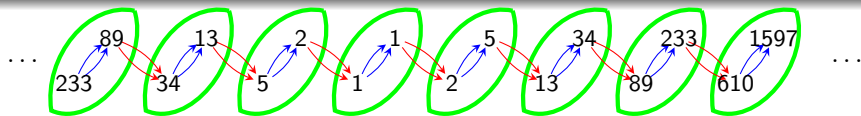
The slices display the frieze vectors
$\ldots, (233, 89), (34, 13), (5, 2), (1, 1), (2, 5), (13, 34), (89, 233), (610, 1597), \ldots$
relative to a cluster with the quiver $1 \Rightarrow 2$.

Frieze vectors algorithm

Proposition 3

A vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ is a frieze vector relative to an acyclic Q iff a_k divides

$$\prod_{k \to j \text{ in } Q} x_j + \prod_{k \leftarrow j \text{ in } Q} x_j$$
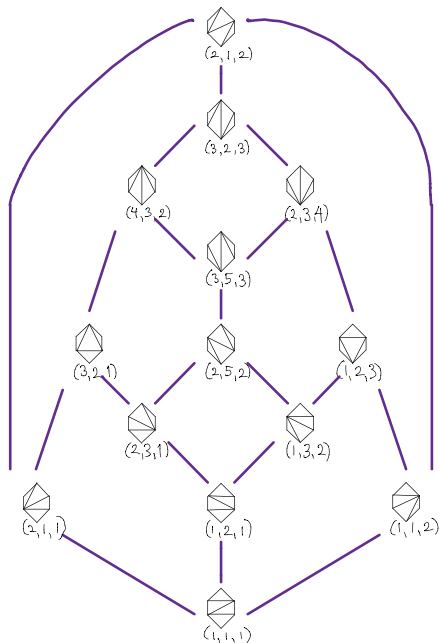
for all $k = 1, \ldots, n$.

Example

A vector $(a_1, a_2, a_3) \in \mathbb{Z}^3_{>0}$ is a positive frieze vector relative to $1 \to 2 \leftarrow 3$ iff

$$\frac{a_2 + 1}{a_1}, \frac{a_1 a_3 + 1}{a_2}, \frac{a_2 + 1}{a_3}$$

are integers.

Frieze vectors

Theorem 4 (G – Schiffler)

Fix $\mathcal{A}(Q)$ and fix an arbitrary cluster $\mathbf{x} = (x_1, \ldots, x_n)$. Then there is a bijection between clusters in $\mathcal{A}(Q)$ and unitary frieze vectors relative to $\mathbf{x}$.

Sketch of Proof: Define

$$\phi : \{ \text{ clusters in } \mathcal{A}(Q)\} \to \{ \text{ unitary frieze vectors } \}$$
$$\mathbf{x}' = \{x_1', \ldots, x_n'\} \mapsto \phi(\mathbf{x}') = \mathcal{F}(\mathbf{x}) = (a_1, \ldots, a_n)$$

where $\mathcal{F}$ is the frieze defined by specializing the cluster variables in $\mathbf{x}'$ to 1. Then ϕ is a bijection. Injectivity follows from Proposition 1. Surjectivity follows from the construction of ϕ.

Friezahedron (work in progress)

In type $\mathbb{A}_n$, $\mathbb{D}_n$, and $\mathbb{E}_6$, it is known that there are finitely many positive integral frieze vectors. Take the convex hull of these points in $\mathbb{R}^n$.

```
sage: V = [ [1, 1, 1],
[1, 1, 2], [1, 2, 1], [1,
2, 3], [1, 3, 2], [2, 1,
1], [2, 1, 2], [2, 3, 1],
[2, 3, 4], [2, 5, 2], [3,
2, 1], [3, 2, 3], [3, 5,
3], [4, 3, 2] ]
sage: P = Polyhedron(V)
```


Frieze vectors relative to (x_1, x_2, x_3),
$Q: 1 \to 2 \leftarrow 3$
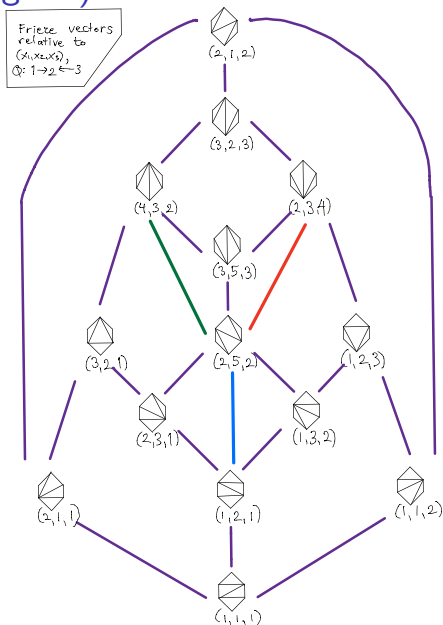
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Comments and questions

Thank you! Hvala!