

Unitary friezes and frieze vectors

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Frieze

A *frieze* is an image that repeats itself along one direction. In architecture, a frieze is a decoration running horizontally below a ceiling or roof.



Conway and Coxeter, 1970s

Definition

A **Conway – Coxeter frieze pattern** is an array of positive integers such that:

- 1 it is bounded above and below by a row of 1s
- 2 every diamond

$$\begin{array}{ccccc} & & b & & \\ & a & & d & \\ & & c & & \end{array}$$

satisfies the diamond rule $ad - bc = 1$.

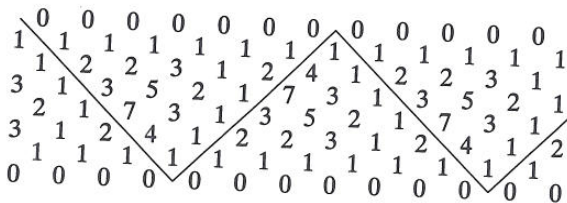
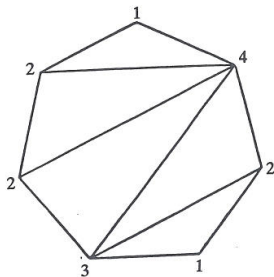
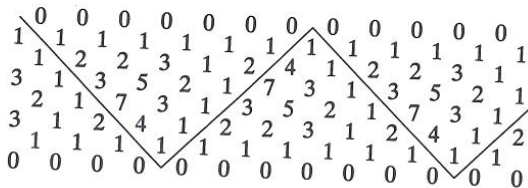


Table 1.

Conway and Coxeter, 1970s

Theorem

A Conway – Coxeter frieze with n nontrivial rows \longleftrightarrow a triangulation of an $(n + 3)$ -gon



Fomin and Zelevinsky, 2001

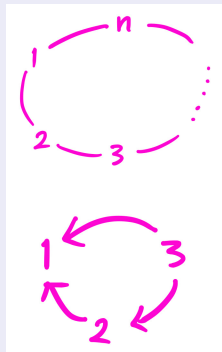
Start with a quiver (directed graph) Q on n vertices with no loops and no 2-cycles.

Example: type $\tilde{A}_{p,q}$

An acyclic quiver Q is of type $\tilde{A}_{p,q}$ if and only if

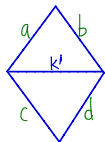
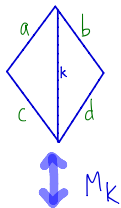
- its underlying graph is a circular graph with $n = p + q$ vertices,
- the quiver Q has p counterclockwise arrows and q clockwise arrows

For example, this is a quiver of type $\tilde{A}_{1,2} \rightarrow$



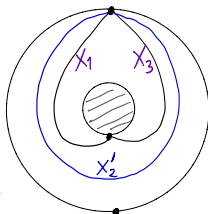
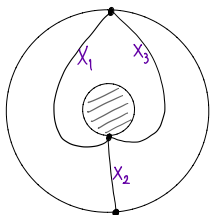
Annulus with $p+q$ marked points on the boundary
(Fomin - Shapiro - Thurston, 2006)

- An arc is an internal curve between marked points
- A triangulation is a maximal collection of non-crossing arcs
- A flip M_K replaces

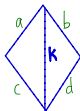


Ptolemy Rule

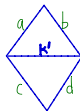
$$k' = \frac{ad+bc}{k}$$



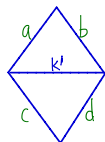
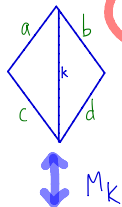
$$x_2' = \frac{x_1 + x_3}{x_2}$$



with

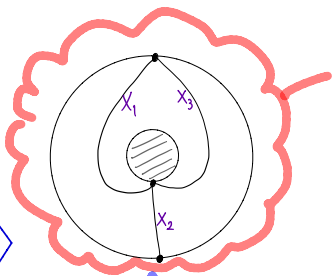


a cluster $\{x_1, x_2, x_3\}$

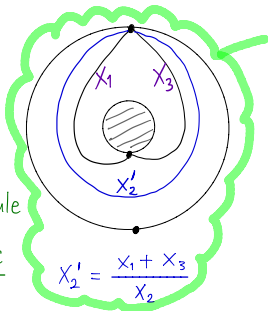


Ptolemy Rule

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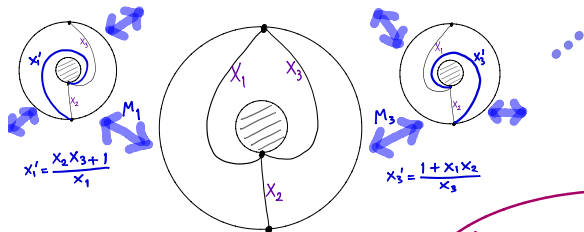


M_2 mutation at x_2

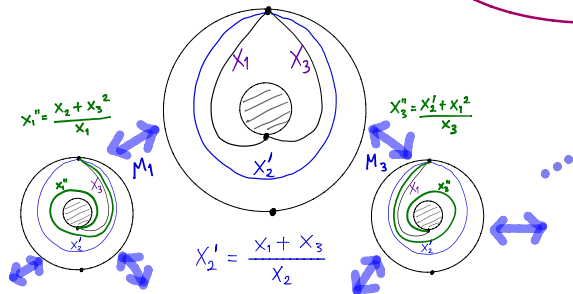


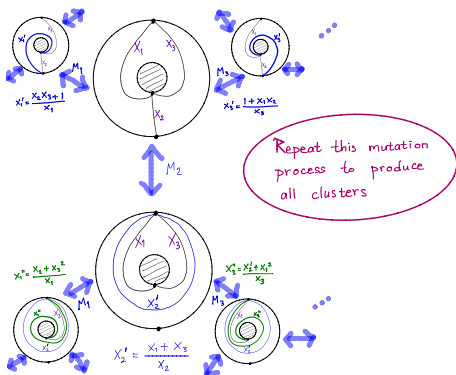
a new cluster $\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$

$$x_2' = \frac{x_1 + x_3}{x_2}$$



Repeat this mutation process to produce all clusters



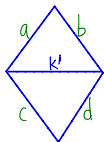
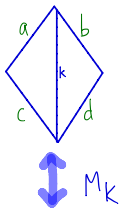


Def (Fomin – Zelevinsky, 2001)

- $\{ \text{cluster variables} \} =$

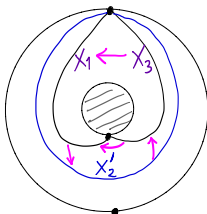
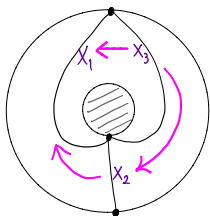
$$\bigcup_{\text{all clusters } \mathbf{x}} \{ \text{elements of } \mathbf{x} \}$$

- The **cluster algebra** $\mathcal{A}(Q)$ is the \mathbb{Z} -algebra of $Q(x_1, \dots, x_n)$ generated by all cluster variables.



Ptolemy Rule

$$k' = \frac{ad+bc}{k}$$



$$x_2' = \frac{x_1 + x_3}{x_2}$$

If j follows i
counterclockwise
along a
triangle

$j \triangle i$ then
draw $j \leftarrow i$



Friezes

Definition

Let Q be a quiver and $\mathcal{A}(Q)$ the cluster algebra from Q .

- A **frieze** of type Q is a ring homomorphism $\mathcal{F} : \mathcal{A}(Q) \rightarrow R$
- We say that \mathcal{F} is **positive integral** if $R = \mathbb{Z}$ and \mathcal{F} maps every cluster variable to a positive integer
- We say that \mathcal{F} is **unitary** if there exists a cluster \mathbf{x} in $\mathcal{A}(Q)$ such that \mathcal{F} maps every cluster variable in \mathbf{x} to 1

Examples:

- The identity frieze $Id : \mathcal{A}(Q) \rightarrow \mathcal{A}(Q)$.
- A frieze $\mathcal{F} : \mathcal{A}(Q) \rightarrow \mathbb{Z}$ defined by fixing a cluster \mathbf{x} and sending each cluster variable in \mathbf{x} to 1.

Friezes examples

$$\begin{array}{ccccccc}
 1 & & 1 & & 1 & & 1 \\
 & x_3 & & \frac{x_1 x_3 + 1 + x_2}{x_2 x_3} & & \frac{x_2 + 1}{x_1} & x_1 \\
 x_2 & & \frac{x_1 x_3 + 1}{x_2} & & \frac{x_2^2 + 2x_2 + 1 + x_1 x_3}{x_1 x_2 x_3} & & x_2 \\
 & x_1 & & \frac{x_1 x_3 + 1 + x_2}{x_1 x_2} & & \frac{x_2 + 1}{x_3} & x_3 \\
 1 & & 1 & & 1 & & 1
 \end{array}$$

Figure: The identity frieze $Id : \mathcal{A}(Q) \rightarrow \mathcal{A}(Q)$ for the type \mathbb{A}_3 quiver $Q = 1 \rightarrow 2 \leftarrow 3$.

$$\begin{array}{ccccccc}
 1 & & 1 & & 1 & & 1 \\
 & 1 & & 3 & & 2 & & 1 \\
 1 & & 2 & & 5 & & 1 & \\
 & 1 & & 3 & & 2 & & 1 \\
 1 & & 1 & & 1 & & 1 &
 \end{array}$$

Figure: Setting $x_1 = x_2 = x_3 = 1$ produces a Conway – Coxeter frieze pattern.

Unitary friezes

Definition

We say that a frieze \mathcal{F} is **unitary** if there exists a cluster \mathbf{x} in $\mathcal{A}(Q)$ such that \mathcal{F} maps every cluster variable in \mathbf{x} to 1.

Proposition 1 (G – Schiffler)

Let \mathcal{F} be a positive unitary integral frieze, i.e., there is a cluster \mathbf{x} such that $\mathcal{F}(u) = 1$ for all $u \in \mathbf{x}$. Then \mathbf{x} is unique.

Sketch of Proof: If u is a cluster variable not in a cluster \mathbf{x} , then the Laurent expansion of u in \mathbf{x} has two or more terms.

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Remark

All positive integral friezes of type \mathbb{A} are unitary (due to Conway and Coxeter), but there are non-unitary positive integral friezes of type \mathbb{D} , $\widetilde{\mathbb{D}}$, \mathbb{E} , and $\widetilde{\mathbb{E}}$ (due to Fontaine and Plamondon).

Friezes of type $\tilde{\mathbb{A}}_{p,q}$

Theorem 2 (G – Schiffler)

All positive integral friezes of type $\tilde{\mathbb{A}}_{p,q}$ are unitary.

Example: There are the two friezes of type $\tilde{\mathbb{A}}_{1,2}$, up to translation.

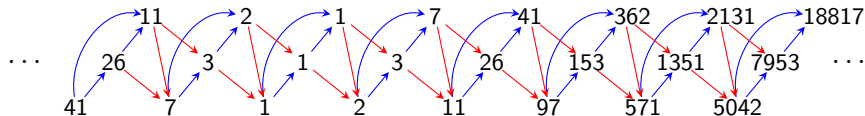


Figure: An $\tilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

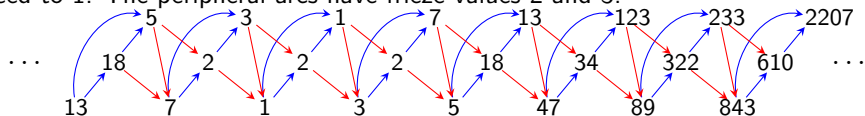


Figure: An $\tilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

Friezes of type $\tilde{\mathbb{A}}_{p,q}$

Algorithm for finding the triangulation where all arcs have weight one:

Pick some acyclic cluster $\mathbf{x}_0 = \{x_1, \dots, x_n\}$. If not all cluster variables of \mathbf{x}_0 have weight 1, we mutate \mathbf{x}_0 at x_k with maximal frieze value. Then:

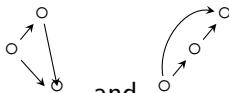
- $\text{frieze}(x'_k) < \mathcal{F}(x_k)$
- Furthermore, if the vertex k is not a sink/source, then $F(x_k) = 1$

If not every cluster variable in \mathbf{x}_1 has weight 1, repeat this procedure, and so on. Since \mathcal{F} is positive integral, this process must stop.

Lemma (for the algorithm)

Let \mathcal{F} be a positive frieze of type $\tilde{\mathbb{A}}_{p,q}$. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a cluster such that $\mathcal{F}(x) = 1$ for each regular (i.e. peripheral) cluster variable. Let k be such that $\mathcal{F}(x_k) \geq \mathcal{F}(x_j)$ for all j , and suppose that $\mathcal{F}(x_k) > 1$. Then $\mathcal{F}(\mu_k(x_k)) < \mathcal{F}(x_k)$ and if $\mu_k(x_k)$ a regular cluster variable then $\mathcal{F}(\mu_k(x_k)) = 1$.

Friezes of type $\tilde{\mathbb{A}}_{p,q}$



Every acyclic shape, for example,
values of a cluster.

and tells us the frieze

Example (A possible step in the algorithm)

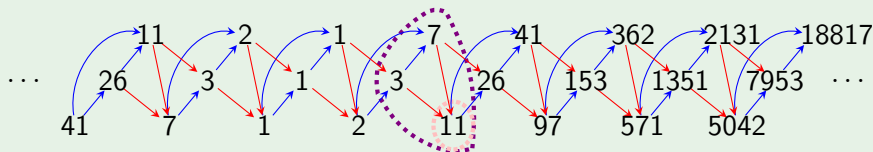
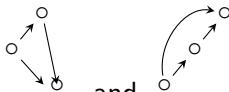


Figure: An $\tilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

Mutating at the position with frieze value 11 produces a new frieze value $\frac{3 \times 7 + 1}{11} = 2 < 11$.

Friezes of type $\tilde{\mathbb{A}}_{p,q}$



Every acyclic shape, for example,

and

tells us the frieze values of a cluster.

Example (A possible step in the algorithm)

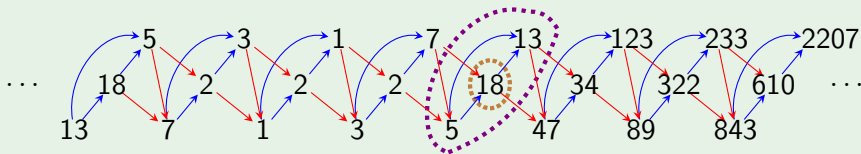


Figure: An $\tilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

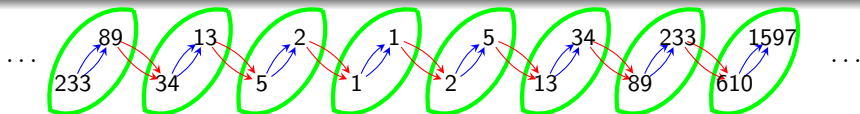
Mutating at the position with frieze value 18 (which is not a sink/source) produces a new frieze value $\frac{5+13}{18} = 1$.

Frieze vectors

Definition

Fix a cluster $\mathbf{x} = (x_1, \dots, x_n)$.

- A vector $(a_1, \dots, a_n) \in \mathbb{Z}_{>0}^n$ can be used to define a frieze $\mathcal{F} : \mathcal{A}(Q) \rightarrow \mathbb{Q}$ by defining $\mathcal{F}(x_i) = a_i$ for all $i = 1, \dots, n$.
- We say that (a_1, \dots, a_n) is a **positive integral frieze vector** relative to \mathbf{x} if \mathcal{F} maps every cluster variable to a positive integer (as opposed to \mathbb{Q}).
- If (a_1, \dots, a_n) determines a unitary frieze, we say that (a_1, \dots, a_n) is a **unitary** frieze vector.



The slices display the frieze vectors

$\dots, (233, 89), (34, 13), (5, 2), (1, 1), (2, 5), (13, 34), (89, 233), (610, 1597), \dots$
relative to a cluster with the quiver $1 \Rightarrow 2$.

Frieze vectors algorithm

Proposition 3

A vector $(a_1, \dots, a_n) \in \mathbb{Z}^n$ is a frieze vector relative to an acyclic Q iff a_k divides

$$\prod_{k \rightarrow j \text{ in } Q} x_j + \prod_{k \leftarrow j \text{ in } Q} x_j$$

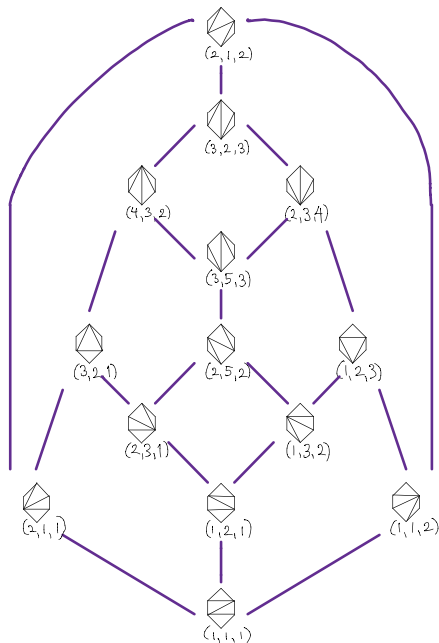
for all $k = 1, \dots, n$.

Example

A vector $(a_1, a_2, a_3) \in \mathbb{Z}_{>0}^3$ is a positive frieze vector relative to $1 \rightarrow 2 \leftarrow 3$ iff

$$\frac{a_2 + 1}{a_1}, \frac{a_1 a_3 + 1}{a_2}, \frac{a_2 + 1}{a_3}$$

are integers.



Frieze vectors

Theorem 4 (G – Schiffler)

Fix $\mathcal{A}(Q)$ and fix an arbitrary cluster $\mathbf{x} = (x_1, \dots, x_n)$. Then there is a bijection between clusters in $\mathcal{A}(Q)$ and unitary frieze vectors relative to \mathbf{x} .

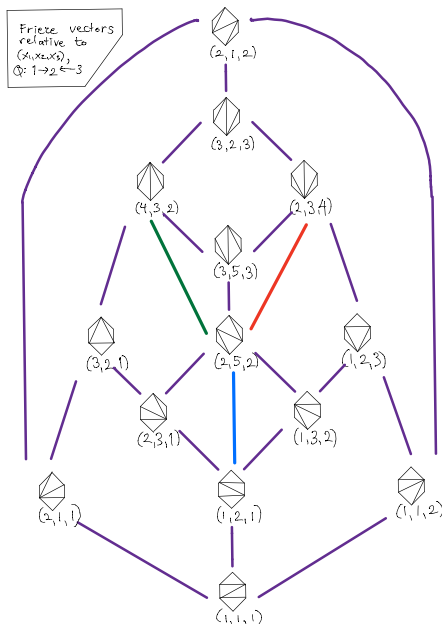
Sketch of Proof: Define

$$\begin{aligned}\phi : \{ \text{clusters in } \mathcal{A}(Q) \} &\rightarrow \{ \text{unitary frieze vectors} \} \\ \mathbf{x}' = \{x'_1, \dots, x'_n\} &\mapsto \phi(\mathbf{x}') = \mathcal{F}(\mathbf{x}) = (a_1, \dots, a_n)\end{aligned}$$

where \mathcal{F} is the frieze defined by specializing the cluster variables in \mathbf{x}' to 1. Then ϕ is a bijection. Injectivity follows from Proposition 1. Surjectivity follows from the construction of ϕ .

Comments and questions

Thank you!
Hvala!



Friezahedron (current work)

In type \mathbb{A}_n , \mathbb{D}_n , and \mathbb{E}_6 , it is known that there are finitely many positive integral frieze vectors. Take the convex hull of these points in \mathbb{R}^n .

sage: $V = [[1, 1, 1], [1, 1, 2], [1, 2, 1], [1, 2, 3], [1, 3, 2], [2, 1, 1], [2, 1, 2], [2, 3, 1], [2, 3, 4], [2, 5, 2], [3, 2, 1], [3, 2, 3], [3, 5, 3], [4, 3, 2]]$

sage: $P = \text{Polyhedron}(V)$

