

Conway – Coxeter friezes, cluster algebras, and SageMath

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Sage Days 95: Women In Sage
Saint-Faustin-Lac-Carré, Québec
Thursday, July 12, 2018

Outline

1. Conway – Coxeter friezes

- ▶ A Conway – Coxeter frieze is a Catalan object
- ▶ Connection other classical objects like continued fractions and binary words
- ▶ Connection to quiver representations and cluster algebras
- ▶ Using SageMath to draw friezes using LaTeX

2. Cluster algebras

- ▶ Commutative algebras with a lot of combinatorial structure
- ▶ Using SageMath to do cluster algebra computation

Frieze

A *frieze* is an image that repeats itself along one direction. The name comes from architecture, where a frieze is a decoration running horizontally below a ceiling or roof.

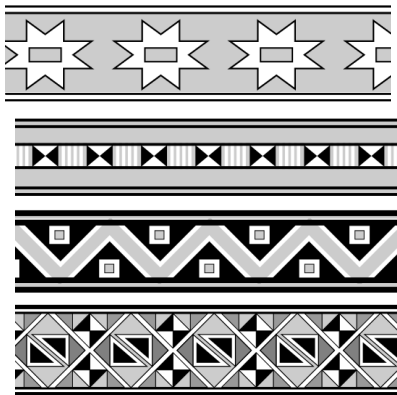


Figure: M. Ascher, *Ethnomathematics*, p162.

Conway-Coxeter frieze

Definition

A (type A) **frieze** is an array such that:

1. it is bounded above and below by a row of 1s
2. every diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies the diamond rule $ad - bc = 1$.

A **Conway-Coxeter frieze** consists of only positive integers.

Example (an integer frieze)

		1	1	1	1	1	1	1	...
Row 2	...	3	1	2	2	1	3	1	
		2	2	1	3	1	2	2	...
	...	1	1	1	1	1	1	1	

Glide symmetry

A *glide symmetry* is a combination of a translation and a reflection.

		1	1	1	1	1	1	...
Row 2	...	3	1	2	2	1	3	1
		2	2	1	3	1	2	2
	...	1	1	1	1	1	1	1

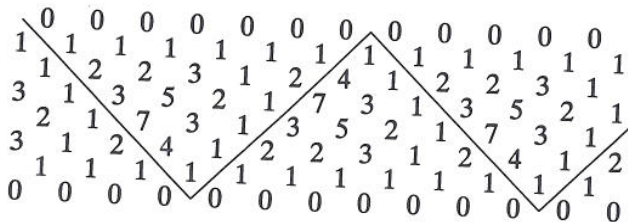


Table 1.

Children practicing arithmetic

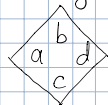
Note: every frieze is completely determined by the 2nd row.

Frieze 1

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	2	3	1	2	4	1	2	2	3	1			
		3	5											
			7											

Rule

Every



satisfies

$$ad - bc = 1$$

Frieze 2

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	2	2	3	1	2	5	1	2	2	2	3	1	
		3	3				9							
			4											

Children practicing arithmetic: Answer Key

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	1	0	1	0	1	0	1	1	1	0	1	1
1	1	2	2	3	1	1	2	4	1	1	2	1	1	1	1
3	1	3	5	3	2	1	1	7	3	1	3	2	2	3	1
2	1	7	3	3	1	1	2	3	5	3	2	1	3	5	2
3	1	2	4	1	1	2	3	2	3	2	1	1	7	3	1
1	1	1	1	1	1	1	2	1	3	1	2	1	4	1	2
0	0	0	0	0	0	0	1	0	1	0	1	0	1	1	0

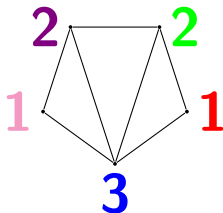
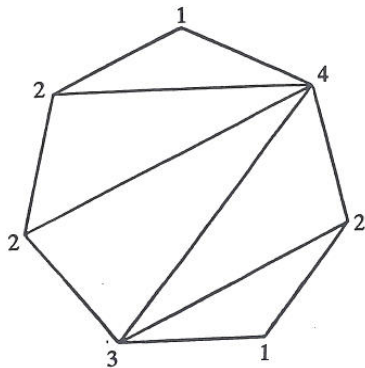
Table 1.

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	3	3	1	2	5	1	1	2	2	2	3	3
1	3	3	5	3	2	1	1	9	4	1	3	3	5	3	2
1	4	7	3	3	1	4	7	3	3	1	4	7	3	3	1
2	1	9	4	1	3	3	5	3	2	1	9	4	1	1	1
1	1	2	5	1	2	2	2	3	1	2	5	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 2.

Frieze Sage demo

What do the numbers around each polygon count?



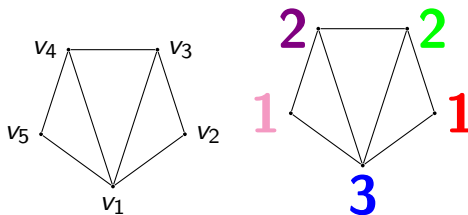
Conway and Coxeter (1970s)

Theorem

A Conway – Coxeter frieze with n nontrivial rows \longleftrightarrow a triangulation of an $(n + 3)$ -gon

Note: Hence Conway – Coxeter friezes are Catalan objects.

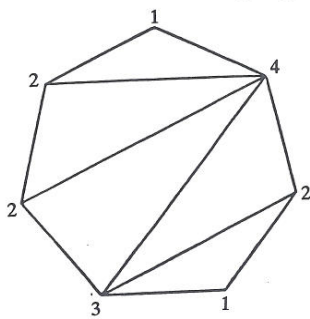
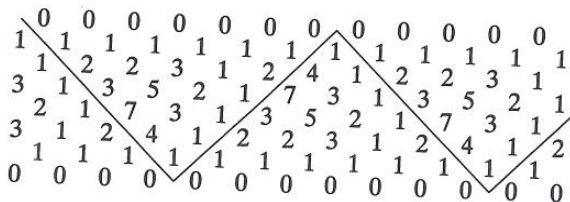
		1	1	1	1	1	1	1	...
Row 2	...	3	1	2	2	1	3	1	
		2	2	1	3	1	2	2	...
	...	1	1	1	1	1	1	1	



Conway and Coxeter (1970s)

Theorem

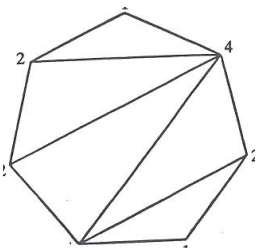
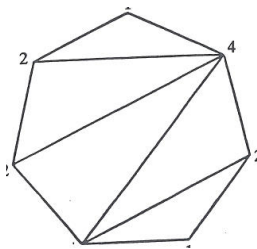
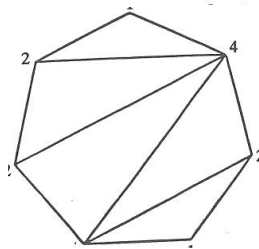
A Conway – Coxeter frieze with n nontrivial rows \longleftrightarrow a triangulation of an $(n + 3)$ -gon



Primary school algorithm

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	2	2	3	1	2	1	4	1	2	1	2	1	3	1
3	2	1	7	3	1	3	7	5	3	2	1	3	7	5	2
3	1	1	2	4	1	2	3	2	5	3	2	1	1	2	1
0	1	0	1	1	0	1	1	0	1	0	1	0	1	1	0

Table 1.

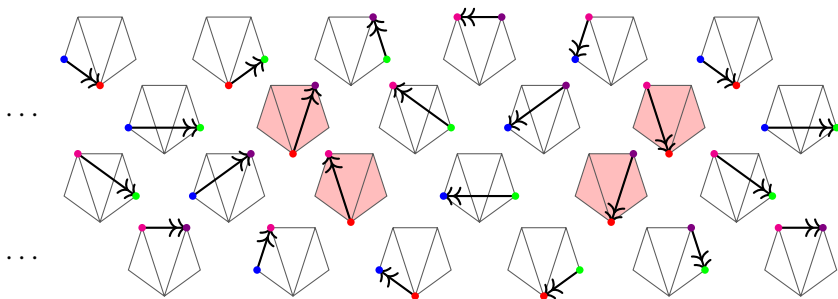


Broline, Crowe, and Isaacs (BCI, 1970s)

Theorem

Entries of a frieze \longleftrightarrow edges between two vertices.

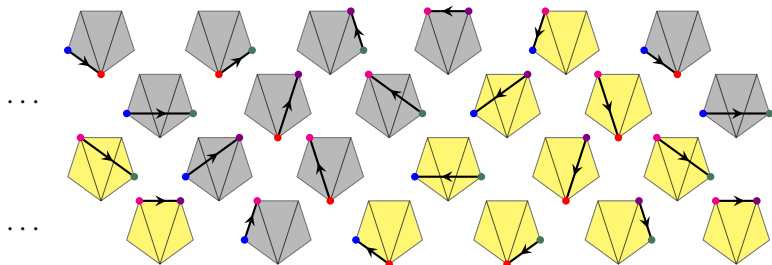
		1	1	1	1	1	1	...
Row 2	...	3	1	2	2	1	3	
		2	2	1	3	1	2	...
	...	1	1	1	1	1	1	



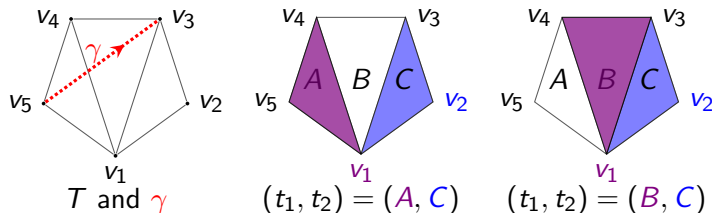
Glide symmetry (again)

Recall: A *glide symmetry* is a combination of a translation and a reflection. If we forget the arrows' orientation, the diagonals have glide symmetry.

		1	1	1	1	1	1	...
Row 2	...	3	1	2	2	1	3	1
		2	2	1	3	1	2	...
	...	1	1	1	1	1	1	1



Broline, Crowe, and Isaacs (BCI, 1970s)



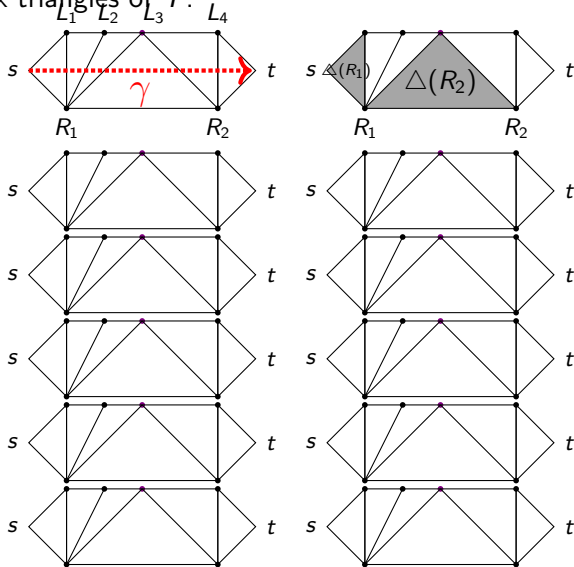
Definition (BCI tuple)

Let R_1, R_2, \dots, R_r be the boundary vertices to the right of γ . A **BCI tuple** for γ is an r -tuple (t_1, \dots, t_r) such that:

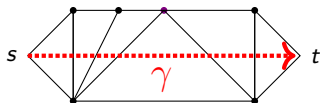
- (B1) the i -th entry t_i is a triangle of T having R_i as a vertex.
- (B2) the entries are pairwise distinct.

How many BCI tuples are there?

Example: A triangulation T of an octagon and a diagonal γ which crosses six triangles of T .



Continued fractions (Çanakçı, Schiffler)



Continued Fraction



3 , 1 , 2

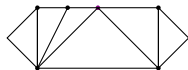
$$3 + \frac{1}{\left(1 + \frac{1}{2}\right)} \stackrel{\text{compute}}{=}$$

Binary number representations

A **binary number representation** is an expression of a nonzero integer in the base-2 numeral system, with digits consisting of 1s and 0s. For example,

- ▶ $1 = 1 * 2^0$ (in decimal) is written as 1 (in binary).
- ▶ $2 = 1 * 2^1$ (in decimal) is written as 10 (in binary).
- ▶ $4 = 1 * 2^2$ (in decimal) is written as 100 (in binary).
- ▶ $5 = 1 * 2^2 + 1 * 2^0$ (in decimal) is written as 101 (in binary).
- ▶ $29 = 16 + 8 + 4 + 1 = 1 * 2^4 + 1 * 2^3 + 1 * 2^2 + 1 * 2^0$ is written as 11101 (in binary).

Subwords of binary numbers



We can think of “11101” as a word in the alphabets $\{0, 1\}$. The following are all the (scattered, non-consecutive) subwords of “11101” which start with “1”:

- ▶ 11101 (empty)
- ▶ 11101: 1
- ▶ 11101: 10
- ▶ 11101: 11
- ▶ 11101: 101
- ▶ 11101: 110
- ▶ 11101: 111
- ▶ 11101: 1101
- ▶ 11101: 1110
- ▶ 11101: 1111
- ▶ 11101: 11101 (the word itself)

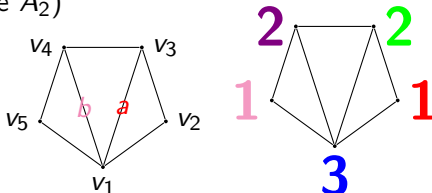
Cluster algebras (Fomin and Zelevinsky, 2000)

A **cluster algebra** is a commutative ring generated by sets called **clusters**. Each element of a cluster is called **cluster variables**.

Theorem (Caldero – Chapoton, 2006): The cluster variables of a cluster algebra from a triangulated polygon (type A) form a frieze.

$$\begin{array}{cccccccccccccccc}
 \dots & & 1 & & 1 & & 1 & & 1 & & 1 & & \dots & & 1 \\
 & \frac{1+a+b}{ab} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} & & b & & \frac{1+a+b}{ab} & & & \\
 \dots & & \frac{1+a}{b} & & b & & \frac{1+a+b}{ab} & & a & & \frac{1+b}{a} & & \frac{1+a+b}{ab} & & \frac{1+a}{b} \\
 & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
 \end{array}$$

(Example: type A_2)



- Remark: If the variables are specialized to 1, we recover the Conway – Coxeter positive integer frieze.

Example: a type A_3 frieze

1		1		1		1	
	x_3		$\frac{x_1 x_3 + 1 + x_2}{x_2 x_3}$		$\frac{x_2 + 1}{x_1}$		x_1
x_2		$\frac{x_1 x_3 + 1}{x_2}$		$\frac{x_2^2 + 2x_2 + 1 + x_1 x_3}{x_1 x_2 x_3}$		x_2	
	x_1		$\frac{x_1 x_3 + 1 + x_2}{x_1 x_2}$		$\frac{x_2 + 1}{x_3}$		x_3
1		1		1		1	

Frieze over the positive integers

Specializing $x_1 = x_2 = x_3 = 1$ gives a Conway – Coxeter positive integer frieze

1	1	1	1	
	1	3	2	1
1	2	5	1	
	1	3	2	1
1	1	1	1	

Frieze over the integers

Specializing $x_1 = x_2 = 1$ and $x_3 = -1$ gives

1	1	1	1	
	-1	-1	2	1
1	0	-3	1	
	1	1	-2	-1
1	1	1	1	

Frieze over the Gaussian integers $\mathbb{Z}[i]$

Specializing $x_1 = 1$, $x_2 = i$, and $x_3 = i$ gives

1	1	1	1
i	$-1 - 2i$	$1 + i$	1
i	$1 - i$	$-3i$	i
1	$2 - i$	$1 - i$	i
1	1	1	1

Frieze over the quadratic integer ring $\mathbb{Z}[\sqrt{-3}]$

Specializing $x_1 = 1$, $x_2 = \frac{1+\sqrt{-3}}{2}$, $x_3 = 1$ gives

$$1$$

$$1$$

$$1$$

$$1$$

$$2 - \sqrt{-3}$$

$$\frac{3 + \sqrt{-3}}{2}$$

$$\frac{1 + \sqrt{-3}}{2}$$

$$1 - \sqrt{-3}$$

$$\frac{7 - \sqrt{-3}}{2}$$

$$1$$

$$2 - \sqrt{-3}$$

$$\frac{3 + \sqrt{-3}}{2}$$

$$1$$

$$1$$

$$1$$

Cluster algebra Sage demo

Website references:

1. **Wikipedia entry**

https://en.wikipedia.org/wiki/Cluster_algebra

2. Cluster Algebras Portal

<http://www.math.lsa.umich.edu/~fomin/cluster.html>

arXiv.org references:

1. Introductory cluster algebra survey by Lauren Williams titled
Cluster algebras: an introduction

<https://arxiv.org/abs/1212.6263>

2. Cluster algebra textbook by Sergey Fomin, Lauren Williams, Andrei Zelevinsky titled **Introduction to cluster algebras**

<https://arxiv.org/abs/1608.05735>

3. Frieze survey by Sophie Morier-Genoud titled **Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics**

<https://arxiv.org/abs/1503.05049>

4. Frieze paper by Emily Gunawan and Ralf Schiffler titled **Frieze vectors and unitary friezes**

<https://arxiv.org/abs/1806.00940>

Thank you