The Box-Ball System (2020 UCONN REU)

Ben Drucker   **Eli Garcia**   **Rose Silver**   Aubrey Rumbolt

Definition. (Young Tableaux)

- A Young tableau (pl. tableaux) is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
- The Young tableau is *standard* if each row is an increasing sequence (going from left to right) and each column is an increasing sequence (going from top to bottom).
- The *reading word* of a standard Young tableau is the permutation formed by concatenating the rows of the tableau from bottom to top.

Example. (Standard Young Tableau)

- $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 \\ \cline{1-1} \end{array}$ is a standard Young tableau. Its reading word is 53412.

- $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array}$ is a nonstandard Young tableau.

$$\pi = 452361 \longrightarrow \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} = P$$

Let P be the empty tableau. Given a permutation $\pi = \pi_1 \cdots \pi_n$, the *Robinson-Schensted (RS) insertion algorithm* is a method of insertion of each element of π into P. The RS insertion algorithm always gives a standard Young tableau.

BOX-BALL SYSTEM AND EXAMPLE

A box-ball system is a collection of discrete time states each containing elements of a permutation, π in one-line notation. We can move forward in this system by rearranging π using the following rule: Place π in a strip of infinite boxes. (This corresponds to the $t = 0$ state of the system.) Then, move each element of π to the nearest empty box to its right, as shown below.
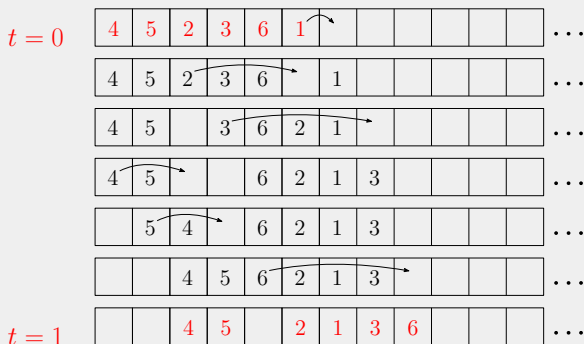
To continue the analysis, keep making BBS moves until we reach a "steady state" (i.e., the system decomposes into *solitons*)
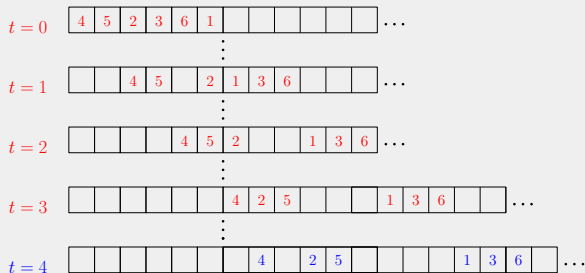
Then, create a (not necessarily standard) tableau by stacking solitons:

Soliton decomposition $\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$ with shape $(3, 2, 1)$.

Selected Results Involving Tableaux

Theorem

The following are equivalent:

1. $SD(\pi) = P(\pi)$.
2. $SD(\pi)$ is a standard tableau.

Conjecture

The following is equivalent to (1) and (2).

3. $\operatorname{sh} SD(\pi) = \operatorname{sh} P(\pi)$.

- We have shown $(1) \iff (2)$ and $(1) \implies (3)$. It remains to prove $(3) \implies (1)$.

SELECTED RESULTS INVOLVING STEADY-STATE TIMES

Theorem: Classification of permutations with steady-state value of $t = 0$

A permutation r has a box-ball steady-state value of $t = 0$ if and only if r is the reading word of a standard tableau.

Theorem: Some permutations with steady-state value of $t = 1$

Let r be the reading word of a standard tableau P. If r' is the permutation resulting from the swapping of a special pattern of elements in r, then r' has a box-ball steady-state value of $t = 1$. (r' is the result of performing a *Knuth move* on r.)

$n - 3$ conjecture

The steady-state time for a permutation in S_n is no more than $n - 3$.

THANK YOU!
QUESTIONS?