Unitary friezes and frieze vectors

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Frieze

A *frieze* is an image that repeats itself along one direction. In architecture, a frieze is a decoration running horizontally below a ceiling or roof.

Conway and Coxeter, 1970s

Definition

A **Conway – Coxeter frieze pattern** is an array of positive integers such that:

1. it is bounded above and below by a row of 1s
2. every diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

   satisfies the diamond rule $ad - bc = 1$.

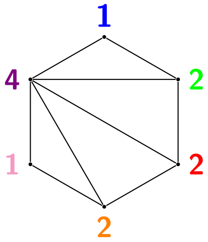
$$\begin{array}{ccccccccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & \\
& 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & & \\
& & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & \\
& & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & \\
& & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

Conway and Coxeter, 1970s

Theorem

A Conway – Coxeter frieze with n nontrivial rows $\longleftrightarrow$ a triangulation of an $(n+3)$-gon.

$$
\begin{array}{cccccccccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & & \\
& 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & & & \\
& & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & & \\
& & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & \\
& & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}
$$



Fomin and Zelevinsky, 2001

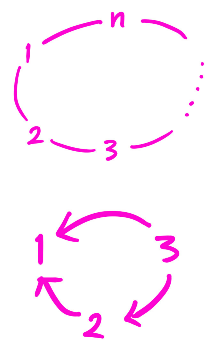
Start with a quiver (directed graph) Q on n vertices with no loops and no 2-cycles.

Example: type $\widetilde{\mathbb{A}}_{p,q}$

An acyclic quiver Q is of type $\widetilde{\mathbb{A}}_{p,q}$ if and only if

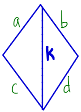
- its underlying graph is a circular graph with $n = p + q$ vertices,
- the quiver Q has p counterclockwise arrows and q clockwise arrows

For example, this is a quiver of type $\widetilde{\mathbb{A}}_{1,2}$ $\rightarrow$

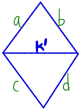


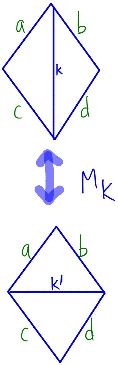
Annulus with p+q marked points on the boundary (Fomin - Shapiro - Thurston, 2006)

- An arc is an internal curve between marked points
- A triangulation is a maximal collection of non-crossing arcs
- A flip M_K replaces



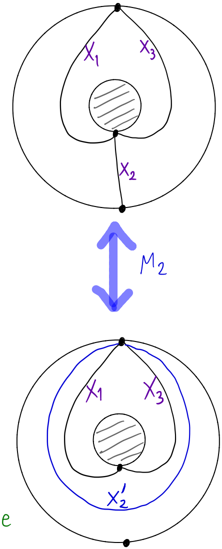
with



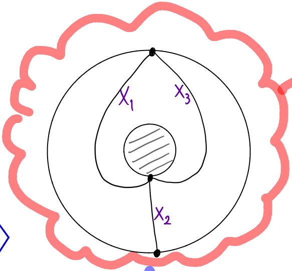


Ptolemy Rule

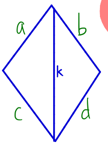
$$k' = \frac{ad+bc}{k}$$



$$X_2' = \frac{X_1 + X_3}{X_2}$$

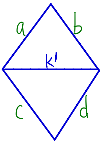


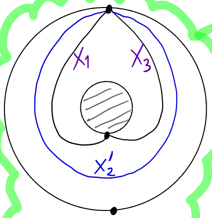
a cluster $\{x_1, x_2, x_3\}$



M_K

M_2 mutation at x_2



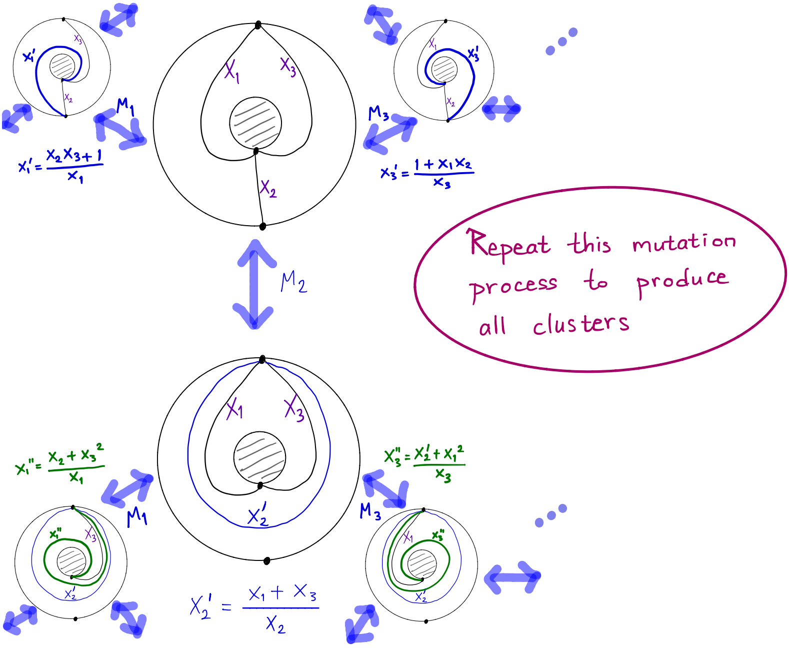


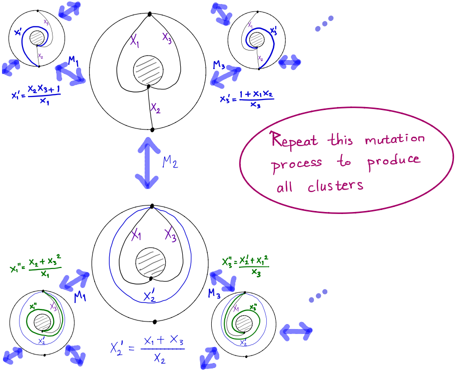
a new cluster $\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$

Ptolemy Rule

$$k' = \frac{ad+bc}{k}$$

$$x_2' = \frac{x_1+x_3}{x_2}$$


x_1'
x_3
x_2
M_1
$x_1' = \frac{x_2 x_3 + 1}{x_1}$
X_1
X_3
X_2
M_2
x_1
x_3'
M_3
$x_3' = \frac{1 + x_1 x_2}{x_3}$
Repeat this mutation process to produce all clusters
$x_1'' = \frac{x_2 + x_3^2}{x_1}$
M_1
x_1''
x_2'
X_2'
$X_2' = \frac{X_1 + X_3}{X_2}$
$x_3'' = \frac{x_2' + x_1^2}{x_3}$
M_3
x_3''

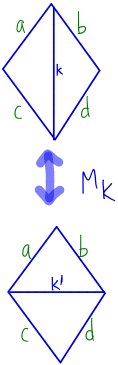


Def (Fomin – Zelevinsky, 2001)

- $\{$ cluster variables $\} =$

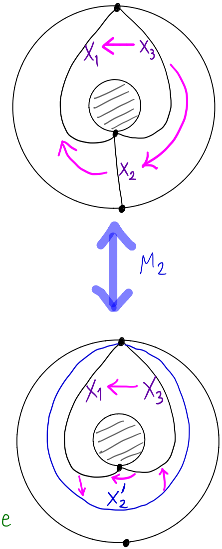
$$\bigcup_{\text{all clusters } \mathbf{x}} \{ \text{ elements of } \mathbf{x} \}$$

- The **cluster algebra** $\mathcal{A}(Q)$ is the $\mathbb{Z}$-algebra of $Q(x_1, \ldots, x_n)$ generated by all cluster variables.



Ptolemy Rule

$$k' = \frac{ad+bc}{k}$$

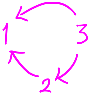


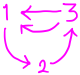
$$X_2' = \frac{X_1 + X_3}{X_2}$$

If j follows i counterclockwise along a triangle

then

draw $j \leftarrow i$





Friezes

Definition

Let Q be a quiver and $\mathcal{A}(Q)$ the cluster algebra from Q.

- A **frieze** of type Q is a ring homomorphism $\mathcal{F} : \mathcal{A}(Q) \to R$
- We say that $\mathcal{F}$ is **positive integral** if $R = \mathbb{Z}$ and $\mathcal{F}$ maps every cluster variable to a positive integer
- We say that $\mathcal{F}$ is **unitary** if there exists a cluster $\mathbf{x}$ in $\mathcal{A}(Q)$ such that F maps every cluster variable in $\mathbf{x}$ to 1

Examples:

- The identity frieze $Id : \mathcal{A}(Q) \to \mathcal{A}(Q)$.
- A frieze $\mathcal{F} : \mathcal{A}(Q) \to \mathbb{Z}$ defined by fixing a cluster $\mathbf{x}$ and sending each cluster variable in $\mathbf{x}$ to 1.

Friezes examples

$$
\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
 & x_3 & & \frac{x_1x_3+1+x_2}{x_2x_3} & & \frac{x_2+1}{x_1} & & x_1 \\
x_2 & & \frac{x_1x_3+1}{x_2} & & \frac{x_2^2+2x_2+1+x_1x_3}{x_1x_2x_3} & & x_2 & \\
 & x_1 & & \frac{x_1x_3+1+x_2}{x_1x_2} & & \frac{x_2+1}{x_3} & & x_3 \\
1 & & 1 & & 1 & & 1 &
\end{array}
$$

Figure: The identity frieze $Id : \mathcal{A}(Q) \to \mathcal{A}(Q)$ for the type $\mathbb{A}_3$ quiver $Q = 1 \to 2 \leftarrow 3$.

$$
\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
 & 1 & & 3 & & 2 & & 1 \\
1 & & 2 & & 5 & & 1 & \\
 & 1 & & 3 & & 2 & & 1 \\
1 & & 1 & & 1 & & 1 &
\end{array}
$$

Figure: Setting $x_1 = x_2 = x_3 = 1$ produces a Conway – Coxeter frieze pattern.

Unitary friezes

Definition

We say that a frieze $\mathcal{F}$ is **unitary** if there exists a cluster $\mathbf{x}$ in $\mathcal{A}(Q)$ such that $\mathcal{F}$ maps every cluster variable in $\mathbf{x}$ to 1.

Proposition 1 (G – Schiffler)

Let $\mathcal{F}$ be a positive unitary integral frieze, i.e., there is a cluster $\mathbf{x}$ such that $\mathcal{F}(u) = 1$ for all $u \in \mathbf{x}$. Then $\mathbf{x}$ is unique.

Sketch of Proof: If u is a cluster variable not in a cluster $\mathbf{x}$, then the Laurent expansion of u in $\mathbf{x}$ has two or more terms.

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Remark

All positive integral friezes of type $\mathbb{A}$ are unitary (due to Conway and Coxeter), but there are non-unitary positive integral friezes of type $\mathbb{D}$, $\widetilde{\mathbb{D}}$, $\mathbb{E}$, and $\widetilde{\mathbb{E}}$ (due to Fontaine and Plamondon).

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Theorem 2 (G – Schiffler)

All positive integral friezes of type $\widetilde{\mathbb{A}}_{p,q}$ are unitary.

Example: There are the two friezes of type $\widetilde{\mathbb{A}}_{1,2}$, up to translation.

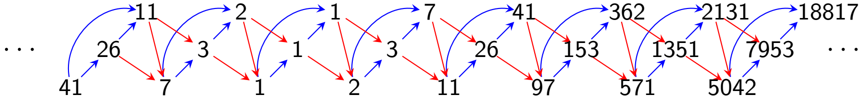


Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

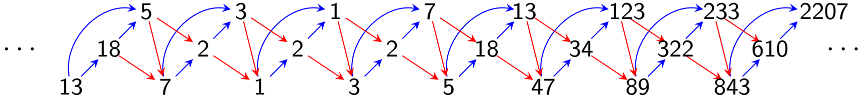


Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Algorithm for finding the triangulation where all arcs have weight one:

Pick some acyclic cluster $\mathbf{x_0} = \{x_1, \ldots, x_n\}$. If not all cluster variables of $\mathbf{x_0}$ have weight 1, we mutate $\mathbf{x}_0$ at x_k with maximal frieze value. Then:

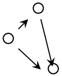
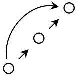
- $\mathit{frieze}(x_k') < \mathcal{F}(x_k)$
- Furthermore, if the vertex k is not a sink/source, then $F(x_k) = 1$

If not every cluster variable in $\mathbf{x}_1$ has weight 1, repeat this procedure, and so on. Since $\mathcal{F}$ is positive integral, this process must stop.

Lemma (for the algorithm)

Let $\mathcal{F}$ be a positive frieze of type $\widetilde{\mathbb{A}}_{p,q}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a cluster such that $\mathcal{F}(x) = 1$ for each regular (i.e. peripheral) cluster variable. Let k be such that $\mathcal{F}(x_k) \geq \mathcal{F}(x_j)$ for all j, and suppose that $\mathcal{F}(x_k) > 1$. Then $\mathcal{F}(\mu_k(x_k)) < \mathcal{F}(x_k)$ and if $\mu_k(x_k)$ a regular cluster variable then $\mathcal{F}(\mu_k(x_k)) = 1$.

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Every acyclic shape, for example,  and  tells us the frieze values of a cluster.

Example (A possible step in the algorithm)

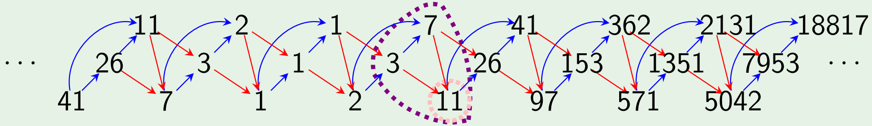
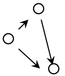
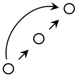


Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

Mutating at the position with frieze value 11 produces a new frieze value $\frac{3\times 7+1}{11} = 2 < 11$.

Friezes of type $\widetilde{\mathbb{A}}_{p,q}$

Every acyclic shape, for example,  and  tells us the frieze values of a cluster.

Example (A possible step in the algorithm)

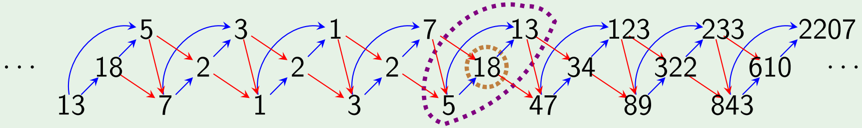


Figure: An $\widetilde{\mathbb{A}}_{1,2}$ frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

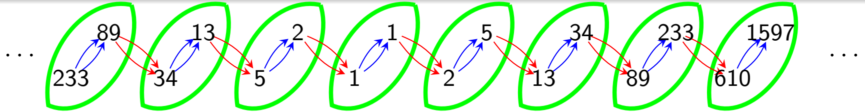
Mutating at the position with frieze value 18 (which is not a sink/source) produces a new frieze value $\frac{5+13}{18} = 1$.

Frieze vectors

Definition

Fix a cluster $\mathbf{x} = (x_1, \ldots, x_n)$.

- A vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ can be used to define a frieze $\mathcal{F} : \mathcal{A}(Q) \to \mathbb{Q}$ by defining $\mathcal{F}(x_i) = a_i$ for all $i = 1, \ldots, n$.
- We say that $(a_1, \ldots, a_n)$ is a **positive integral frieze vector** relative to $\mathbf{x}$ if $\mathcal{F}$ maps every cluster variable to a positive integer (as opposed to $\mathbb{Q}$).
- If $(a_1, \ldots, a_n)$ determines a unitary frieze, we say that $(a_1, \ldots, a_n)$ is a **unitary** frieze vector.



The slices display the frieze vectors
$\ldots, (233, 89), (34, 13), (5, 2), (1, 1), (2, 5), (13, 34), (89, 233), (610, 1597), \ldots$
relative to a cluster with the quiver $1 \Rightarrow 2$.

Frieze vectors algorithm

Proposition 3

A vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ is a frieze vector relative to an acyclic Q iff a_k divides

$$\prod_{k \to j \text{ in } Q} x_j + \prod_{k \leftarrow j \text{ in } Q} x_j$$

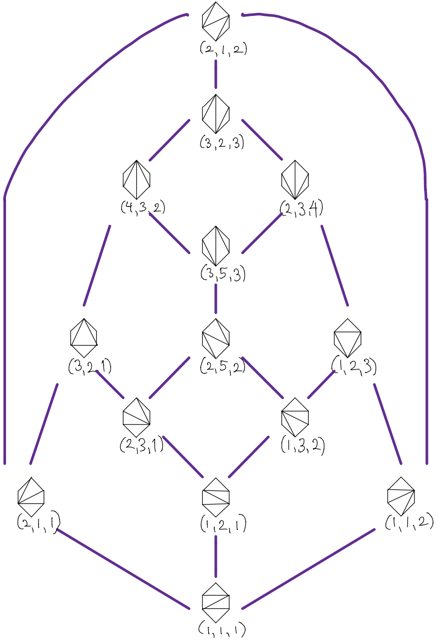
for all $k = 1, \ldots, n$.

Example

A vector $(a_1, a_2, a_3) \in \mathbb{Z}^3_{>0}$ is a positive frieze vector relative to $1 \to 2 \leftarrow 3$ iff

$$\frac{a_2 + 1}{a_1}, \frac{a_1 a_3 + 1}{a_2}, \frac{a_2 + 1}{a_3}$$

are integers.



Frieze vectors

Theorem 4 (G – Schiffler)

Fix $\mathcal{A}(Q)$ and fix an arbitrary cluster $\mathbf{x} = (x_1, \ldots, x_n)$. Then there is a bijection between clusters in $\mathcal{A}(Q)$ and unitary frieze vectors relative to $\mathbf{x}$.

Sketch of Proof: Define

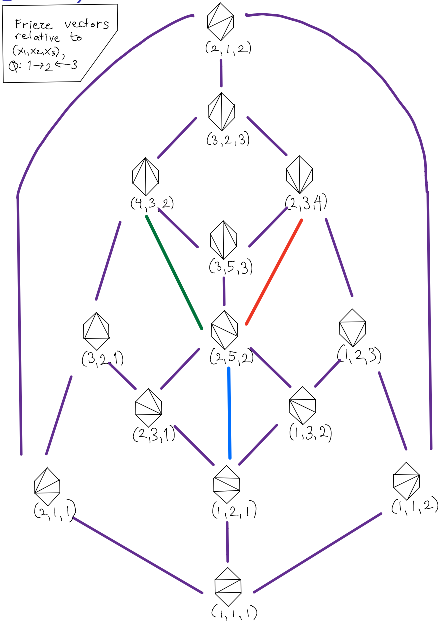
$$\phi : \{ \text{ clusters in } \mathcal{A}(Q)\} \to \{ \text{ unitary frieze vectors } \}$$
$$\mathbf{x}' = \{x'_1, \ldots, x'_n\} \mapsto \phi(\mathbf{x}') = \mathcal{F}(\mathbf{x}) = (a_1, \ldots, a_n)$$

where $\mathcal{F}$ is the frieze defined by specializing the cluster variables in $\mathbf{x}'$ to 1. Then ϕ is a bijection. Injectivity follows from Proposition 1. Surjectivity follows from the construction of ϕ.

Friezahedron (work in progress)

In type $\mathbb{A}_n$, $\mathbb{D}_n$, and $\mathbb{E}_6$, it is known that there are finitely many positive integral frieze vectors. Take the convex hull of these points in $\mathbb{R}^n$.

```
 sage:  V = [ [1, 1, 1],
[1, 1, 2], [1, 2, 1], [1,
2, 3], [1, 3, 2], [2, 1,
1], [2, 1, 2], [2, 3, 1],
[2, 3, 4], [2, 5, 2], [3,
2, 1], [3, 2, 3], [3, 5,
3], [4, 3, 2] ]
 sage:  P = Polyhedron(V)
```



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Comments and questions

Thank you!

Hvala!