Friezes, triangulations, continued fractions, and binary numbers

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Slides available at
https://egunawan.github.io/talks/smith18

Friezes

A *frieze* is an image that repeats itself along one direction. The name comes from architecture, where a frieze is a decoration running horizontally below a ceiling or roof.

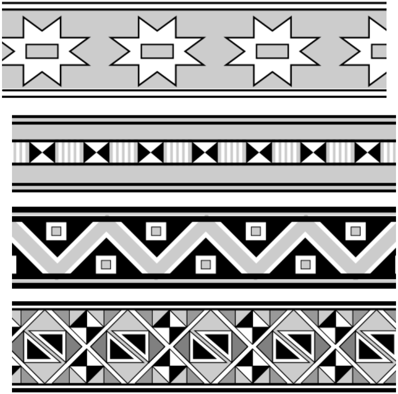


Figure: M. Ascher, *Ethnomathematics*, p162.

Conway – Coxeter friezes

Definition

A (Conway – Coxeter) *frieze* is an array of positive integers such that:

1. it is bounded above and below by a row of 1s
2. every diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

   satisfies the rule $ad - bc = 1$.

Example

$$\begin{array}{lcccccccccccccccc}
 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots \\
\text{Row 2} & \cdots & & \mathbf{3} & & \mathbf{1} & & \mathbf{2} & & \mathbf{2} & & \mathbf{1} & & 3 & & 1 & \\
 & & 2 & & 2 & & 1 & & 3 & & 1 & & 2 & & 2 & & \cdots \\
 & \cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

Note: every frieze is completely determined by the 2nd row.

Glide symmetry

A *glide symmetry* is a combination of a translation and a reflection.

		1		**1**		**1**		**1**		**1**		**1**		**1**		…
Row 2	…		**3**		**1**		**2**		**2**		**1**		**3**		**1**	
		2		**2**		**1**		**3**		**1**		**2**		**2**		…
	…		**1**		**1**		**1**		**1**		**1**		**1**		**1**	

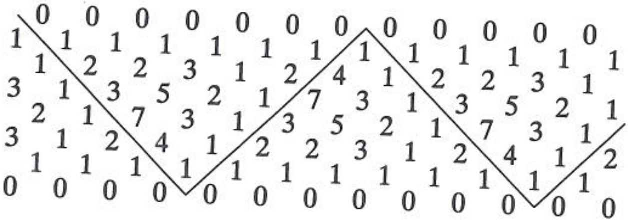


Table 1.

Children practicing arithmetic

Frieze 1

0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1 1

1 2 2 3 1 2 4 1 2 2 3 1

3 5

7

Rule

Every

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies

$$ad - bc = 1$$

Frieze 2

0 0 0 0 0 0 0 0 0 0 0 0 0

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

2 2 2 3 1 2 5 1 2 2 2 3 1

1 3 3 9

4

Children practicing arithmetic: Answer Key

0	0	0	0	0	0	0	0	0	0	0	0	
1	1	1	1	1	1	1	1	1	1	1	1	1
1	2	2	3	1	2	4	1	2	2	3	1	
3	1	3	5	2	1	7	3	1	3	5	2	1
2	1	7	3	1	3	5	2	1	7	3	1	
3	1	2	4	1	2	2	3	1	2	4	1	2
1	1	1	1	1	1	1	1	1	1	1	1	
0	0	0	0	0	0	0	0	0	0	0	0	0

Table 1.

0	0	0	0	0	0	0	0	0	0	0	0	
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	3	1	2	5	1	2	2	2	3	
1	3	3	5	2	1	9	4	1	3	3	5	2
1	4	7	3	1	4	7	3	1	4	7	3	
2	1	9	4	1	3	3	5	2	1	9	4	1
1	2	5	1	2	2	2	3	1	2	5	1	
1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	0	0	0	0	0	0	0	0	0	0	

Table 2.

What do the numbers around each polygon count?

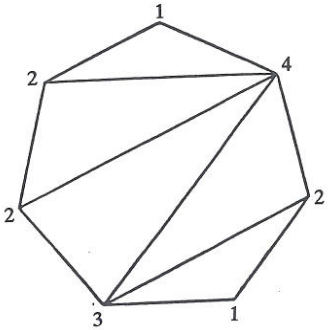
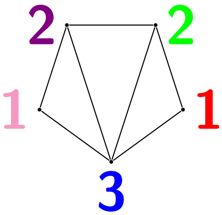


Figure 3: A triangulation of a heptagon.



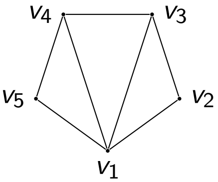
Conway and Coxeter (1970s)

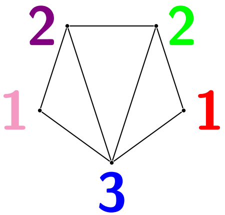
Theorem

A Conway - Coxeter frieze with n nontrivial rows $\longleftrightarrow$ a triangulations of an $(n+3)$-gon

Note: Hence Conway - Coxeter friezes are Catalan objects.

		1		1		1		1		1		1		1		$\cdots$
Row 2	$\cdots$		3		1		2		2		1		3		1	
		2		2		1		3		1		2		2		$\cdots$
	$\cdots$		1		1		1		1		1		1		1	





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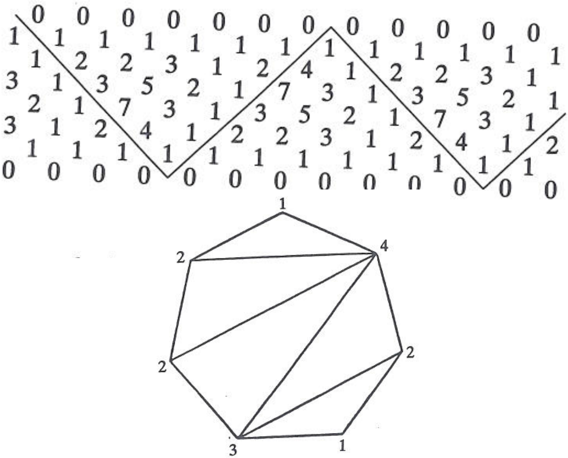


Figure 3: A triangulation of a heptagon.

Primary school algorithm

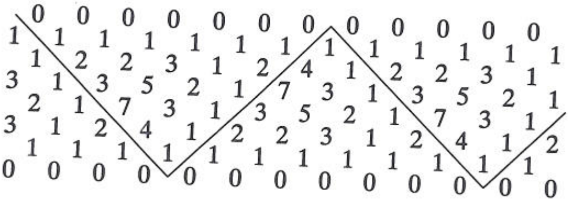
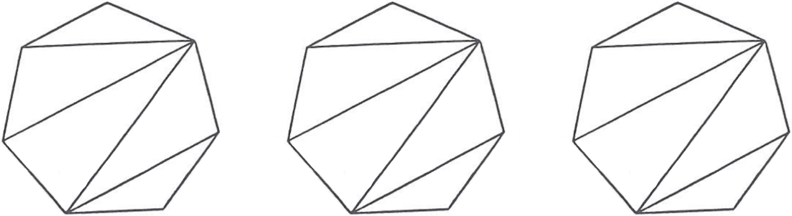


Table 1.

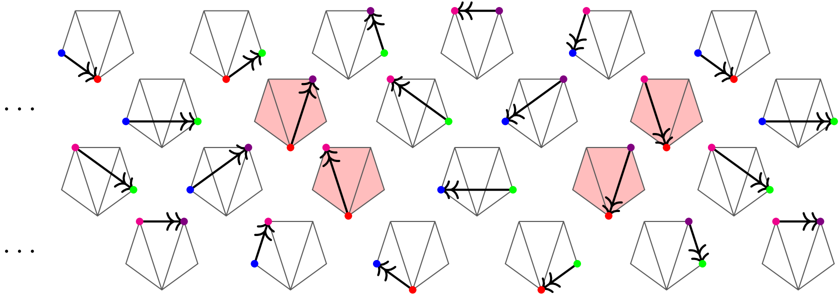


Broline, Crowe, and Isaacs (BCI, 1970s)

Theorem

Entries of a finite frieze $\longleftrightarrow$ edges between two vertices.

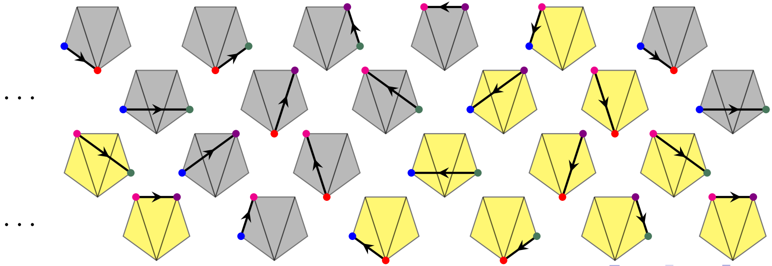
		1		1		1		1		1		1		$\cdots$
Row 2	$\cdots$		3		**1**		2		2		**1**		3	
		2		2		**1**		3		**1**		2		$\cdots$
	$\cdots$		1		1		1		1		1		1	



Glide symmetry (again)

A *glide symmetry* is a combination of a translation and a reflection. If we forget the arrows' orientation, the diagonals have glide symmetry.

		1		**1**		**1**		**1**		**1**		**1**		**1**		⋯
Row 2	⋯		**3**		**1**		**2**		**2**		**1**		**3**		**1**	
		2		**2**		**1**		**3**		**1**		**2**		**2**		⋯
	⋯		**1**		**1**		**1**		**1**		**1**		**1**		**1**	

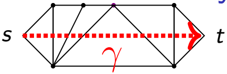


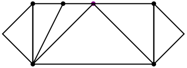
Binary number representations

A *binary number representation* is an expression of a nonzero integer in the base-2 numeral system, with digits consisting of 1s and 0s. For example,

- $1 = 1 * 2^0$ (in decimal) is written as 1 (in binary).
- $2 = 1 * 2^1$ (in decimal) is written as 10 (in binary).
- $4 = 1 * 2^2$ (in decimal) is written as 100 (in binary).
- $5 = 1 * 2^2 + 1 * 2^0$ (in decimal) is written as 101 (in binary).
- $29 = 16 + 8 + 4 + 1 = 1 * 2^4 + 1 * 2^3 + 1 * 2^2 + 1 * 2^0$ is written as 11101 (in binary).

Subwords of binary numbers (G.)

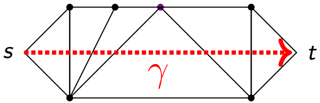




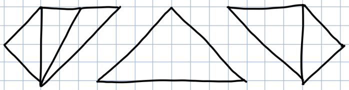
We can think of "11101" as a word in the alphabets $\{0, 1\}$. All subwords of "11101" which start with "1":

- "11101" (itself)
- "11101"
- "11101"
- "11101"
- "11101"
- "11101"
- "11101"
- "11101"
- "11101"
- "11101"
- "11101"

Continued fractions (Çanakçı, Schiffler)



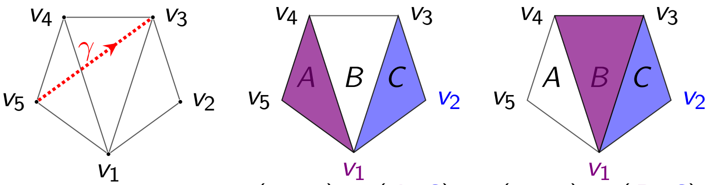
Continued Fraction



3 , 1 , 2

$$3 + \frac{1}{\left(1 + \frac{1}{2}\right)} \overset{\text{compute}}{=}$$

Broline, Crowe, and Isaacs (BCI, 1970s)



T and γ $\quad$ $(t_1, t_2) = (A, C)$ $\quad$ $(t_1, t_2) = (B, C)$

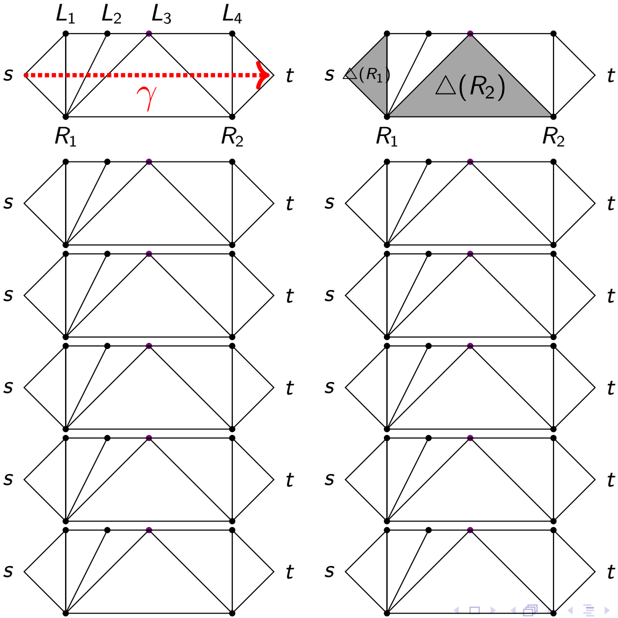
Definition (BCI tuple)

Let R_1, R_2, ..., R_r be the boundary vertices to the right of γ. A *BCI tuple* for γ is an r-tuple $(t_1, \ldots, t_r)$ such that:

(B1) the i-th entry t_i is a triangle of T having R_i as a vertex.

(B2) the entries are pairwise distinct.

How many BCI tuples are there?



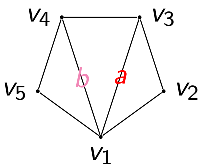
Cluster algebras (Fomin – Zelevinsky, 2000)

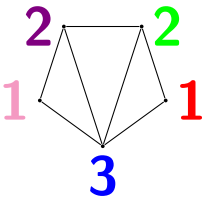
Definition A *cluster algebra* is a commutative ring with a distinguished set of generators, called *cluster variables*.

Theorem (Caldero – Chapoton (2006): The cluster variables of a cluster algebra from a triangulated polygon (type A) form a frieze.

$$\begin{array}{ccccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & \\
\cdots & & \frac{1+a}{b} & & b & & \frac{1+a+b}{a\,b} & & a & & \frac{1+b}{a} & & \frac{1+a}{b} \\
& 1 & & 1 & & 1 & & 1 & & 1 & & 1 &
\end{array}$$

(Example: type A_2)





- Remark: If the variables are specialized to 1, we recover the Conway – Coxeter integer frieze.

Example: a type A_3 frieze

$$
\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
 & x_3 & & \frac{x_1x_3+1+x_2}{x_2x_3} & & \frac{x_2+1}{x_1} & & x_1 \\
x_2 & & \frac{x_1x_3+1}{x_2} & & \frac{x_2^2+2x_2+1+x_1x_3}{x_1x_2x_3} & & x_2 & \\
 & x_1 & & \frac{x_1x_3+1+x_2}{x_1x_2} & & \frac{x_2+1}{x_3} & & x_3 \\
1 & & 1 & & 1 & & 1 &
\end{array}
$$

Frieze over the positive integers

Specializing $x_1 = x_2 = x_3 = 1$ gives a Conway – Coxeter positive integer frieze

$$
\begin{array}{cccccccc}
1 & & 1 & & 1 & & 1 & \\
& 1 & & 3 & & 2 & & 1 \\
1 & & 2 & & 5 & & 1 & \\
& 1 & & 3 & & 2 & & 1 \\
1 & & 1 & & 1 & & 1 &
\end{array}
$$

Thank you