

# Cluster algebras and friezes

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# Frieze

In architecture, a *frieze* is an image that repeats itself along one direction.



# Conway and Coxeter, 1970s

## Definition

A **Conway – Coxeter frieze pattern** is an array of positive integers such that:

- 1 it is bounded above and below by a row of 1s
- 2 every diamond

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfies the diamond rule  $ad - bc = 1$ .

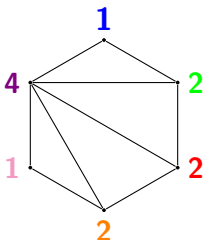


# Conway and Coxeter, 1970s

## Theorem

A Conway – Coxeter frieze pattern with  $n$  nontrivial rows  $\longleftrightarrow$  a triangulation of an  $(n + 3)$ -gon.

1	1	1	1	1	1	1	1	1	1	1	1	1	
	1	4	1	2	2	2	1	4	1	2	2	2	
		3	3	1	3	3	1	3	3	1	3	1	
			2	2	1	4	1	2	2	1	4	1	2
				1	1	1	1	1	1	1	1	1	1



# Fomin and Zelevinsky, 2001

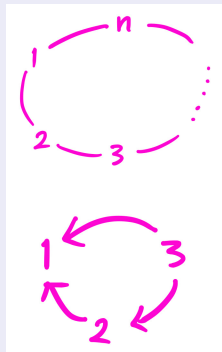
Start with a quiver (directed graph)  $Q$  on  $n$  vertices with no loops and no 2-cycles.

Example: type  $\tilde{A}_{p,q}$

An acyclic quiver  $Q$  is of type  $\tilde{A}_{p,q}$  if and only if

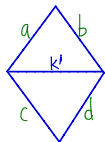
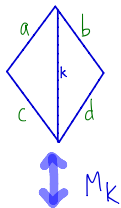
- its underlying graph is a circular graph with  $n = p + q$  vertices,
- the quiver  $Q$  has  $p$  counterclockwise arrows and  $q$  clockwise arrows

For example, this is a quiver of type  $\tilde{A}_{1,2} \rightarrow$



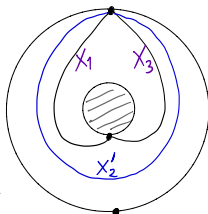
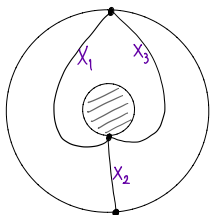
# Annulus with $p+q$ marked points on the boundary (Fomin - Shapiro - Thurston, 2006)

- An arc is an internal curve between marked points
- A triangulation is a maximal collection of non-crossing arcs
- A flip  $M_K$  replaces

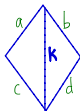


Ptolemy Rule

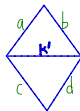
$$k' = \frac{ad+bc}{k}$$



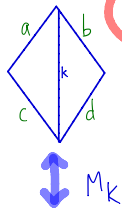
$$x_2' = \frac{x_1 + x_3}{x_2}$$



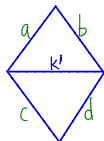
with



a cluster  $\{x_1, x_2, x_3\}$



$M_K$

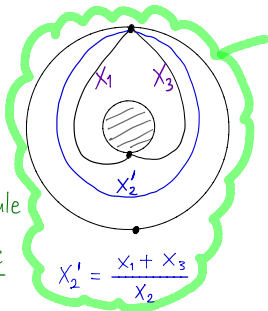


Ptolemy Rule

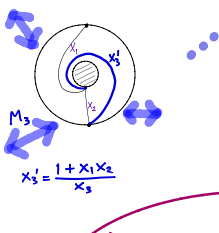
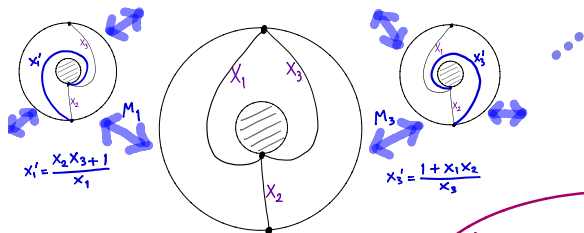
$$k' = \frac{ad+bc}{k}$$

$M_2$  mutation at  $x_2$

a new cluster  $\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$



$$x_2' = \frac{x_1 + x_3}{x_2}$$



Repeat this mutation process to produce all clusters

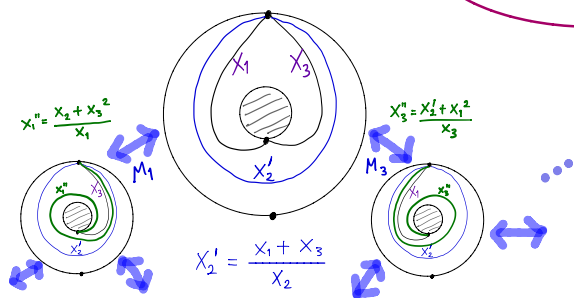
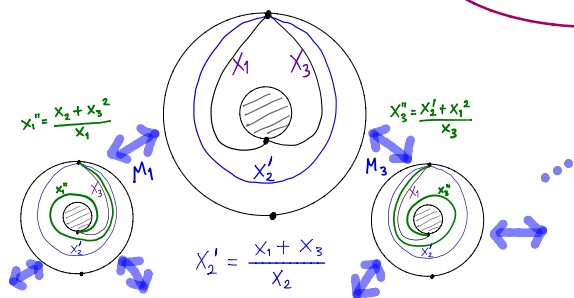
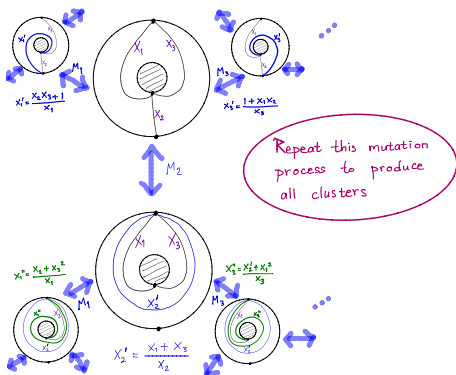


Diagram illustrating the fourth mutation step. A cluster  $X_2'$  is mutated to  $X_2''$  using the formula:

$$X_2'' = \frac{X_1 + X_3}{X_2}$$

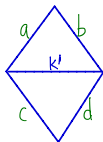
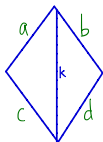




Def (Fomin – Zelevinsky, 2001)

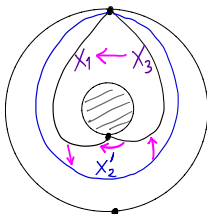
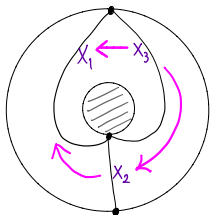
- $\{ \text{cluster variables} \} =$   

$$\bigcup_{\text{all clusters } x} \{ \text{elements of } x \}$$
- The **cluster algebra**  $\mathcal{A}(Q)$  is the  $\mathbb{Z}$ -subalgebra of  $\mathbb{Q}(x_1, \dots, x_n)$  generated by all cluster variables.



Ptolemy Rule

$$k' = \frac{ad+bc}{k}$$



$$x_2' = \frac{x_1 + x_3}{x_2}$$

If  $j$  follows  $i$   
counterclockwise  
along a  
triangle

$j \triangle i$  then  
draw  $j \leftarrow i$



# Friezes

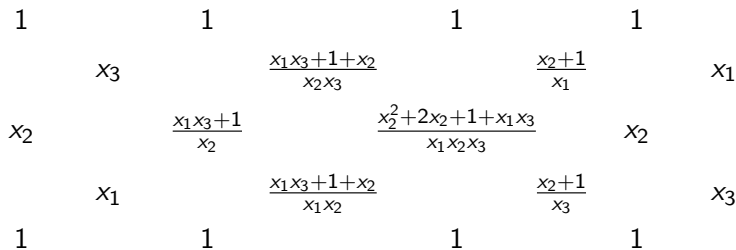
## Definition

Let  $Q$  be a quiver and  $\mathcal{A}(Q)$  the cluster algebra from  $Q$ . A **frieze** of type  $Q$  is a ring homomorphism  $\mathcal{F} : \mathcal{A}(Q) \rightarrow \mathbb{Z}$  which maps every cluster variable to a positive integer.

## Example:

- A frieze  $\mathcal{F} : \mathcal{A}(Q) \rightarrow \mathbb{Z}$  defined by fixing a cluster  $\mathbf{x}$  and sending each cluster variable in  $\mathbf{x}$  to 1.

## Friezes examples



**Figure:** The cluster variables of the cluster algebra  $\mathcal{A}(Q) \rightarrow \mathcal{A}(Q)$  for the type  $\mathbb{A}_3$  quiver  $Q = 1 \rightarrow 2 \leftarrow 3$ .

1		1		1		1	
	1		3		2		1
1		2		5		1	
	1		3		2		1
1		1		1		1	

**Figure:** Setting  $x_1=x_2=x_3=1$  produces a Conway – Coxeter frieze pattern.

# Unitary friezes

## Definition

We say that a frieze  $\mathcal{F}$  is **unitary** if there exists a cluster  $\mathbf{x}$  in  $\mathcal{A}(Q)$  such that  $\mathcal{F}$  maps every cluster variable in  $\mathbf{x}$  to 1.

## Remark

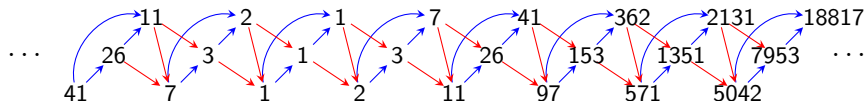
All positive integral friezes of type  $\mathbb{A}$  are unitary (due to Conway and Coxeter), but there are non-unitary positive integral friezes of type  $\mathbb{D}$ ,  $\widetilde{\mathbb{D}}$ ,  $\mathbb{E}$ , and  $\widetilde{\mathbb{E}}$  (due to Fontaine and Plamondon).

# Friezes of type $\tilde{\mathbb{A}}_{p,q}$

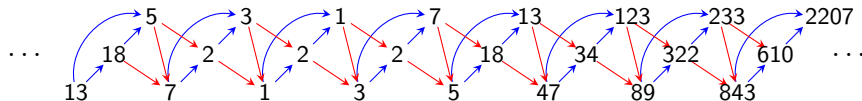
## Theorem 2 (G – Schiffler)

All positive integral friezes of type  $\tilde{\mathbb{A}}_{p,q}$  are unitary.

**Example:** There are the two friezes of type  $\tilde{\mathbb{A}}_{1,2}$ , up to translation.



**Figure:** An  $\tilde{\mathbb{A}}_{1,2}$  frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.



**Figure:** An  $\tilde{\mathbb{A}}_{1,2}$  frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

# Friezes of type $\tilde{\mathbb{A}}_{p,q}$

Every acyclic shape, for example,

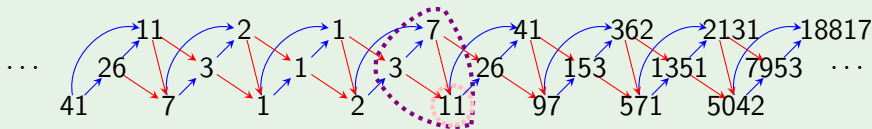


and



tells us the frieze

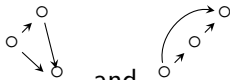
## Example (A possible step in the algorithm)



**Figure:** An  $\tilde{\mathbb{A}}_{1,2}$  frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

Mutating at the position with frieze value 11 produces a new frieze value  $\frac{3 \times 7 + 1}{11} = 2 < 11$ .

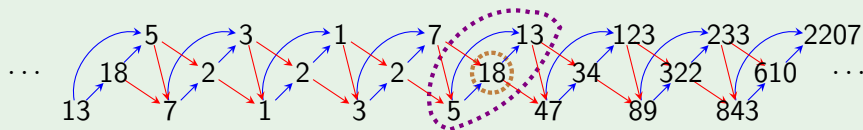
# Friezes of type $\tilde{\mathbb{A}}_{p,q}$



Every acyclic shape, for example,  
values of a cluster.

and tells us the frieze

## Example (A possible step in the algorithm)



**Figure:** An  $\tilde{\mathbb{A}}_{1,2}$  frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

Mutating at the position with frieze value 18 (which is not a sink/source) produces a new frieze value  $\frac{5+13}{18} = 1$ .

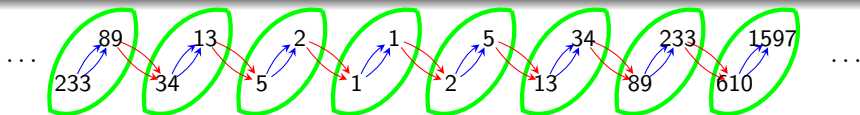


# Frieze vectors

## Definition

Fix a cluster  $\mathbf{x} = (x_1, \dots, x_n)$ .

- A vector  $(a_1, \dots, a_n) \in \mathbb{Z}_{>0}^n$  can be used to define a homomorphism  $\mathcal{F} : \mathcal{A}(Q) \rightarrow \mathbb{Q}$  by defining  $\mathcal{F}(x_i) = a_i$  for all  $i = 1, \dots, n$ .
- We say that  $(a_1, \dots, a_n)$  is a **frieze vector** relative to  $\mathbf{x}$  if  $\mathcal{F}$  maps every cluster variable to a positive integer.
- If  $(a_1, \dots, a_n)$  determines a unitary frieze, we say that  $(a_1, \dots, a_n)$  is a **unitary** frieze vector.



The slices display the frieze vectors

$\dots, (233, 89), (34, 13), (5, 2), (1, 1), (2, 5), (13, 34), (89, 233), (610, 1597), \dots$   
relative to a cluster with the quiver  $1 \Rightarrow 2$ .

# Frieze vectors algorithm

## Proposition 3

A vector  $(a_1, \dots, a_n) \in \mathbb{Z}^n$  is a frieze vector relative to an acyclic  $Q$  iff  $a_k$  divides

$$\prod_{k \rightarrow j \text{ in } Q} x_j + \prod_{k \leftarrow j \text{ in } Q} x_j$$

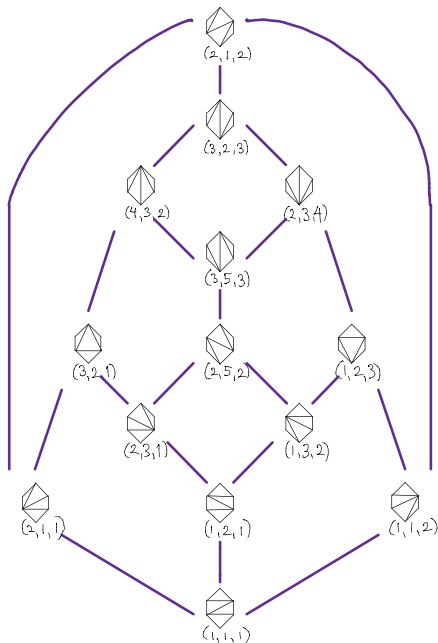
for all  $k = 1, \dots, n$ .

## Example

A vector  $(a_1, a_2, a_3) \in \mathbb{Z}_{>0}^3$  is a positive frieze vector relative to  $1 \rightarrow 2 \leftarrow 3$  iff

$$\frac{a_2 + 1}{a_1}, \frac{a_1 a_3 + 1}{a_2}, \frac{a_2 + 1}{a_3}$$

are integers.



# Frieze vectors

## Theorem 4 (G – Schiffler)

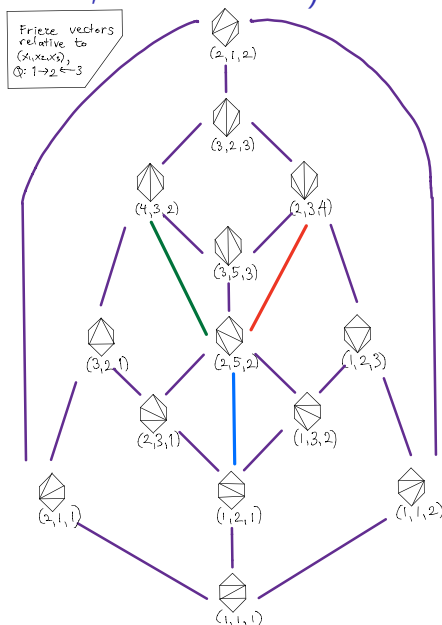
Fix  $\mathcal{A}(Q)$  and fix an arbitrary cluster  $\mathbf{x} = (x_1, \dots, x_n)$ . Then there is a bijection between clusters in  $\mathcal{A}(Q)$  and unitary frieze vectors relative to  $\mathbf{x}$ .

# Friezahedron (further questions, with Castillo)

In type  $\mathbb{A}_n$ ,  $\mathbb{D}_n$ , and  $\mathbb{E}_6$ , it is known that there are finitely many positive integral frieze vectors. Take the convex hull of these points in  $\mathbb{R}^n$ .

**sage:**  $V = [ [1, 1, 1], [1, 1, 2], [1, 2, 1], [1, 2, 3], [1, 3, 2], [2, 1, 1], [2, 1, 2], [2, 3, 1], [2, 3, 4], [2, 5, 2], [3, 2, 1], [3, 2, 3], [3, 5, 3], [4, 3, 2] ]$

**sage:**  $P = \text{Polyhedron}(V)$



# Finite-type friezes

Classify quivers with finitely many friezes (up to cluster automorphism).

- Quivers that have finitely many friezes (up to cluster automorphism):  $\mathbb{A}$ ,  $\widetilde{\mathbb{A}}_{p,q}$ ,  $\mathbb{D}$ , and  $\mathbb{E}_6$ .
- Quivers that I think may have finitely many friezes (up to cluster automorphism): rank 2,  $\mathbb{E}_6$ ,  $\mathbb{E}_7$ ,  $\widetilde{\mathbb{D}}_n$ ,  $\widetilde{\mathbb{E}}_6$ ,  $\widetilde{\mathbb{E}}_7$ ,  $\widetilde{\mathbb{E}}_8$ , and quivers with triangulation model.

# References



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[To appear in Journal of Combinatorics.](#)

Comments and questions

Thank you!