

# Unitary friezes and frieze vectors

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# Frieze

A *frieze* is an image that repeats itself along one direction. In architecture, a frieze is a decoration running horizontally below a ceiling or roof.



# Conway and Coxeter, 1970s

## Definition

A **Conway – Coxeter frieze pattern** is an array of positive integers such that:

- 1 it is bounded above and below by a row of 1s
- 2 every diamond

$$\begin{array}{ccccc} & & b & & \\ & a & & d & \\ & & c & & \end{array}$$

satisfies the diamond rule  $ad - bc = 1$ .

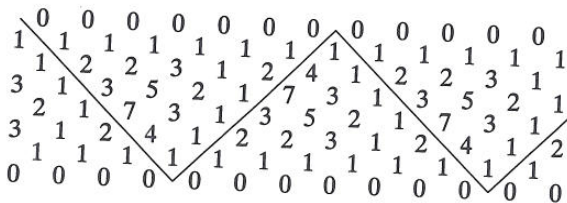
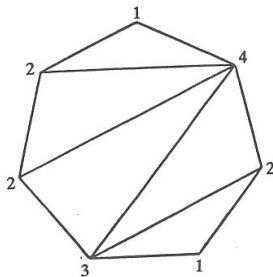
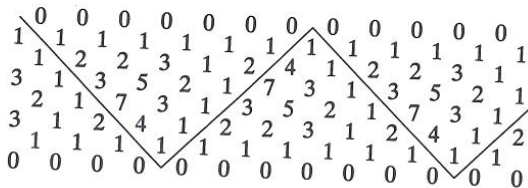


Table 1.

# Conway and Coxeter, 1970s

## Theorem

A Conway – Coxeter frieze with  $n$  nontrivial rows  $\longleftrightarrow$  a triangulation of an  $(n + 3)$ -gon



# Fomin and Zelevinsky, 2001

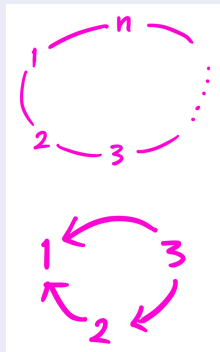
Start with a quiver (directed graph)  $Q$  on  $n$  vertices with no loops and no 2-cycles.

Example: type  $\tilde{A}_{p,q}$

An acyclic quiver  $Q$  is of type  $\tilde{A}_{p,q}$  if and only if

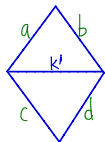
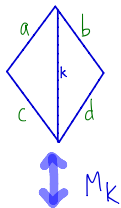
- its underlying graph is a circular graph with  $n = p + q$  vertices,
- the quiver  $Q$  has  $p$  arrows oriented counterclockwise and  $q$  arrows oriented clockwise

For example, this is a quiver of type  $\tilde{A}_{1,2} \rightarrow$



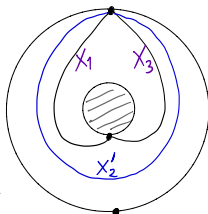
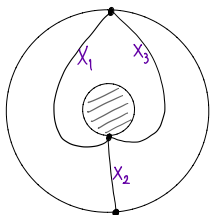
Annulus with  $p+q$  marked points on the boundary  
(Fomin - Shapiro - Thurston, 2006)

- An arc is an internal curve between marked points
- A triangulation is a maximal collection of non-crossing arcs
- A flip  $M_K$  replaces

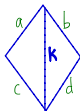


Ptolemy Rule

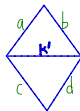
$$k' = \frac{ad+bc}{k}$$



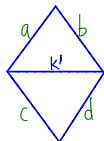
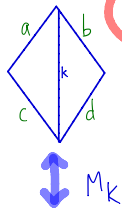
$$x_2' = \frac{x_1 + x_3}{x_2}$$



with

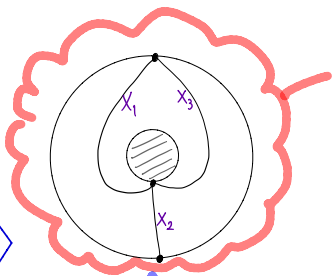


a cluster  $\{x_1, x_2, x_3\}$

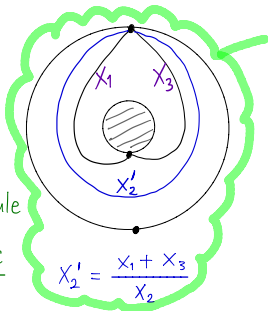


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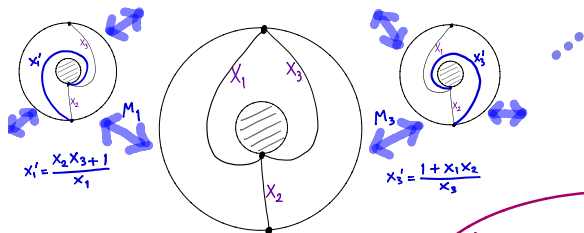


$M_2$  mutation at  $x_2$

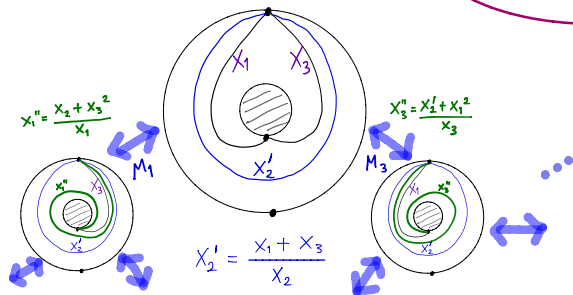


a new cluster  $\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$

$$x_2' = \frac{x_1 + x_3}{x_2}$$

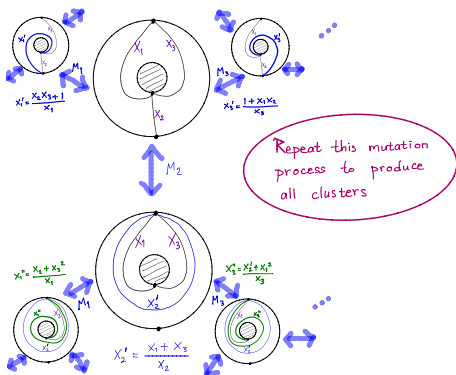


Repeat this mutation process to produce all clusters



$$x_2' = \frac{x_1 + x_3}{x_2}$$

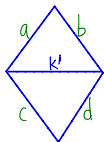
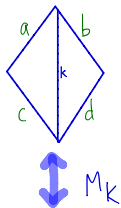




Def (Fomin – Zelevinsky, 2001)

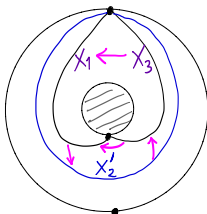
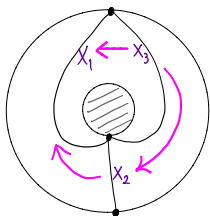
- $\{ \text{cluster variables} \} =$   

$$\bigcup_{\text{all clusters } \mathbf{x}} \{ \text{elements of } \mathbf{x} \}$$
- The **cluster algebra**  $\mathcal{A}(Q)$  is the  $\mathbb{Z}$ -algebra of  $Q(x_1, \dots, x_n)$  generated by all cluster variables.



Ptolemy Rule

$$k' = \frac{ad+bc}{k}$$



$$x_2' = \frac{x_1 + x_3}{x_2}$$

If  $j$  follows  $i$   
counterclockwise  
along a  
triangle

$j \triangle i$  then  
draw  $j \leftarrow i$



# Friezes

## Definition

Let  $Q$  be a quiver and  $\mathcal{A}(Q)$  the cluster algebra from  $Q$ .

- A **frieze** of type  $Q$  is a ring homomorphism  $\mathcal{F} : \mathcal{A}(Q) \rightarrow R$
- We say that  $\mathcal{F}$  is **positive integral** if  $R = \mathbb{Z}$  and  $\mathcal{F}$  maps every cluster variable to a positive integer
- We say that  $\mathcal{F}$  is **unitary** if there exists a cluster  $\mathbf{x}$  in  $\mathcal{A}(Q)$  such that  $\mathcal{F}$  maps every cluster variable in  $\mathbf{x}$  to 1

## Examples:

- The identity frieze  $Id : \mathcal{A}(Q) \rightarrow \mathcal{A}(Q)$ .
- A frieze  $\mathcal{F} : \mathcal{A}(Q) \rightarrow \mathbb{Z}$  defined by fixing a cluster  $\mathbf{x}$  and sending each cluster variable in  $\mathbf{x}$  to 1.

## Friezes examples

$$\begin{array}{ccccccc}
 1 & & 1 & & 1 & & 1 \\
 & x_3 & & \frac{x_1 x_3 + 1 + x_2}{x_2 x_3} & & \frac{x_2 + 1}{x_1} & x_1 \\
 x_2 & & \frac{x_1 x_3 + 1}{x_2} & & \frac{x_2^2 + 2x_2 + 1 + x_1 x_3}{x_1 x_2 x_3} & & x_2 \\
 & x_1 & & \frac{x_1 x_3 + 1 + x_2}{x_1 x_2} & & \frac{x_2 + 1}{x_3} & x_3 \\
 1 & & 1 & & 1 & & 1
 \end{array}$$

Figure: The identity frieze  $Id : \mathcal{A}(Q) \rightarrow \mathcal{A}(Q)$  for the type  $\mathbb{A}_3$  quiver  $Q = 1 \rightarrow 2 \leftarrow 3$ .

$$\begin{array}{ccccccc}
 1 & & 1 & & 1 & & 1 \\
 & 1 & & 3 & & 2 & & 1 \\
 1 & & 2 & & 5 & & 1 & \\
 & 1 & & 3 & & 2 & & 1 \\
 1 & & 1 & & 1 & & 1 & 
 \end{array}$$

Figure: Setting  $x_1 = x_2 = x_3 = 1$  produces a Conway – Coxeter frieze pattern.

# Unitary friezes

## Definition

We say that a frieze  $\mathcal{F}$  is **unitary** if there exists a cluster  $\mathbf{x}$  in  $\mathcal{A}(Q)$  such that  $\mathcal{F}$  maps every cluster variable in  $\mathbf{x}$  to 1.

## Proposition 1 (G – Schiffler)

Let  $\mathcal{F}$  be a positive unitary integral frieze, i.e., there is a cluster  $\mathbf{x}$  such that  $\mathcal{F}(u) = 1$  for all  $u \in \mathbf{x}$ . Then  $\mathbf{x}$  is unique.

**Sketch of Proof:** If  $u$  is a cluster variable not in a cluster  $\mathbf{x}$ , then the Laurent expansion of  $u$  in  $\mathbf{x}$  has two or more terms.

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## Remark

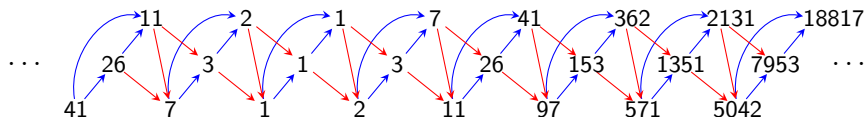
All positive integral friezes of type  $\mathbb{A}$  are unitary (due to Conway and Coxeter), but there are non-unitary positive integral friezes of type  $\mathbb{D}$ ,  $\widetilde{\mathbb{D}}$ ,  $\mathbb{E}$ , and  $\widetilde{\mathbb{E}}$  (due to Fontaine and Plamondon).

# Friezes of type $\tilde{\mathbb{A}}_{p,q}$

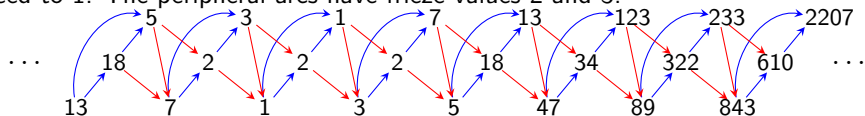
## Theorem 2 (G – Schiffler)

All positive integral friezes of type  $\tilde{\mathbb{A}}_{p,q}$  are unitary.

**Example:** There are the two friezes of type  $\tilde{\mathbb{A}}_{1,2}$ , up to translation.



**Figure:** An  $\tilde{\mathbb{A}}_{1,2}$  frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.



**Figure:** An  $\tilde{\mathbb{A}}_{1,2}$  frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

## Friezes of type $\tilde{\mathbb{A}}_{p,q}$

Algorithm for finding the triangulation where all arcs have weight one:

Pick some acyclic cluster  $\mathbf{x}_0 = \{x_1, \dots, x_n\}$ . If not all cluster variables of  $\mathbf{x}_0$  have weight 1, we mutate  $\mathbf{x}_0$  at  $x_k$  with maximal frieze value. Then:

- $\text{frieze}(x'_k) < \mathcal{F}(x_k)$
- Furthermore, if the vertex  $k$  is not a sink/source, then  $F(x_k) = 1$

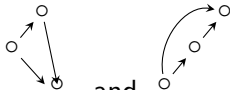
If not every cluster variable in  $\mathbf{x}_1$  has weight 1, repeat this procedure, and so on. Since  $\mathcal{F}$  is positive integral, this process must stop.

### Lemma (for the algorithm)

Let  $\mathcal{F}$  be a positive frieze of type  $\tilde{\mathbb{A}}_{p,q}$ . Let  $\mathbf{x} = (x_1, \dots, x_n)$  be a cluster such that  $\mathcal{F}(x) = 1$  for each regular (i.e. peripheral) cluster variable. Let  $k$  be such that  $\mathcal{F}(x_k) \geq \mathcal{F}(x_j)$  for all  $j$ , and suppose that  $\mathcal{F}(x_k) > 1$ . Then  $\mathcal{F}(\mu_k(x_k)) < \mathcal{F}(x_k)$  and if  $\mu_k(x_k)$  a regular cluster variable then  $\mathcal{F}(\mu_k(x_k)) = 1$ .



## Friezes of type $\tilde{\mathbb{A}}_{p,q}$

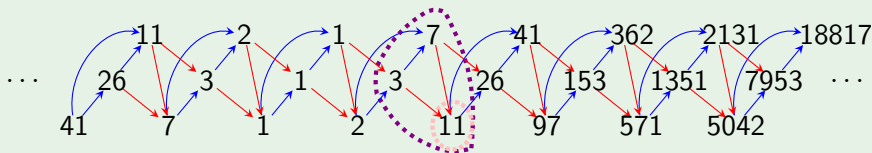


Every acyclic shape, for example,

and

tells us the frieze

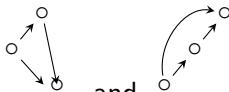
### Example (A possible step in the algorithm)



**Figure:** An  $\tilde{\mathbb{A}}_{1,2}$  frieze obtained by specializing the cluster variables of an acyclic seed to 1. The peripheral arcs have frieze values 2 and 3.

Mutating at the position with frieze value 11 produces a new frieze value  $\frac{3 \times 7 + 1}{11} = 2 < 11$ .

## Friezes of type $\tilde{\mathbb{A}}_{p,q}$

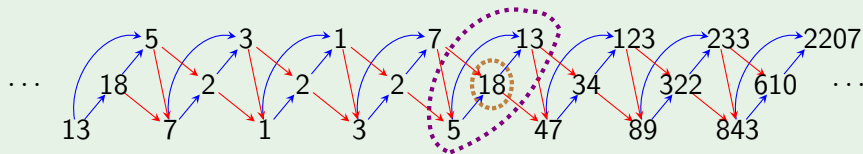


Every acyclic shape, for example,

and

tells us the frieze values of a cluster.

### Example (A possible step in the algorithm)



**Figure:** An  $\tilde{\mathbb{A}}_{1,2}$  frieze obtained by specializing the cluster variables of a non-acyclic seed to 1. The peripheral arcs have frieze values 1 and 5.

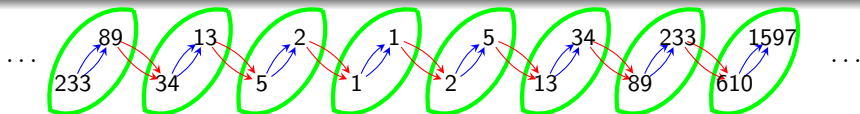
Mutating at the position with frieze value 18 (which is not a sink/source) produces a new frieze value  $\frac{5+13}{18} = 1$ .

# Frieze vectors

## Definition

Fix a cluster  $\mathbf{x} = (x_1, \dots, x_n)$ .

- A vector  $(a_1, \dots, a_n) \in \mathbb{Z}_{>0}^n$  can be used to define a frieze  $\mathcal{F} : \mathcal{A}(Q) \rightarrow \mathbb{Q}$  by defining  $\mathcal{F}(x_i) = a_i$  for all  $i = 1, \dots, n$ .
- We say that  $(a_1, \dots, a_n)$  is a **positive integral frieze vector** relative to  $\mathbf{x}$  if  $\mathcal{F}$  maps every cluster variable to a positive integer (as opposed to  $\mathbb{Q}$ ).
- If  $(a_1, \dots, a_n)$  determines a unitary frieze, we say that  $(a_1, \dots, a_n)$  is a **unitary** frieze vector.



The slices display the frieze vectors

$\dots, (233, 89), (34, 13), (5, 2), (1, 1), (2, 5), (13, 34), (89, 233), (610, 1597), \dots$   
relative to a cluster with the quiver  $1 \Rightarrow 2$ .



# Frieze vectors

## Theorem 4 (G – Schiffler)

Fix  $\mathcal{A}(Q)$  and fix an arbitrary cluster  $\mathbf{x} = (x_1, \dots, x_n)$ . Then there is a bijection between clusters in  $\mathcal{A}(Q)$  and unitary frieze vectors relative to  $\mathbf{x}$ .

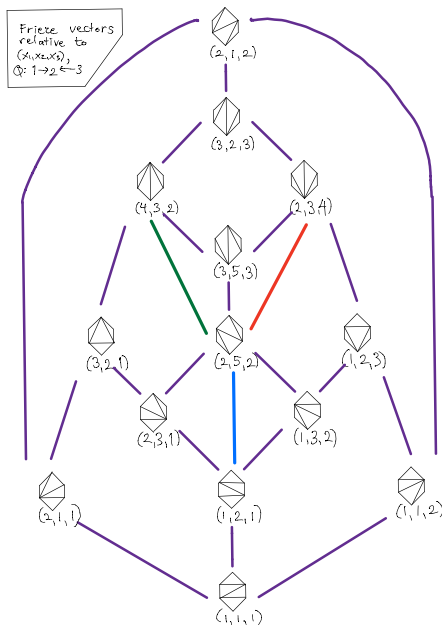
**Sketch of Proof:** Define

$$\begin{aligned}\phi : \{ \text{clusters in } \mathcal{A}(Q) \} &\rightarrow \{ \text{unitary frieze vectors} \} \\ \mathbf{x}' = \{x'_1, \dots, x'_n\} &\mapsto \phi(\mathbf{x}') = \mathcal{F}(\mathbf{x}) = (a_1, \dots, a_n)\end{aligned}$$

where  $\mathcal{F}$  is the frieze defined by specializing the cluster variables in  $\mathbf{x}'$  to 1. Then  $\phi$  is a bijection. Injectivity follows from Proposition 1. Surjectivity follows from the construction of  $\phi$ .

# Comments and questions

Thank you!  
Hvala!



# Friezahedron (current work)

In type  $\mathbb{A}_n$ ,  $\mathbb{D}_n$ , and  $\mathbb{E}_6$ , it is known that there are finitely many positive integral frieze vectors.

We can take the convex hull of these points in  $\mathbb{R}^n$ .

**sage:**  $V = [$   
 $[1, 1, 1], [1, 1, 2],$   
 $[1, 2, 1], [1, 2, 3],$   
 $[1, 3, 2], [2, 1, 1],$   
 $[2, 1, 2], [2, 3, 1], [2,$   
 $3, 4], [2, 5, 2], [3, 2,$   
 $1], [3, 2, 3], [3, 5, 3],$   
 $[4, 3, 2] ]$

**sage:**  $P = \text{Polyhedron}(V)$

