Task 4.1 Adding numerical integration methods

Three numerical integration methods were implemented within the **Cosmology** class: the Rectangle rule, the Trapezoid rule, and Simpson's rule. These methods were applied to calculate the cosmological distance $D(z)$, where z is the redshift. The distance $D(z)$ is calculated in **Megaparsecs [Mpc]** based on the constant H_0 and the speed of light $c = 3 \times 10^5$ km/s. The following results were obtained for $z = 1$, with $H_0 = 72$ km/s/Mpc, $\Omega_m = 0.3$, and $\Omega_\lambda = 0.7$:

- **Distance (Rectangle Rule)**: 3214.33 Mpc
- **Distance (Trapezoid Rule)**: 3214.28 Mpc
- **Distance (Simpson's Rule)**: 3214.28 Mpc

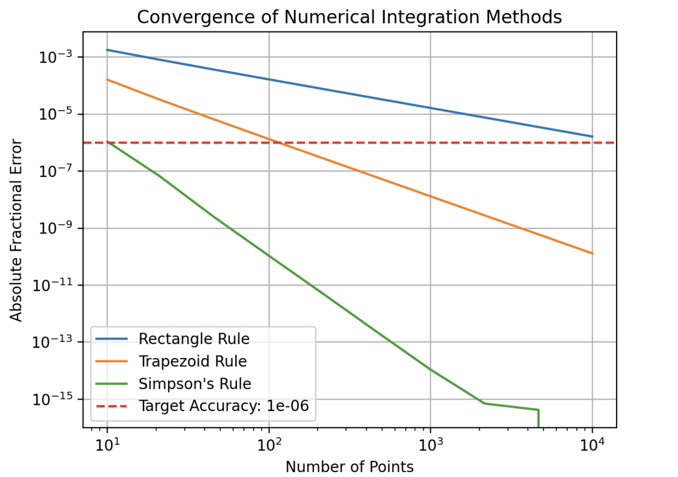
These results are consistent with the expected value of approximately 3200 Mpc, validating the implementation of the numerical integration methods.

Task 4.2 Convergence testing

We compare the convergence and accuracy of three numerical integration methods—Rectangle Rule, Trapezoid Rule, and Simpson's Rule—by calculating the cosmological distance to redshift $z = 1$. We test how the accuracy of each method changes as the number of integration steps increases. Additionally, we determine a good target accuracy for the calculation based on physical reasoning, which is plotted alongside the results.

I evaluated the performance of the Rectangle, Trapezoid, and Simpson's Rule by varying the number of integration points from $n = 10$ to $n = 10^4$, and calculated the absolute fractional error for each method.

Given the typical precision of cosmological measurements, an absolute fractional error of 10^{-6} strikes a good balance between precision and practicality. This level of accuracy ensures that we are well within the margin of error required for meaningful results while not pushing the computations to the point of diminishing returns. This choice reflects physical reasoning because achieving better precision would likely exceed the uncertainties inherent in observational data, making further computational effort redundant.



The results are plotted in the graph titled **Convergence of Numerical Integration Methods**, which shows the absolute fractional error as a function of the number of steps for each method:

Simpson's Rule (green line) converges quickly, achieving very high accuracy with a relatively small number of steps.

The error drops below 10^{-14} with around 1000 steps, making it the most accurate method in this comparison.

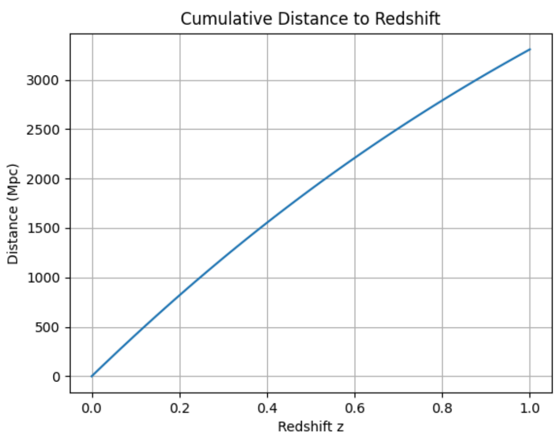
Trapezoid Rule (orange line) converges more slowly than Simpson's Rule but still performs reasonably well (better than the rectangle rule), reaching fractional errors below 10^{-8} with approximately 1000 steps.

Rectangle Rule (blue line) is the least accurate and converges slowly. It doesn't reach an accuracy better than 10^{-6} within the tested range of steps, requiring significantly more steps to achieve similar precision to the other methods.

Therefore, **Simpson's Rule** is the recommended method for this calculation. It has the fastest convergence rate and achieves high accuracy with a relatively small number of steps compared to the other methods. From the convergence analysis, Simpson's rule reaches a fractional error below 10^{-14} with around 1000 steps, which is even lower than the desired target accuracy. So, I recommend **Simpson's Rule** with **1000 steps**.

Task 4.3 Cumulative version

A cumulative version of the Trapezoid rule was implemented to compute the cosmological distance for a range of redshifts from $z = 0$ to $z = 1$. The resulting plot shows the cumulative distance as a function of redshift:

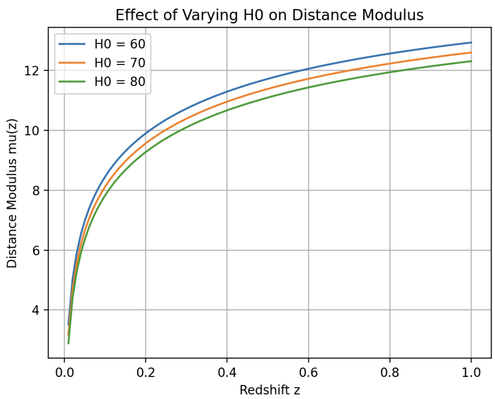


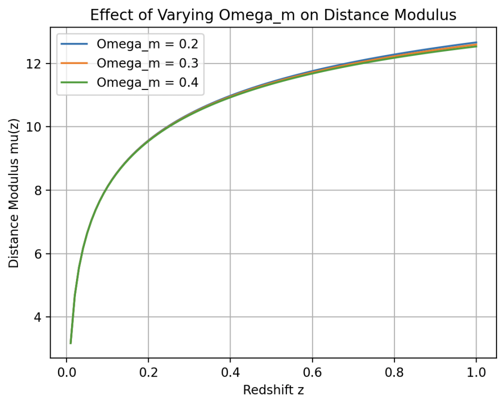
The plot demonstrates that the cosmological distance increases non-linearly with redshift, as expected in the cosmological model.

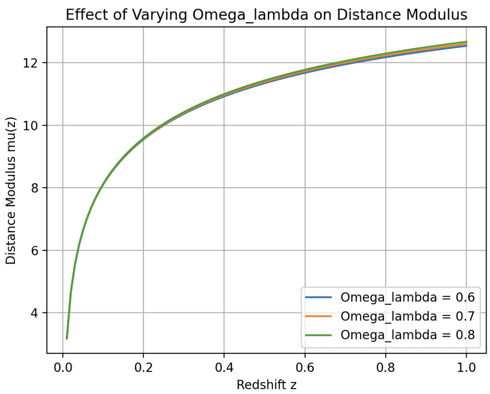
Additionally, the SciPy interpolator was used to calculate the distance modulus $\mu(z)$, which is a logarithmic measure of the distance. The distance modulus for a range of redshift values was computed and plotted, considering variations in H_0, Ω_m, and Ω_λ.

Task 4.4 Exploration

In this task, the effect of varying the parameters H_0, Ω_m, and Ω_λ on the distance modulus $\mu(z)$ was explored. The following three plots examine the effect of varying a single parameter (H_0, Ω_m, and Ω_λ) on the distance modulus as a function of redshift.:







These plots indicate that higher values of H_0 result in lower distance moduli, as a larger H_0 constant leads to smaller distances for the same redshift. Variations in Ω_m and Ω_λ also have noticeable effects: higher Ω_λ produces slightly higher moduli, while higher Ω_m results in slightly lower moduli.

Additional test code

The function check_scipy_custom is an additional tool for comparing results from our custom functions with those from SciPy libraries. With n = 1000, the result is:

```
Distance (Rectangle Rule): 3214.33 Mpc
Distance (Trapezoid Rule): 3214.28 Mpc
Distance (Simpson's Rule): 3214.28 Mpc
Rectangle Rule Distance (Custom): 3214.33179 Mpc
Rectangle Rule Distance (SciPy): 3214.27944 Mpc
Difference: 0.05235 Mpc

Trapezoid Rule Distance (Custom): 3214.27949 Mpc
Trapezoid Rule Distance (SciPy): 3214.27944 Mpc
Difference: 0.00004 Mpc

Simpson's Rule Distance (Custom): 3214.27944 Mpc
Simpson's Rule Distance (SciPy): 3211.06516 Mpc
Difference: 3.21428 Mpc

Cumulative Trapezoid Rule Validation:
z=0.00: Custom = 0.00000 Mpc, SciPy = 0.00000 Mpc, Difference = 0.00000 Mpc
z=0.10: Custom = 407.51018 Mpc, SciPy = 407.51018 Mpc, Difference = 0.00000 Mpc
z=0.20: Custom = 795.34185 Mpc, SciPy = 795.34185 Mpc, Difference = 0.00000 Mpc
z=0.30: Custom = 1163.09627 Mpc, SciPy = 1163.09627 Mpc, Difference = 0.00000 Mpc
z=0.40: Custom = 1510.84358 Mpc, SciPy = 1510.84358 Mpc, Difference = 0.00000 Mpc
z=0.50: Custom = 1839.02809 Mpc, SciPy = 1839.02809 Mpc, Difference = 0.00000 Mpc
z=0.60: Custom = 2148.36909 Mpc, SciPy = 2148.36909 Mpc, Difference = 0.00000 Mpc
z=0.70: Custom = 2439.77001 Mpc, SciPy = 2439.77001 Mpc, Difference = 0.00000 Mpc
z=0.80: Custom = 2714.24233 Mpc, SciPy = 2714.24233 Mpc, Difference = 0.00000 Mpc
z=0.90: Custom = 2972.84636 Mpc, SciPy = 2972.84636 Mpc, Difference = 0.00000 Mpc
```

With n = 10000, the result is:

```
Distance (Rectangle Rule): 3214.33 Mpc
Distance (Trapezoid Rule): 3214.28 Mpc
Distance (Simpson's Rule): 3214.28 Mpc
Rectangle Rule Distance (Custom): 3214.28467 Mpc
Rectangle Rule Distance (SciPy): 3214.27944 Mpc
```

```
Difference: 0.00523 Mpc

Trapezoid Rule Distance (Custom): 3214.27944 Mpc
Trapezoid Rule Distance (SciPy): 3214.27944 Mpc
Difference: 0.00000 Mpc

Simpson's Rule Distance (Custom): 3214.27944 Mpc
Simpson's Rule Distance (SciPy): 3213.95802 Mpc
Difference: 0.32143 Mpc

Cumulative Trapezoid Rule Validation:
z=0.00: Custom = 0.00000 Mpc, SciPy = 0.00000 Mpc, Difference = 0.00000 Mpc
z=0.10: Custom = 407.15214 Mpc, SciPy = 407.15214 Mpc, Difference = 0.00000 Mpc
z=0.20: Custom = 794.66169 Mpc, SciPy = 794.66169 Mpc, Difference = 0.00000 Mpc
z=0.30: Custom = 1162.13031 Mpc, SciPy = 1162.13031 Mpc, Difference = 0.00000 Mpc
z=0.40: Custom = 1509.62702 Mpc, SciPy = 1509.62702 Mpc, Difference = 0.00000 Mpc
z=0.50: Custom = 1837.59394 Mpc, SciPy = 1837.59394 Mpc, Difference = 0.00000 Mpc
z=0.60: Custom = 2146.74753 Mpc, SciPy = 2146.74753 Mpc, Difference = 0.00000 Mpc
z=0.70: Custom = 2437.98807 Mpc, SciPy = 2437.98807 Mpc, Difference = 0.00000 Mpc
z=0.80: Custom = 2712.32391 Mpc, SciPy = 2712.32391 Mpc, Difference = 0.00000 Mpc
z=0.90: Custom = 2970.81233 Mpc, SciPy = 2970.81233 Mpc, Difference = 0.00000 Mpc
```

These results show that our custom functions to integrate over redshift using the rectangle rule, trapezoid rule, Simpson’s rule, and cumulative trapezoid rule are accurate. Additionally, with a higher number of steps, any discrepancies can be minimised.