Math 228A: Problem Set 7

Eric Huynh

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1. Problem 1 (QR Algorithm)

   Here we implemented the QR algorithm with and without shifts and estimated the convergence for real symmetric matrices. We also seek to comment on the behavior of convergence in both methods. Our main test matrix (given in the problem statement) is the MATLAB embedded matrix given by the function: **hilb(4)**

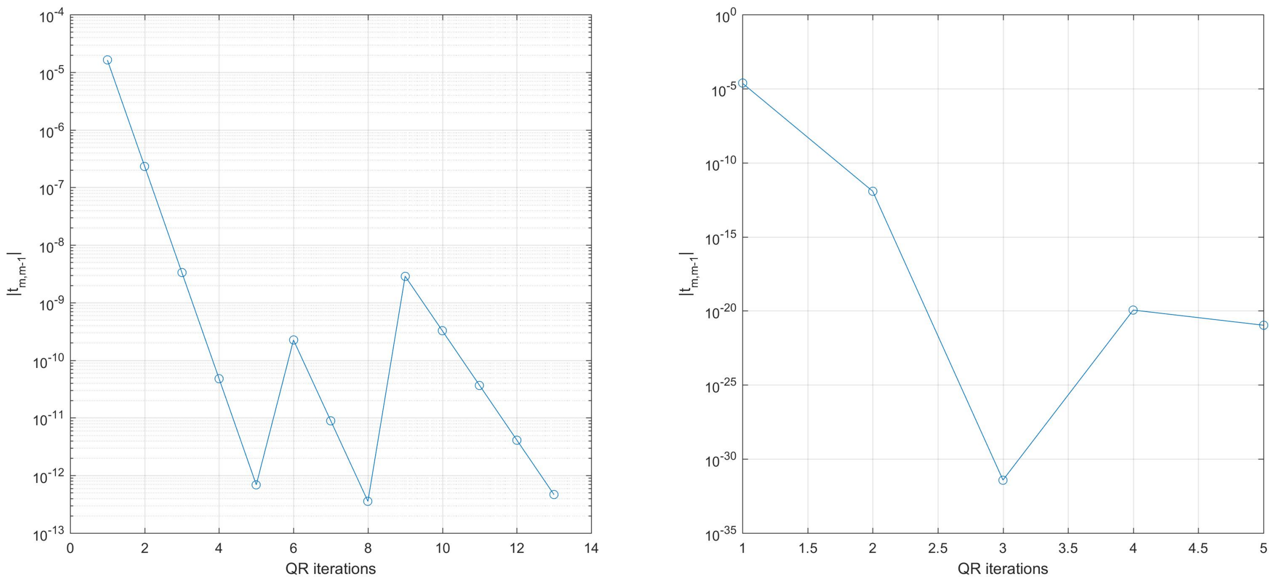
$$\textbf{hilb(4)} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

   Our other test problem is a 15 by 15 real symmetric matrix:

$$\mathbf{D} = diag(15 : -1 : 1) + ones(15, 15) = \begin{bmatrix} 16 & 1 & 1 & \dots & 1 \\ 1 & 15 & 1 & \dots & 1 \\ 1 & 1 & 14 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 2 \end{bmatrix}$$

   Using the Unshifted and Shifted (using the Wilkinson Shift) QR alg we obtain the following eigenvalues for each problem:

unshifted **D**	shifted **D**	unshifted **hilb(4)**	shifted **hilb(4)**
1.214655366493386	1.214655366493358	0.000096702304023	0.000096702304023
2.256950983586661	2.256950983586650	0.006738273605761	0.006738273605761
3.287775587950246	3.287775587950209	0.169141220221451	0.169141220221450
4.314311852797403	4.314311852797298	1.500214280059250	1.500214280059241
5.338959876555060	5.338959876554912		
6.362944488529090	6.362944488528962		
7.387092745473991	7.387092745473670		
8.412112071301131	8.412112071300840		
9.438745755142579	9.438745755142200		
10.467921659050885	10.467921659050392		
11.500983018469753	11.500983018469270		
12.540186373729746	12.540186373729309		
13.590131956253426	13.590131956252876		
14.664096999369015	14.664096999367917		
24.223131265302381	24.223131265302232		



(a) Unshifted-QR

(b) Wilkinson Shifted QR

Figure 1: Convergence of **hilb(4)**

Next we plot our stored convergence criterion : $|t_{m,m-1}|$ as we iterate against the number of QR iterations per eigenvalue. (figures above)

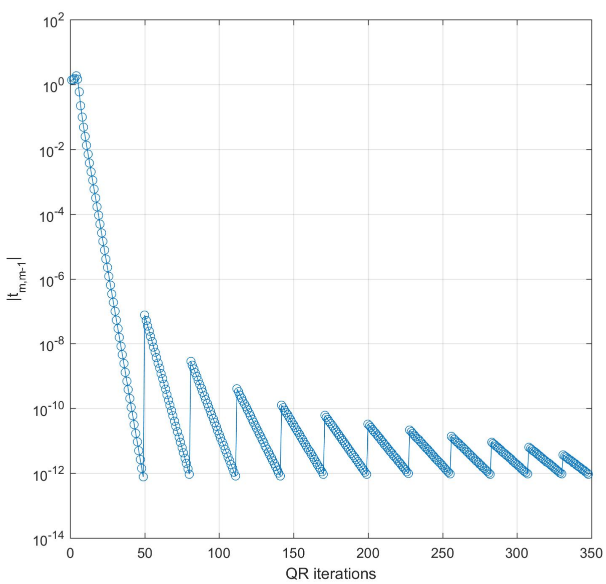
We notice that the Unshifted QR generally has linear convergence performance for each eigenvalue. This is especially apparent in Figure 1 (a) and Figure 2 (a).

Also we notice that the Wilkinson Shifted QR displays cubic convergence, where some eigenvalues converge in even one or two steps.

Now we turn to answer the question: ”Is it meaningful to speak of a certain number of QR iterations per eigenvalue?” In short, yes, it is because we very clearly see that, in Figure (a), our eigenvalues converge with fewer QR iterations after finding some of the other eigenvalues.

One explanation why this effect occurs is due to the convergence of pure QR to the largest eigenvalue and the subsequent reduction of the tridiagonal matrix to find the next eigenvalue. The theory set out by Trefethen and Bau’s Theorem 28.4 states that the unshifted algorithm (pure QR) applied to a real symmetric matrix, will have linear convergence with constant $max_j \frac{|\lambda_{j+1}|}{|\lambda_j|}$.

Thus for every reduction of the largest eigenvalue, and shrinking of the matrix, we find that the linear convergence is changes by a constant factor so that we converge in fewer iterations.



(a) Unshifted-QR

(b) Wilkinson Shifted QR

Figure 2: Convergence of $\mathbf{D}$ = diag(15:-1:1) + ones(15,15)