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Del in cylindrical and spherical coordinates

 $This is a list of some \underline{vector\ calculus}\ formulae\ for\ working\ with\ common\ \underline{curvilinear}\ \underline{coordinate\ systems}.$

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Notes

- This article uses the standard notation ISO 80000-2, which supersedes ISO 31-11, for spherical coordinates (other sources may reverse the definitions of θ and φ):
 - The polar angle is denoted by θ : it is the angle between the z-axis and the radial vector connecting the origin to the point in question.
 - The <u>azimuthal angle</u> is denoted by φ: it is the angle between the x-axis and the projection of the radial vector onto the xy-plane.
- The function <u>atan2(y, x)</u> can be used instead of the mathematical function <u>arctan(y/x)</u> owing to its <u>domain</u> and <u>image</u>. The classical arctan function has an image of (−π/2, +π/2), whereas atan2 is defined to have an image of (−π, π].

Coordinate conversions

Conversion between Cartesian, cylindrical, and spherical coordinates^[1]

		From		
		Cartesian	Cylindrical	Spherical
То	Cartesian	x = x $y = y$ $z = z$	$x = \rho \cos \varphi$ $y = \rho \sin \varphi$ $z = z$	$x = r \sin \theta \cos \varphi$ $y = r \sin \theta \sin \varphi$ $z = r \cos \theta$
	Cylindrical	$ ho = \sqrt{x^2 + y^2}$ $ ho = \arctan\left(\frac{y}{x}\right)$ $ ho = z$	$ \rho = \rho \\ \varphi = \varphi \\ z = z $	$\rho = r \sin \theta$ $\varphi = \varphi$ $z = r \cos \theta$
	Spherical	$r = \sqrt{x^2 + y^2 + z^2}$ $ heta = rctanigg(rac{\sqrt{x^2 + y^2}}{z}igg)$ $arphi = rctanigg(rac{y}{x}igg)$	$r = \sqrt{ ho^2 + z^2}$ $ heta = \arctan\left(rac{ ho}{z} ight)$ $arphi = arphi$	$egin{aligned} r &= r \ heta &= heta \ heta &= heta \ arphi &= arphi \end{aligned}$

Unit vector conversions

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of destination coordinates $^{[1]}$

	Cartesian	Cylindrical	Spherical
Cartesian	N/A	$ \hat{\mathbf{x}} = \cos \varphi \hat{\boldsymbol{\rho}} - \sin \varphi \hat{\boldsymbol{\varphi}} \hat{\mathbf{y}} = \sin \varphi \hat{\boldsymbol{\rho}} + \cos \varphi \hat{\boldsymbol{\varphi}} \hat{\mathbf{z}} = \hat{\mathbf{z}} $	$\begin{split} \hat{\mathbf{x}} &= \sin \theta \cos \varphi \hat{\mathbf{r}} + \cos \theta \cos \varphi \hat{\boldsymbol{\theta}} - \sin \varphi \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{y}} &= \sin \theta \sin \varphi \hat{\mathbf{r}} + \cos \theta \sin \varphi \hat{\boldsymbol{\theta}} + \cos \varphi \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} &= \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\boldsymbol{\theta}} \end{split}$
Cylindrical	$\hat{ ho} = rac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{oldsymbol{arphi}} = rac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$ $\hat{oldsymbol{z}} = \hat{oldsymbol{z}}$	N/A	$\hat{ ho} = \sin \theta \hat{\mathbf{r}} + \cos \theta \hat{\theta}$ $\hat{\varphi} = \hat{\varphi}$ $\hat{\mathbf{z}} = \cos \theta \hat{\mathbf{r}} - \sin \theta \hat{\theta}$
Spherical	$\hat{\mathbf{r}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$ $\hat{\boldsymbol{\theta}} = \frac{(x\hat{\mathbf{x}} + y\hat{\mathbf{y}})z - (x^2 + y^2)\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}\sqrt{x^2 + y^2}}$ $\hat{\boldsymbol{\varphi}} = \frac{-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}}{\sqrt{x^2 + y^2}}$	$\hat{\mathbf{r}} = \frac{\rho \hat{\rho} + z \hat{\mathbf{z}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\theta} = \frac{z \hat{\rho} - \rho \hat{\mathbf{z}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\varphi} = \hat{\varphi}$	N/A

Conversion between unit vectors in Cartesian, cylindrical, and spherical coordinate systems in terms of source coordinates

	Cartesian	Cylindrical	Spherical
Cartesian	N/A	$\hat{\mathbf{x}} = rac{x\hat{ ho} - y\hat{oldsymbol{arphi}}}{\sqrt{x^2 + y^2}}$ $\hat{\mathbf{y}} = rac{y\hat{ ho} + x\hat{oldsymbol{arphi}}}{\sqrt{x^2 + y^2}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$\begin{split} \hat{\mathbf{x}} &= \frac{x \left(\sqrt{x^2 + y^2} \hat{\mathbf{r}} + z \hat{\boldsymbol{\theta}} \right) - y \sqrt{x^2 + y^2 + z^2} \hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2} \sqrt{x^2 + y^2 + z^2}} \\ \hat{\mathbf{y}} &= \frac{y \left(\sqrt{x^2 + y^2} \hat{\mathbf{r}} + z \hat{\boldsymbol{\theta}} \right) + x \sqrt{x^2 + y^2 + z^2} \hat{\boldsymbol{\varphi}}}{\sqrt{x^2 + y^2} \sqrt{x^2 + y^2 + z^2}} \\ \hat{\mathbf{z}} &= \frac{z \hat{\mathbf{r}} - \sqrt{x^2 + y^2} \hat{\boldsymbol{\theta}}}{\sqrt{x^2 + y^2 + z^2}} \end{split}$
Cylindrical	$\hat{ ho} = \cos \varphi \hat{\mathbf{x}} + \sin \varphi \hat{\mathbf{y}}$ $\hat{\varphi} = -\sin \varphi \hat{\mathbf{x}} + \cos \varphi \hat{\mathbf{y}}$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	N/A	$\hat{\rho} = \frac{\rho \hat{\mathbf{r}} + z \hat{\boldsymbol{\theta}}}{\sqrt{\rho^2 + z^2}}$ $\hat{\varphi} = \hat{\varphi}$ $\hat{\mathbf{z}} = \frac{z \hat{\mathbf{r}} - \rho \hat{\boldsymbol{\theta}}}{\sqrt{\rho^2 + z^2}}$
Spherical	$\hat{\mathbf{r}} = \sin \theta \left(\cos \varphi \hat{\mathbf{x}} + \sin \varphi \hat{\mathbf{y}} \right) + \cos \theta \hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos \theta \left(\cos \varphi \hat{\mathbf{x}} + \sin \varphi \hat{\mathbf{y}} \right) - \sin \theta \hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = -\sin \varphi \hat{\mathbf{x}} + \cos \varphi \hat{\mathbf{y}}$	$\hat{\mathbf{r}} = \sin \theta \hat{\boldsymbol{\rho}} + \cos \theta \hat{\mathbf{z}}$ $\hat{\boldsymbol{\theta}} = \cos \theta \hat{\boldsymbol{\rho}} - \sin \theta \hat{\mathbf{z}}$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	N/A

Del formula

Table with the $\underline{\text{del}}$ operator in cartesian, cylindrical and spherical coordinates

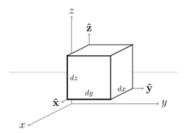
lable with the <u>del</u> operator in cartesian, cylindrical and spherical coordinates					
Operation	Cartesian coordinates (x, y, z)	Cylindrical coordinates $(ho, arphi, z)$	Spherical coordinates (r, θ, φ) , where θ is the polar and φ is the azimuthal angle $^{\alpha}$		
A <u>vector field</u> A	$A_{oldsymbol{x}}\hat{oldsymbol{x}}+A_{oldsymbol{y}}\hat{oldsymbol{y}}+A_{oldsymbol{z}}\hat{oldsymbol{z}}$	$A_{ ho}\hat{oldsymbol{ ho}} + A_{arphi}\hat{oldsymbol{arphi}} + A_{oldsymbol{z}}\hat{oldsymbol{z}}$	$A_{ au}\hat{\mathbf{r}}+A_{ heta}\hat{oldsymbol{ heta}}+A_{arphi}\hat{oldsymbol{\phi}}$		
Gradient ∇f ^[1]	$\frac{\partial f}{\partial x}\hat{\mathbf{x}} + \frac{\partial f}{\partial y}\hat{\mathbf{y}} + \frac{\partial f}{\partial z}\hat{\mathbf{z}}$	$rac{\partial f}{\partial ho}\hat{ ho} + rac{1}{ ho}rac{\partial f}{\partial arphi}\hat{arphi} + rac{\partial f}{\partial z}\hat{f z}$	$rac{\partial f}{\partial r}\hat{\mathbf{r}}+rac{1}{r}rac{\partial f}{\partial heta}\hat{ heta}+rac{1}{r\sin heta}rac{\partial f}{\partial arphi}\hat{oldsymbol{arphi}}$		
$\frac{\textbf{Divergence}}{\nabla \cdot \mathbf{A}^{[1]}}$	$rac{\partial A_x}{\partial x} + rac{\partial A_y}{\partial y} + rac{\partial A_z}{\partial z}$	$\frac{1}{\rho}\frac{\partial\left(\rho A_{\rho}\right)}{\partial\rho}+\frac{1}{\rho}\frac{\partial A_{\varphi}}{\partial\varphi}+\frac{\partial A_{z}}{\partial z}$	$rac{1}{r^2}rac{\partial \left(r^2A_{ au} ight)}{\partial r}+rac{1}{r\sin heta}rac{\partial}{\partial heta}\left(A_ heta\sin heta ight)+rac{1}{r\sin heta}rac{\partial A_arphi}{\partialarphi}$		
$\underline{\mathbf{Curl}} \; \nabla \times \mathbf{A}^{[1]}$	$\begin{split} & \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z}\right) \hat{\mathbf{x}} \\ & + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \hat{\mathbf{y}} \\ & + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \hat{\mathbf{z}} \end{split}$	$egin{aligned} \left(rac{1}{ ho}rac{\partial A_x}{\partial arphi}-rac{\partial A_{arphi}}{\partial z} ight)\hat{ ho} \ +\left(rac{\partial A_ ho}{\partial z}-rac{\partial A_x}{\partial ho} ight)\hat{oldsymbol{arphi}} \ +rac{1}{ ho}\left(rac{\partial (ho A_{arphi})}{\partial ho}-rac{\partial A_ ho}{\partial arphi} ight)\hat{oldsymbol{z}} \end{aligned}$	$egin{aligned} &rac{1}{r\sin heta}\left(rac{\partial}{\partial heta}(A_{arphi}\sin heta)-rac{\partial A_{ar{ heta}}}{\partialarphi} ight)\hat{\mathbf{r}}\ &+rac{1}{r}\left(rac{1}{\sin heta}rac{\partial A_{r}}{\partialarphi}-rac{\partial}{\partial r}\left(rA_{arphi} ight) ight)\hat{ heta}\ &+rac{1}{r}\left(rac{\partial}{\partial r}\left(rA_{ar{ heta}} ight)-rac{\partial A_{r}}{\partial heta} ight)\hat{arphi} \end{aligned}$		
Laplace operator $\nabla^2 f \equiv \Delta f^{[1]}$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2}\frac{\partial}{\partial r}\bigg(r^2\frac{\partial f}{\partial r}\bigg) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\bigg(\sin\theta\frac{\partial f}{\partial\theta}\bigg) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$		
$\begin{tabular}{ll} \mbox{Vector} \\ \mbox{Laplacian} \\ \mbox{$\nabla^2 {\bf A} \equiv \Delta {\bf A}$} \end{tabular}$	$ abla^2 A_x \hat{f x} + abla^2 A_y \hat{f y} + abla^2 A_z \hat{f z}$	$\begin{array}{l} -\frac{-\text{View by clicking [show]}}{\left(\nabla^2 A_{\rho} - \frac{A_{\rho}}{\rho^2} - \frac{2}{\rho^2} \frac{\partial A_{\varphi}}{\partial \varphi}\right) \hat{\rho}} \\ + \left(\nabla^2 A_{\varphi} - \frac{A_{\varphi}}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_{\rho}}{\partial \varphi}\right) \hat{\varphi} \\ + \nabla^2 A_{z} \hat{z} \end{array}$	$\begin{split} &\left(\nabla^2 A_\tau - \frac{2A_\tau}{r^2} - \frac{2}{r^2 \sin\theta} \frac{\partial \left(A_\theta \sin\theta\right)}{\partial \theta} - \frac{2}{r^2 \sin\theta} \frac{\partial A_\varphi}{\partial \varphi}\right) \hat{\mathbf{r}} \\ &+ \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2 \sin^2\theta} + \frac{2}{r^2} \frac{\partial A_\tau}{\partial \theta} - \frac{2\cos\theta}{r^2 \sin^2\theta} \frac{\partial A_\varphi}{\partial \varphi}\right) \hat{\boldsymbol{\theta}} \\ &+ \left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2 \sin^2\theta} + \frac{2}{r^2 \sin\theta} \frac{\partial A_\tau}{\partial \varphi} + \frac{2\cos\theta}{r^2 \sin^2\theta} \frac{\partial A_\theta}{\partial \varphi}\right) \hat{\boldsymbol{\phi}} \end{split}$		
$\frac{\text{Material}}{\text{derivative}^{\alpha[2]}} \\ \frac{(A \cdot \nabla)B}{(A \cdot \nabla)B}$	$\mathbf{A} \cdot abla B_x \hat{\mathbf{x}} + \mathbf{A} \cdot abla B_y \hat{\mathbf{y}} + \mathbf{A} \cdot abla B_z \hat{\mathbf{z}}$	$\begin{split} &\left(A_{\rho}\frac{\partial B_{\rho}}{\partial \rho} + \frac{A_{\varphi}}{\rho}\frac{\partial B_{\rho}}{\partial \varphi} + A_{z}\frac{\partial B_{\rho}}{\partial z} - \frac{A_{\varphi}B_{\varphi}}{\rho}\right)\hat{\rho} \\ &+ \left(A_{\rho}\frac{\partial B_{\varphi}}{\partial \rho} + \frac{A_{\varphi}}{\rho}\frac{\partial B_{\varphi}}{\partial \varphi} + A_{z}\frac{\partial B_{\varphi}}{\partial z} + \frac{A_{\varphi}B_{\rho}}{\rho}\right)\hat{\varphi} \\ &+ \left(A_{\rho}\frac{\partial B_{z}}{\partial \rho} + \frac{A_{\varphi}}{\rho}\frac{\partial B_{z}}{\partial \varphi} + A_{z}\frac{\partial B_{z}}{\partial z}\right)\hat{\mathbf{z}} \end{split}$			
Tensor divergence ∇ · T	$\begin{aligned} & - \text{View by clicking [show]} - \\ & \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \right) \hat{\mathbf{x}} \\ & + \left(\frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{xy}}{\partial z} \right) \hat{\mathbf{y}} \\ & + \left(\frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z} \right) \hat{\mathbf{z}} \end{aligned}$	$\begin{split} & - \text{View by clicking [show]} - \\ & \left[\frac{\partial T_{\rho\rho}}{\partial \rho} + \frac{1}{\rho} \frac{\partial T_{\varphi\rho}}{\partial \varphi} + \frac{\partial T_{z\rho}}{\partial z} + \frac{1}{\rho} (T_{\rho\rho} - T_{\varphi\phi}) \right] \hat{\rho} \\ & + \left[\frac{\partial T_{\rho\varphi}}{\partial \rho} + \frac{1}{\rho} \frac{\partial T_{\varphi\varphi}}{\partial \varphi} + \frac{\partial T_{z\varphi}}{\partial z} + \frac{1}{\rho} (T_{\rho\varphi} + T_{\varphi\rho}) \right] \hat{\varphi} \\ & + \left[\frac{\partial T_{\rhoz}}{\partial \rho} + \frac{1}{\rho} \frac{\partial T_{\varphiz}}{\partial \varphi} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{\rhoz}}{\rho} \right] \hat{\mathbf{z}} \end{split}$	$\begin{split} & - \text{View by clicking [show]} - \\ & \left[\frac{\partial T_{rr}}{\partial r} + 2\frac{T_{rr}}{r} + \frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta} + \frac{\cot\theta}{r}T_{\theta r} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi r}}{\partial \varphi} - \frac{1}{r}(T_{\theta \theta} + T_{\varphi \varphi}) \right] \hat{\mathbf{r}} \\ & + \left[\frac{\partial T_{r\theta}}{\partial r} + 2\frac{T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta \theta}}{\partial \theta} + \frac{\cot\theta}{r}T_{\theta \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi \theta}}{\partial \varphi} + \frac{T_{\theta r}}{r} - \frac{\cot\theta}{r}T_{\varphi \varphi} \right] \hat{\theta} \\ & + \left[\frac{\partial T_{r\varphi}}{\partial r} + 2\frac{T_{r\varphi}}{r} + \frac{1}{r}\frac{\partial T_{\theta \varphi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\varphi \varphi}}{\partial \varphi} + \frac{T_{\varphi r}}{r} + \frac{\cot\theta}{r}(T_{\theta \varphi} + T_{\varphi \theta}) \right] \hat{\varphi} \end{split}$		
Differential displacement $d\ell^{[1]}$	$dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$	$d ho\hat{oldsymbol{ ho}} + hodarphi\hat{oldsymbol{arphi}} + dz\hat{oldsymbol{z}}$	$dr\hat{f r} + rd heta\hat{m heta} + r\sin hetadarphi\hat{m \phi}$		
Differential normal area dS	$dy dz \hat{\mathbf{x}} \\ + dx dz \hat{\mathbf{y}} \\ + dx dy \hat{\mathbf{z}}$	$ hodarphidz\hat{oldsymbol{ ho}}\ +d hodz\hat{oldsymbol{arphi}}\ + hod hodarphi\hat{f z}$	$r^2 \sin heta d heta darphi \hat{\mathbf{r}} \ + r \sin heta dr darphi \hat{oldsymbol{ heta}} \ + r dr d heta \hat{oldsymbol{arphi}}$		
Differential volume $dV^{[1]}$	dxdydz	$ ho\mathrm{d} ho\mathrm{d}arphi\mathrm{d} z$	$r^2 \sin heta dr d heta darphi$		

 $^{^{\}alpha}$ This page uses θ for the polar angle and φ for the azimuthal angle, which is common notation in physics. The source that is used for these formulae uses θ for the azimuthal angle and φ for the polar angle, which is common mathematical notation. In order to get the mathematics formulae, switch θ and φ in the formulae shown in the table above.

Non-trivial calculation rules

- 1. div grad $f \equiv \nabla \cdot \nabla f \equiv \nabla^2 f$
- 2. curl grad $f \equiv \nabla \times \nabla f = \mathbf{0}$
- 3. div curl $\mathbf{A} \equiv \nabla \cdot (\nabla \times \mathbf{A}) = 0$
- 4. $\operatorname{\mathbf{curl}} \operatorname{\mathbf{curl}} \mathbf{A} \equiv \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) \nabla^2 \mathbf{A} \left(\operatorname{\underline{Lagrange's formula}} \operatorname{for del} \right)$
- 5. $\nabla^2(fg) = f\nabla^2g + 2\nabla f \cdot \nabla g + g\nabla^2 f$

Cartesian derivation

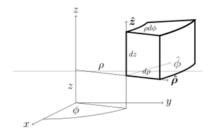


$$\begin{split} \operatorname{div} \mathbf{A} &= \lim_{V \to 0} \frac{\iint_{\partial V} \mathbf{A} \cdot d\mathbf{S}}{\iiint_{V} dV} = \frac{A_{x}(x + dx) dy dz - A_{x}(x) dy dz + A_{y}(y + dy) dx dz - A_{y}(y) dx dz + A_{x}(z + dz) dx dy - A_{x}(z) dx dy}{dx dy dz} \\ &= \frac{\partial A_{x}}{\partial x} + \frac{\partial A_{y}}{\partial y} + \frac{\partial A_{x}}{\partial z} \end{split}$$

$$\begin{split} (\operatorname{curl} \mathbf{A})_x &= \lim_{S^{\downarrow \hat{\mathbf{x}}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_S dS} = \frac{A_x(y + dy)dz - A_x(y)dz + A_y(z)dy - A_y(z + dz)dy}{dydz} \\ &= \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial z} \end{split}$$

The expressions for $(\operatorname{curl} \mathbf{A})_y$ and $(\operatorname{curl} \mathbf{A})_z$ are found in the same way.

Cylindrical derivation



$$\begin{split} \operatorname{div} \mathbf{A} &= \lim_{V \to 0} \frac{\iint_{\partial V} \mathbf{A} \cdot d\mathbf{S}}{\iiint_{V} dV} = \frac{A_{\rho}(\rho + d\rho)(\rho + d\rho)d\phi \, dz - A_{\rho}(\rho)\rho d\phi \, dz + A_{\phi}(\phi + d\phi)d\rho \, dz - A_{\phi}(\phi)d\rho dz + A_{z}(z + dz)d\rho \, (\rho + d\rho/2)d\phi - A_{z}(z)d\rho \, (\rho + d\rho/2)d\phi}{(\rho + d\rho/2) \, d\phi \, d\rho \, dz} \\ &= \frac{1}{\rho} \frac{\partial (\rho A_{\rho})}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_{\phi}}{\partial \phi} + \frac{\partial A_{z}}{\partial z} \end{split}$$

$$\begin{split} (\operatorname{curl} \mathbf{A})_{\rho} &= \lim_{S^{\perp \hat{\rho}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\boldsymbol{\ell}}{\iint_{S} dS} = \frac{A_{\phi}(z)(\rho + d\rho)d\phi - A_{\phi}(z + dz)(\rho + d\rho)d\phi + A_{z}(\phi + d\phi)dz - A_{z}(\phi)dz}{(\rho + d\rho)d\phi dz} \\ &= -\frac{\partial A_{\phi}}{\partial z} + \frac{1}{\rho} \frac{\partial A_{z}}{\partial \phi} \end{split}$$

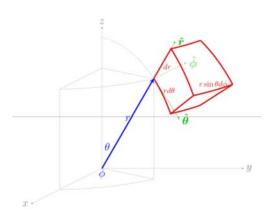
$$\begin{aligned} (\operatorname{curl} \mathbf{A})_{\phi} &= \lim_{S^{\perp \hat{\varphi}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_{S} dS} = \frac{A_{z}(\rho) dz - A_{z}(\rho + d\rho) dz + A_{\rho}(z + dz) d\rho - A_{\rho}(z) d\rho}{d\rho dz} \\ &= -\frac{\partial A_{z}}{\partial \rho} + \frac{\partial A_{\rho}}{\partial z} \end{aligned}$$

$$\begin{split} (\operatorname{curl} \mathbf{A})_z &= \lim_{S^{\perp \hat{x}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_S dS} = \frac{A_{\rho}(\phi) d\rho - A_{\rho}(\phi + d\phi) d\rho + A_{\phi}(\rho + d\rho)(\rho + d\rho) d\phi - A_{\phi}(\rho) \rho d\phi}{(\rho + d\rho/2) d\rho d\phi} \\ &= -\frac{1}{\rho} \frac{\partial A_{\rho}}{\partial \phi} + \frac{1}{\rho} \frac{\partial (\rho A_{\phi})}{\partial \rho} \end{split}$$

$$\operatorname{curl} \mathbf{A} = (\operatorname{curl} \mathbf{A})_{\rho} \, \hat{\rho} + (\operatorname{curl} \mathbf{A})_{\phi} \, \hat{\phi} + (\operatorname{curl} \mathbf{A})_{z} \, \hat{z} = \left(\frac{1}{\rho} \frac{\partial A_{z}}{\partial \phi} - \frac{\partial A_{\phi}}{\partial z}\right) \hat{\rho} + \left(\frac{\partial A_{\rho}}{\partial z} - \frac{\partial A_{z}}{\partial \rho}\right) \hat{\phi} + \frac{1}{\rho} \left(\frac{\partial (\rho A_{\phi})}{\partial \rho} - \frac{\partial A_{\rho}}{\partial \phi}\right) \hat{z}$$

Spherical derivation

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$$\begin{split} \operatorname{div} \mathbf{A} &= \lim_{V \to 0} \frac{\iint_{\partial V} \mathbf{A} \cdot d\mathbf{S}}{\iiint_{V} dV} \\ &= \frac{A_{r}(r + dr)(r + dr)d\theta \left(r + dr\right) \sin\theta d\phi - A_{r}(r)r d\theta r \sin\theta d\phi + A_{\theta}(\theta + d\theta) r dr d\phi - A_{\theta}(\theta) \sin(\theta) r dr d\phi + A_{\phi}(\phi + d\phi)(r + dr/2) dr d\theta - A_{\phi}(\phi)(r + dr/2) dr d\theta}{dr r d\theta r \sin\theta d\phi} \\ &= \frac{1}{r^{2}} \frac{\partial (r^{2}A_{r})}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (A_{\theta} \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial A_{\phi}}{\partial \phi} \end{split}$$

$$\begin{split} (\operatorname{curl} \mathbf{A})_r &= \lim_{S^{\perp r} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_S dS} = \frac{A_{\theta}(\phi) \, r d\theta + A_{\phi}(\theta + d\theta) \, r \sin(\theta + d\theta) d\phi - A_{\theta}(\phi + d\phi) \, r d\theta - A_{\phi}(\theta) \, r \sin(\theta) d\phi}{r d\theta \, r \sin \theta d\phi} \\ &= \frac{1}{r \sin \theta} \frac{\partial (A_{\phi} \sin \theta)}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial A_{\theta}}{\partial \phi} \end{split}$$

$$\begin{aligned} (\operatorname{curl} \mathbf{A})_{\theta} &= \lim_{S^{\perp \hat{\theta}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_{S} dS} = \frac{A_{\phi}(r) \, r \sin \theta d\phi + A_{r}(\phi + d\phi) dr - A_{\phi}(r + dr)(r + dr) \sin \theta d\phi - A_{r}(\phi) dr}{dr \, r \sin \theta d\phi} \\ &= \frac{1}{r \sin \theta} \frac{\partial A_{r}}{\partial \phi} - \frac{1}{r} \frac{\partial (r A_{\phi})}{\partial r} \end{aligned}$$

$$\begin{split} (\operatorname{curl} \mathbf{A})_{\phi} &= \lim_{S^{\perp \frac{1}{\theta}} \to 0} \frac{\int_{\partial S} \mathbf{A} \cdot d\ell}{\iint_{S} dS} = \frac{A_{r}(\theta) dr + A_{\theta}(r + dr)(r + dr) d\theta - A_{r}(\theta + d\theta) dr - A_{\theta}(r) r d\theta}{(r + dr/2) dr d\theta} \\ &= \frac{1}{r} \frac{\partial (rA_{\theta})}{\partial r} - \frac{1}{r} \frac{\partial A_{r}}{\partial \theta} \end{split}$$

$$\operatorname{curl} \mathbf{A} = (\operatorname{curl} \mathbf{A})_r \, \hat{r} + (\operatorname{curl} \mathbf{A})_\theta \, \hat{\theta} + (\operatorname{curl} \mathbf{A})_\phi \, \hat{\phi} = \frac{1}{r \sin \theta} \left(\frac{\partial (A_\phi \sin \theta)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right) \hat{r} + \frac{1}{r} \left(\frac{1}{\sin \theta} \, \frac{\partial A_r}{\partial \phi} - \frac{\partial (rA_\phi)}{\partial r} \right) \hat{\theta} + \frac{1}{r} \left(\frac{\partial (rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \hat{\phi}$$

Unit vector conversion formula

The unit vector of a coordinate parameter u is defined in such a way that a small positive change in u causes the position vector \vec{r} to change in \hat{u} direction.

Therefore, $\frac{\partial \vec{r}}{\partial u} = \frac{\partial s}{\partial u} \hat{u}$ where s is the arc length parameter.

For two sets of coordinate systems
$$u_i$$
 and v_j , according to chain rule, $d\vec{r} = \sum_i \frac{\partial \vec{r}}{\partial u_i} du_i = \sum_i \frac{\partial s}{\partial u_i} \hat{u}_i du_i = \sum_j \frac{\partial s}{\partial v_j} \hat{v}_j dv_j = \sum_j \frac{\partial s}{\partial u_j} \hat{v}_j \sum_i \frac{\partial v_j}{\partial u_i} du_i = \sum_i \sum_j \frac{\partial s}{\partial v_j} \frac{\partial v_j}{\partial u_i} \hat{v}_j du_i$

Now, let all of $du_i = 0$ but one and then divide both sides by the corresponding differential of that coordinate parameter, we find:

$$\frac{\partial s}{\partial u_i} \hat{u_i} = \sum_i \frac{\partial s}{\partial v_j} \frac{\partial v_j}{\partial u_i} \hat{v_j}$$

See also

- De
- Orthogonal coordinates
- Curvilinear coordinates
- Vector fields in cylindrical and spherical coordinates

References

- 1. Griffiths, David J. (2012). Introduction to Electrodynamics. Pearson. ISBN 978-0-321-85656-2.
- 2. Weisstein, Eric W. "Convective Operator" (http://mathworld.wolfram.com/ConvectiveOperator.html). Mathworld. Retrieved 23 March 2011

External links

■ Maxima Computer Algebra system scripts (http://www.csulb.edu/~woollett/) to generate some of these operators in cylindrical and spherical coordinates

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