Appendix D

Vector Analysis

The following conventions are used in this appendix and throughout the book:

f, g, ϕ, ψ are scalar functions of $\mathbf{x}, t$;
$\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are vector functions of $\mathbf{x}, t$;
$A = |\mathbf{A}| \equiv \sqrt{\mathbf{A} \cdot \mathbf{A}}$ is the magnitude or length of the vector $\mathbf{A}$;
$\hat{\mathbf{e}}_A \equiv \mathbf{A}/A$ is a unit vector in the $\mathbf{A}$ direction;
$\mathbf{x}$ is the vector from the origin to the point (x, y, z);
$\mathbf{T}, \mathbf{W}, \mathbf{AB}$, etc., are dyad (second rank tensor) functions of $\mathbf{x}, t$ that will
   be called simply tensors;
$\mathbf{I}$ is the identity tensor or unit dyad;
$\mathbf{T}^{\mathsf{T}}$ is the transpose of tensor $\mathbf{T}$ (interchange of indices of the tensor
   elements), a tensor;
$tr(\mathbf{T})$ is the trace of the tensor $\mathbf{T}$ (sum of its diagonal elements), a scalar;
$det(\mathbf{T}) \equiv \|\mathbf{T}\|$ is the determinant of the tensor $\mathbf{T}$ (determinant of the
   matrix of tensor elements), a scalar.

D.1   Vector Algebra

Basic algebraic relations:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}, \quad \text{commutative addition} \tag{D.1}$$

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}, \quad \text{associative addition} \tag{D.2}$$

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}), \quad \text{difference} \tag{D.3}$$

$$f\mathbf{A} = \mathbf{A}f, \quad \text{commutative scalar multiplication} \tag{D.4}$$

$$(f + g)\mathbf{A} = f\mathbf{A} + g\mathbf{A}, \quad \text{distributive scalar multiplication} \tag{D.5}$$

$$f(\mathbf{A} + \mathbf{B}) = f\mathbf{A} + f\mathbf{B}, \quad \text{distributive scalar multiplication} \tag{D.6}$$

$$f(g\mathbf{A}) = (fg)\mathbf{A}, \quad \text{associative scalar multiplication} \tag{D.7}$$

Dot product:

$$\mathbf{A} \cdot \mathbf{B} = 0 \text{ implies } \mathbf{A} = \mathbf{0} \text{ or } \mathbf{B} = \mathbf{0}, \text{ or } \mathbf{A} \perp \mathbf{B} \tag{D.8}$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}, \quad \text{commutative dot product} \tag{D.9}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}, \quad \text{distributive dot product} \tag{D.10}$$

$$(f\mathbf{A}) \cdot (g\mathbf{B}) = fg(\mathbf{A} \cdot \mathbf{B}), \quad \text{associative scalar, dot product} \tag{D.11}$$

Cross product:

$$\mathbf{A} \times \mathbf{B} = \mathbf{0} \text{ implies } \mathbf{A} = \mathbf{0} \text{ or } \mathbf{B} = \mathbf{0}, \text{ or } \mathbf{A} \parallel \mathbf{B} \tag{D.12}$$

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \quad \mathbf{A} \times \mathbf{A} = \mathbf{0}, \quad \text{anti-commutative cross product} \tag{D.13}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \quad \text{distributive cross product} \tag{D.14}$$

$$(f\mathbf{A}) \times (g\mathbf{B}) = fg(\mathbf{A} \times \mathbf{B}), \quad \text{associative scalar, cross product} \tag{D.15}$$

Scalar relations:

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{A} \times \mathbf{B} \cdot \mathbf{C} = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}, \quad \text{dot-cross product} \tag{D.16}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{D.17}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) + (\mathbf{B} \times \mathbf{C}) \cdot (\mathbf{A} \times \mathbf{D}) + (\mathbf{C} \times \mathbf{A}) \cdot (\mathbf{B} \times \mathbf{D}) = 0 \tag{D.18}$$

Vector relations:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}), \quad bac - cab \text{ rule}$$

$$= (\mathbf{C} \times \mathbf{B}) \times \mathbf{A} = \mathbf{A} \cdot (\mathbf{C}\mathbf{B} - \mathbf{B}\mathbf{C}) \tag{D.19}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0} \tag{D.20}$$

$$(\mathbf{A} \cdot \mathbf{B})\mathbf{C} = \mathbf{A} \cdot (\mathbf{B}\mathbf{C}), \quad \text{associative dot product} \tag{D.21}$$

$$(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) = \mathbf{C}(\mathbf{A} \times \mathbf{B} \cdot \mathbf{D}) - \mathbf{D}(\mathbf{A} \times \mathbf{B} \cdot \mathbf{C})$$

$$= \mathbf{B}(\mathbf{C} \times \mathbf{D} \cdot \mathbf{A}) - \mathbf{A}(\mathbf{C} \times \mathbf{D} \cdot \mathbf{B}) \tag{D.22}$$

Projection of a vector $\mathbf{A}$ in directions relative to a vector $\mathbf{B}$:

$$\mathbf{A} = A_{\parallel}(\mathbf{B}/B) + \mathbf{A}_{\perp} = A_{\parallel}\hat{\mathbf{b}} + \mathbf{A}_{\perp} \tag{D.23}$$

$$\hat{\mathbf{b}} \equiv \mathbf{B}/B, \quad \text{unit vector in } \mathbf{B} \text{ direction} \tag{D.24}$$

$$A_{\parallel} \equiv \mathbf{B} \cdot \mathbf{A}/B = \hat{\mathbf{b}} \cdot \mathbf{A}, \quad \text{component of } \mathbf{A} \text{ along } \mathbf{B} \tag{D.25}$$

$$\mathbf{A}_{\perp} \equiv -\mathbf{B} \times (\mathbf{B} \times \mathbf{A})/B^2, \quad \text{component of } \mathbf{A} \text{ perpendicular to } \mathbf{B}$$

$$= -\hat{\mathbf{b}} \times (\hat{\mathbf{b}} \times \mathbf{A}) \tag{D.26}$$

D.2 Tensor Algebra

Scalar relations:

$$\mathsf{I} : \mathbf{A}\mathbf{B} \equiv (\mathsf{I} \cdot \mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B} \tag{D.27}$$

$$\mathbf{A}\mathbf{B} : \mathbf{C}\mathbf{D} \equiv \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C})\mathbf{D} = (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D})$$

dot-product

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$

cross-product

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{e}}_{\mathbf{A}\times\mathbf{B}} AB\sin\theta$$

dot-cross-product

$$\mathbf{A}\times\mathbf{B} \cdot \mathbf{C} = \pm \text{ volume of}$$
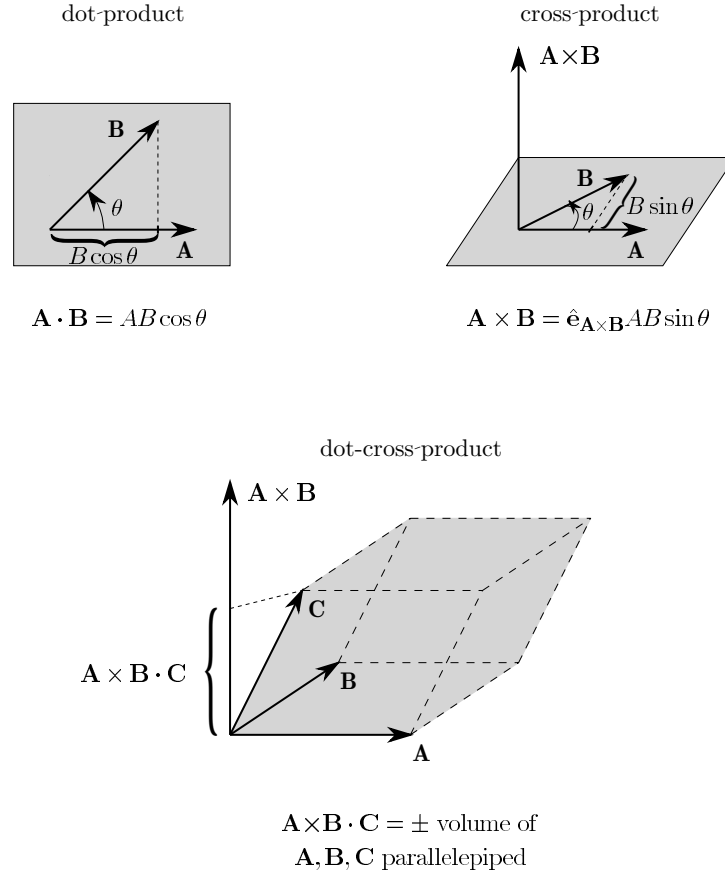$$\mathbf{A}, \mathbf{B}, \mathbf{C} \text{ parallelepiped}$$

Figure D.1: Schematic illustration of dot, cross and dot-cross products of vectors.

$$= \mathbf{D} \cdot \mathbf{AB} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{CD} \cdot \mathbf{A} \tag{D.28}$$

$$\mathbf{I} : \mathbf{T} = tr(\mathbf{T}), \quad \mathbf{T} : \mathbf{T} \equiv |\mathbf{T}|^2 \tag{D.29}$$

$$\mathbf{T} : \mathbf{AB} = (\mathbf{T} \cdot \mathbf{A}) \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{T} \cdot \mathbf{A} \tag{D.30}$$

$$\mathbf{AB} : \mathbf{T} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{T}) = \mathbf{B} \cdot \mathbf{T} \cdot \mathbf{A} \tag{D.31}$$

$$\mathbf{B} \times \mathbf{T} : \mathbf{W} = -(\mathbf{T} \cdot \mathbf{W})^{\mathsf{T}} : \mathbf{B} \times \mathbf{I} \tag{D.32}$$

Vector relations:

$$\mathbf{I} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{I} = \mathbf{A} \tag{D.33}$$

$$\mathbf{A} \cdot \mathbf{T}^{\mathsf{T}} = \mathbf{T} \cdot \mathbf{A}, \quad \mathbf{T}^{\mathsf{T}} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{T} \tag{D.34}$$

$$\mathbf{A} \cdot (\mathbf{CB} - \mathbf{BC}) = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$$
$$= \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}), \quad bac - cab \text{ rule} \tag{D.35}$$

$$(\mathbf{A} \times \mathbf{C}) \cdot \mathbf{T} = \mathbf{A} \cdot (\mathbf{C} \times \mathbf{T}) = -\mathbf{C} \cdot (\mathbf{A} \times \mathbf{T}) \tag{D.36}$$

$$\mathbf{T} \cdot (\mathbf{A} \times \mathbf{C}) = (\mathbf{T} \times \mathbf{A}) \cdot \mathbf{C} = -(\mathbf{T} \times \mathbf{C}) \cdot \mathbf{A} \tag{D.37}$$

$$\mathbf{A} \cdot (\mathbf{T} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{T}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \cdot \mathbf{T}) \tag{D.38}$$

$$(\mathbf{A} \times \mathbf{T}) \cdot \mathbf{C} = \mathbf{A} \times (\mathbf{T} \cdot \mathbf{C}) = -(\mathbf{T} \cdot \mathbf{C}) \times \mathbf{A} \tag{D.39}$$

$$\mathbf{A} \cdot (\mathbf{T} \times \mathbf{C}) - \mathbf{C} \cdot (\mathbf{T} \times \mathbf{A}) = [\mathbf{I} \, tr(\mathbf{T}) - \mathbf{T}] \cdot (\mathbf{A} \times \mathbf{C}) \tag{D.40}$$

$$(\mathbf{A} \times \mathbf{T}) \cdot \mathbf{C} - (\mathbf{C} \times \mathbf{T}) \cdot \mathbf{A} = (\mathbf{A} \times \mathbf{C}) \cdot [\mathbf{I} \, tr(\mathbf{T}) - \mathbf{T}] \tag{D.41}$$

Tensor relations:

$$\mathbf{I} \cdot \mathbf{AB} = (\mathbf{I} \cdot \mathbf{A})\mathbf{B} = \mathbf{AB}, \quad \mathbf{AB} \cdot \mathbf{I} = \mathbf{A}(\mathbf{B} \cdot \mathbf{I}) = \mathbf{AB} \tag{D.42}$$

$$\mathbf{I} \times \mathbf{A} = \mathbf{I} \times \mathbf{A} \tag{D.43}$$

$$\mathbf{A} \times (\mathbf{BC}) = (\mathbf{A} \times \mathbf{B})\mathbf{C}, \quad (\mathbf{AB}) \times \mathbf{C} = \mathbf{A}(\mathbf{B} \times \mathbf{C}) \tag{D.44}$$

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{I} = \mathbf{I} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{BA} - \mathbf{AB} \tag{D.45}$$

$$(\mathbf{A} \times \mathbf{T})^{\mathsf{T}} = -\mathbf{T}^{\mathsf{T}} \times \mathbf{A}, \quad (\mathbf{T} \times \mathbf{A})^{\mathsf{T}} = -\mathbf{A} \times \mathbf{T}^{\mathsf{T}} \tag{D.46}$$

$$(\mathbf{A} \times \mathbf{T}) - (\mathbf{A} \times \mathbf{T})^{\mathsf{T}} = \mathbf{I} \times [\mathbf{A} \, tr(\mathbf{T}) - \mathbf{T} \cdot \mathbf{A}] \tag{D.47}$$

$$(\mathbf{T} \times \mathbf{A}) - (\mathbf{T} \times \mathbf{A})^{\mathsf{T}} = \mathbf{I} \times [\mathbf{A} \, tr(\mathbf{T}) - \mathbf{A} \cdot \mathbf{T}] \tag{D.48}$$

$$\mathbf{T}_S = \tfrac{1}{2}(\mathbf{T} + \mathbf{T}^{\mathsf{T}}), \quad \text{symmetric part of tensor } \mathbf{T} \tag{D.49}$$

$$\mathbf{T}_A = \tfrac{1}{2}(\mathbf{T} - \mathbf{T}^{\mathsf{T}}), \quad \text{anti-symmetric part of tensor } \mathbf{T} \tag{D.50}$$

$$\mathbf{B} \times \mathbf{T}_S \times \mathbf{B} = B^2 \mathbf{T}_S - (\mathbf{BB} \cdot \mathbf{T}_S + \mathbf{T}_S \cdot \mathbf{BB})$$
$$- (\mathbf{I}B^2 - \mathbf{BB})(\mathbf{I}B^2 - \mathbf{BB}) \cdot \mathbf{T}_S/B^2 - \mathbf{BB}(\mathbf{BB} \cdot \mathbf{T}_S)/B^2 \tag{D.51}$$

D.3   Derivatives

Temporal derivatives:

$$\frac{d\mathbf{A}}{dt} \text{ is a vector tangent to the curve defined by} \mathbf{A}(t) \tag{D.52}$$

$$\frac{d}{dt}(f\mathbf{A}) = \frac{df}{dt}\mathbf{A} + f\frac{d\mathbf{A}}{dt} \tag{D.53}$$

$$\frac{d}{dt}(\mathbf{A} + \mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt} \tag{D.54}$$

$$\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) = \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} \tag{D.55}$$

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt} \tag{D.56}$$

Definitions of partial derivatives in space ($\boldsymbol{\nabla} \equiv \partial/\partial\mathbf{x} = del$ or *nabla* is the differential vector operator):

$$\boldsymbol{\nabla}f \equiv \frac{\partial f}{\partial\mathbf{x}}, \quad gradient \text{ of scalar function } f, \text{ a vector — vector in direction}$$

of and measure of the greatest rate of spatial change of f (D.57)

$$\boldsymbol{\nabla} \cdot \mathbf{A} \equiv \frac{\partial}{\partial\mathbf{x}} \cdot \mathbf{A}, \quad divergence \text{ of vector function } \mathbf{A}, \text{ a scalar —}$$

divergence ($\boldsymbol{\nabla} \cdot \mathbf{A} > 0$) or convergence ($\boldsymbol{\nabla} \cdot \mathbf{A} < 0$) of $\mathbf{A}$ lines (D.58)

$$\boldsymbol{\nabla} \times \mathbf{A} \equiv \frac{\partial}{\partial\mathbf{x}} \times \mathbf{A}, \quad curl \text{ (or rotation) of vector function } \mathbf{A}, \text{ a vector}[1]—$$

vorticity of $\mathbf{A}$ lines (D.59)

$$\nabla^2 f \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}f, \quad del \text{ } square \text{ or } Laplacian \text{ (divergence of gradient)}$$

derivative of scalar function f, a scalar, which is sometimes

written as Δf — three-dimensional measure of curvature of f

(f is larger where $\nabla^2 f < 0$ and smaller where $\nabla^2 f > 0$) (D.60)

$$\nabla^2 \mathbf{A} \equiv (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}), \quad \text{Laplacian derivative}$$

of vector function $\mathbf{A}$, a vector (D.61)

For the general vector coordinate $\mathbf{x} \equiv x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$ and $|\mathbf{x}| \equiv \sqrt{x^2 + y^2 + z^2}$:

$$\boldsymbol{\nabla} \cdot \mathbf{x} = 3, \quad \boldsymbol{\nabla} \cdot (\mathbf{x}/|\mathbf{x}|) = 2/|\mathbf{x}| \tag{D.62}$$

$$\boldsymbol{\nabla} \times \mathbf{x} = \mathbf{0}, \quad \boldsymbol{\nabla} \times (\mathbf{x}/|\mathbf{x}|) = \mathbf{0} \tag{D.63}$$

$$\boldsymbol{\nabla}|\mathbf{x}| = \mathbf{x}/|\mathbf{x}|, \quad \boldsymbol{\nabla}(1/|\mathbf{x}|) = -\mathbf{x}/|\mathbf{x}|^3 \tag{D.64}$$

$$\boldsymbol{\nabla}\mathbf{x} = \mathbf{I} \tag{D.65}$$

$$(\mathbf{A} \cdot \boldsymbol{\nabla})(\mathbf{x}/|\mathbf{x}|) = [\mathbf{A} - (\mathbf{x} \cdot \mathbf{A})\mathbf{x}/|\mathbf{x}|^2]/|\mathbf{x}| \equiv \mathbf{A}_\perp/|\mathbf{x}| \tag{D.66}$$

$$\nabla^2(1/|\mathbf{x}|) \equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}(1/|\mathbf{x}|) = -\boldsymbol{\nabla} \cdot (\mathbf{x}/|\mathbf{x}|^3) = -4\pi\delta(\mathbf{x}) \tag{D.67}$$

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[1]Rigorously speaking, the cross product of two vectors and the curl of a vector are pseudo-vectors because they are anti-symmetric contractions of second rank tensors — see tensor references at end of this appendix.

First derivatives with scalar functions:

$$\boldsymbol{\nabla}(f + g) = \boldsymbol{\nabla}f + \boldsymbol{\nabla}g \tag{D.68}$$

$$\boldsymbol{\nabla}(fg) = (\boldsymbol{\nabla}f)g + f\boldsymbol{\nabla}g = \boldsymbol{\nabla}(gf) \tag{D.69}$$

$$\boldsymbol{\nabla}(f\mathbf{A}) = (\boldsymbol{\nabla}f)\mathbf{A} + f\boldsymbol{\nabla}\mathbf{A} \tag{D.70}$$

$$\boldsymbol{\nabla}\cdot f\mathbf{A} = \boldsymbol{\nabla}f\cdot\mathbf{A} + f\boldsymbol{\nabla}\cdot\mathbf{A} \tag{D.71}$$

$$\boldsymbol{\nabla}\times f\mathbf{A} = \boldsymbol{\nabla}f\times\mathbf{A} + f\boldsymbol{\nabla}\times\mathbf{A} \tag{D.72}$$

$$\boldsymbol{\nabla}\cdot f\mathbf{T} = \boldsymbol{\nabla}f\cdot\mathbf{T} + f\boldsymbol{\nabla}\cdot\mathbf{T} \tag{D.73}$$

$$\boldsymbol{\nabla}\times f\mathbf{T} = \boldsymbol{\nabla}f\times\mathbf{T} + f\boldsymbol{\nabla}\times\mathbf{T} \tag{D.74}$$

First derivative scalar relations:

$$\boldsymbol{\nabla}\cdot(\mathbf{A} + \mathbf{B}) = \boldsymbol{\nabla}\cdot\mathbf{A} + \boldsymbol{\nabla}\cdot\mathbf{B} \tag{D.75}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\boldsymbol{\nabla}\times\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\nabla}\times\mathbf{B} \tag{D.76}$$

$$(\mathbf{B}\cdot\boldsymbol{\nabla})(\mathbf{A}\cdot\mathbf{C}) = \mathbf{C}\cdot(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} + \mathbf{A}\cdot(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{C}$$
$$\equiv \mathbf{CB}\colon\boldsymbol{\nabla}\mathbf{A} + \mathbf{AB}\colon\boldsymbol{\nabla}\mathbf{C} \tag{D.77}$$

$$\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{C} - \mathbf{C}\cdot\boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{A} \equiv (\mathbf{CA} - \mathbf{AC})\colon\boldsymbol{\nabla}\mathbf{B} = (\mathbf{A}\times\mathbf{C})\cdot\boldsymbol{\nabla}\times\mathbf{B} \tag{D.78}$$

$$2\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{C} \equiv 2\mathbf{CA}\colon\boldsymbol{\nabla}\mathbf{B} = \mathbf{A}\cdot\boldsymbol{\nabla}(\mathbf{B}\cdot\mathbf{C}) + \mathbf{C}\cdot\boldsymbol{\nabla}(\mathbf{B}\cdot\mathbf{A})$$
$$- \mathbf{B}\cdot\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{C}) + (\mathbf{B}\times\mathbf{C})\cdot(\boldsymbol{\nabla}\times\mathbf{A})$$
$$+ (\mathbf{B}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\mathbf{C}) + (\mathbf{A}\times\mathbf{C})\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \tag{D.79}$$

$$\mathbf{I}\colon\boldsymbol{\nabla}\mathbf{B} = \boldsymbol{\nabla}\cdot\mathbf{B} \tag{D.80}$$

$$\mathbf{A}\times\mathbf{I}\colon\boldsymbol{\nabla}\mathbf{B} = \mathbf{A}\cdot\boldsymbol{\nabla}\times\mathbf{B} \tag{D.81}$$

$$\mathbf{A}\cdot\boldsymbol{\nabla}\cdot\mathbf{T} = \boldsymbol{\nabla}\cdot(\mathbf{A}\cdot\mathbf{T}) - \boldsymbol{\nabla}\mathbf{A}\colon\mathbf{T} = \boldsymbol{\nabla}\cdot(\mathbf{A}\cdot\mathbf{T}) - \mathbf{T}\colon\boldsymbol{\nabla}\mathbf{A} \tag{D.82}$$

First derivative vector relations:

$$\boldsymbol{\nabla}\times(\mathbf{A} + \mathbf{B}) = \boldsymbol{\nabla}\times\mathbf{A} + \boldsymbol{\nabla}\times\mathbf{B} \tag{D.83}$$

$$\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{A}) + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}$$
$$= (\boldsymbol{\nabla}\mathbf{A})\cdot\mathbf{B} + (\boldsymbol{\nabla}\mathbf{B})\cdot\mathbf{A} \tag{D.84}$$

$$\boldsymbol{\nabla}(B^2/2) \equiv \boldsymbol{\nabla}(\mathbf{B}\cdot\mathbf{B}/2) = \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B}) + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{B} = (\boldsymbol{\nabla}\mathbf{B})\cdot\mathbf{B} \tag{D.85}$$

$$(\mathbf{B}\cdot\boldsymbol{\nabla})(\mathbf{A}\times\mathbf{C}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A}\times\mathbf{C} + \mathbf{A}\times(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{C} \tag{D.86}$$

$$\boldsymbol{\nabla}\cdot\mathbf{AB} = (\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{B} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} = (\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{B} + \mathbf{A}\cdot(\boldsymbol{\nabla}\mathbf{B}) \tag{D.87}$$

$$\boldsymbol{\nabla}\cdot\mathbf{I} = \mathbf{0} \tag{D.88}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{I}\times\mathbf{A}) = \boldsymbol{\nabla}\times\mathbf{A} \tag{D.89}$$

$$\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B}) = (\boldsymbol{\nabla}\mathbf{B})\cdot\mathbf{A} - \mathbf{A}\cdot(\boldsymbol{\nabla}\mathbf{B}) = (\boldsymbol{\nabla}\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B} \tag{D.90}$$

$$\boldsymbol{\nabla}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}(\boldsymbol{\nabla}\cdot\mathbf{A}) + (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{B}$$
$$= \boldsymbol{\nabla}\cdot(\mathbf{BA} - \mathbf{AB}) \tag{D.91}$$

$$\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B}\times\mathbf{C} + \mathbf{C}\times\boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{A} = \mathbf{C}\times[\mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{B})] \tag{D.92}$$

$$\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{B}\times\mathbf{C} - \mathbf{C}\cdot\boldsymbol{\nabla}\mathbf{B}\times\mathbf{A} = [(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{I} - \boldsymbol{\nabla}\mathbf{B}]\cdot(\mathbf{A}\times\mathbf{C}) \tag{D.93}$$

$$\mathbf{A}\times\boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{C} - \mathbf{C}\times\boldsymbol{\nabla}\mathbf{B}\cdot\mathbf{A} = (\mathbf{A}\times\mathbf{C})\cdot[(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{I} - \boldsymbol{\nabla}\mathbf{B}] \tag{D.94}$$

First derivative tensor relations:

$$\mathbf{I} \cdot \boldsymbol{\nabla}\mathbf{B} = \boldsymbol{\nabla}\mathbf{B}, \quad \boldsymbol{\nabla}\mathbf{B} \cdot \mathbf{I} = \boldsymbol{\nabla}\mathbf{B} \tag{D.95}$$

$$\boldsymbol{\nabla}\times\mathbf{AB} = (\boldsymbol{\nabla}\times\mathbf{A})\mathbf{B} - \mathbf{A}\times\boldsymbol{\nabla}\mathbf{B} \tag{D.96}$$

$$\boldsymbol{\nabla}(\mathbf{A}\times\mathbf{B}) = \boldsymbol{\nabla}\mathbf{A}\times\mathbf{B} - \boldsymbol{\nabla}\mathbf{B}\times\mathbf{A} \tag{D.97}$$

$$\mathbf{A}\times\boldsymbol{\nabla}\mathbf{B} + \boldsymbol{\nabla}\mathbf{B}\times\mathbf{A}$$
$$= \mathbf{I}\times[(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{A} - (\boldsymbol{\nabla}\mathbf{B})\cdot\mathbf{A}] + [\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})]\mathbf{I} - \mathbf{A}(\boldsymbol{\nabla}\times\mathbf{B})$$
$$= \mathbf{I}\times[(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{A} - \mathbf{A}\cdot(\boldsymbol{\nabla}\mathbf{B})] + [\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})]\mathbf{I} - (\boldsymbol{\nabla}\times\mathbf{B})\mathbf{A} \tag{D.98}$$

$$\boldsymbol{\nabla}\mathbf{B}\times\mathbf{A} + (\mathbf{A}\times\boldsymbol{\nabla}\mathbf{B})^{\mathsf{T}} = [\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})]\mathbf{I} - \mathbf{A}(\boldsymbol{\nabla}\times\mathbf{B}) \tag{D.99}$$

$$\mathbf{A}\times\boldsymbol{\nabla}\mathbf{B} + (\boldsymbol{\nabla}\mathbf{B}\times\mathbf{A})^{\mathsf{T}} = [\mathbf{A}\cdot(\boldsymbol{\nabla}\times\mathbf{B})]\mathbf{I} - (\boldsymbol{\nabla}\times\mathbf{B})\mathbf{A} \tag{D.100}$$

$$\mathbf{A}\times\boldsymbol{\nabla}\mathbf{B} - (\mathbf{A}\times\boldsymbol{\nabla}\mathbf{B})^{\mathsf{T}} = \mathbf{I}\times[(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{A} - (\boldsymbol{\nabla}\mathbf{B})\cdot\mathbf{A}] \tag{D.101}$$

$$\boldsymbol{\nabla}\mathbf{B}\times\mathbf{A} - (\boldsymbol{\nabla}\mathbf{B}\times\mathbf{A})^{\mathsf{T}} = [(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{A} - \mathbf{A}\cdot(\boldsymbol{\nabla}\mathbf{B})]\times\mathbf{I} \tag{D.102}$$

Second derivative relations:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}f \equiv \nabla^2 f \tag{D.103}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}f = \mathbf{0} \tag{D.104}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}f\times\boldsymbol{\nabla}g = 0 \tag{D.105}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{A} \equiv \nabla^2\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}) \tag{D.106}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{A} = 0 \tag{D.107}$$

$$\boldsymbol{\nabla}\cdot(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{A} = (\mathbf{B}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\cdot\mathbf{A}) - (\boldsymbol{\nabla}\times\mathbf{A})\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \tag{D.108}$$

$$\boldsymbol{\nabla}\times[(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{A}]$$
$$= (\mathbf{A}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\times\mathbf{A}) + (\boldsymbol{\nabla}\cdot\mathbf{A})(\boldsymbol{\nabla}\times\mathbf{A}) - [(\boldsymbol{\nabla}\times\mathbf{A})\cdot\boldsymbol{\nabla}]\mathbf{A} \tag{D.109}$$

Derivatives of projections of $\mathbf{A}$ in $\mathbf{B}$ direction $[\hat{\mathbf{b}} \equiv \mathbf{B}/B,\ \mathbf{A} = A_\parallel\hat{\mathbf{b}} + \mathbf{A}_\perp,$ $A_\parallel \equiv \hat{\mathbf{b}}\cdot\mathbf{A},\ \mathbf{A}_\perp \equiv -\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\times\mathbf{A}),\ (\hat{\mathbf{b}}\cdot\boldsymbol{\nabla})\hat{\mathbf{b}} = -\hat{\mathbf{b}}\times(\boldsymbol{\nabla}\times\hat{\mathbf{b}}) \equiv \boldsymbol{\kappa}]$:

$$\boldsymbol{\nabla}\cdot\mathbf{A} = (A_\parallel/B)(\boldsymbol{\nabla}\cdot\mathbf{B}) + (\mathbf{B}\cdot\boldsymbol{\nabla})(A_\parallel/B) + \boldsymbol{\nabla}\cdot\mathbf{A}_\perp \tag{D.110}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A}_\perp = -\mathbf{A}_\perp\cdot[\boldsymbol{\nabla}\ln B + (\hat{\mathbf{b}}\cdot\boldsymbol{\nabla})\hat{\mathbf{b}}] - (1/B)\,\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{A}) \tag{D.111}$$

$$\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\mathbf{A}\cdot\hat{\mathbf{b}} \equiv \hat{\mathbf{b}}\hat{\mathbf{b}} \boldsymbol{:} \boldsymbol{\nabla}\mathbf{A} = (\hat{\mathbf{b}}\cdot\boldsymbol{\nabla})A_\parallel - \mathbf{A}_\perp\cdot(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla})\hat{\mathbf{b}}$$
$$= \mathbf{A}\cdot\boldsymbol{\nabla}\ln B - (1/B)\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{A}) + \boldsymbol{\nabla}\cdot\mathbf{A} - (A_\parallel/B)(\boldsymbol{\nabla}\cdot\mathbf{B}) \tag{D.112}$$

$$\text{For } \mathbf{A}_\perp = (1/B^2)\,\mathbf{B}\times\boldsymbol{\nabla}f, \quad \hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{A}_\perp) = (\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}f)(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}}) \tag{D.113}$$

D.4 Integrals

For a volume V enclosed by a closed, continuous surface S with differential volume element d^3x and differential surface element $d\mathbf{S} \equiv \hat{\mathbf{n}}\,dS$ where $\hat{\mathbf{n}}$ is the unit normal outward from the volume V, for well-behaved functions $f, g, \mathbf{A}, \mathbf{B}$ and $\mathbf{T}$:

$$\int_V d^3x\,\boldsymbol{\nabla}f = \oiint_S d\mathbf{S}\,f, \tag{D.114}$$

$$\int_V d^3x \, \boldsymbol{\nabla} \cdot \mathbf{A} = \oiint_S d\mathbf{S} \cdot \mathbf{A}, \qquad \text{divergence or Gauss' theorem,} \quad \text{(D.115)}$$

$$\int_V d^3x \, \boldsymbol{\nabla} \cdot \mathbf{T} = \oiint_S d\mathbf{S} \cdot \mathbf{T}, \qquad \text{(D.116)}$$

$$\int_V d^3x \, \boldsymbol{\nabla} \times \mathbf{A} = \oiint_S d\mathbf{S} \times \mathbf{A}, \qquad \text{(D.117)}$$

$$\int_V d^3x \, f \nabla^2 g = \int_V d^3x \, \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + \oiint_S d\mathbf{S} \cdot f \boldsymbol{\nabla} g,$$
$$\text{Green's first identity,} \quad \text{(D.118)}$$

$$\int_V d^3x \, (f \nabla^2 g - g \nabla^2 f) = \oiint_S d\mathbf{S} \cdot (f \boldsymbol{\nabla} g - g \boldsymbol{\nabla} f),$$
$$\text{Green's second identity,} \quad \text{(D.119)}$$

$$\int_V d^3x \, [\mathbf{A} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) - \mathbf{B} \cdot \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})]$$
$$= \oiint d\mathbf{S} \cdot [\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{B})],$$
$$\text{vector form of Green's second identity.} \quad \text{(D.120)}$$

The gradient, divergence and curl partial differential operators can be defined using integral relations in the limit of small surfaces ΔS encompassing small volumes ΔV, as follows:

$$\boldsymbol{\nabla} f \equiv \lim_{\Delta V \to 0} \left(\frac{1}{\Delta V} \oiint_{\Delta S} d\mathbf{S} \, f \right) \qquad \text{gradient,} \qquad \text{(D.121)}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A} \equiv \lim_{\Delta V \to 0} \left(\frac{1}{\Delta V} \oiint_{\Delta S} d\mathbf{S} \cdot \mathbf{A} \right) \qquad \text{divergence,} \qquad \text{(D.122)}$$

$$\boldsymbol{\nabla} \times \mathbf{A} \equiv \lim_{\Delta V \to 0} \left(\frac{1}{\Delta V} \oiint_{\Delta S} d\mathbf{S} \times \mathbf{A} \right) \qquad \text{curl.} \qquad \text{(D.123)}$$

For S representing an open surface bounded by a closed, continuous contour C with line element $d\boldsymbol{\ell}$ which is defined to be positive when the right-hand-rule sense of the line integral around C points in the $d\mathbf{S}$ direction:

$$\iint_S d\mathbf{S} \times \boldsymbol{\nabla} f = \oint_C d\boldsymbol{\ell} \, f, \qquad \text{(D.124)}$$

$$\iint_S d\mathbf{S} \cdot \boldsymbol{\nabla} \times \mathbf{A} = \oint_C d\boldsymbol{\ell} \cdot \mathbf{A}, \qquad \text{Stokes' theorem,} \quad \text{(D.125)}$$

$$\iint_S (d\mathbf{S} \times \boldsymbol{\nabla}) \times \mathbf{A} = \oint_C d\boldsymbol{\ell} \times \mathbf{A}, \qquad \text{(D.126)}$$

$$\iint_S d\mathbf{S} \cdot (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = \oint_C d\boldsymbol{\ell} \cdot f \boldsymbol{\nabla} g = \oint_C f \, dg = - \oint_C g \, df,$$
$$\text{Green's theorem.} \quad \text{(D.127)}$$

The appropriate differential line element $d\boldsymbol{\ell}$, surface area $d\mathbf{S}$, and volume d^3x can be defined in terms of any three differential line elements $d\boldsymbol{\ell}(i)$, $i = 1, 2, 3$

that are linearly independent [i.e., $d\boldsymbol{\ell}(1) \cdot d\boldsymbol{\ell}(2) \times d\boldsymbol{\ell}(3) \neq 0$] by

$$d\boldsymbol{\ell} = d\boldsymbol{\ell}(i), \quad i = 1, 2, \text{ or } 3, \quad \text{differential line element,} \quad \text{(D.128)}$$

$$d\mathbf{S} = d\boldsymbol{\ell}(i) \times d\boldsymbol{\ell}(j), \qquad\qquad \text{differential surface area,} \quad \text{(D.129)}$$

$$d^3x = d\boldsymbol{\ell}(1) \cdot d\boldsymbol{\ell}(2) \times d\boldsymbol{\ell}(3), \qquad \text{differential volume.} \qquad \text{(D.130)}$$

In exploring properties of fluids and plasmas we often want to know how the differential line, surface and volume elements change as they move with the fluid flow velocity $\mathbf{V}$. In particular, when taking time derivatives of integrals, we need to know what the time derivatives of these differentials are as they are carried along with a fluid. To determine this, note first that if the flow is uniform then all points in the fluid would be carried along in the same direction at the same rate; hence, the time derivatives of the differentials would vanish. However, if the flow is nonuniform, the differential line elements and hence all the differentials would change in time. To calculate the time derivatives of the differentials, consider the motion of two initially close points $\mathbf{x}_1, \mathbf{x}_2$ as they are carried along with a fluid flow velocity $\mathbf{V}(\mathbf{x}, t)$. Using the Taylor series expansion $\mathbf{V}(\mathbf{x}_2, t) = \mathbf{V}(\mathbf{x}_1, t) + (\mathbf{x}_2 - \mathbf{x}_1) \cdot \boldsymbol{\nabla}\mathbf{V} + \cdots$ and integrating the governing equation $d\mathbf{x}/dt = \mathbf{V}$ over time, we obtain

$$\mathbf{x}_2 - \mathbf{x}_1 = \mathbf{x}_2(t=0) - \mathbf{x}_1(t=0) + \int_0^t dt' \, (\mathbf{x}_2' - \mathbf{x}_1') \cdot \boldsymbol{\nabla}\mathbf{V} + \cdots \qquad \text{(D.131)}$$

in which $\mathbf{x}_2(t=0)$ and $\mathbf{x}_1(t=0)$ are the initial positions at $t=0$. Taking the time derivative of this equation and identifying the differential line element $d\boldsymbol{\ell}$ as $\mathbf{x}_2 - \mathbf{x}_1$ in the limit where the points $\mathbf{x}_2$ and $\mathbf{x}_1$ become infinetesimally close, we find

$$d\dot{\boldsymbol{\ell}} \equiv \frac{d}{dt}(d\boldsymbol{\ell}) = d\boldsymbol{\ell} \cdot \boldsymbol{\nabla}\mathbf{V}. \qquad \text{(D.132)}$$

The time derivative of the differential surface area $d\mathbf{S}$ can be calculated by taking the time derivative of (D.129) and using this last equation to obtain

$$
\begin{aligned}
d\dot{\mathbf{S}} &\equiv \frac{d}{dt}(d\mathbf{S}) = d\dot{\boldsymbol{\ell}}(1) \times d\boldsymbol{\ell}(2) + d\boldsymbol{\ell}(1) \times d\dot{\boldsymbol{\ell}}(2) \\
&= d\boldsymbol{\ell}(1) \cdot \boldsymbol{\nabla}\mathbf{V} \times d\boldsymbol{\ell}(2) - d\boldsymbol{\ell}(2) \cdot \boldsymbol{\nabla}\mathbf{V} \times d\boldsymbol{\ell}(1) \\
&= (\boldsymbol{\nabla} \cdot \mathbf{V}) \, d\mathbf{S} - \boldsymbol{\nabla}\mathbf{V} \cdot d\mathbf{S} \qquad \text{(D.133)}
\end{aligned}
$$

in which (D.93) and (D.33) have been used in obtaining the last form. Similarly, the time derivative of the differential volume element moving with the fluid is

$$
\begin{aligned}
\frac{d}{dt}(d^3x) &= d\dot{\boldsymbol{\ell}}(3) \cdot d\mathbf{S} + d\boldsymbol{\ell}(3) \cdot d\dot{\mathbf{S}} \\
&= d\boldsymbol{\ell}(3) \cdot \boldsymbol{\nabla}\mathbf{V} \cdot d\mathbf{S} + d\boldsymbol{\ell}(3) \cdot (\boldsymbol{\nabla} \cdot \mathbf{V})d\mathbf{S} - d\boldsymbol{\ell}(3) \cdot \boldsymbol{\nabla}\mathbf{V} \cdot d\mathbf{S} \\
&= (\boldsymbol{\nabla} \cdot \mathbf{V}) \, d^3x, \qquad \text{(D.134)}
\end{aligned}
$$

which shows that the differential volume in a compresssible fluid increases or decreases according to whether the fluid is rarefying ($\boldsymbol{\nabla} \cdot \mathbf{V} > 0$) or compressing ($\boldsymbol{\nabla} \cdot \mathbf{V} < 0$).

D.5   Vector Field Representations

Any vector field $\mathbf{B}$ can be expressed in terms of a scalar potential Φ_M and a vector potential $\mathbf{A}$:

$$\mathbf{B} = -\boldsymbol{\nabla}\Phi_M + \boldsymbol{\nabla}\times\mathbf{A}, \quad \text{potential representation.} \tag{D.135}$$

The $\boldsymbol{\nabla}\Phi_M$ part of $\mathbf{B}$ represents the longitudinal or irrotational ($\boldsymbol{\nabla}\times\boldsymbol{\nabla}\Phi_M = \mathbf{0}$) component while the $\boldsymbol{\nabla}\times\mathbf{A}$ part represents the transverse or solenoidal component ($\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{A} = 0$). A vector field $\mathbf{B}$ that satisfies $\boldsymbol{\nabla}\times\mathbf{B} = \mathbf{0}$ is called a longitudinal or irrotational field; one that satisfies $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ is called a solenoidal or transverse field. For a $\mathbf{B}(\mathbf{x})$ that vanishes at infinity, the potentials Φ_M and $\mathbf{A}$ are given by Green's function solutions

$$\Phi_M(\mathbf{x}) = \int d^3x' \, \frac{(\boldsymbol{\nabla}\cdot\mathbf{B})_{\mathbf{x}'}}{4\pi|\mathbf{x}-\mathbf{x}'|}, \quad \mathbf{A}(\mathbf{x}) = \int d^3x' \, \frac{(\boldsymbol{\nabla}\times\mathbf{B})_{\mathbf{x}'}}{4\pi|\mathbf{x}-\mathbf{x}'|}. \tag{D.136}$$

When there is symmetry in a coordinate ζ (i.e., a two or less dimensional system), a solenoidal vector field $\mathbf{B}$ can be written in terms of a stream function ψ in such a way that it automatically satisfies the solenoidal condition $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$:

$$\mathbf{B} = \boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi = |\boldsymbol{\nabla}\zeta|\,\hat{\mathbf{e}}_\zeta\times\boldsymbol{\nabla}\psi = -\,\boldsymbol{\nabla}\times\psi\boldsymbol{\nabla}\zeta, \quad \text{stream function form.} \tag{D.137}$$

For this situation the vector potential becomes

$$\mathbf{A} = -\,\psi\,\boldsymbol{\nabla}\zeta = -\,\psi\,|\boldsymbol{\nabla}\zeta|\,\hat{\mathbf{e}}_\zeta. \tag{D.138}$$

For a fully three-dimensional situation with no symmetry, a solenoidal vector field $\mathbf{B}$ can in general be written as

$$\mathbf{B} = \boldsymbol{\nabla}\alpha\times\boldsymbol{\nabla}\beta, \quad \text{Clebsch representation,} \tag{D.139}$$

In this representation α and β are stream functions that are constant along the vector field $\mathbf{B}$ since $\mathbf{B}\cdot\boldsymbol{\nabla}\alpha = 0$ and $\mathbf{B}\cdot\boldsymbol{\nabla}\beta = 0$.

D.6   Properties Of Curve Along A Vector Field

The motion of a point $\mathbf{x}$ along a vector field $\mathbf{B}$ is described by

$$\frac{d\mathbf{x}}{d\ell} = \frac{\mathbf{B}}{B} = \hat{\mathbf{b}} \equiv \hat{\boldsymbol{\tau}}, \quad \text{tangent vector} \tag{D.140}$$

in which $d\ell$ is a differential distance along $\mathbf{B}$. The unit vector $\hat{\mathbf{b}}$ is tangent to the vector field $\mathbf{B}(\mathbf{x})$ at the point $\mathbf{x}$ and so is often written as $\hat{\boldsymbol{\tau}}$ — a unit tangent vector.

The curvature vector $\boldsymbol{\kappa}$ of the vector field $\mathbf{B}$ is defined by

$$\boldsymbol{\kappa} \equiv \frac{d^2\mathbf{x}}{d\ell^2} = \frac{d\hat{\mathbf{b}}}{d\ell} = (\hat{\mathbf{b}}\cdot\boldsymbol{\nabla})\hat{\mathbf{b}} = -\hat{\mathbf{b}}\times(\boldsymbol{\nabla}\times\hat{\mathbf{b}}), \quad \text{curvature vector} \tag{D.141}$$

in which (D.85) has been used in the obtaining the last expression. The unit vector in the curvature vector direction is defined by

$$\hat{\boldsymbol{\kappa}} \equiv (\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})\hat{\mathbf{b}} \, / \, |(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})\hat{\mathbf{b}}|, \quad \text{curvature unit vector.} \qquad (\text{D.142})$$

The local radius of curvature vector $\mathbf{R}_C$ is in the opposite direction from the curvature vector $\boldsymbol{\kappa}$ and is defined by

$$\mathbf{R}_C \equiv -\boldsymbol{\kappa}/|\boldsymbol{\kappa}|^2, \quad \boldsymbol{\kappa} = -\mathbf{R}_C/R_C^2, \quad \text{radius of curvature.} \qquad (\text{D.143})$$

Hence, $|\mathbf{R}_C| \equiv R_C = 1/|\boldsymbol{\kappa}|$ is the magnitude of the local radius of curvature — the radius of the circle tangent to the vector field $\mathbf{B}(\mathbf{x})$ at the point $\mathbf{x}$.

A triad of orthogonal unit vectors (see Fig. D.2) can be constructed from the tangent unit vector $\hat{\boldsymbol{\tau}}$ and an arbitrary unit vector $\hat{\mathbf{n}}$ normal (or perpendicular) to the vector field $\mathbf{B}(\mathbf{x})$ at the point $\mathbf{x}$:

$$\hat{\boldsymbol{\tau}} \equiv \hat{\mathbf{b}}, \ \hat{\mathbf{n}} \text{ and } \hat{\mathbf{b}} \equiv \hat{\boldsymbol{\tau}} \times \hat{\mathbf{n}} = \hat{\mathbf{b}} \times \hat{\mathbf{n}}, \quad \text{Frenet unit vector triad} \qquad (\text{D.144})$$

in which $\hat{\mathbf{b}}$ is the binormal unit vector, the third orthogonal unit vector. The component of a vector $\mathbf{C}$ in the direction of the vector field $\mathbf{B}$ is called the parallel component: $C_\parallel \equiv \hat{\boldsymbol{\tau}} \cdot \mathbf{C} = \hat{\mathbf{b}} \cdot \mathbf{C}$. The component in the $\hat{\mathbf{n}}$ direction is called the normal component: $C_{\text{N}} \equiv \hat{\mathbf{n}} \cdot \mathbf{C}$. The component in the $\hat{\mathbf{b}}$ direction, which is perpendicular to the $\hat{\boldsymbol{\tau}} \times \hat{\mathbf{n}}$ plane, is called the binormal component: $C_{\text{B}} \equiv \hat{\mathbf{b}} \cdot \mathbf{C} = \hat{\boldsymbol{\tau}} \times \hat{\mathbf{n}} \cdot \mathbf{C}$.

Consider for example the components of the curvature vector $\boldsymbol{\kappa}$. Since $\hat{\mathbf{b}} \cdot \boldsymbol{\kappa} = 0$, the curvature vector has no parallel component ($\kappa_\parallel = 0$) — the curvature vector for the vector field $\mathbf{B}(\mathbf{x})$ is perpendicular to it at the point $\mathbf{x}$. The components of the curvature vector $\boldsymbol{\kappa}$ relative to a surface $\psi(\mathbf{x}) = \text{con-}$ stant in which the vector field lies (i.e., $\mathbf{B} \cdot \boldsymbol{\nabla}\psi = 0$) can be specified as follows. Define the normal to be in the direction of the gradient of ψ: $\hat{\mathbf{n}} \equiv \boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|$. Then, the components of the curvature vector perpendicular to (normal) and lying within (geodesic) the ψ surface are given by

$$\kappa_n = \hat{\mathbf{n}} \cdot \boldsymbol{\kappa} = \boldsymbol{\nabla}\psi \cdot \boldsymbol{\kappa}/|\boldsymbol{\nabla}\psi|, \quad \text{normal curvature,} \qquad (\text{D.145})$$

$$\kappa_g = \hat{\mathbf{b}} \cdot \boldsymbol{\kappa} = (\hat{\mathbf{b}} \times \boldsymbol{\nabla}\psi) \cdot \boldsymbol{\kappa}/|\hat{\mathbf{b}} \times \boldsymbol{\nabla}\psi|, \quad \text{geodesic curvature.} \qquad (\text{D.146})$$

The torsion $\boldsymbol{\tau}$ (twisting) of a vector field $\mathbf{B}$ is defined by

$$\boldsymbol{\tau} \equiv -\frac{d\hat{\mathbf{b}}}{d\ell} = -(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})(\hat{\mathbf{b}} \times \hat{\mathbf{n}}), \quad \text{torsion vector.} \qquad (\text{D.147})$$

The binormal component of the torsion vector vanishes ($\tau_{\text{B}} \equiv \hat{\mathbf{b}} \cdot \boldsymbol{\tau} = 0$). The normal component of the torsion vector locally defines the scale length L_τ along the vector $\mathbf{B}$ over which the vector field $\mathbf{B}(\mathbf{x})$ twists through an angle of one radian:

$$L_\tau \equiv 1/|\tau_{\text{N}}|, \quad \tau_{\text{N}} \equiv -\hat{\mathbf{n}} \cdot \frac{d\hat{\mathbf{b}}}{d\ell} = -\hat{\mathbf{n}} \cdot (\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})(\hat{\mathbf{b}} \times \hat{\mathbf{n}}), \quad \text{torsion length.} \quad (\text{D.148})$$
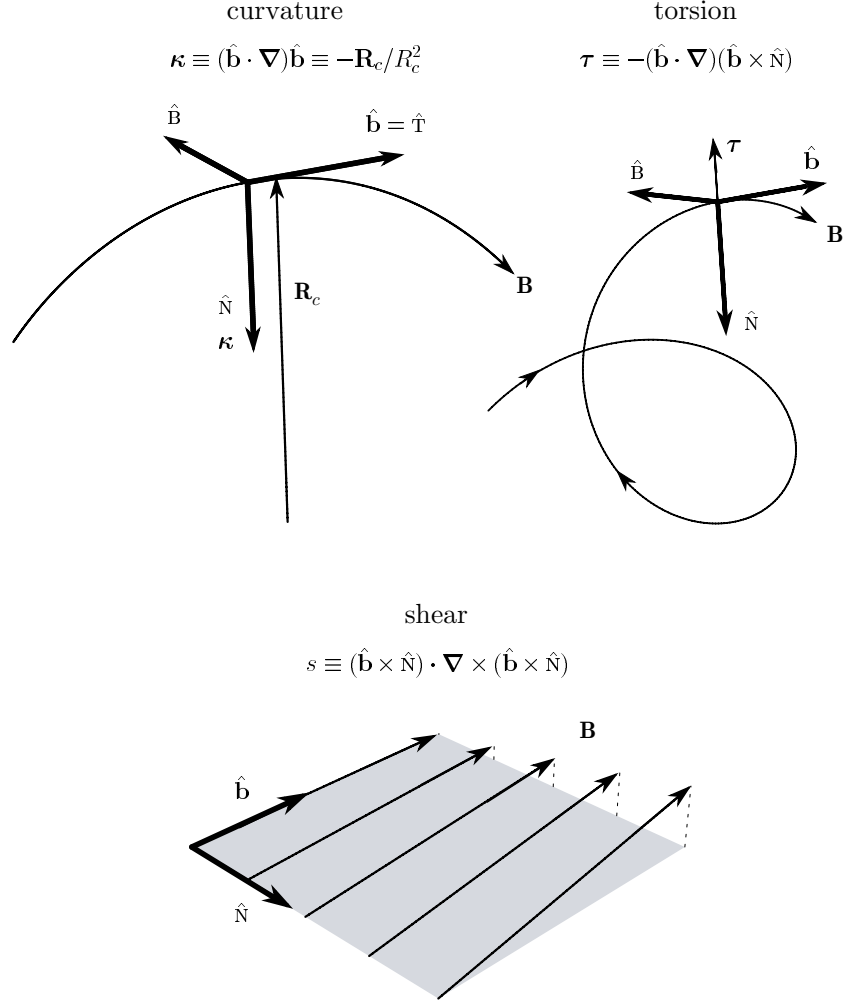
curvature

$$\boldsymbol{\kappa} \equiv (\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})\hat{\mathbf{b}} \equiv -\mathbf{R}_c/R_c^2$$

torsion

$$\boldsymbol{\tau} \equiv -(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})(\hat{\mathbf{b}} \times \hat{\mathrm{N}})$$

shear

$$s \equiv (\hat{\mathbf{b}} \times \hat{\mathrm{N}}) \cdot \boldsymbol{\nabla} \times (\hat{\mathbf{b}} \times \hat{\mathrm{N}})$$

Figure D.2: Properties (curvature, torsion, shear) of a spatially inhomogeneous vector field $\mathbf{B}(\mathbf{x})$. The unit vector $\hat{\mathbf{b}} \equiv \mathbf{B}/B = d\mathbf{x}/d\ell = \hat{\boldsymbol{\tau}}$ is locally tangent to the vector field $\mathbf{B}$. The unit normal $\hat{\mathbf{n}}$ is perpendicular to the vector field $\mathbf{B}$, shown here in the curvature direction. The binormal $\hat{\mathbf{B}}$ is orthogonal to both $\hat{\mathbf{b}}$ and $\hat{\mathbf{n}}$.

If the unit normal $\hat{\mathbf{n}}$ is taken to be in the $\boldsymbol{\nabla}\psi$ direction, the parallel component of the torsion vector is equal to the geodesic curvature $[\tau_\parallel \equiv \hat{\mathbf{b}} \cdot \boldsymbol{\tau} = (\hat{\mathbf{b}} \times \boldsymbol{\nabla}\psi) \cdot (\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})\hat{\mathbf{b}} / |\hat{\mathbf{b}} \times \boldsymbol{\nabla}\psi| \equiv \kappa_g]$.

The local shear ς (differential twisting motion, or nonplanar differential tangential motion in the plane defined by $\hat{\mathbf{b}} = \hat{\boldsymbol{\tau}}$ and $\hat{\mathbf{n}}$) in a vector field $\mathbf{B}$ is given by the binormal component of the curl or rotation in the binormal unit vector:

$$\varsigma \equiv \hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{b}} = (\hat{\mathbf{b}} \times \hat{\mathbf{n}}) \cdot \boldsymbol{\nabla} \times (\hat{\mathbf{b}} \times \hat{\mathbf{n}}) \equiv 1/L_S, \quad \text{local shear.} \tag{D.149}$$

The shear length L_S is defined as the scale length over which the vector field $\mathbf{B}(\mathbf{x})$ shears through an angle of one radian. The parallel component of the total curl or rotation of a vector field $\mathbf{B}$ is given by a combination of its torsion and shear, and $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{n}}$:

$$\begin{aligned}
\sigma \equiv \hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{b}} &= (\hat{\mathbf{b}} \times \hat{\mathbf{n}}) \cdot \boldsymbol{\nabla} \times (\hat{\mathbf{b}} \times \hat{\mathbf{n}}) - 2\hat{\mathbf{n}} \cdot (\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})(\hat{\mathbf{b}} \times \hat{\mathbf{n}}) + \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{n}} \\
&= \varsigma + 2\tau_{\text{N}} + \hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{n}}, \qquad \text{total rotation in } \mathbf{B} \text{ field.} \tag{D.150}
\end{aligned}$$

If the normal $\hat{\mathbf{n}}$ is taken to be in the $\boldsymbol{\nabla}\psi$ direction, $\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{n}} = 0$ and then

$$\varsigma = \frac{(\hat{\mathbf{b}} \times \boldsymbol{\nabla}\psi) \cdot \boldsymbol{\nabla} \times (\hat{\mathbf{b}} \times \boldsymbol{\nabla}\psi)}{|\hat{\mathbf{b}} \times \boldsymbol{\nabla}\psi|^2} \equiv \frac{1}{L_S}, \quad \text{local shear with } \hat{\mathbf{n}} \equiv \boldsymbol{\nabla}\psi/|\boldsymbol{\nabla}\psi|, \tag{D.151}$$

and

$$\sigma \equiv \hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{b}} = \varsigma + 2\tau_{\text{N}}. \tag{D.152}$$

In the absence of shear ($\varsigma = 0$), this last relation yields $\tau_{\text{N}} = (1/2)\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathbf{b}}$ — the torsion for "rigid body rotation" is just half the parallel component of the rotation in the vector field $\mathbf{B}$.

In most applied mathematics books the normal $\hat{\mathbf{n}}$ is taken to be in the curvature vector direction (i.e., $\hat{\mathbf{n}} \equiv \hat{\boldsymbol{\kappa}}$) instead of the $\boldsymbol{\nabla}\psi$ direction. Then, the parallel component of the torsion vector also vanishes $[\tau_\parallel \equiv \hat{\mathbf{b}} \cdot \boldsymbol{\tau} = \hat{\mathbf{b}} \times \hat{\boldsymbol{\kappa}} \cdot (\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})\hat{\mathbf{b}} = \hat{\mathbf{b}} \times \hat{\boldsymbol{\kappa}} \cdot \boldsymbol{\kappa} = 0]$ and

$$\boldsymbol{\tau} \equiv \tau_{\text{N}}\hat{\mathbf{N}}, \quad \text{for } \hat{\mathbf{n}} \equiv \hat{\boldsymbol{\kappa}}. \tag{D.153}$$

For this case the interrelationships between the triad of unit vectors $\hat{\boldsymbol{\tau}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}$ are given by the *Frenet-Serret formulas*:

$$\begin{aligned}
\frac{d\hat{\boldsymbol{\tau}}}{d\ell} &= \kappa_{\text{N}}\hat{\mathbf{N}}, & \hat{\boldsymbol{\tau}} &\equiv \mathbf{B}/B \equiv \hat{\mathbf{b}}, \\
\frac{d\hat{\mathbf{N}}}{d\ell} &= -\kappa_{\text{N}}\hat{\boldsymbol{\tau}} + \tau_{\text{N}}\hat{\mathbf{B}}, & \hat{\mathbf{N}} &\equiv \hat{\boldsymbol{\kappa}} = (\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})\hat{\mathbf{b}} / |(\hat{\mathbf{b}} \cdot \boldsymbol{\nabla})\hat{\mathbf{b}}|, & \tag{D.154} \\
\frac{d\hat{\mathbf{B}}}{d\ell} &= -\tau_{\text{N}}\hat{\mathbf{N}}, & \hat{\mathbf{B}} &\equiv \hat{\boldsymbol{\tau}} \times \hat{\mathbf{N}} = \hat{\mathbf{b}} \times \hat{\boldsymbol{\kappa}}.
\end{aligned}$$

The local shear ς and total rotation σ in the vector field $\mathbf{B}$ for this case are as given above in (D.149) and (D.150), respectively, for a general unit normal $\hat{\mathbf{n}}$.

D.7 Base Vectors and Vector Components

The three vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$, which are not necessarily orthogonal, can be used as a basis for a three-dimensional coordinate system if they are linearly independent (i.e., $\mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^2 \neq 0$). The three reciprocal base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are defined by

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j, \tag{D.155}$$

where

$$\delta^i_j \equiv \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad \text{Kronecker delta.} \tag{D.156}$$

The reciprocal base vectors can be written in terms of the original base vectors:

$$\mathbf{e}_1 = \frac{\mathbf{e}^2 \times \mathbf{e}^3}{\mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3)}, \quad \mathbf{e}_2 = \frac{\mathbf{e}^3 \times \mathbf{e}^1}{\mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3)}, \quad \mathbf{e}_3 = \frac{\mathbf{e}^1 \times \mathbf{e}^2}{\mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3)}. \tag{D.157}$$

Or, in general index notation

$$\mathbf{e}_i = \epsilon_{ijk} \frac{\mathbf{e}^j \times \mathbf{e}^k}{\mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3)}, \quad i, j, k = \text{permutations of } 1, 2, 3 \tag{D.158}$$

in which

$$\epsilon_{ijk} = \begin{cases} +1 & \text{when } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1 & \text{when } i, j, k \text{ is an odd permutation of } 1, 2, 3 \\ 0 & \text{when any two indices are equal} \end{cases} \quad \text{Levi-Civita symbol.} \tag{D.159}$$

The reciprocal Levi–Civita symbol ϵ^{ijk} is the same, i.e., $\epsilon^{ijk} = \epsilon_{ijk}$. These formulas are also valid if the subscripts and subscripts are reversed. Thus, the "original" base vectors could be the reciprocal base vectors $\mathbf{e}_i$ and the "reciprocal" base vectors could be the original base vectors $\mathbf{e}^i$ since both sets of base vectors are linearly independent. Either set can be used as a basis for representing three-dimensional vectors.

The identity tensor can be written in terms of the base or reciprocal vectors as follows:

$$\begin{aligned} \mathbf{I} &\equiv \sum_i \mathbf{e}^i \mathbf{e}_i = \mathbf{e}^1 \mathbf{e}_1 + \mathbf{e}^2 \mathbf{e}_2 + \mathbf{e}^3 \mathbf{e}_3 \\ &\equiv \sum_i \mathbf{e}_i \mathbf{e}^i = \mathbf{e}_1 \mathbf{e}^1 + \mathbf{e}_2 \mathbf{e}^2 + \mathbf{e}_3 \mathbf{e}^3. \end{aligned} \quad \text{identity tensor} \tag{D.160}$$

This definition can be used to write any vector or operator in terms of either its base or reciprocal vector components:

$$\begin{aligned} \mathbf{A} = \mathbf{A} \cdot \mathbf{I} &= (\mathbf{A} \cdot \mathbf{e}^1) \mathbf{e}_1 + (\mathbf{A} \cdot \mathbf{e}^2) \mathbf{e}_2 + (\mathbf{A} \cdot \mathbf{e}^3) \mathbf{e}_3 = \sum_i A^i \mathbf{e}_i, \quad A^i \equiv \mathbf{A} \cdot \mathbf{e}^i, \\ &= (\mathbf{A} \cdot \mathbf{e}_1) \mathbf{e}^1 + (\mathbf{A} \cdot \mathbf{e}_2) \mathbf{e}^2 + (\mathbf{A} \cdot \mathbf{e}_3) \mathbf{e}^3 = \sum_j A_j \mathbf{e}^j, \quad A_j \equiv \mathbf{A} \cdot \mathbf{e}_j, \end{aligned} \tag{D.161}$$

$$\begin{aligned} \boldsymbol{\nabla} \equiv \mathbf{I} \cdot \boldsymbol{\nabla} &= \mathbf{e}^1 (\mathbf{e}_1 \cdot \boldsymbol{\nabla}) + \mathbf{e}^2 (\mathbf{e}_2 \cdot \boldsymbol{\nabla}) + \mathbf{e}^3 (\mathbf{e}_3 \cdot \boldsymbol{\nabla}) \\ &= \mathbf{e}_1 (\mathbf{e}^1 \cdot \boldsymbol{\nabla}) + \mathbf{e}_2 (\mathbf{e}^2 \cdot \boldsymbol{\nabla}) + \mathbf{e}_3 (\mathbf{e}^3 \cdot \boldsymbol{\nabla}). \end{aligned} \tag{D.162}$$

The dot product between two vectors $\mathbf{A}$ and $\mathbf{B}$ is given in terms of their base and reciprocal vector components by

$$\mathbf{A} \cdot \mathbf{B} = \sum_i A^i B_i = \sum_i A_i B^i = \sum_{ij} (\mathbf{e}_i \cdot \mathbf{e}_j) A^i B^j = \sum_{ij} (\mathbf{e}^i \cdot \mathbf{e}^j) A_i B_j. \quad \text{(D.163)}$$

Similarly, the cross product between two vectors is given by

$$\mathbf{A} \times \mathbf{B} = \sum_{ij} A^i B^j \, \mathbf{e}_i \times \mathbf{e}_j = \sum_{ijk} A^i B^j \, \mathbf{e}^k \, (\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3)$$

$$= \sum_{ij} A_i B_j \, \mathbf{e}^i \times \mathbf{e}^j = \sum_{ijk} \epsilon^{ijk} A_i B_j \, \mathbf{e}_k \, (\mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3)$$

$$= (\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3) \begin{Vmatrix} \mathbf{e}^1 & \mathbf{e}^2 & \mathbf{e}^3 \\ A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \end{Vmatrix} = (\mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3) \begin{Vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{Vmatrix}. \quad \text{(D.164)}$$

The dot-cross product of three vectors is given by

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \sum_{ijk} A^i B^j C^k \, \mathbf{e}_i \cdot \mathbf{e}_j \times \mathbf{e}_k = \sum_{ijk} \epsilon_{ijk} A^i B^j C^k \, (\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3)$$

$$= \sum_{ijk} A_i B_j C_k \, \mathbf{e}^i \cdot \mathbf{e}^j \times e^k = \sum_{ijk} \epsilon^{ijk} A_i B_j C_k \, (\mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3)$$

$$= (\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3) \begin{Vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{Vmatrix} = (\mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3) \begin{Vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{Vmatrix}. \quad \text{(D.165)}$$

For the simplest situation where the three base vectors $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ are orthogonal ($\mathbf{e}^1 \cdot \mathbf{e}^2 = \mathbf{e}^2 \cdot \mathbf{e}^3 = \mathbf{e}^1 \cdot \mathbf{e}^3 = 0$), the reciprocal vectors point in the same directions as the original base vectors. Thus, after normalizing the base and reciprocal vectors they become equal:

$$\begin{aligned} \hat{\mathbf{e}}_1 &= \mathbf{e}_1/|\mathbf{e}_1| = \hat{\mathbf{e}}^1 = \mathbf{e}^1/|\mathbf{e}^1| && \text{orthogonal} \\ \hat{\mathbf{e}}_2 &= \mathbf{e}_2/|\mathbf{e}_2| = \hat{\mathbf{e}}^2 = \mathbf{e}^2/|\mathbf{e}^2| && \text{unit} \\ \hat{\mathbf{e}}_3 &= \mathbf{e}_3/|\mathbf{e}_3| = \hat{\mathbf{e}}^3 = \mathbf{e}^3/|\mathbf{e}^3| && \text{vectors.} \end{aligned} \quad \text{(D.166)}$$

The simplifications of (**??**)–(**??**) are given in (D.196)–(D.201) in the section (D.9) below on orthogonal coordinate systems.

D.8 Curvilinear Coordinate Systems

Consider transformation from the Cartesian coordinate system $\mathbf{x} = (x, y, z)$ to a curvilinear coordinate system labeled by the three independent functions u^1, u^2, u^3:

$$\mathbf{x} = \mathbf{x}(u^1, u^2, u^3): \quad x = x(u^1, u^2, u^3), \quad y = y(u^1, u^2, u^3), \quad z = z(u^1, u^2, u^3). \quad \text{(D.167)}$$

The transformation is invertible if the partial derivatives $\partial \mathbf{x}/\partial u^i$ for $i = 1, 2, 3$ are continuous and the Jacobian determinant (i.e., $\partial \mathbf{x}/\partial u^1 \cdot \partial \mathbf{x}/\partial u^2 \times \partial \mathbf{x}/\partial u^3$) formed from these nine partial derivatives does not vanish in the domain of interest. The inverse transformation is then given by

$$u^i = u^i(\mathbf{x}): \quad u^1 = u^1(x, y, z), \quad u^2 = u^2(x, y, z), \quad u^3 = u^3(x, y, z). \quad \text{(D.168)}$$

In a curvilinear coordinate system there are three coordinate surfaces:

$$\begin{aligned}
u^1(\mathbf{x}) &= c^1 \quad (u^2, u^3 \text{ variable}), \\
u^2(\mathbf{x}) &= c^2 \quad (u^1, u^3 \text{ variable}), \\
u^3(\mathbf{x}) &= c^3 \quad (u^1, u^2 \text{ variable}).
\end{aligned} \quad \text{(D.169)}$$

There are also three coordinate curves given by

$$\begin{aligned}
u^2(\mathbf{x}) &= c^2, \quad u^3(\mathbf{x}) = c^3 \quad (u^1 \text{ variable}), \\
u^3(\mathbf{x}) &= c^3, \quad u^1(\mathbf{x}) = c^1 \quad (u^2 \text{ variable}), \\
u^1(\mathbf{x}) &= c^1, \quad u^2(\mathbf{x}) = c^2 \quad (u^3 \text{ variable}).
\end{aligned} \quad \text{(D.170)}$$

The direction in which u^i increases along a coordinate curve is taken to be the positive direction for u^i. If the curvilinear coordinate curves intersect at right angles (i.e., $\boldsymbol{\nabla} u^i \cdot \boldsymbol{\nabla} u^j = 0$ except for $i = j$), then the system is orthogonal. The familiar Cartesian, cylindrical and spherical coordinate systems are all orthogonal. They are discussed at the end of the next section which covers orthogonal coordinates.

A nonorthogonal curvilinear coordinate system can be constructed from an invertible set of functions $u^1(\mathbf{x}), u^2(\mathbf{x}), u^3(\mathbf{x})$ as follows. A set of base vectors $\mathbf{e}^i$ can be defined by

$$\mathbf{e}^i = \boldsymbol{\nabla} u^i, \quad i = 1, 2, 3 \quad \text{contravariant base vectors.} \quad \text{(D.171)}$$

These so-called *contravariant* (superscript index) base vectors point in the direction of the gradient of the curvilinear coordinates u^i, and hence in the directions perpendicular to the $u^i(\mathbf{x}) = c^i$ surfaces. The set of reciprocal base vectors $\mathbf{e}_i$ is given by

$$\mathbf{e}_i = \epsilon_{ijk} \frac{\mathbf{e}^j \times \mathbf{e}^k}{\mathbf{e}^1 \cdot \mathbf{e}^2 \times e^3} = \frac{\epsilon_{ijk}}{J^{-1}} \boldsymbol{\nabla} u^j \times \boldsymbol{\nabla} u^k, \quad \text{covariant base vectors,} \quad \text{(D.172)}$$

in which

$$J^{-1} \equiv \boldsymbol{\nabla} u^1 \cdot \boldsymbol{\nabla} u^2 \times \boldsymbol{\nabla} u^3 = \mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3 \quad \text{inverse Jacobian} \quad \text{(D.173)}$$

is the Jacobian of the "inverse" transformation from the u^i curvilinear coordinate system back to the original Cartesian coordinate system.

An alternative form for the reciprocal base vectors can be obtained from the definition of the derivative of one of the curvilinear coordinates $u^i(\mathbf{x})$ in terms of the gradient: $du^i = \boldsymbol{\nabla} u^i \cdot d\mathbf{x} = \boldsymbol{\nabla} u^i \cdot \sum_j (\partial \mathbf{x}/\partial u^j) \, dx^j$, which becomes an

identity if and only if $\boldsymbol{\nabla} u^i \cdot (\partial \mathbf{x} / \partial u^j) = \delta^i_j$. Since this last relation is the same as the defining relation for reciprocal base vectors ($\mathbf{e}^i \cdot \mathbf{e}_j = \delta^i_j$), it follows that

$$\mathbf{e}_i = \frac{\partial \mathbf{x}}{\partial u^i}, \quad i = 1, 2, 3 \quad \text{covariant base vectors.} \tag{D.174}$$

The so-called *covariant* (subscript index) base vectors point in the direction of the local tangent to the u^i variable coordinate curve (from the $\partial \mathbf{x} / \partial u^i$ definition), i.e., parallel to the u^i coordinate curve. Alternatively, the covariant base vectors can be thought of as pointing in the direction of the cross product of contravariant base vectors for the two coordinate surfaces other than the u^i coordinate being considered (from the $\boldsymbol{\nabla} u^j \times \boldsymbol{\nabla} u^k$ definition). That these two directional definitions coincide follows from the properties of curvilinear surfaces and curves. The contravariant base vectors $\mathbf{e}^i$ can also be defined as the reciprocal base vectors of covariant base vectors $\mathbf{e}_i$:

$$\mathbf{e}^i = \epsilon^{ijk} \frac{\mathbf{e}_j \times \mathbf{e}_k}{\mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_2} = \frac{\epsilon^{ijk}}{J} \frac{\partial \mathbf{x}}{\partial u^j} \times \frac{\partial \mathbf{x}}{\partial u^k}; \quad i, j, k = \text{permutations of } 1, 2, 3$$
$$\text{contravariant base vectors} \tag{D.175}$$

in which

$$J = \frac{\partial \mathbf{x}}{\partial u^1} \cdot \frac{\partial \mathbf{x}}{\partial u^2} \times \frac{\partial \mathbf{x}}{\partial u^3} = \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 \quad \text{Jacobian} \tag{D.176}$$

is the Jacobian of the transformation from the Cartesian coordinate system to the curvilinear coordinate system specified by the functions u^i.

The geometrical properties of a nonorthogonal curvilinear coordinate system are characterized by the dot products of the base vectors:

$$g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j = \frac{\partial \mathbf{x}}{\partial u^i} \cdot \frac{\partial \mathbf{x}}{\partial u^j} \quad \text{covariant metric elements,}$$
$$g^{ij} \equiv \mathbf{e}^i \cdot \mathbf{e}^j = \boldsymbol{\nabla} u^i \cdot \boldsymbol{\nabla} u^j \quad \text{contravariant metric elements.} \tag{D.177}$$

These symmetric tensor metric elements can be used to write the covariant components of a vector in terms of its contravariant components and vice versa:

$$A_i \equiv \mathbf{A} \cdot \mathbf{e}_i = \mathbf{A} \cdot \mathbf{I} \cdot \mathbf{e}_i = \sum_j (\mathbf{A} \cdot \mathbf{e}^j)(\mathbf{e}_j \cdot \mathbf{e}_i) = \sum_j g_{ij} A^j$$
$$A^i \equiv \mathbf{A} \cdot \mathbf{e}^i = \mathbf{A} \cdot \mathbf{I} \cdot \mathbf{e}^i = \sum_j (\mathbf{A} \cdot \mathbf{e}_j)(\mathbf{e}^j \cdot \mathbf{e}^i) = \sum_j g^{ij} A_i. \tag{D.178}$$

Similarly, they can also be used to write the covariant base vectors in terms of the contravariant base vectors and vice versa:

$$\mathbf{e}_i = \sum_j g_{ij} \, \mathbf{e}^j, \quad \mathbf{e}^i = \sum_j g^{ij} \, \mathbf{e}_i. \tag{D.179}$$

From the dot product between these relations and their respective reciprocal base vectors it can be shown that

$$\sum_j g_{ij} \, g^{jk} = \sum_j g^{kj} g_{ji} = \delta^k_i. \tag{D.180}$$

The determinant of the matrix comprised of the metric coefficients is called g:

$$g \equiv \|g_{ij}\| = \|g^{ij}\|^{-1},\tag{D.181}$$

in which the second relation follows from interpreting the summation relations at the end of the preceding paragraph in terms of matrix operations: $[g_{ij}][g^{ik}] = [\mathbf{I}]$, which yields $[g_{ij}] = [g^{jk}]^{-1}$. Since the determinant of the inner product of two matrices is given by the product of the determinants of the two matrices,

$$g = \|g_{ij}\| = \left\| \frac{\partial \mathbf{x}}{\partial u^i} \cdot \frac{\partial \mathbf{x}}{\partial u^j} \right\| = \left\| \frac{\partial \mathbf{x}}{\partial u^i} \right\| \left\| \frac{\partial \mathbf{x}}{\partial u^j} \right\| = \left(\frac{\partial \mathbf{x}}{\partial u^1} \cdot \frac{\partial \mathbf{x}}{\partial u^2} \times \frac{\partial \mathbf{x}}{\partial u^3} \right)^2 = J^2.\tag{D.182}$$

Thus, the determinant of the metric coefficients is related to the Jacobian and inverse Jacobian as follows:

$$J = \sqrt{g} = \mathbf{e}_1 \cdot \mathbf{e}_2 \times \mathbf{e}_3 = \frac{\partial \mathbf{x}}{\partial u^1} \cdot \frac{\partial \mathbf{x}}{\partial u^2} \times \frac{\partial \mathbf{x}}{\partial u^3} \qquad \text{Jacobian,}$$

$$J^{-1} = 1/\sqrt{g} = \mathbf{e}^1 \cdot \mathbf{e}^2 \times \mathbf{e}^3 = \boldsymbol{\nabla} u^1 \cdot \boldsymbol{\nabla} u^2 \times \boldsymbol{\nabla} u^3 \quad \text{inverse Jacobian.}\tag{D.183}$$

The various partial derivatives in space can be worked out in terms of covariant derivatives $(\partial/\partial u^i)$ using the properties of the covariant and contravariant base vectors for a general, nonorthogonal curvilinear coordinate system as follows:

$$\boldsymbol{\nabla} f = \sum_i \boldsymbol{\nabla} u^i \frac{\partial f}{\partial u^i} = \sum_i \mathbf{e}^i \frac{\partial f}{\partial u^i} \qquad\qquad\qquad \text{gradient,}\tag{D.184}$$

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{A} &= \boldsymbol{\nabla} \cdot (\mathbf{A} \cdot \mathbf{I}) = \boldsymbol{\nabla} \cdot \sum_i \sqrt{g} \, (\mathbf{A} \cdot \mathbf{e}^i) \frac{\mathbf{e}_i}{\sqrt{g}} = \sum_i \frac{\mathbf{e}_i}{\sqrt{g}} \cdot \boldsymbol{\nabla}(\sqrt{g} A^i) \\
&= \sum_i \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} (\sqrt{g} \, \mathbf{A} \cdot \mathbf{e}^i) = \sum_i \frac{1}{J} \frac{\partial}{\partial u^i} (J \, \mathbf{A} \cdot \boldsymbol{\nabla} u^i) \qquad \text{divergence,}
\end{aligned}\tag{D.185}$$

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{A} &= \boldsymbol{\nabla} \times (\mathbf{A} \cdot \mathbf{I}) = \boldsymbol{\nabla} \times \sum_j (\mathbf{A} \cdot \mathbf{e}_j) \mathbf{e}^j = \sum_j \boldsymbol{\nabla} A_j \times \boldsymbol{\nabla} u^j \\
&= \sum_{ij} \frac{\partial A_j}{\partial u^i} \boldsymbol{\nabla} u^i \times \boldsymbol{\nabla} u^j = \sum_{ijk} \frac{\epsilon^{ijk}}{\sqrt{g}} \frac{\partial(\mathbf{A} \cdot \mathbf{e}_j)}{\partial u^i} \\
&= \frac{1}{\sqrt{g}} \left\| \begin{matrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial u^1} & \frac{\partial}{\partial u^2} & \frac{\partial}{\partial u^3} \\ A_1 & A_2 & A_3 \end{matrix} \right\| \qquad\qquad\qquad \text{curl,}
\end{aligned}\tag{D.186}$$

$$
\begin{aligned}
\nabla^2 f \quad &\equiv \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f = \sum_i \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left(\sqrt{g}\, \mathbf{e}^i \cdot \sum_j \mathbf{e}^j \frac{\partial f}{\partial u^j} \right) \\
&= \sum_{ij} \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i} \left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial u^j} \right) = \sum_{ij} \frac{1}{J} \frac{\partial}{\partial u^i} \left(J\, \boldsymbol{\nabla} u^i \cdot \boldsymbol{\nabla} u^j \frac{\partial f}{\partial u^j} \right) \quad \text{Laplacian.}
\end{aligned}
$$
$$(\text{D}.187)$$

Differential line, surface and volume elements can be written in terms of differentials of the coordinates u^i of a general, nonorthogonal curvilinear coordinate system as follows. Total vector differential and line elements are:

$$
dx = \sum_i \frac{\partial \mathbf{x}}{\partial u^i}\, dx^i = \sum_i \mathbf{e}_i\, dx^i
$$
$$
|d\boldsymbol{\ell}| \equiv \sqrt{d\mathbf{x} \cdot d\mathbf{x}} \; = \sqrt{\sum_{ij} g_{ij}\, du^i du^j} \quad \text{metric of coordinates.}
$$
$$(\text{D}.188)$$

Differential line elements $d\boldsymbol{\ell}(i)$ along curve u^i $(du^j = du^k = 0)$ for $i, j, k =$ permutations of 1, 2, 3 are

$$
d\boldsymbol{\ell}(i) = \mathbf{e}_i\, du^i = \frac{\epsilon_{ijk}}{\sqrt{g}}\, \boldsymbol{\nabla} u^j \times \boldsymbol{\nabla} u^k\, du^i
$$
$$
|d\boldsymbol{\ell}(i)| = \sqrt{\mathbf{e}_i \cdot \mathbf{e}_i}\, du^i = \sqrt{g_{ii}}\, du^i
$$
$$(\text{D}.189)$$

The differential surface element $d\mathbf{S}(i)$ in the $u^i = c^i$ surface $(du^i = 0)$ for i, j, k = permutations of 1, 2, 3 is

$$
d\mathbf{S}(i) \equiv d\boldsymbol{\ell}(j) \times d\boldsymbol{\ell}(k) = \sqrt{g}\, \epsilon_{ijk} \boldsymbol{\nabla} u^i\, du^j du^k
$$
$$
|d\mathbf{S}(i)| = \sqrt{g_{jj} g_{kk} - g_{jk}^2}\, du^j du^k = \sqrt{g^{ii} g}\, du^j du^k
$$
$$(\text{D}.190)$$

The differential volume element is

$$
d^3 x \equiv d\boldsymbol{\ell}(1) \cdot d\boldsymbol{\ell}(2) \times d\boldsymbol{\ell}(3) = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)\, du^1 du^2 du^3 = \sqrt{g}\, du^1 du^2 du^3 .
$$
$$(\text{D}.191)$$

D.9   Orthogonal Coordinate Systems

Consider transformation from the Cartesian coordinate system $\mathbf{x} = (x, y, z)$ to an orthogonal curvilinear coordinate system defined by three independent functions $u^i = u^i(x, y, z)$ for $i = 1, 2, 3$. [Here, the superscripts 1,2,3 are not powers; rather, they represent labels for the three functions. The functions are labeled in this way to maintain consistency with the general (nonorthogonal) curvilinear coordinate literature.] The coordinate surfaces are defined by $u^i = c_i$, where c_i are constants. The three orthogonal unit vectors that point in directions locally perpendicular to the coordinate surfaces are

$$
\hat{\mathbf{e}}_i \equiv \boldsymbol{\nabla} u^i / |\boldsymbol{\nabla} u^i| \quad \text{orthogonal unit vectors.} \qquad (\text{D}.192)
$$

For the simplest orthogonal coordinate system, the Cartesian coordinate system, $\hat{\mathbf{e}}_1 = \boldsymbol{\nabla} x = \hat{\mathbf{x}}$, $\hat{\mathbf{e}}_2 = \boldsymbol{\nabla} y = \hat{\mathbf{y}}$, $\hat{\mathbf{e}}_3 = \boldsymbol{\nabla} z = \hat{\mathbf{z}}$.

Because of the normalization and assumed orthogonality of these unit vectors,

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} \equiv \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j, \end{cases} \quad \text{Kronecker delta.} \tag{D.193}$$

The cross products of unit vectors are governed by the right-hand rule which is embodied in the mathematical relation

$$\hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \epsilon_{ijk}\, \hat{\mathbf{e}}_k \tag{D.194}$$

in which the Levi-Civita symbol ϵ_{ijk} is defined by

$$\epsilon_{ijk} \equiv \begin{cases} +1, & \text{for } i, j, k = 1, 2, 3 \text{ or } 2, 3, 1 \text{ or } 3, 1, 2 \quad \text{(even permutations)} \\ -1, & \text{for } i, j, k = 2, 1, 3 \text{ or } 1, 3, 2 \text{ or } 3, 2, 1 \quad \text{(odd permutations)} \\ 0, & \text{for any two indices the same.} \end{cases} \tag{D.195}$$

A vector $\mathbf{A}$ can be represented in terms of its components in the orthogonal directions (parallel to $\boldsymbol{\nabla} u^i$) of the unit vectors $\hat{\mathbf{e}}_i$:

$$\mathbf{A} = \sum_i A_i \hat{\mathbf{e}}_i = A_1 \hat{\mathbf{e}}_1 + A_2 \hat{\mathbf{e}}_2 + A_3 \hat{\mathbf{e}}_3, \quad A_i \equiv \mathbf{A} \cdot \hat{\mathbf{e}}_i \tag{D.196}$$

For an orthogonal coordinate system the identity dyad or tensor is

$$\mathbf{I} = \sum_i \hat{\mathbf{e}}_i \hat{\mathbf{e}}_i = \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 \quad \text{identity tensor.} \tag{D.197}$$

Thus, the vector differential operator becomes

$$\begin{aligned} \boldsymbol{\nabla} = \mathbf{I} \cdot \boldsymbol{\nabla} &= \sum_i \hat{\mathbf{e}}_i \left(\hat{\mathbf{e}}_i \cdot \boldsymbol{\nabla} \right) = \hat{\mathbf{e}}_1 \left(\hat{\mathbf{e}}_1 \cdot \boldsymbol{\nabla} \right) + \hat{\mathbf{e}}_2 \left(\hat{\mathbf{e}}_2 \cdot \boldsymbol{\nabla} \right) + \hat{\mathbf{e}}_3 \left(\hat{\mathbf{e}}_3 \cdot \boldsymbol{\nabla} \right) \\ &= \sum_i \boldsymbol{\nabla} u^i \frac{\partial}{\partial u^i} = \boldsymbol{\nabla} u^1 \frac{\partial}{\partial u^1} + \boldsymbol{\nabla} u^2 \frac{\partial}{\partial u^2} + \boldsymbol{\nabla} u^3 \frac{\partial}{\partial u^3}. \end{aligned} \tag{D.198}$$

Here and below the sum over i is over the three components 1,2,3.

Using the relations for the dot and cross products of the unit vectors $\hat{\mathbf{e}}_i$ given in (D.193) and (D.194) the dot, cross and dot-cross products of vectors become

$$\mathbf{A} \cdot \mathbf{B} = \sum_i A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3, \tag{D.199}$$

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \sum_{ij} A_i B_j\, \hat{\mathbf{e}}_i \times \hat{\mathbf{e}}_j = \sum_{ijk} \epsilon_{ijk}\, A_i B_j\, \hat{\mathbf{e}}_k = \left\| \begin{matrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{matrix} \right\| \\ &= \hat{\mathbf{e}}_1 (A_2 B_3 - A_3 B_2) + \hat{\mathbf{e}}_2 (A_3 B_1 - A_1 B_2) + \hat{\mathbf{e}}_3 (A_1 B_2 - A_2 B_1). \end{aligned} \tag{D.200}$$

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \sum_{ijk} \epsilon_{ijk}\, A_i B_j C_k = \left\| \begin{matrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{matrix} \right\|. \tag{D.201}$$

The differential line element in the ith direction is given by

$$d\boldsymbol{\ell}(i) = \hat{\mathbf{e}}_i \, h_i \, du^i, \text{ with } h_i \equiv 1/|\boldsymbol{\nabla} u^i|, \quad \text{differential line element.} \qquad \text{(D.202)}$$

Thus, the differential surface vector for the $u^i = c_i$ surface, which is defined by $d\mathbf{S}(i) = d\boldsymbol{\ell}(j) \times d\boldsymbol{\ell}(k)$, becomes

$$d\mathbf{S}(i) = \hat{\mathbf{e}}_i \, h_j h_k \, du^j du^k, \text{ for } i \neq j \neq k, \quad \text{differential surface area.} \qquad \text{(D.203)}$$

Since the differential volume element is $d^3 x = d\boldsymbol{\ell}(i) \cdot d\mathbf{S}(i) = d\boldsymbol{\ell}(1) \cdot d\boldsymbol{\ell}(2) \times d\boldsymbol{\ell}(3)$ and the Jacobian of the transformation is given by $J = 1/(\boldsymbol{\nabla} u^1 \cdot \boldsymbol{\nabla} u^2 \times \boldsymbol{\nabla} u^3) = h_1 h_2 h_3$,

$$d^3 x = h_1 h_2 h_3 \, du^1 du^2 du^3, \quad \text{differential volume.} \qquad \text{(D.204)}$$

For orthogonal coordinate systems the various partial derivatives in space are

$$\boldsymbol{\nabla} f = \sum_i \frac{\hat{\mathbf{e}}_i}{h_i} \frac{\partial f}{\partial u^i} = \sum_i \hat{\mathbf{e}}_i \, (\hat{\mathbf{e}}_i \cdot \boldsymbol{\nabla}) \, f, \qquad \text{(D.205)}$$

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \sum_i \frac{1}{J} \frac{\partial}{\partial u^i} \left(\frac{J}{h_i} \mathbf{A} \cdot \hat{\mathbf{e}}_i \right) = \sum_i \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u^i} \left(\frac{h_1 h_2 h_3}{h_i} \mathbf{A} \cdot \hat{\mathbf{e}}_i \right), \text{(D.206)}$$

$$\boldsymbol{\nabla} \times \mathbf{A} = \sum_{ijk} \frac{\epsilon_{ijk} h_k \hat{\mathbf{e}}_k}{J} \frac{\partial}{\partial u^i} \left(h_j \mathbf{A} \cdot \hat{\mathbf{e}}_j \right) = \sum_{ijk} \frac{\epsilon_{ijk} h_k \hat{\mathbf{e}}_k}{h_1 h_2 h_3} \frac{\partial}{\partial u^i} \left(h_j \mathbf{A} \cdot \hat{\mathbf{e}}_j \right), \text{(D.207)}$$

$$\nabla^2 f = \sum_i \frac{1}{J} \frac{\partial}{\partial u^i} \left(\frac{J}{h_i^2} \frac{\partial f}{\partial u^i} \right) = \sum_i \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u^i} \left(\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial f}{\partial u^i} \right). \qquad \text{(D.208)}$$

The three most common orthogonal coordinate systems are the Cartesian, cylindrical, and spherical coordinate systems. Their coordinate surfaces and unit vectors are shown in Fig. D.3. They will be defined in this book by

$$\text{Cartesian}: \quad u^i = (x, y, z)$$
$$h_x = 1, \quad h_y = 1, \quad h_z = 1 \quad \Longrightarrow \quad J = 1; \qquad \text{(D.209)}$$

$$\text{cylindrical}: \quad u^i = (r, \theta, z)$$
$$r \equiv \sqrt{x^2 + y^2}, \quad \theta \equiv \arctan(y/x), \quad z \equiv z,$$
$$x = r \cos\theta, \quad y = r \sin\theta, \quad z = z,$$
$$h_r = 1, \quad h_\theta = r, \quad h_z = 1 \quad \Longrightarrow \quad J = r; \qquad \text{(D.210)}$$

$$\text{spherical}: \quad u^i = (r, \vartheta, \varphi)$$
$$r \equiv \sqrt{x^2 + y^2 + z^2}, \quad \vartheta \equiv \arctan(\sqrt{x^2 + y^2}/r), \quad \varphi \equiv \arctan(y/x),$$
$$x = r \sin\vartheta \cos\varphi, \quad y = r \sin\vartheta \sin\varphi, \quad z = r \cos\vartheta,$$
$$h_r = 1, \quad h_\theta = r, \quad h_\varphi = r \sin\vartheta \quad \Longrightarrow \quad J = r^2 \sin\vartheta. \qquad \text{(D.211)}$$
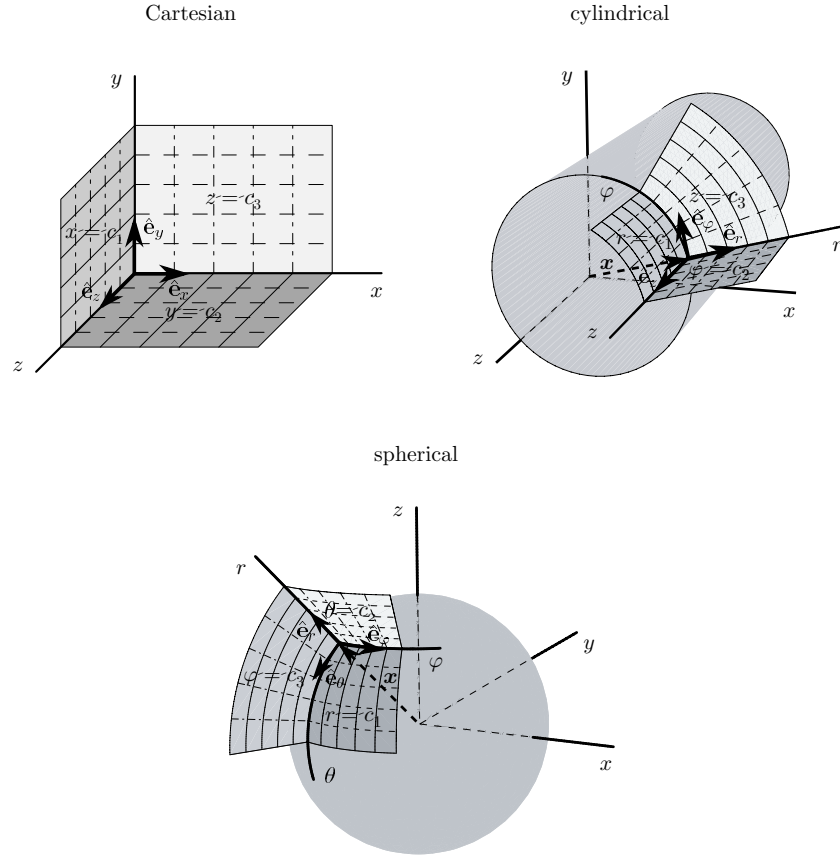
Cartesian

cylindrical

spherical

Figure D.3: Orthogonal unit vectors and constant coordinate surfaces for the three most common orthogonal coordinate systems.

Note that with these definitions the cylindrical angle θ is the same as the azimuthal (longitudinal) spherical angle φ, but that the radial coordinate r is different in the cylindrical and spherical coordinate systems. The spherical angle ϑ is a latitude angle — see Fig. D.3. Explicit forms for the various partial derivatives in space, (D.205) – (D.208), are given in Appendix Z.

REFERENCES

Intermediate level discussions of vector analysis are provided in

Greenberg, *Advanced Engineering Mathematics*, Chapters 13-16 (1998) [**?**]

Kusse and Westwig, *Mathematical Physics* (1998) [**?**]

Danielson, *Vectors and Tensors in Engineering and Physics*, 2nd Ed. (1997) [**?**]

More advanced treatments are available in

Arfken, *Mathematical Methods for Physicists* (??) [**?**]

Greenberg, *Foundations of Applied Mathematics*, Chapters 8,9 (1978) [**?**]

Morse and Feshbach, *Methods of Theoretical Physics*, Part I, Chapter 1 (1953) [**?**]