Fluid mechanics II
by
Dr. Ehsan Amani

Chapter 3

Fluid kinematics

- **Fluid mechanics I: Velocity field kinematic information**

$$\vec{V}(x, y, z, t) = u(x, y, z, t)\hat{\imath} + v(x, y, z, t)\hat{\jmath} + w(x, y, z, t)\hat{k} \qquad (1.3)$$

  - **Streamlines, path lines, streaklines**
  - **Fluid particle (element) acceleration**

$$\vec{a}_p(t) = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} = \frac{\partial \vec{V}}{\partial t} + \left(\vec{V}.\vec{\nabla}\right)\vec{V} \qquad (2.3)$$

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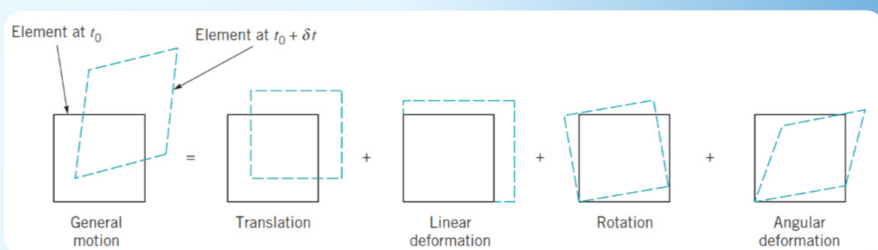
Fluid element kinematics

- **Fluid element motion and deformation**
  - **Translation**
  - **Linear deformation**
  - **Rotation**
  - **Angular deformation**



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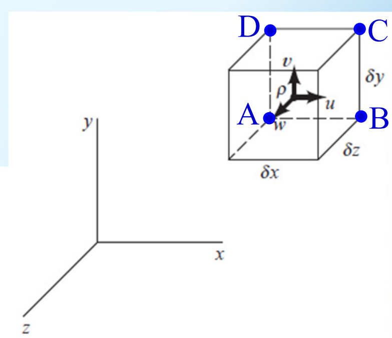
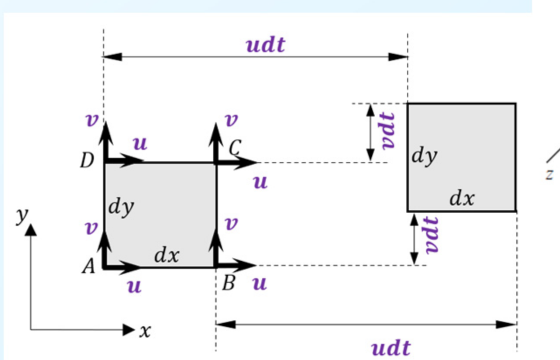
Fluid element kinematics

- **Pure translation**
  - $\vec{V}(x, y, z, t) = \vec{V} = cte$





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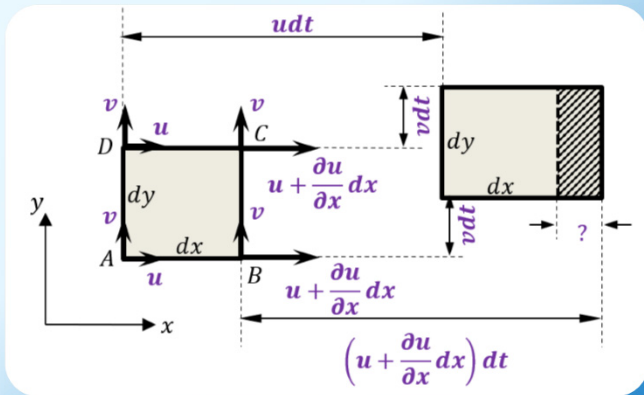
Fluid element kinematics

- **Linear deformation**
  - $v = cte, w = cte$
  - $u = u(x)$

$$\dot{\varepsilon}_{xx} = \frac{\overbrace{\frac{\partial u}{\partial x} dxdt}^{\text{length change during } dt}}{\underbrace{dxdt}_{\text{Initial length}}} = \frac{\partial u}{\partial x} \quad (3.3)$$

strain rate (−)



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Fluid element kinematics

- **Linear deformation**
  - $v = cte, w = cte$
  - $u = u(x)$

$$\dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x} \quad (3.3)$$

  - Similarly, if $v = v(y)$ or $w = w(z)$

$$\dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y}, \quad \dot{\varepsilon}_{zz} = \frac{\partial w}{\partial z} \quad (4.3)$$

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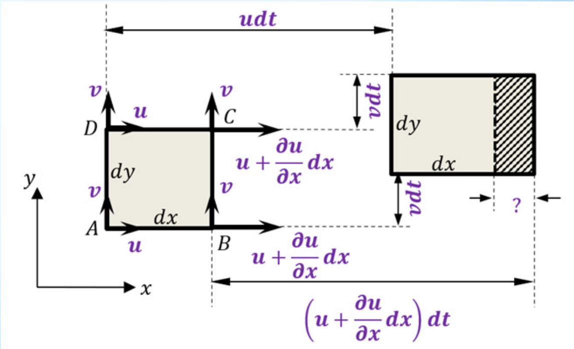
Fluid element kinematics

- **Fluid element volume change rate**
  - $v = cte, w = cte$
  - $u = u(x)$



$$\frac{1}{d\mho}\frac{d(d\mho)}{\partial t} = \frac{1}{dxdydz}\frac{\left(\frac{\partial u}{\partial x}dxdt\right)dydz}{dt} = \frac{\partial u}{\partial x}$$

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Fluid element kinematics

- **Fluid element volume change rate**
  - $v = cte, w = cte$
  - $u = u(x)$

$$\frac{1}{d\mho}\frac{d(d\mho)}{\partial t} = \frac{\partial u}{\partial x} = \dot{\varepsilon}_{xx}$$

  - Similarly, if $v = v(y)$

$$\frac{1}{d\mho}\frac{d(d\mho)}{\partial t} = \frac{\partial v}{\partial y} = \dot{\varepsilon}_{yy}$$

  - What if $u = u(x)$ and $v = v(y)$ simultaneously?

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Fluid element kinematics

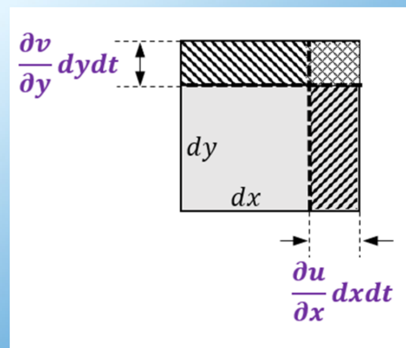
- **Fluid element volume change rate**
  - What if $u = u(x)$ and $v = v(y)$ simultaneously?

$$\frac{1}{d\mho}\frac{d(d\mho)}{\partial t} = \frac{1}{dxdydz}\frac{dx\left(1+\frac{\partial u}{\partial x}dt\right)dy\left(1+\frac{\partial v}{\partial y}dt\right)dz - dxdydz}{dt} =$$

$$\frac{1+\frac{\partial u}{\partial x}dt+\frac{\partial v}{\partial y}dt+\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}(dt)^2-1}{dt} =$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}dt \quad ; dt \to 0$$

$$\frac{1}{d\mho}\frac{d(d\mho)}{\partial t} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$



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Fluid element kinematics

- **Fluid element volume change rate**
  - Corollary: Superposition is held for element deformation
  - In general case

$$\frac{1}{d\mho}\frac{d(d\mho)}{\partial t} = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = \vec{\nabla}.\vec{V} = \dot{\varepsilon}_{xx}+\dot{\varepsilon}_{yy}+\dot{\varepsilon}_{zz} \qquad (5.3)$$

  - Exercise: Eq. (5.3) was derived using a cubic fluid element (Cartesian coordinates). Can you derive this conclusion for an arbitrary fluid element? You may need to consult advanced fluid mechanics books.

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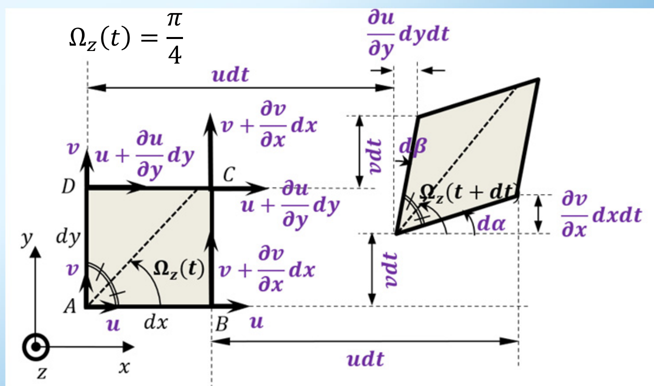
Fluid element kinematics

- **Rotation and angular deformation**
  - $w = cte, u = u(y)$ and $v = v(x)$

$$d\Omega_z = \Omega_z(t+dt) - \Omega_z(t) = d\alpha + \frac{\frac{\pi}{2} - (d\alpha + d\beta)}{2} - \frac{\pi}{4} = \frac{d\alpha - d\beta}{2} = \frac{dt}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$d\alpha \cong \tan d\alpha = \frac{\frac{\partial v}{\partial x}dxdt}{dx} = \frac{\partial v}{\partial x}dt$$

$$d\beta \cong \tan d\beta = \frac{\frac{\partial u}{\partial y}dydt}{dy} = \frac{\partial u}{\partial y}dt$$



Fluid element kinematics

- **Rotation and angular deformation**
  - **Rotation rate about z-axis**

rotation rate about z-axis (rad/s)

$$\dot{\Omega}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (6.3)$$

  - **Similarly,**

$$\dot{\Omega}_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad \dot{\Omega}_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad (7.3)$$

  - **Rotation rate vector and vorticity**

$$\vec{\Omega} = \dot{\Omega}_x\hat{\imath} + \dot{\Omega}_y\hat{\jmath} + \dot{\Omega}_z\hat{k} = \frac{1}{2}\vec{\nabla} \times \vec{V} \qquad (8.3)$$

Vorticity (rad/s)

$$\vec{\omega} = 2\vec{\Omega} = \vec{\nabla} \times \vec{V} = \mathrm{curl}\vec{V} \qquad (9.3)$$

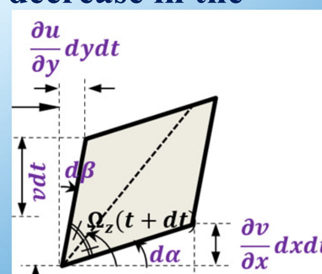
Fluid element kinematics

- **Rotation and angular deformation**
  - **If $\vec{\omega} = 0$ everywhere, the flow is called irrotational**
  - **Shear strain rate: The rate of decrease in the fluid element angle**

$$\dot{\gamma}_{xy} = \frac{d\alpha + d\beta}{dt} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \dot{\gamma}_{yx}$$

$$\dot{\gamma}_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \dot{\gamma}_{zx} \qquad (10.3)$$

$$\dot{\gamma}_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \dot{\gamma}_{zy}$$



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Fluid element kinematics

- **Rotation and angular deformation**
  - **Strain-rate tensor**

Strain-rate tensor (rad/s)

$$S_{ij} \equiv \begin{bmatrix} \dot{\varepsilon}_{xx} & \frac{\dot{\gamma}_{xy}}{2} & \frac{\dot{\gamma}_{xz}}{2} \\ \frac{\dot{\gamma}_{yx}}{2} & \dot{\varepsilon}_{yy} & \frac{\dot{\gamma}_{yz}}{2} \\ \frac{\dot{\gamma}_{zx}}{2} & \frac{\dot{\gamma}_{zy}}{2} & \dot{\varepsilon}_{zz} \end{bmatrix} \qquad (11.3)$$

Angular deformation (strain rate)

Linear deformation (strain rate)

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right); i,j = 1,2,3 \qquad (12.3)$$

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The fluid flow governing equations

- **Derivation using tensorial notation**
  - **Independent of a specific coordinates system**
  - **Using a fluid mass element or (usually stagnant) volume element**
  - **Needs the knowledge of tensor algebra and calculus**

Lecture Notes: III.1

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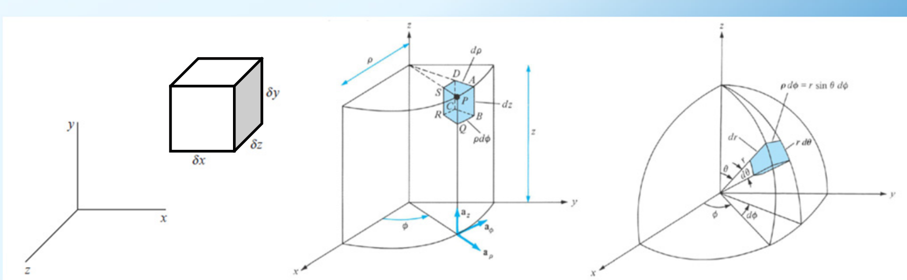
The fluid flow governing equations

- **Derivation for a specific coordinates system**
  1. **Considering a general stagnant volume element or fluid mass element in the coordinates system**
     - **At a general position, e.g., (x, y, z), and time t**
     - **Each element dimension equals the differential distance in that coordinate direction**



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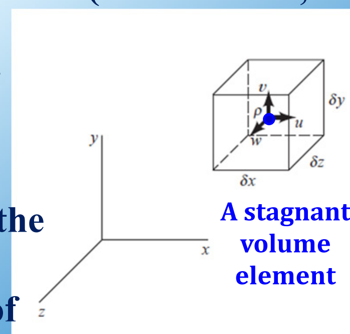
The fluid flow governing equations

- **Derivation for a specific coordinates system**
  1. **Considering a general stagnant volume element or fluid mass element in the coordinates system**
  2. **Choosing an arbitrary reference point** $(u, v, w, \rho, p, \ldots)$ **within the element (at the center, corners, etc.)**
  3. **Writing the principle law for the element**
  4. **When necessary, using the Taylor expansion to express the properties at different points within the element in terms of those at the reference point**



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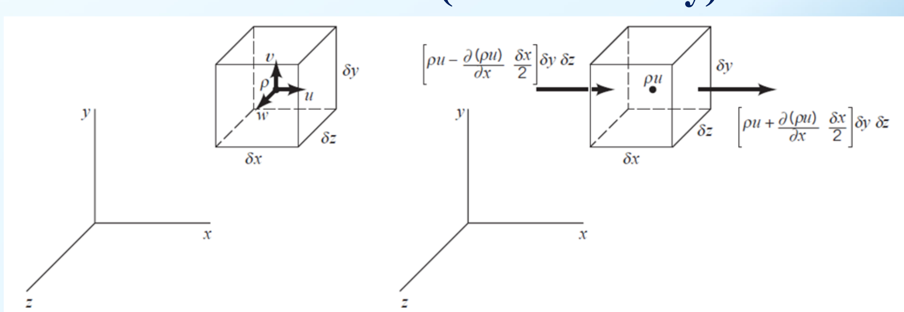
The fluid flow governing equations

- **Mass conservation (the continuity)**



⇨ Lecture Notes: III.2

$$\underbrace{\frac{\partial \rho}{\partial t}}_{\text{Rate of change of element mass (per unit volume)}} + \underbrace{\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}}_{\text{Net outlet mass flow rate from the element}} = 0 \qquad (15.3)$$

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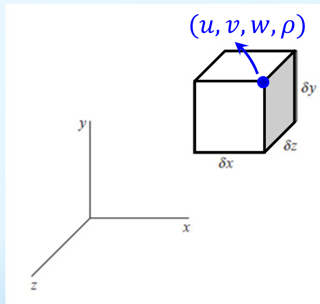
The fluid flow governing equations

- **Mass conservation (the continuity)**
  - **Exercise: Using the different reference point within the element shown below, derive Eq. (15.3).**



  - **Exercise: Assuming the Cartesian element to be a fluid (mass) element, derive Eq. (15.3).**

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The fluid flow governing equations

- **Mass conservation (the continuity)**
  - **General tensorial form**

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \qquad (15.3)$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.(\rho \vec{V}) = 0 \qquad (16.3)$$

  - **See chap3-Del Operations … .pdf**
  - **Exercise: show that Eq. (15.3) can be written as:**

$$\frac{D\rho}{Dt} + \rho \vec{\nabla}.\vec{V} = 0 \qquad (17.3)$$

  - **For incompressible flows ($\frac{D\rho}{Dt} = 0$):**

$\vec{\nabla}.\vec{V} = 0$ (17.3) **Cartesian coordinates:** $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ (17.3)'

  - **For steady flows ($\frac{\partial}{\partial t} = 0$):** $\vec{\nabla}.(\rho \vec{V}) = 0$ (20.3)

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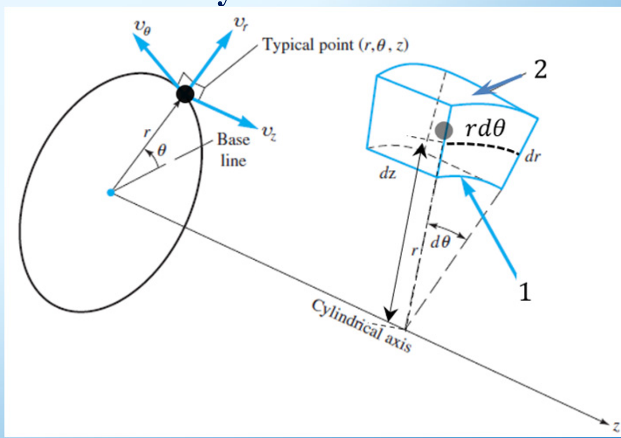
The fluid flow governing equations

- Mass conservation (the continuity)
  - Sample problem: Derive the continuity for the cylindrical coordinates system.



⇨ Lecture Notes: III.3

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The fluid flow governing equations

- Stream function
  - For steady 2D flows
  - Velocity components can be expressed in terms of a single variable, stream function ψ
  - A change of variable automatically satisfying the continuity
  - Removing the need for explicitly considering the continuity

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The fluid flow governing equations

- **Stream function**
  - **For steady 2D plane flows**

continuity $\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \qquad \psi(x,y): \rho u \equiv \frac{\partial \psi}{\partial y}, -\rho v \equiv \frac{\partial \psi}{\partial x} \qquad (21.3)$

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \ \checkmark$$

  - **Physical interpretations:**
    - **Each $\psi = cte$ curve is a streamline**

Lecture Notes: III.4

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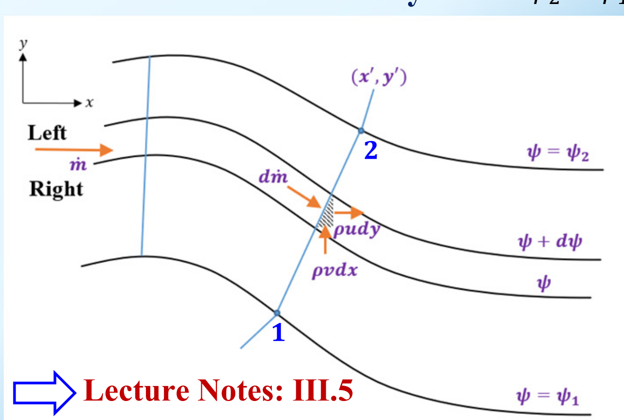
The fluid flow governing equations

- **Physical interpretations**
  - **Each $\psi = cte$ curve is a streamline**
  - **The (mass) flow rate and direction between points 1 and 2 can be determined by** $\dot{m} = \psi_2 - \psi_1 \quad (22.3)$



ψ is larger at the left-hand-side of the flow direction

Lecture Notes: III.5

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The fluid flow governing equations

- **Stream function**
  - **For incompressible flows, usually:**

$$u \equiv \frac{\partial \psi}{\partial y}, -v \equiv \frac{\partial \psi}{\partial x} \qquad (23.3)$$

$$Q = \dot{m}/\rho = \psi_2 - \psi_1 \qquad (24.3)$$

  - **Exercise: Show that for steady 2D plane flows using polar coordinates**

$$\rho r u \equiv \frac{\partial \psi}{\partial \theta}, -\rho v \equiv \frac{\partial \psi}{\partial r} \qquad (25.3)$$

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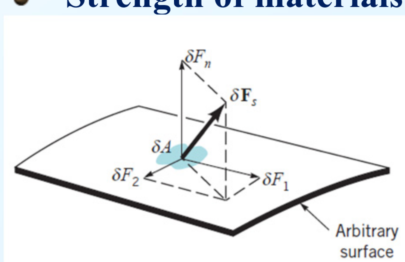
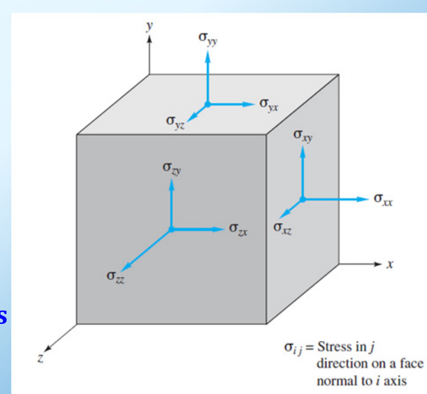
The fluid flow governing equations

- **Momentum equation**
  - **Strength of materials I: Stress**





$$\sigma_{nn} = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$$ **Normal stress**

$$\sigma_{n1} = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A}$$ **Shear stress**

$$\sigma_{n2} = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$$ **Shear stress**

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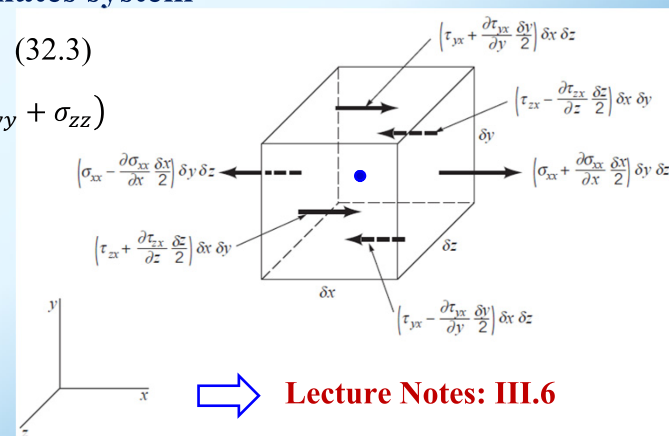
The fluid flow governing equations

- **Momentum equation**
  - **Derivation using a fluid element in Cartesian coordinates system**

$$\sigma_{nn} = -p + \tau_{nn} \qquad (32.3)$$

$$p \equiv -\frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$$



⇨ **Lecture Notes: III.6**

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The fluid flow governing equations

- **Momentum equation**
  - **Derivation using a fluid element in Cartesian coordinates system**

$$\begin{cases}
\hat{\imath}: & \rho\dfrac{Du}{Dt} = -\dfrac{\partial p}{\partial x} + \dfrac{\partial \tau_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + f_x \\
\hat{\jmath}: & \rho\dfrac{Dv}{Dt} = -\dfrac{\partial p}{\partial y} + \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \tau_{yy}}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + f_y \\
\hat{k}: & \underbrace{\rho\dfrac{Dw}{Dt}}_{\substack{\text{Inertial force} \\ \text{(per unit volume)}}} = \underbrace{-\dfrac{\partial p}{\partial z}}_{\substack{\text{pressure force} \\ (\Leftarrow)}} + \underbrace{\dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \tau_{zz}}{\partial z}}_{\substack{\text{viscous force} \\ (\Leftarrow)}} + \underbrace{f_z}_{\substack{\text{body force} \\ (\Leftarrow)}}
\end{cases} \qquad (33.3)$$

  - **Exercise: Derive Eq. (33.3) using an stagnant volume element in Cartesian coordinates system [2].**

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The fluid flow governing equations

- **Momentum equation**

$$\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla} p + \vec{\nabla} . \underline{\tau} + \vec{f} \qquad (34.3)$$

  - **See Chap3-table.pdf**

§B.5 THE EQUATION OF MOTION IN TERMS OF τ

$$[\rho D\mathbf{v}/Dt = -\nabla p + [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g}]$$

$u \leftrightarrow v_x$
$v \leftrightarrow v_y$
$w \leftrightarrow v_z$

Cartesian coordinates (x, y, z):[a]

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \left[\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right] + \rho g_x \qquad \text{(B.5-1)}$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \left[\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right] + \rho g_y \qquad \text{(B.5-2)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \left[\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right] + \rho g_z \qquad \text{(B.5-3)}$$

[a] These equations have been written without making the assumption that $\boldsymbol{\tau}$ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, τ_{xy} and τ_{yx} may be interchanged.

Cylindrical coordinates (r, θ, z):[b]

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}\right] + \rho g_r \qquad \text{(B.5-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right] + \rho g_\theta \qquad \text{(B.5-5)}$$

The fluid flow governing equations

- **Momentum equation**
  - **Exercise: Using the continuity, show that Eq. (33.3) can be written as (conservative form):**

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u u}{\partial x} + \frac{\partial \rho v u}{\partial y} + \frac{\partial \rho w u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho u v}{\partial x} + \frac{\partial \rho v v}{\partial y} + \frac{\partial \rho w v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y$$

$$\frac{\partial \rho w}{\partial t} + \frac{\partial \rho u w}{\partial x} + \frac{\partial \rho v w}{\partial y} + \frac{\partial \rho w w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z$$

(33.3)’

The fluid flow governing equations

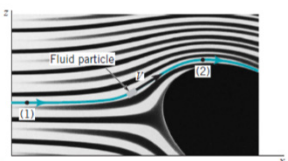
- **Limit cases**
  - **Inviscid flow: Negligible viscous force**

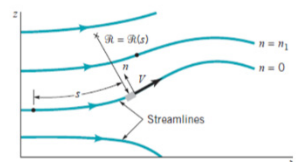
$$\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}p + \vec{f} \qquad (35.3)$$ **Euler's equation**

  - **+ steady + gravity body force only (in the negative z-direction)** $\vec{f} = \rho\vec{g} = -\rho g\hat{k} = -\rho g\vec{\nabla}z$

$$\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}p - \rho g\vec{\nabla}z \qquad (36.3) \qquad \rho\vec{a} = -\vec{\nabla}p - \rho g\vec{\nabla}z$$

  - **Fluid mechanics I: Integrating Eq. (36.3) along or perpendicular to a streamline**





Bernoulli equation

$p + \frac{1}{2}\rho V^2 + \gamma z$ = constant along streamline

$p + \rho\int\frac{V^2}{\Re}dn + \gamma z$ = constant across the streamline

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The fluid flow governing equations

- **Limit cases**
  - **Incompressible, inviscid, steady, irrotational**

Ideal flow (Incompressible, inviscid)

Potential flow (Incompressible, inviscid, steady, irrotational)

⇨ **Lecture Notes: III.7**

$$\frac{p}{\rho} + gz + \frac{V^2}{2} = cte \qquad (43.3)$$ **Entire the flow**

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The fluid flow governing equations

- **Stokes law:**
  - **For the detailed proof, consult Ref. [3], chap 10**
  - **Empirical: For many fluids (Newtonian fluid), the viscous stress tensor, $\underline{\tau}$, is a linear function of strain rate components, S_{ij}.**
  - **Isotropic material**
  - **Galilean invariance**

$$\tau_{ij} = 2\mu S_{ij} - \frac{2}{3}(\mu - K)(\vec{\nabla}.\vec{V})\delta_{ij}; i,j = 1,2,3 \qquad (46.3)$$

Dynamic viscosity **Bulk viscosity (almost always = 0)** **Identity tensor**

$$\delta_{ij} = \begin{cases} 1 & ; i = j \\ 0 & ; \text{otherwise} \end{cases}$$

  - **Exercise: Show that for parallel flow ($\vec{V} = u(y)\hat{\imath}$):** $\tau_{xx} = \tau_{yy} = \tau_{zz} = \tau_{xz} = \tau_{yz} = 0$, $\tau_{xy} = \mu \frac{\partial u}{\partial y}$

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The fluid flow governing equations

- **Stokes law:**
  - **General coordinates: chap3-tables.pdf**

$\mu' = -\frac{2}{3}\mu + \kappa$

§B.1 NEWTON'S LAW OF VISCOSITY

$[\boldsymbol{\tau} = +\mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^\dagger) + (\frac{2}{3}\mu + \kappa)(\nabla \cdot \mathbf{v})\boldsymbol{\delta}]$

Cartesian coordinates (x, y, z):

$$\tau_{xx} = +\mu\left[2\frac{\partial v_x}{\partial x}\right] + (\tfrac{2}{3}\mu + \kappa)(\nabla \cdot \mathbf{v})$$

$$\tau_{yy} = +\mu\left[2\frac{\partial v_y}{\partial y}\right] + (\tfrac{2}{3}\mu + \kappa)(\nabla \cdot \mathbf{v})$$

$$\tau_{zz} = +\mu\left[2\frac{\partial v_z}{\partial z}\right] + (\tfrac{2}{3}\mu + \kappa)(\nabla \cdot \mathbf{v})$$

$$\tau_{xy} = \tau_{yx} = +\mu\left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]$$

$$\tau_{yz} = \tau_{zy} = +\mu\left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]$$

$$\tau_{zx} = \tau_{xz} = +\mu\left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]$$

Cylindrical coordinates (r, θ, z):

$$\tau_{rr} = +\mu\left[2\frac{\partial v_r}{\partial r}\right] + (\tfrac{2}{3}\mu + \kappa)(\nabla \cdot \mathbf{v})$$

$$\tau_{\theta\theta} = +\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\right] + (\tfrac{2}{3}\mu + \kappa)(\nabla \cdot \mathbf{v})$$

$$\tau_{zz} = +\mu\left[2\frac{\partial v_z}{\partial z}\right] + (\tfrac{2}{3}\mu + \kappa)(\nabla \cdot \mathbf{v})$$

$$\tau_{r\theta} = \tau_{\theta r} = +\mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = +\mu\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]$$

$$\tau_{zr} = \tau_{rz} = +\mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]$$

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The fluid flow governing equations

- **Navier-Stokes equations**
  - **Exercise: Assuming constant properties, ρ and μ, inserting from §B.1 into §B.5, show that**

$$\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}+\frac{\partial v_z}{\partial z}=0$$

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=-\frac{\partial P}{\partial y}+\mu\left(\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right)+\rho g_y \quad (48.3)$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left(\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right)+\rho g_z$$

  - **Chap3-tables.pdf: Different coordinates systems**

§B.6 EQUATION OF MOTION FOR A NEWTONIAN FLUID WITH CONSTANT ρ AND μ

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}]$$

Cartesian coordinates (x, y, z):

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The fluid flow governing equations

- **Navier-Stokes equations**
  - **Incompressible flow vs. variable density flow**

Unknowns	Equations
u	x-momentum
v	y-momentum
w	z-momentum
p	continuity

Unknowns	Equations
u	x-momentum
v	y-momentum
w	z-momentum
p	continuity
ρ	state
T	energy

- **More complexities:**
  - **Non-Newtonian**
  - **Reacting**
  - **Multiphase**
  - **…**

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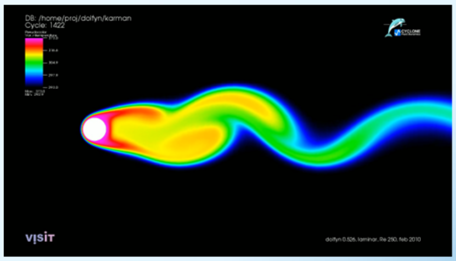
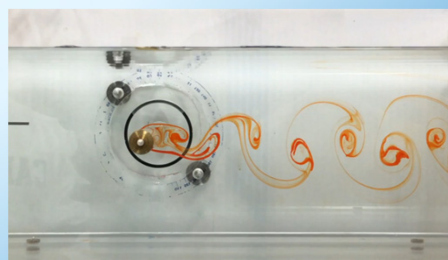
Computational Fluid Dynamics (CFD)

- Navier-Stokes equations
  - Vortex shedding



Simulation ▲



▲ experiment

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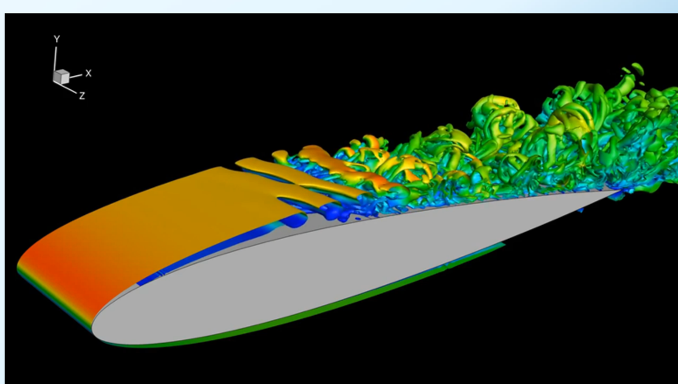
Computational Fluid Dynamics (CFD)

- Navier-Stokes equations
  - Turbulence



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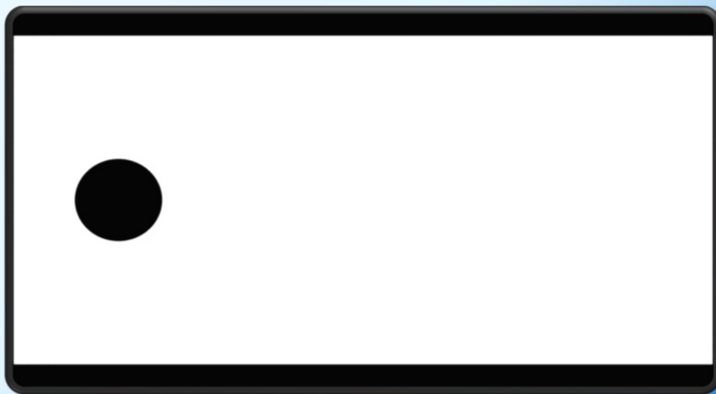
Computational Fluid Dynamics (CFD)

- (Mass + Momentum + energy + state)
  - Supersonic flow and shocks



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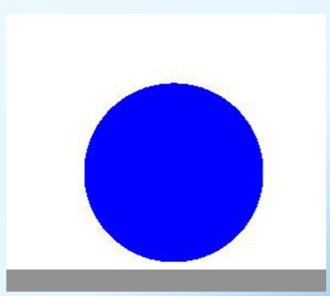
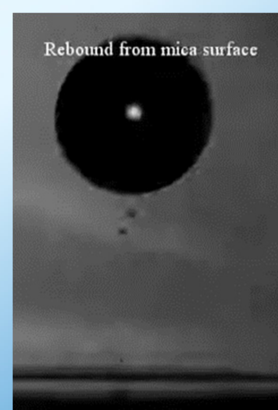
Computational Fluid Dynamics (CFD)

- Navier-Stokes equations
  - Multiphase flow – droplet wall impact



Simulation ▲

Experiment ►



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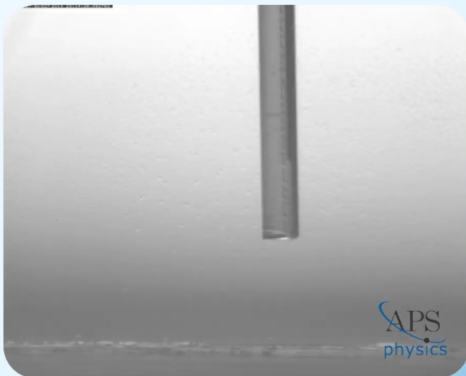
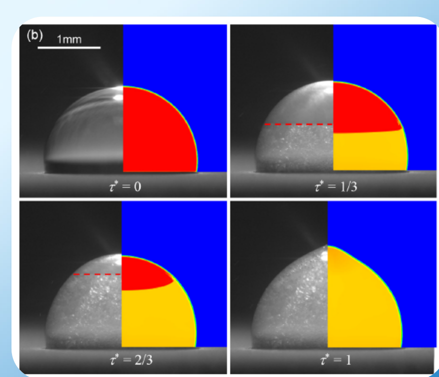
Computational Fluid Dynamics (CFD)

- (Mass + Momentum + energy + state + …)
  - Multiphase flow – droplet solidification





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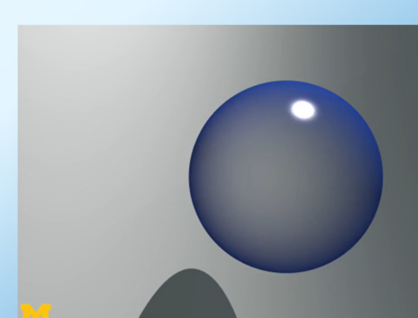
Computational Fluid Dynamics (CFD)

- (Mass + Momentum + energy + state + …)
  - Multiphase flow – cavitation and bubble collapse

Experiment ▲



Simulation ▲

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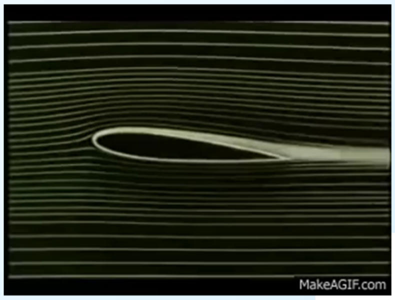
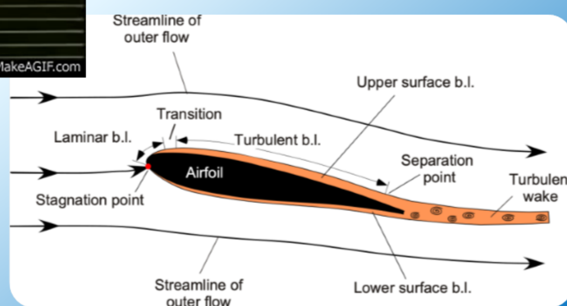
Analytical solutions

- **Boundary layer**





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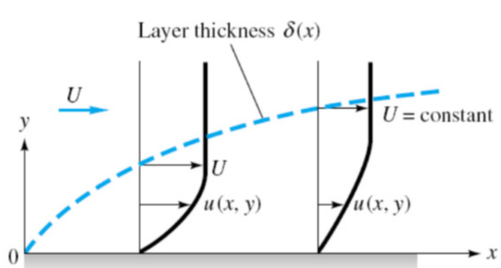
Analytical solutions

- **Sample problem: Boundary layer approximation**

A reasonable approximation for the two-dimensional incompressible laminar boundary layer on the flat surface in below figure is

$$u = U\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \quad \text{for } y \le \delta \quad \text{where } \delta = Cx^{1/2},\ C = \text{const}$$

(a) Assuming a no-slip condition at the wall, find an expression for the velocity component $v(x,y)$ for $y \le \delta$. (b) Then find the maximum value of v at the station x = 1 m, for the particular case of airflow, when U = 3 m/s and δ =1.1 cm.



Lecture Notes: III.8

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Analytical solutions

- **Poiseuille flow**
  - **Bioengineering application of laminar flow in pipes**

Blood flow in blood vessels ▶

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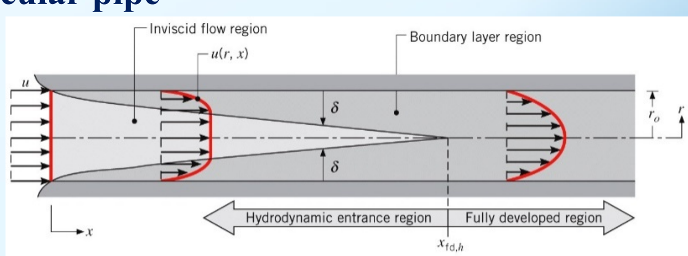
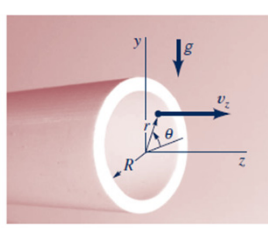
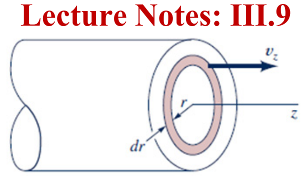
Analytical solutions

- **Sample problem (Poiseuille flow): The laminar, incompressible, steady, fully-developed flow in a circular pipe**





Lecture Notes: III.9



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Analytical solutions

- **Tables**

§B.1 NEWTON'S LAW OF VISCOSITY

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

§B.6 EQUATION OF MOTION FOR A NEWTONIAN FLUID WITH CONSTANT ρ AND μ

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r \quad \text{(B.6-4)}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta \quad \text{(B.6-5)}$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \quad \text{(B.6-6)}$$

The end of chapter 3