# Examples of Dynamical Reduction Using Symmetry for Simple Mechanical Systems with Virtual Holonomic Constraints

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# Agenda

# Underactuated Robotics


**Figure:** Modern underactuated robot[1]

- Modern robotic systems are **underactuated**: there are more degrees of freedom than actuators [2].
- Robots with $\geq 2$ degrees of underactuation may have complex dynamics

[1] WikimediaCommons. *GSMA Mobile World Congress 2021*. https://commons.wikimedia.org/wiki/File:MWC21_-_25.jpg. 2021

[2] R. Tedrake. *Underactuated Robotics: Algorithms for Walking, Running, Swimming, Flying, and Manipulation*. Course Notes for MIT 6.832. 2022. URL: %5Curl%7Bhttps://underactuated.csail.mit.edu%7D

# Motivation

Robotics engineers need mathematical methods to control underactuated robots that are:

- **Fast**: the robot can respond in real time
- **Reliable**: the robots follows the planned trajectory
- **Stable**: the robot can deal with disturbances
- **Efficient**: minimize computation for small embedded computers


**Figure:** Robotic arm used in industrial applications[3]

[3] J. Baxt. *File:CPCCG screening robot.jpg*. https://upload.wikimedia.org/wikipedia/en/thumb/2/27/CPCCG_screening_robot.jpg/1600px-CPCCG_screening_robot.jpg?20090730203425. 2021

# Problem Definition

This project considers a particular class of systems with the following traits:

- **Simple**[4]: mechanical system such as a pendulum
- **Underactuated**: $\text{coordinates} - \text{actuators} \geq 2$
- **Symmetric**: 1 or more symmetries in the Lagrangian
- **Constrained**: subject to Virtual Holonomic Constraints

[4] "Simple Mechanical Control System", as defined in Bullo, Lewis *Geometric Control of Mechanical Systems*

# Virtual Holonomic Constraints

A **holonomic constraint** is a precise restriction on the position variables of a mechanical system[5].

It can always be expressed as a homogeneous function:

$$f(x_1, x_2, \ldots) = 0 \qquad\qquad f(x, y) := x^2 + y^2 - L^2 = 0 \tag{1}$$

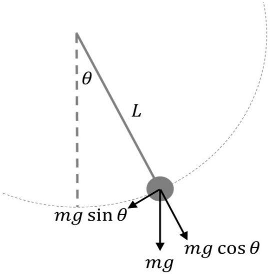


**Figure:** The pendulum's length $L$ is holonomically constrained[6]

[5] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*. Vol. 17. Springer Science & Business Media, 2013

[6] O. S. U. Anna Davis. *pendulumswing.jpg*. https://ximera.osu.edu/ode/main/simplePendulum/simplePendulum. 2013

# Virtual Holonomic Constraints

A **Virtual Holonomic Constraint** is similar, however, the constraint is applied through **feedback control**[7].

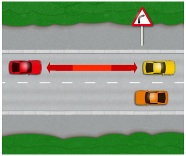

**Figure:** Vehicles on a highway[8]

- **Holonomic**: the cars maintain a fixed distance because they are mechanically linked with a chain
- **Virtual Holonomic**: the vehicles maintain a fixed distance because Red's adaptive cruise control actively tunes the throttle to always match Yellow's speed

[7] M. Maggiore and L. Consolini. "Virtual holonomic constraints for Euler–Lagrange systems". In: *IEEE Transactions on Automatic Control* 58.4 (2012), pp. 1001–1008

[8] D. T. Tips. https://www.drivingtesttips.biz/wp-content/uploads/2014/05/2-second-rule-300x250.jpg

# Methodology

A simple, underactuated, symmetric system subject to VHCs can be dynamically reduced to two independent "vertical" and "horizontal" vector fields, resulting in simpler control dynamics[9].

We compute and simulate the reduced dynamics of two such systems:

- Offset Robotic Arm
- Spherical Pendulum

[9] P. J. McCarthy, M. Maggiore, C. Nielsen, and L. Consolini. "A Noether Theorem and Symmetry Reduction for Forced Affine Connection Systems on 2-Manifolds". Preprint. 2020

# Offset Robotic Arm

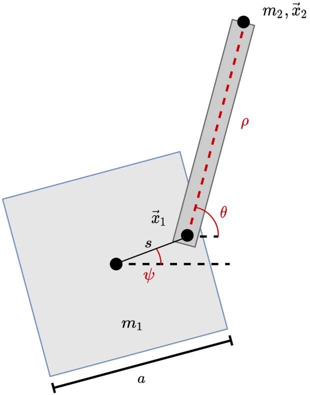


**Figure:** Offset robotic arm schematic diagram

- Degrees of freedom: rod rotates in $\theta$, block rotates in $\psi$
- Actuators: block actuates in $\psi$

# Offset Robotic Arm

After deriving the Lagrangian of the system, it was shown that the system is symmetric in $\theta - \psi$. Based on the symmetry, the chosen holonomic constraint is

$$h(\theta, \psi) := \theta - \psi = 0. \tag{2}$$

Under this constraint, the configuration manifold is

$$\mathcal{C} = \{\theta, \phi \in Q : \theta = \psi\} \tag{3}$$

This is parametrized and solved to give the constrained dynamics: 1 second-order DE.

# Offset Robotic Arm

The constrained dynamics are then reduced further.

- Use McCarthy's method to compute the "vertical" vector field (from the symmetry and the VHC)

$$V := \frac{\partial \Phi}{\partial g} \tag{4}$$

- Choose a "horizontal" vector field based on the tangent space of the constraint space.

$$H := \nabla_q h(q) \tag{5}$$

**Result**: the constrained dynamics can be expressed simply using the $V$ and $H$ vectors.

# Offset Robotic Arm

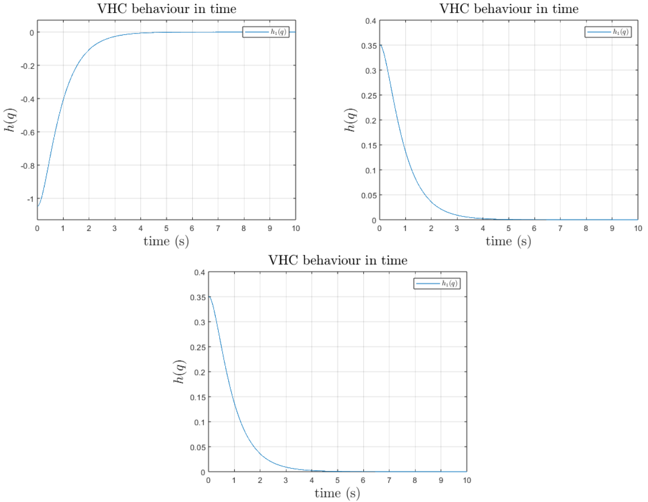


**Figure:** Plot showing convergence of the VHCs (control error) towards zero on the Offset Robotic Arm model. Here, the constraint is defined using the equivalent $h := \arg\left\{\exp\left(i[\theta - \psi]\right)\right\}$.

# Spherical Pendulum

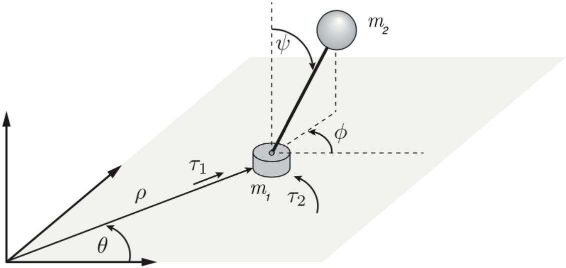


**Figure:** Spherical pendulum schematic diagram[10]

- Degrees of freedom: $\rho, \theta, \phi, \psi$
- Actuators: radial $\tau_1$, tangential $\tau_2$

[10] P. J. McCarthy, M. Maggiore, C. Nielsen, and L. Consolini. "A Noether Theorem and Symmetry Reduction for Forced Affine Connection Systems on 2-Manifolds". Preprint. 2020

# Offset Robotic Arm

The configuration manifold is $Q = \mathbb{R}_{>0} \times \mathbb{S}^1 \times \mathbb{S}^2$.
After deriving the Lagrangian of the system, it was shown that the system is symmetric in $\theta - \phi$. Based on the symmetry, the chosen holonomic constraint is

$$h(\rho, \theta, \phi, \psi) := \begin{pmatrix} \theta - \phi \\ \psi - \frac{\pi}{4} \tanh \rho \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6}$$

Under this constraint, the constrained configuration manifold is

$$\mathcal{C} = \left\{ \rho, \theta, \phi \psi \in Q : \theta = \phi, psi = \frac{\pi}{4} \tanh \rho \right\} \subseteq Q. \tag{7}$$

A similar process as above is taken to compute the constrained and reduced dynamics.

# Spherical Pendulum

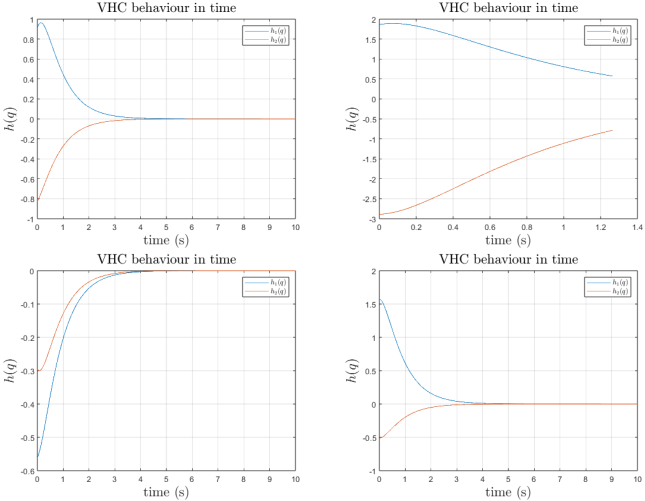


**Figure:** Plot showing convergence of the VHCs (control error) towards zero on the Spherical Pendulum model. Here, the constraint is defined $h_1 := \theta - \phi$, $h_2 = \psi - \frac{\pi}{4} \tanh \rho$.

# Conclusion

- Example symmetric mechanical systems were analyzed:
  - Derived symmetry and Euler-Lagrange dynamics
  - Selected appropriate VHCs according to the symmetry behaviour
  - Created models in Matlab to show convergence towards the VHC conditions
  - Computed the reduced dynamics
- McCarthy's dynamical reduction technique can prove useful in simplifying the dynamics of robotic applications subject to VHCs.

# Personal Takeaways

- Underactuated robotics blends areas in math, physics, and engineering
- Dive into the literature to learn the conventions and state of your field
- Documenting on-the-go is essential
- Understand the purpose and audience of the project, and work towards fulfilling that purpose for your audience

# Bibliography I


- Anna Davis, O. S. U. *pendulumswing.jpg*. https://ximera.osu.edu/ode/main/simplePendulum/simplePendulum. 2013.
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# Bibliography II


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