Inference from astrometry (in the first Gaia data release)

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MPIA, Heidelberg

Astrostats workshop, ESAC
29 November 2017

How do you learn most effectively?

Mentimeter

47

What do you enjoy most?

Mentimeter

47

Overview of the day

- Astrometry, Gaia, and the first Gaia data release
- Inferring distance from parallax

break

- Exercises
- Multiple stars and correlations
- Inferring distance to and size of a star cluster (plus Gaia archive)

lunch

- Exercises
- Hierarchical model
- Inferring distance and velocity from parallax and proper motion

break

- Exercises

Code for exercises

My original R code

https://github.com/ehalley/Astrostats-2017-ESAC

Alex Bombrun's python version

https://github.com/bombrun/AstrometryNotebooks

Astrometry and Gaia

Triangulation and parallax

Earth
and Sun

star of
interest

background
stars

Stellar parallaxes

- 1" ⇒ 1pc
  - 206 000 AU, 3.1×10^{13} km
- Alpha Centauri
  - parallax 0.75" (750 mas)
  - ~2500 times smaller than angular size of Moon (~ 0.5°)
- Stars in our Galaxy up to ~10^5pc (~0.01 mas)

Image credit: Babak Tafreshi (TWAN)

Parallax displacements are correlated

Parallax displacements

Linear motion in 3D space

rect.	angular coords
x	α right ascension
y	δ declination
z	ϖ parallax
V_x	μ_α proper motion in right ascension
V_y	μ_δ proper motion in declination
V_z	V_r radial velocity

Linear motion on sky seen by observer

Image credit: Hipparcos/ESA

Solving the astrometry

Asrometric accuracy

- parallax accuracy essentially independent of size of parallax (likewise for position and proper motion)
- astrometric uncertainties dominated by photon statistics for G > 12
- parallax (and PM) uncertainty: $\sigma(\varpi) \sim 1/\sqrt{\text{flux}}$
  - depends primarily on apparent magnitude and no. observations

Gaia spacecraft

Graphics: ESA, Astrium

Payload

Graphics: ESA, Astrium

Focal plane

Graphics: ESA, Astrium

Launch (19 Dec. 2013)

Earth-Sun L2 orbit

Gaia Data Release 1

Astrometry in GDR1

- Results of global astrometric solution
  - 2D positions of 1.1 billion sources down to G≃20.7 with a median positional precision of 2.4 mas

    the *secondary* set
  - 2D positions, 2D proper motions, and parallaxes of 2 million sources down to G≃12 with median precisions of 0.3 mas and 1.3 mas/yr

    the *primary set* or *TGAS* (Tycho-Gaia Astrometric Solution)
- *Caveat: this is a preliminary release based on incomplete processing*

TGAS principle

Gaia
data
(~2015)

TGAS principle

Astrometric solution

- Data: 5–25 Gaia transits over 14 (11) months
- Set prior on 5 astrometric parameters for 2 million primary sources
  - Tycho-2/Hipparcos positions and uncertainties
  - broad priors on parallaxes and PMs (Tycho-2/Hipparcos *not* used)
- Run iterative astrometric solution over primary set
  - get useful 5-parametric solutions
  - fix attitude and CCD parameters
- Apply solution to secondary set
  - only positions meaningful (with larger errors due parallax/PM neglect)

Limitations of GDR-1

- short time span and few observations
- stars assumed to have linear motions (e.g. no binaries)
- simplified modelling of satellite attitude
- no PSF calibration; no account of colour or CTI
- and others

TGAS - parallax precisions

Galactic coordinates, Aitoff, l=b=0 at centre, 0.84 deg^2 cells

2 million sources. 50% with G<11.0, 90% with G<12.1
median precision: 0.3 mas
systematic error: 0.3 mas

TGAS - proper motion precisions

Galactic coordinates, Aitoff, l=b=0 at centre, 0.84 deg^2 cells

2 million sources. 50% with G<11.0, 90% with G<12.1
median precision: 1.3 mas/yr
systematic error: 0.3 mas/yr

Some other aspects of the data

- Bright limit: G ~ 6 mag
- Photometric precision: 0.5–10 mmag for sources with 10 transits
- Maximum source density in secondary set is about 1 million/deg^2
  - minimum separation to detect equal G binary: 0.23" / 0.70”
- Some filtering of data
  - sources with too few observations or long data gaps
  - for TGAS: σ(parallax) > 1 mas or σ(pos) > 20 mas
  - for secondaries: σ(pos) > 100 mas or excess noise > 20 mas or N<5

Summary of Gaia data release 1

- positions for 1.1 billion sources, G<20.7, accuracy ~10 mas
- positions, proper motions, parallaxes for 2 million sources, G<12, median precision 0.3 mas and 1.3 mas/yr
  - accuracy limit ~0.3 mas and 0.1−0.3 mas/yr
  - averaging will *not* give you a $1/\sqrt{N}$ uncertainty reduction (local error correlations)
- http://www.cosmos.esa.int/web/gaia/dr1

Inferring distance from parallax

Parallax and its meaning

ϖ	parallax
σ_ϖ	parallax uncertainty
r	distance

$$\varpi/\mathrm{as} = \frac{1}{r/\mathrm{pc}}$$

$$\text{likelihood:} \quad P(\varpi \,|\, r, \sigma_\varpi) = \frac{1}{\sqrt{2\pi}\sigma_\varpi} \exp\left[-\frac{1}{2\sigma_\varpi^2}\left(\varpi - \frac{1}{r}\right)^2\right]$$

$$r = 500$$
$$\sigma_\varpi = 0.001$$

Parallax and its meaning

Intution: $\varpi \pm \sigma_\varpi \quad \rightarrow \quad r = \dfrac{1}{\varpi} \pm \dfrac{\sigma_\varpi}{\varpi^2}$

Clearly wrong, because 1sigma intervals project to

$$r = \left[\frac{1}{\varpi - \sigma_\varpi}, \frac{1}{\varpi}\right] \quad \text{and} \quad r = \left[\frac{1}{\varpi}, \frac{1}{\varpi + \sigma_\varpi}\right]$$

$r = 500$

$\sigma_\varpi = 0.001$

Estimating distance is an inference problem

likelihood

$$P(\varpi \,|\, r, \sigma_\varpi) \;=\; \frac{1}{\sqrt{2\pi}\sigma_\varpi} \exp\left[-\frac{1}{2\sigma_\varpi^2}\left(\varpi - \frac{1}{r}\right)^2\right]$$

posterior

$$P(r \,|\, \varpi, \sigma_\varpi) \;\propto\; P(\varpi \,|\, r, \sigma_\varpi)\, P(r)$$

$r = 100$

parallax

f = 0.1

f = 0.2

f = 0.5

$$f = \frac{\sigma_\varpi}{\varpi}$$

uniform prior $r > 0$ $\quad \varpi = 0.01$

distance

Issues in distance estimation (single object)

- What data do I use? [likelihood]
  - parallax, magnitude, colour, …
  - what assumptions do these entail (e.g. HRD)?
- What prior do I adopt?
- What estimators do I use to summarize the posterior?
  - mean, median, mode
  - standard deviation, quantiles, FWHM

Example distribution summaries

Possible distance priors

Prior PDF

uniform in r
uniform space density
exponentially decreasing space density

Test using simulations (GUMS)

Fractional distance error vs. fractional parallax uncertainty

Astraatmadja & CBJ 2016 a

Truncated uniform distance prior: posterior

$$\varpi = 0.01$$

different curves are for different values of f

Truncated uniform density prior: posterior

$$\varpi = 0.01$$

different curves are for different values of f

Exp. decreasing density prior: posterior

$$\varpi = 0.01$$

different curves are for different values of f

Exp. decreasing density prior: post. mode

$$\frac{r^3}{L} - 2r^2 + \frac{\varpi}{\sigma_\varpi^2} r - \frac{1}{\sigma_\varpi^2} = 0$$

lines labelled with $\log_{10}\varpi$

Milky Way prior

- Astraatmadja & CBJ 2016a,b for TGAS
- use MW star count model to determine intrinsic density of stars in 3D as function of absolute magnitude, i.e. $\rho(r, l, b \mid M_G)$
- use models of
  - stellar CMD
  - dust absorption
  - Gaia selection effects (magnitude limits)

to determine $P(r \mid l, b)$

Milky Way prior

Astraatmadja & CBJ 2016 a

Milky Way posteriors for ϖ = 1 mas

Astraatmadja & CBJ 2016 a

Fractional parallax uncertainty in TGAS

Astraatmadja & CBJ 2016 b

Parallaxes and parallax accuracy in TGAS

Astraatmadja & CBJ 2016 b

Key points

- Inverting parallaxes to estimate distances generally a bad idea
  - highly nonlinear (uncertainties)
  - we have extra constraints (positivity, physical information)
  - non-positive parallaxes contain information
- Priors play increasingly important role as quality of data degrades
- Often don't want to compute distances
  - no intermediate inference: go directly to quantities of interest
  - project models into astrometric space where possible (simpler noise model)

Resources

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DISTANCE

23

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I. DISTANCES OF TWO MILLION STARS
ATALOGUE

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Inferring distance from multiple parallaxes (and correlations)

Parallaxes of nearby sources are correlated

N-dimensional likelihood with correlations

$$P(\boldsymbol{\varpi}|\mathbf{r}, C) = \frac{1}{(2\pi)^{N/2}|C|^{1/2}} \exp\left[-\frac{1}{2}\left(\boldsymbol{\varpi} - \frac{1}{\mathbf{r}}\right)^{\mathsf{T}} C^{-1}\left(\boldsymbol{\varpi} - \frac{1}{\mathbf{r}}\right)\right]$$

$$\frac{1}{\mathbf{r}} = \left(\frac{1}{r_1}, \frac{1}{r_2}, \ldots\right)$$

$$\rho(\varpi_1, \varpi_2) = a \exp(-\theta/\Theta_{\mathrm{len}})$$

$$\theta = \left([(\alpha_1 - \alpha_2)\cos(0.5(\delta_1 + \delta_2))]^2 + (\delta_1 - \delta_2)^2\right)^{1/2}$$

θ is angular separation
(small angles, not at poles)

Inferring distance to and size of a stellar cluster

Star cluster (Pleiades)

Notation

$\mathcal{N}(x; \mu, V)$ Gaussian of mean μ, variance V

vectors are bold

$\mathbf{r}$ is 3D position vector

$$\frac{1}{\mathbf{r}} = \left(\frac{1}{r_1}, \frac{1}{r_2}, \ldots\right)$$

non-rigorous notation!

C is a covariance matrix

Simple model for a cluster

cluster with centre r_c
and isotropic
Gaussian distribution
with s.d. s_c

$$P(x_i|s_c) = \mathcal{N}(x_i; \mu = 0, \sigma = s_c)$$

Inference of cluster distance and size

$$P(r_c, s_c|\boldsymbol{\varpi}, C) = \frac{P(\boldsymbol{\varpi}|C, r_c, s_c)P(r_c, s_c)}{P(\boldsymbol{\varpi}|C)}$$

Bayes' theorem

$$P(\boldsymbol{\varpi}|C, r_c, s_c) = \int P(\boldsymbol{\varpi}, \mathbf{r}|C, r_c, s_c)d\mathbf{r}$$

marginalize over unknown true distances

$$= \int P(\boldsymbol{\varpi}|\mathbf{r}, C)P(\mathbf{r}|r_c, s_c)d\mathbf{r}$$

prod. rule & cond. ind.

$$P(\boldsymbol{\varpi}|\mathbf{r}, C) = \mathcal{N}(\boldsymbol{\varpi}; \frac{1}{\mathbf{r}}, C)$$

N-dim. Gaussian likelihood

$$P(\mathbf{r}|r_c, s_c) = \mathcal{N}(\mathbf{r}; \mathbf{r}_c, \mathrm{diag}(s_c^2))$$

Gaussian cluster model

Inference of cluster distance and size

approximation: parallax measurements are independent

$$C = \mathrm{diag}(\boldsymbol{\sigma}_{\varpi}^2)$$

can now write likelihood as

$$P(\boldsymbol{\varpi}|C, r_c, s_c) = \prod_i \int_0^\infty P(\varpi_i|r_i, \sigma_{\varpi i}) P(r_i|r_c, s_c) dr_i$$

here swapped order of product and integral
because the integrals are independent

Inference of cluster distance and size

using cosine rule can write each likelihood as

$$P(\varpi_i|\sigma_{\varpi i}, r_c, s_c) = \int_0^\infty P(\varpi_i|r_i, \sigma_{\varpi i}) P(x_i|s_c) \left(\frac{r - r_c \cos\theta}{x} \right) dr_i$$

General approach to inference problems

- identify data and parameters: what do you actually want to infer?
- identify generative model (predicts data given parameters)
- identify measurement model $\longrightarrow$ likelihood
- identify assumptions/background knowledge $\longrightarrow$ prior
- manipulation tools
  - Bayes' theorem
  - marginalization and product rule to introduce known quantities
  - conditional independence to remove terms
  - Jacobians to convert PDFs to be over different variables

Open cluster analysis in GDR1

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Astronomy & Astrophysics
Special issue

Gaia Data Release 1

Gaia Data Release 1

Open cluster astrometry: performance, limitations, and future prospects★

Gaia Collaboration, F. van Leeuwen[1], A. Vallenari[2], C. Jordi[3], L. Lindegren[4], U. Bastian[5], T. Prusti[6], J. H. J. de Bruijne[6], A. G. A. Brown[7], C. Babusiaux[8], C. A. L. Bailer-Jones[9], M. Biermann[5], D. W. Evans[1], L. Eyer[10], F. Jansen[11], S. A. Klioner[12], U. Lammers[13], X. Luri[3], F. Mignard[14], C. Panem[15], D. Pourbaix[16,17], S. Randich[18], P. Sartoretti[8], H. I. Siddiqui[19], C. Soubiran[20], V. Valette[15], N. A. Walton[1], C. Aerts[21,22], F. Arenou[8], M. Cropper[23],

(589 authors)

D.6. The cluster IC 2391

Table D.6. Identifiers and positions for members of the cluster IC 2391.

SourceId	HD	α (degr)	δ (degr)	G	SourceId	HD	α (degr)	δ (degr)	G
5513364924200650752	68276	122.3267	–51.0123	7.620	5317906150880166528	74374	130.3446	–53.6358	9.367
5319716806015487488	70560	125.0151	–53.9216	9.253	5318499818442087936	74516	130.5413	–52.9676	7.395
5515719116036643072		126.4811	–47.5548	11.003	5317887321743547264	74561	130.5757	–53.9022	9.236
5515330266875244544		127.0046	–48.6839	9.923	5317884435525524224		130.7515	–53.9020	10.841
5321517668619002240		127.1900	–52.0907	10.247	5318486074546754048	74678	130.7643	–53.0779	7.664
5316373981428187904	72323	127.3967	–55.4168	7.703	5318536995675325952	74714	130.8244	–52.6030	9.014
5515501481453454208	72516	127.8404	–47.4435	8.627	5318702128577954816	74734	130.8866	–52.0037	7.881
5321225473403915136		128.5853	–52.8346	10.115	5318316096918509568		130.9678	–53.2332	9.579
5318069600154320384		129.1009	–54.0182	10.000	5318512256667367040		131.0216	–52.8880	10.546
5322839178517948800	73462	129.1030	–50.2554	9.325	5316501971457325952		131.0555	–57.2547	11.133
5321190151592875904	73777	129.4459	–52.8700	9.510	5329039874346093056		131.0715	–48.5083	9.301
5318077502894130944		129.4647	–53.7626	11.159	5318521671232021632		131.1088	–52.7089	11.214
5318077915210988928	73904	129.5995	–53.7217	7.658	5318630900840369152		131.4129	–52.4331	9.714
5321582711603786880	74044	129.8187	–52.3137	8.442	5318296271349488640		131.4496	–53.4306	10.048
5318113546259668864	74071	129.8492	–53.4397	5.539	5318229269859699072		131.5635	–53.7562	10.190
5321795845062877312		129.9808	–51.3655	10.042	5317423293481147264		131.8926	–54.4835	11.827
5318117325830884224	74169	129.9972	–53.2609	7.241	5318412682141952512	75466	132.0007	–52.8501	6.305
5318567953805446400	74145	130.0067	–52.7034	8.468	5317551352220546048	76472	133.5550	–53.3850	9.393
5318096125872352768		130.0258	–53.6352	10.255	5325336375583055232		134.0447	–49.4907	10.068
5318554106825052160	74275	130.2020	–52.8018	7.294	5317261802706432000	76840	134.1002	–54.3225	9.186
5318093239654329472		130.2044	–53.6292	10.773	5303472346348134656		135.5162	–58.1471	11.095
5317797574106967552		130.2551	–54.5172	10.279					

results

Name ClustId	α (degr) δ (degr)	ϖ σ_ϖ	$\mu_{\alpha*}$ $\sigma_{\mu\alpha*}$	μ_δ $\sigma_{\mu\delta}$	c_{12} c_{13}	c_{23} r(max)°	nMemb st.dev.
Hyades	66.85	21.39	104.92	-28.00	0.	0.	103
C0424+157	17.04	0.21	0.12	0.09	0.	17.2	1.00
ComaBer	185.983	11.73	-12.14	-8.90	0.14	0.32	50
C1222+263	26.093	0.05	0.14	0.16	-0.07	9.42	0.86
Pleiades	56.438	7.48	20.38	-45.39	-0.17	-0.03	152
C0344+239	23.844	0.03	0.07	0.08	0.02	6.49	1.10
Praesepe	130.081	5.47	-36.06	-13.15	0.40	0.06	79
C0937+201	19.675	0.05	0.07	0.08	-0.05	4.36	1.09
alphaPer	52.069	5.91	23.06	-25.36	-0.29	-0.01	116
C0318+484	49.060	0.03	0.06	0.07	0.09	5.11	1.04
IC2391	130.064	7.01	-24.35	23.76	0.53	-0.04	44
C0838-528	-52.919	0.11	0.14	0.22	-0.01	6.07	1.60
IC2602	159.809	6.74	-17.67	11.06	0.36	-0.19	66
C1041-641	-64.496	0.05	0.09	0.13	0.05	5.35	1.18
Blanco 1	0.855	4.34	18.20	2.66	-0.46	0.07	43
C0001-302	-30.079	0.11	0.12	0.11	-0.44	3.77	1.41
NGC2451A	115.799	5.59	-21.82	15.59	0.63	0.13	37
C0743-378	-38.579	0.11	0.11	0.16	-0.01	4.70	1.38
NGC6475	268.530	3.57	3.10	-5.32	0.01	0.37	78
C1750-348	-34.849	0.02	0.06	0.04	0.10	2.37	0.62
NGC7092	323.437	2.99	-7.34	-19.94	-0.58	0.09	23
C2130+482	48.438	0.12	0.11	0.13	0.21	2.14	1.30
NGC2516	119.438	2.99	-4.07	11.21	0.77	-0.08	87
C0757-607	-60.688	0.08	0.06	0.06	0.05	2.66	1.27
NGC2232	97.149	3.00	-4.34	-1.71	0.32	-0.32	31
C0624-047	-5.111	0.06	0.10	0.08	-0.06	2.64	0.86
IC4665	266.618	2.83	-0.78	-8.37	-0.12	0.19	16
C1743+057	5.583	0.05	0.07	0.06	0.09	1.92	0.60
NGC6633	276.900	2.37	1.45	-1.77	-0.36	0.44	47
C1825+065	6.698	0.03	0.05	0.04	-0.03	1.83	0.63
Coll140	111.127	2.70	-8.02	4.88	0.69	-0.14	30
C0722-321	-32.183	0.08	0.07	0.07	-0.23	2.61	0.84
NGC2422	114.139	2.27	-6.89	0.90	0.59	-0.25	34
C0734-143	-14.156	0.07	0.07	0.07	-0.33	1.83	0.83
NGC3532	166.270	2.42	-10.54	5.19	0.61	-0.04	128
C1104-584	-58.753	0.03	0.03	0.04	0.26	2.19	0.78
NGC2547	123.079	2.79	-8.70	4.16	0.63	-0.01	40
C0809-491	-49.230	0.06	0.06	0.07	-0.11	2.38	0.91

Gaia archive

https://gea.esac.esa.int/archive/

gaiadr1.tgas_source

https://gea.esac.esa.int/archive/

→ EUROPEAN SPACE AGENCY ABOUT ESAC Coryn Bailer-Jones (cbailerj)

gaia archive esa

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hip	tycho2_id	solution_id	source_id	random_index	ref_epoch (Time[Julian Years])	ra (deg)	ra_error (Angle[mas])	dec (deg)	dec_error (Angle[mas])	parallax (Angle[mas])
	1239-555-1	1635378410781933568	4443720911387225 6	644430	2015.0	54.4539656791039	0.2774889696181444	17.83844542369319	0.1898546377428496	3.390413981
	648-424-1	1635378410781933568	1405107140865472 0	1759009	2015.0	48.03784643783192	0.2277959105344771 8	9.296703175911553	0.1731528612334293	3.808440838
	657-117-1	1635378410781933568	1195406221631795 2	1199910	2015.0	54.87692163395508	0.4275747356508141 6	9.336504770280854	0.2047213748360375	8.578651224
	1255-780-1	1635378410781933568	4767314055319180 8	413082	2015.0	63.630601976854074	0.3573790876721525	18.429802420265247	0.2512296985607489	1.407662947
	62-1194-1	1635378410781933568	3956420793878272	186001	2015.0	47.95976020257937	0.209417031268184	5.123588099630956	0.4833351663668273	3.584772840
	62-1301-1	1635378410781933568	3974494016256000	1857965	2015.0	47.842745763563144	0.1353752142873221	5.309319468445373	0.3578946641429233	4.715483097
	62-854-1	1635378410781933568	3987241479189760	928982	2015.0	48.435421127602886	0.3169949704327086	5.350176818688011	0.31668887539401164	1.834543257
	62-1249-1	1635378410781933568	4000469978459904	464491	2015.0	47.99511095238111	0.1619524101365146	5.433935186823232	0.4103148161545495	3.796396857
	61-937-1	1635378410781933568	3886464366552960	1195199	2015.0	46.2944654992699	0.23335540816550918	5.470902829300608	0.20923458219144236	4.267537362
	62-1166-1	1635378410781933568	3993185713923968	1449102	2015.0	48.32940711710609	0.18266277083761928	5.483694715142871	0.17203699018439841	1.814794712
	58-475-1	1635378410781933568	3767304794045824	724551	2015.0	45.77320771034733	0.21248517744105372	4.8982016962026655	0.15489044971099786	1.437898159
	61-898-1	1635378410781933568	3919896391979648	1774363	2015.0	46.47871037630939	0.32584050435200496	5.668903013612662	0.2050071181135407	2.765401754
	61-1037-1	1635378410781933568	3940546594734720	443590	2015.0	46.41958750383744	0.2588930871671032	5.961213959645435	0.22547329616489936	5.065070446
	61-908-1	1635378410781933568	3928520686308992	221795	2015.0	46.18728607847513	0.29411247119150297	5.706110790085455	0.1964329220321841	1.862629478
	62-1065-1	1635378410781933568	3968137465028096	110897	2015.0	47.76381286845869	0.25909336656495446	5.126881785934897	0.6115593700160574	2.187232399
	635-878-1	1635378410781933568	2435153685553190 4	840838	2015.0	37.35270433354835	0.3987100534641945	10.375500915309024	0.21694859070371758	3.187185421
	635-927-1	1635378410781933568	2425433315572390 4	420419	2015.0	37.14874061068336	0.5915180098424486	10.194042690538478	0.36305170367679473	4.980414409
	631-526-1	1635378410781933568	2385888078867699 2	210209	2015.0	35.53425286304 12	0.4464504511530576	9.454478229266465	0.32103120020634596	1.997630676

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Uncertainty and error

A hierarchical model

A hierarchical model

- put prior on length scale parameter, then marginalize over it

Inferring distance and velocity from parallax and proper motion

3D transformation

$$\mathbf{x} = (\varpi,\ \mu_{\alpha^*},\ \mu_\delta)^\mathsf{T}$$

measurements

$$\boldsymbol{\theta} = (r,\ v,\ \phi)^\mathsf{T}$$

parameters

$$\mathbf{m} = \left(\frac{10^3}{r},\ \frac{10^3}{c_2}\frac{v\sin\phi}{r},\ \frac{10^3}{c_2}\frac{v\cos\phi}{r}\right)^\mathsf{T}$$

generative model

$$\boldsymbol{\theta} \to \mathbf{x}$$

$$c_2 = \mathrm{AU\,km^{-1}\,yr^{-1}} = 4.74047$$

Classical estimate of parameters & variance

inverse of generative model

$\mathbf{x} \rightarrow \boldsymbol{\theta}$

$$\boldsymbol{\theta}_{\mathrm{nom}} = \left(\frac{10^3}{\varpi},\ c_2 \frac{\sqrt{\mu_{\alpha^*}^2 + \mu_\delta^2}}{\varpi},\ \arctan\left(\frac{\mu_{\alpha^*}}{\mu_\delta}\right) \right)^{\mathsf{T}}$$

$$C_{\boldsymbol{\theta}} = J\, C_x\, J^{\mathsf{T}}$$

$$J = \left(\frac{\partial \boldsymbol{\theta}}{\partial \mathbf{x}}\right) = \begin{pmatrix} \frac{\partial r}{\partial \varpi} & \frac{\partial r}{\partial \mu_{\alpha^*}} & \frac{\partial r}{\partial \mu_\delta} \\ \frac{\partial v}{\partial \varpi} & \frac{\partial v}{\partial \mu_{\alpha^*}} & \frac{\partial v}{\partial \mu_\delta} \\ \frac{\partial \phi}{\partial \varpi} & \frac{\partial \phi}{\partial \mu_{\alpha^*}} & \frac{\partial \phi}{\partial \mu_\delta} \end{pmatrix} = \begin{pmatrix} -10^3 \frac{1}{\varpi^2} & 0 & 0 \\ -c_2 \frac{\mu}{\varpi^2} & c_2 \frac{\mu_{\alpha^*}}{\varpi \mu} & c_2 \frac{\mu_\delta}{\varpi \mu} \\ 0 & \frac{\mu_\delta}{\mu^2} & -\frac{\mu_{\alpha^*}}{\mu^2} \end{pmatrix}$$

$$\mu^2 = \mu_{\alpha^*}^2 + \mu_\delta^2$$

Inference

$$P(\boldsymbol{\theta}|\mathbf{x}) = \frac{1}{Z} P(\mathbf{x}|\boldsymbol{\theta}) P(\boldsymbol{\theta})$$

$$P(\mathbf{x}|\boldsymbol{\theta}) = \frac{1}{(2\pi)^{3/2}|C_x|^{1/2}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{m}(\boldsymbol{\theta}))^{\mathsf{T}} C_x^{-1} (\mathbf{x}-\mathbf{m}(\boldsymbol{\theta}))\right]$$

$$P(\boldsymbol{\theta}) = P(r)P(v)P(\phi)$$

$$P(r) = \begin{cases} \frac{1}{2L^3} r^2 e^{-r/L} & \text{if} \quad r > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(v) = \begin{cases} \frac{1}{B(\alpha,\beta)} \left(\frac{v}{v_{\max}}\right)^{\alpha-1} \left(1 - \frac{v}{v_{\max}}\right)^{\beta-1} & \text{if} \quad 0 \leq v \leq v_{\max} \\ 0 & \text{otherwise} \end{cases}$$

$$P(\phi) = \frac{1}{2\pi}$$

Sample posterior (emcee used in my code)

Inference of all six Galactic coordinates

TGAS + RAVE data
TYC 6676-44-1

Burger 2017 (BSc thesis, Heidelberg)