ME 388F Homework 2

Erik Hansen

January 30, 2025

1 1D, Mono-energetic, Purely-Absorbing Transport Derivations

(a) Definitions:

- μ is the cosine of the angle between a neutron's direction and the positive x axis.
- x (units cm) is the length measured from an arbitrary point along the non-homogeneous dimension of the system.
- ψ (units $\frac{\text{cm}}{\mu\text{cm}^3\text{MeVs}}$) is the angular flux of neutrons integrated over the angle around the x-axis.
- Σ_t (units $\frac{1}{\text{cm}}$) is the collision cross section of the medium the neutrons pass through. A collision would cause the neutron to change its direction or energy.
- Q (units $\frac{\text{cm}}{\mu\text{cm}^3\text{MeVs}}$) is an external source of neutrons with direction μ at this energy, integrated over azimuthal directions.
- ψ_i is the angular flux at the specified point x_i.
- x_i is the position that determines the boundary condition for this problem. The constraint $\psi(x_i) = \psi_i$ determines the angular flux at all other locations.

(b) First, we note that

$$\psi(x) = \frac{Q}{\Sigma_t} + (\psi_i - \frac{Q}{\Sigma_t})e^{-\frac{\Sigma_t}{\mu}(x-x_i)} \tag{1}$$

and the antiderivative of $\psi(x)$ is

$$f(x) = \int \psi(x)dx = \frac{Q}{\Sigma_t}x - \frac{\mu}{\Sigma_t}(\psi_i - \frac{Q}{\Sigma_t})e^{-\frac{\Sigma_t}{\mu}(x-x_i)} + C = \frac{Q}{\Sigma_t}x - \frac{\mu}{\Sigma_t}\left(\psi(x) - \frac{Q}{\Sigma_t}\right) + C \tag{2}$$

where C is an integration constant. This implies that

$$\int_{x_i}^{x_e} \psi(x)dx = \frac{Q}{\Sigma_t}(x_e - x_i) - \frac{\mu}{\Sigma_t}\left(\psi(x_e) - \psi(x_i)\right) \tag{3}$$

and the average value is

$$\langle\psi\rangle = \frac{1}{x_e - x_i}\int_{x_i}^{x_e} \psi(x)dx = \frac{Q}{\Sigma_t} - \frac{\mu}{\Sigma_t(x_e - x_i)}\left(\psi(x_e) - \psi(x_i)\right) \tag{4}$$

(c) This solution is valid for solutions with negative μ because it was never assumed that $x_e > x_i$. If μ were negative, then $x_e - x_i$ would also be negative, and so the exponent in the expression for $\psi(x)$ would be the same sign. So in either case, $\psi(x_e)$ exhibits the same pattern relative to the boundary condition $\psi(x_i)$.

2 1D, Mono-energetic, Purely-Absorbing, Source-Free Transport Coding

Note that all code used is provided in the s25me388f_hansen repository on GitHub.

(a) If we were to solve for $\mu = 0$, in the transport equation we would find

$$\mu \frac{\partial \psi}{\partial x} + \Sigma_t \psi = Q = \Sigma_t \psi = 0 \tag{5}$$

which would imply that $\psi = 0$ for $\mu = 0$. So in this system there are no neutrons moving perpendicular to the x axis.

(b) With this equation, if we use a forward or backward difference scheme, the equation for the angular flux at each node will only depend on the node before it or the node after it (respectively). We place our left boundary condition for ψ_{1+} at $x_b = 0$ our first equation, so that our first pivot is one and the rest of our first row is zeros. As a result, the matrix in the $Ax = b$ expression for this system is upwind. The vector b in this case will the prescribed value of ψ_{1+} followed by all zeros, given there are no sources.

(c) In hw2.py in the hw2 folder on GitHub, please find the solver for the positive flux in the SourceFree1D class method problem23_forwardflux_banded. To explore Sree Gudala's Canvas discussion post, I used a backwards difference method so my setup for each non-initial node was

$$\mu_r \frac{\psi_{i+1} - \psi_i}{X/N} + \Sigma_t \psi_{i+1} = 0 \tag{6}$$

$$-\psi_i + (1 + \frac{\Sigma_t X}{\mu_r N})\psi_{i+1} = 0 \tag{7}$$

where X is the length scale of our problem and $N = N_x - 1$ is the number of cells, and to be consistent with the lecture note definition, $\tau_r = \frac{\Sigma_t X}{\mu_r N}$. Given the main diagonal solves for ψ_{i+1}, I used a banded matrix structure with $1 + \tau_r$ on the main diagonal and -1 on the subdiagonal to calculate the solution. The average flux inside a cell is calculated using the formula derived in Problem 1 for neighboring cells except in the case with $\Sigma_t = 0$, where we use the mean of the average flux of the two neighboring cells.

(d) In contrast to the positive case, we place the right boundary condition at the last row in our matrix equation. As a result, given that this matrix system recursively solves for leftward elements, the matrix is now downwind - all subdiagonals are zero, and only the main and first superdiagonal are nonzero. The vector b is now a list of zeros except for the last element which is equal to $\psi(X, \mu)$.

(e) See the method problem25_backwardflux. We used the same difference scheme to set up our equation, but in this case we solve for ψ_i instead of ψ_{i+1}, so we used a banded matrix with 1 on the main diagonal and $-(1+\tau_l)$ on the superdiagonal, where $\tau_l = \frac{\Sigma_t X}{\mu_l N}$.

(f) We compute the scalar flux in the method problem26_scalarflux by adding the rightward flux average and the leftward flux average.

3 Code Application:

See plots and captions below. All cases are constructed with a box length equal to one and 10000 cells.

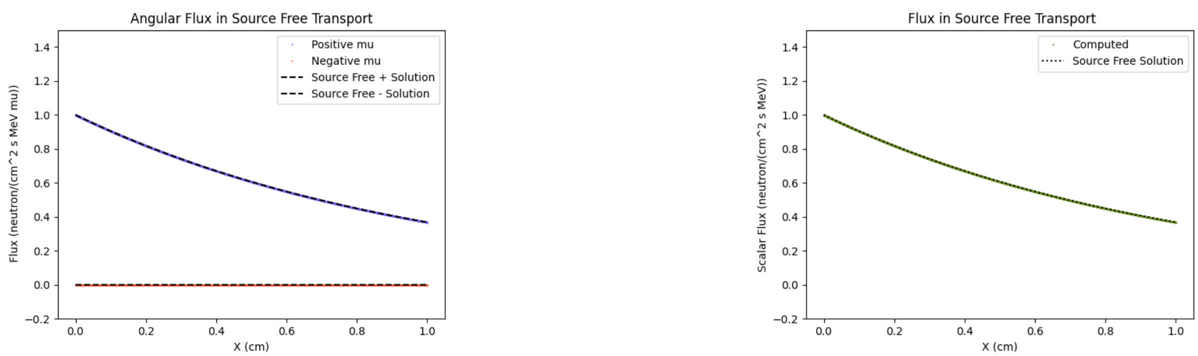

Figure 1: Problem 3a: $\mu_R = 1, \mu_L = -1, \psi(0,\mu_R) = 1, \psi(X,\mu_L) = 0, \Sigma_t = 1$. We see the predicted exponential decay of the forward flux with no backwards flux in this case.

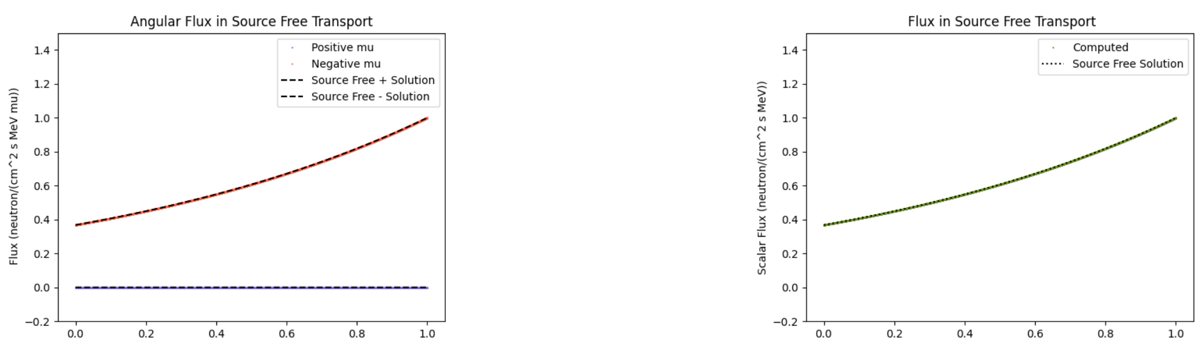

Figure 2: Problem 3b: $\mu_R = 1, \mu_L = -1, \psi(0,\mu_R) = 0, \psi(X,\mu_L) = 1, \Sigma_t = 1$. In this case, we see zero forward flux and an exponential decay as the backward flux passes through the material, opposite to the previous case.

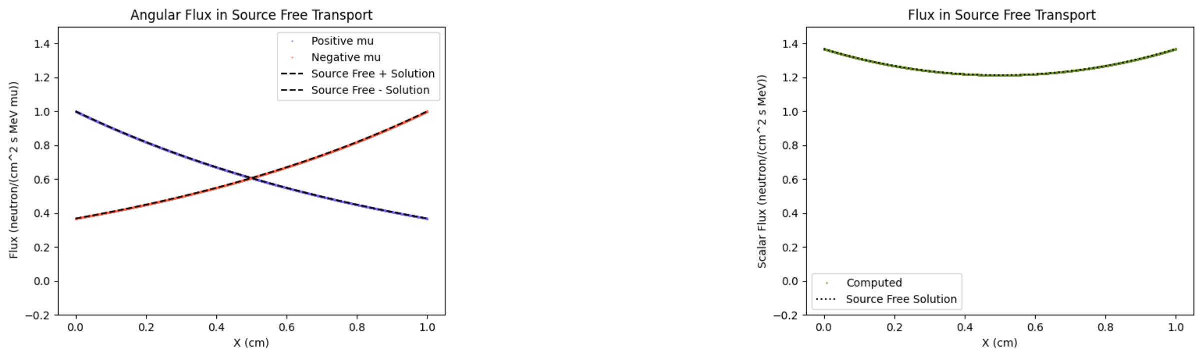

Figure 3: Problem 3c: $\mu_R = 1, \mu_L = -1, \psi(0, \mu_R) = 1, \psi(X, \mu_L) = 1, \Sigma_t = 1$. In this case, we see both angular fluxes from 3a) and 3b), and the scalar flux is the sum of the two fluxes.

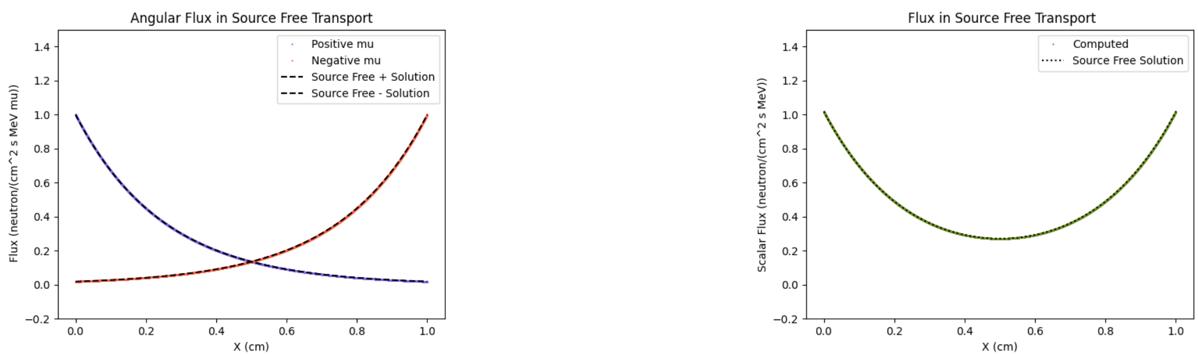

Figure 4: Problem 3d: $\mu_R = 0.25, \mu_L = -0.25, \psi(0, \mu_R) = 1, \psi(X, \mu_L) = 1, \Sigma_t = 1$. When the radiation is directed at an angle to the forward and backward directions, there is a much faster dropoff in the angular fluxes through the material. This makes sense because as μ decreases, the decay rate $|\tau|$ increases.

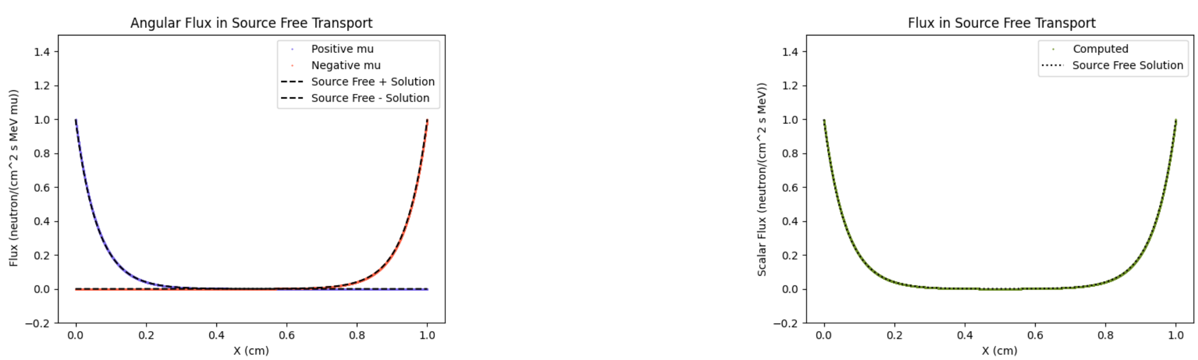

Figure 5: Problem 3e: $\mu_R = 0.25, \mu_L = -0.25, \psi(0, \mu_R) = 1, \psi(X, \mu_L) = 1, \Sigma_t = 4$. With a larger cross section, the decay rate increases even further, and the angular flux is dissipated even faster in both directions.

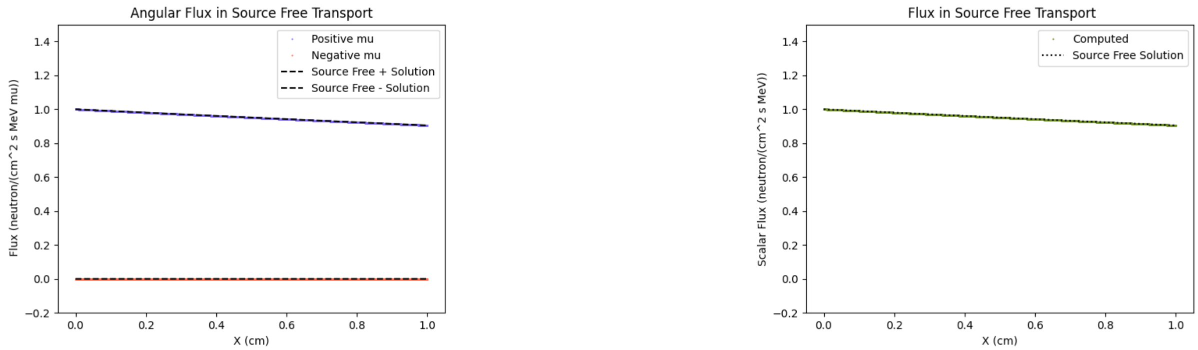

Figure 6: Problem 3f: $\mu_R = 1.0, \mu_L = -1.0, \psi(0, \mu_R) = 1, \psi(X, \mu_L) = 0, \Sigma_t = 0.1$. Compared to the first problem, we still see no backward flux and a decaying forward flux, but the diminished cross section causes a much smaller decay rate.

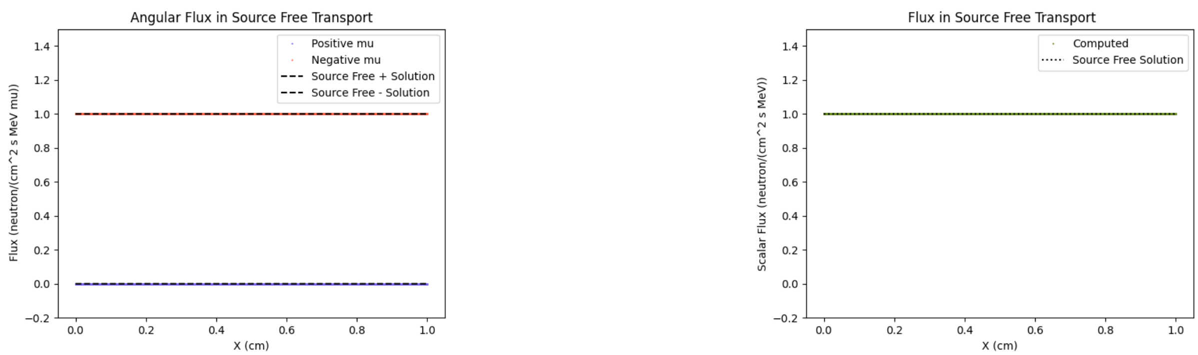

Figure 7: Problem 3f: $\mu_R = 1.0, \mu_L = -1.0, \psi(0, \mu_R) = 0, \psi(X, \mu_L) = 1, \Sigma_t = 0$. There was actually an issue in this problem because τ goes to zero in this limit when coding the solution. However, in this case we note that the differential equation reads $\mu \frac{d\psi}{dx} = 0$, indicating that the flux must be constant, which happens here.

4 1D, Mono-energetic, with Scattering Transport Coding

(a) We implement the scattering initial conditions in the same way as in the no source case. We faced an issue coding this method where we only have the average angular flux and source in each cell and it's undefined whether to use the value in a cell to the right or left of the given node to compute the angular fluxes at each node. The solutions provided assume that the rightward flux feels the source to the left of a given node and the leftward flux feels the source to the right of the given node. This has a physical motivation with the direction where the neutron source came from. For example, the source of rightward neutrons at node two must be the cell between nodes one and two and the source of leftward neutrons at node two must be the cell between nodes two and three.

We applied a matrix method of solution in the problem4_matrix and in the same place as the Problem 2 and 3 code for a matrix-based solution. To do this, we wrote our source in the elements of the vector b except for the initial condition, and applied the same matrix solver as previously.

To apply the source iteration method, we assume at first that the source is zero and compute the average angular flux as above. After this, we compute the average angular flux and the source $Q = \frac{1}{2}\Sigma_s \langle \phi \rangle$, and redo the calculation until the average angular flux between two successive iterations agrees to within 10^{-10} at each node. This took about 10 to 12 iterations per problem.

As a way of checking the code, we performed the calculation using the recursion relations given in the problem4_given and problem4_given_iteration methods. These last methods do not support $\Sigma_t = 0$.

(b) See below for comparisons with the no-scattering solutions. In all cases, we took $\Sigma_s = 0.1$ as directed, and we provide a separate plot of an analytical solution computed using Mathematica (see last section).

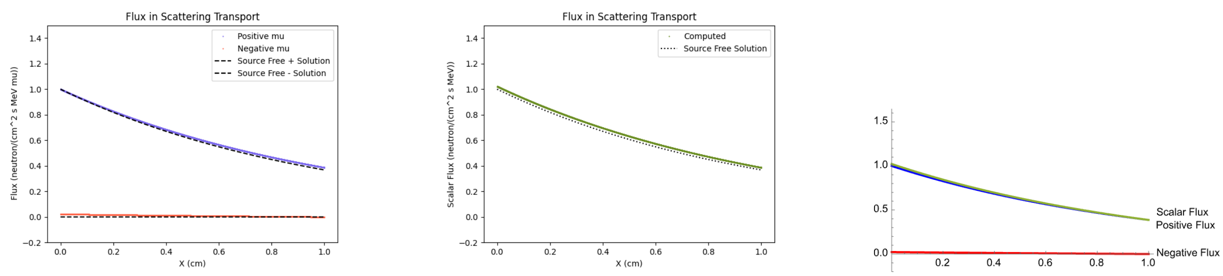

Figure 8: Problem 4b-a: $\mu_R = 1.0, \mu_L = -1.0, \psi(0,\mu_R) = 1, \psi(X,\mu_L) = 0, \Sigma_t = 1$. With the presence of scattering, we see that the backward flux is no longer always zero but instead exhibits exponential decay to zero. This makes sense because some neutrons are reflected, and with an exponentially decreasing flux of positive neutrons, we would expect the reflected neutron flux to exponentially decrease. Furthermore, the forward angular flux is slightly higher because reflected neutrons may reflect again.

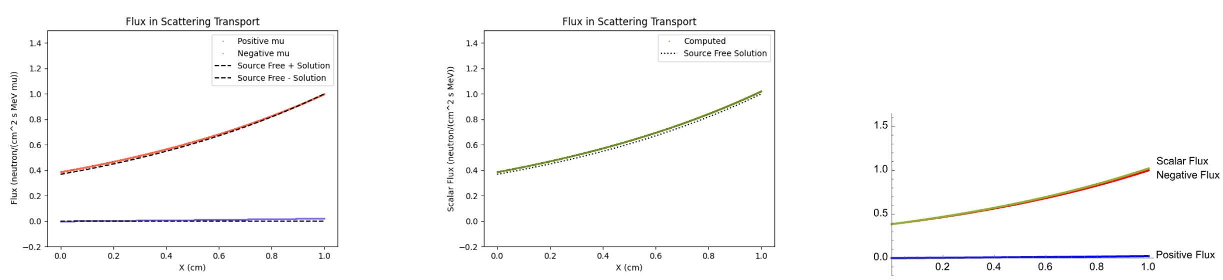

Figure 9: Problem 4b-b: $\mu_R = 1.0, \mu_L = -1.0, \psi(0,\mu_R) = 0, \psi(X,\mu_L) = 1, \Sigma_t = 1$. We see a similar pattern as in the previous case, where we have nonzero forward flux and slightly higher backward flux.

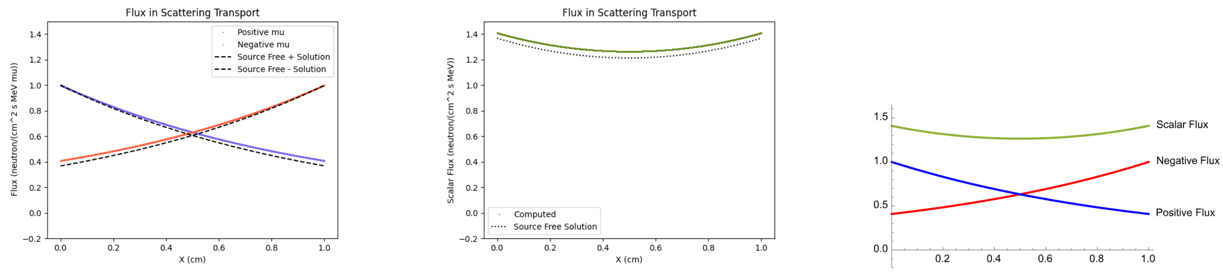

Figure 10: Problem 4b-c: $\mu_R = 1.0, \mu_L = -1.0, \psi(0,\mu_R) = 1, \psi(X,\mu_L) = 1, \Sigma_t = 1$. We see the same pattern as the above cases represented here, where neither angular flux decays as much as it did in the source free case because of reflection present.

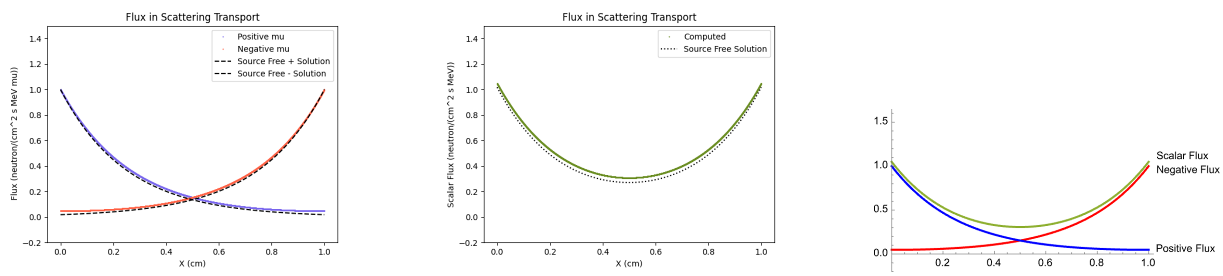

Figure 11: Problem 4b-d: $\mu_R = 0.25, \mu_L = -0.25, \psi(0,\mu_R) = 1, \psi(X,\mu_L) = 1, \Sigma_t = 1$. We see the same differences in this plot as we do in Problem 4b-c.

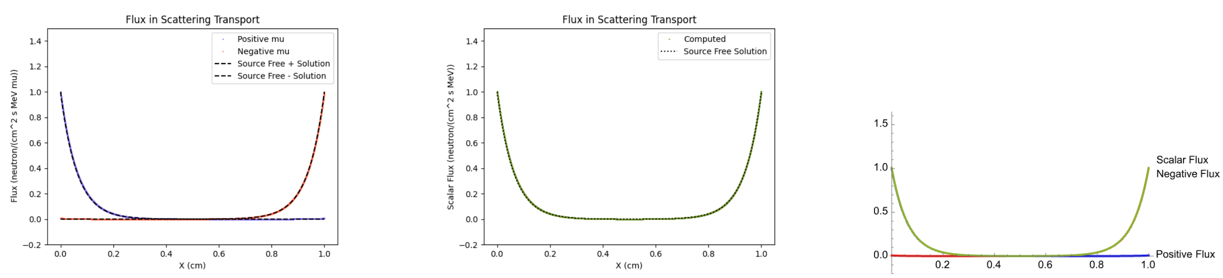

Figure 12: Problem 4b-e: $\mu_R = 0.25, \mu_L = -0.25, \psi(0,\mu_R) = 1, \psi(X,\mu_L) = 1, \Sigma_t = 4$. Here we continue to see the large exponential decay as Problem 3e. However, it does seem like some scattering is notable at the edges of the domain. (Note that plots continue on next page).

5 References

I discussed this homework with Max Hoffing, Sree Gudala, and Alex Macris. I used the Wolfram Alpha Documentation for Mathematica to assist with displaying my analytical solution and the SciPy LinAlg package solve_banded Documentation to write my solver, in addition to the homework instructions and class slides.

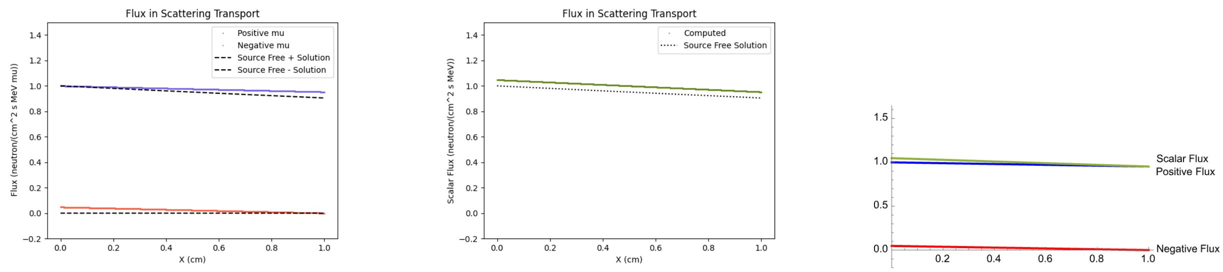

Figure 13: Problem 4b-f: $\mu_R = 1.0, \mu_L = -1.0, \psi(0, \mu_R) = 1, \psi(X, \mu_L) = 0, \Sigma_t = 0.1$. Because the exponential decay is reduced due to the lower transport cross section, less neutrons escape and reflect, which causes a greater increase in the forward and backward neutron angular fluxes relative to the higher transport cross section case considered in Problem 4a.

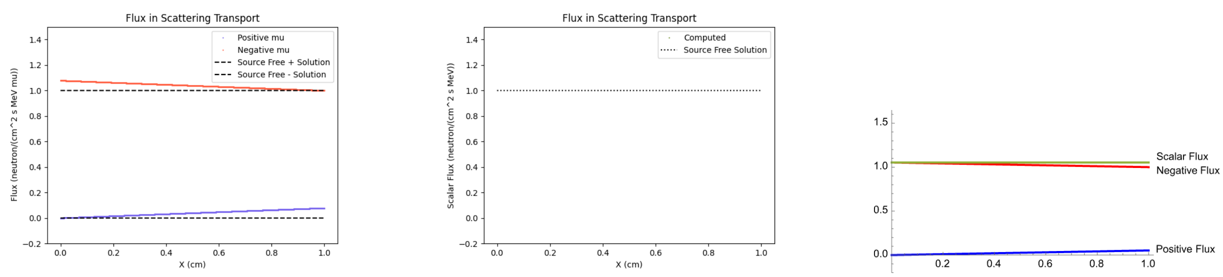

Figure 14: Problem 4b-g: $\mu_R = 1.0, \mu_L = -1.0, \psi(0, \mu_R) = 0, \psi(X, \mu_L) = 1, \Sigma_t = 0$. As before, there is no transport cross section in this case. However, scattering causes both the positive and negative fluxes to increase in their direction of travel. This happens in such a way that the scalar flux is constant across the medium given that the backward and forward fluxes have the same source. (I can't explain this bit physically, but as could be solved analytically below, the increase along the direction of travel is linear with distance).

6 Scattering Analytical Solutions

Using the given source, and given only the two opposite directions that we consider here, we obtain a transport equation

$$\mu \frac{\partial \psi_R}{\partial x} + \Sigma_t \psi_R = \frac{1}{2} \Sigma_s (\psi_R + \psi_L) \tag{8}$$

$$-\mu \frac{\partial \psi_L}{\partial x} + \Sigma_t \psi_L = \frac{1}{2} \Sigma_s (\psi_R + \psi_L) \tag{9}$$

This equation is solved in Mathematica for the given initial conditions and parameter values.