ME 388F Homework 3

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1 Description of Code

I implemented a discrete ordinates S_N and spherical harmonics for two moments P_2 solver in my GitHub repository s25me388f_hansen. The value of the angular and scalar fluxes within a cell were obtained by averaging the values at two nodes, as in a divided differences finite volume scheme, and integrating using Gaussian quadrature (for a discrete ordinates calculation). From there, we rearranged the equations to form a block matrix as shown below. For discrete ordinates, the equations used were

$$\mu_k \frac{\psi_k(x_{i+1}) - \psi_k(x_i)}{\Delta x} + \frac{1}{2}\Sigma_t \left(\psi_k(x_{i+1}) + \psi_k(x_i)\right) = \frac{1}{2}(q_0 + \Sigma_s^0 \phi) \tag{1}$$

and for the P_2 solver are

$$\frac{\phi_1(x_{i+1}) - \phi_1(x_i)}{\Delta x} + \frac{1}{2}\Sigma_t(\phi_0(x_{i+1}) + \phi_0(x_i)) = q_0 + \Sigma_s^0 \phi \tag{2}$$

$$\frac{1}{3}\left(\frac{\phi_0(x_{i+1}) - \phi_0(x_i)}{\Delta x}\right) + \frac{1}{2}\Sigma_t(\phi_1(x_{i+1}) + \phi_1(x_i)) = 0 \tag{3}$$

We used a source iteration method where the equations were solved with no scalar flux, the scalar flux is computed, and the equations are resolved with the adjusted source until the scalar flux has converged. Marshuk boundary conditions were used for all problems with spherical harmonics. These equations were rearranged into the below block matrix, which is then stored into a banded format for the SciPy banded solver.

2 Questions

1. Of the infinite homogeneous problems, only the $\Sigma_s = 1$ problem was not solved by the method utilized for the spherical harmonics and discrete ordinates. This makes sense because the analytic solution to this problem is an infinite scalar flux, and our code fails to reach this scalar flux. In later problems, the asymmetry of the boundary conditions prevents an infinite solution from being reached analytically, which enables the code to solve the problem.

C_1	0	0	0	0
A	B	0	0	0
0	A	B	0	0
0	0	A	B	0
0	0	0	A	B
0	0	0	0	C_2

Table 1: Block matrix structure. Each column represents variables in one spatial cell. With N_μ angular variables, the matrices C_1 and C_2 are $N_\mu/2 \times N_\mu$ matrices representing boundary conditions, and A and B are $N_\mu \times N_\mu$ matrices representing the equations coupling the cells.

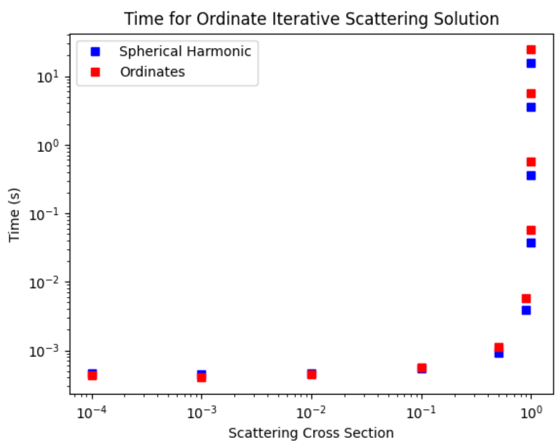

Figure 1: Time to infinite homogeneous solution as a function of the scattering cross section. Note that the $\Sigma_s^0 = 1$ case is shown, but does not converge. Instead, the time to 10^6 iterations cutoff is shown.

2. Both the ordinate and spherical harmonics solution converged to a flat solution, and both are equal to $\frac{Q}{\Sigma_t - \Sigma_s^0}$. This wasn't the case in earlier versions of my code since the source terms were not normalized correctly for discrete ordinates (we have to divide by two to match the form of spherical harmonics). We also plot the time to solution as a function of the scattering cross section in Figure 1.

3. The code I produced ran for the cases of 2560 spatial cells and 16 angular ordinates (for S_N) seemed to approach similar scalar fluxes, as will be shown later and a more detailed comparison will be made.

4. For Problem 2, the angular cross sections for the spectral case always seem to have the cross sections zero, but with a linear increase in the cross section. The angular fluxes increase steadily as the scattering cross section is increased.

   For the discrete ordinate cross sections, the cross section on the left is sharply discontinuous for small scattering cross sections, with negative μ values being zero and positive being one. which discredits the spherical harmonic solution there. However, as the scattering cross section is increased, the negative values increase to approximately be linear, although with a discontinuous slope.

In Problem 3, as the scattering cross section increases, the slope of the spherical harmonic solution decreases. For large discrete ordinate scattering cross sections, we see the angular fluxes being nearly constant at the edges and in the center. However, the discrete ordinate solution starts discontinuous, with positive and negative values being as prescribed at the boundaries, with opposite the boundary conditions at one half, as can be seen in Figure 2.

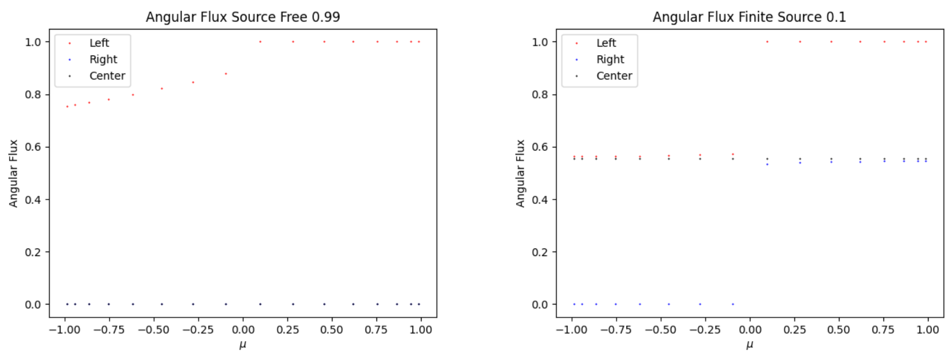

Figure 2: Merging of the left cell angular flux in the source free transport problem. Discontinuous cross section for Problem 3 with points around the middle opposite the prescribed boundary condition.

5. I was struck by how similar some of the solutions were for the finite source problem which seemed to approach the infinite source solution but turned into a parabola, as can be seen with the two below examples in Figure 3.

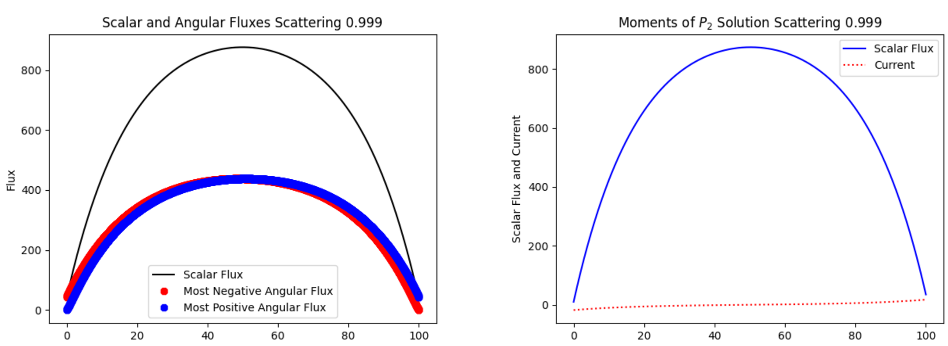

Figure 3: S_{16} and P_2 solution for the finite source problem with 2560 mesh cells.

6. As with Problem 2, I was able to achieve similar solutions for the scalar flux with both discrete ordinates and spherical harmonics. However, I noticed for Problem 3 that there was some trouble around the boundary conditions for small resolutions, and there seemed to be exponential growth of the solution for small scattering cross section with discrete ordinates. See Figure 4.

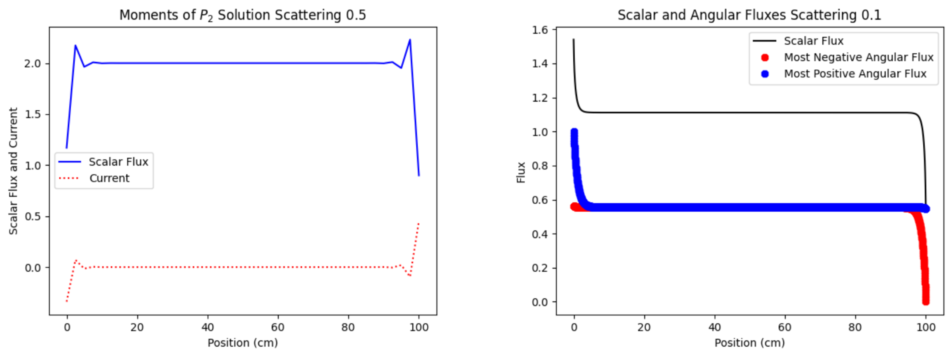

Figure 4: Spherical harmonic Problem 3 solution with 40 cells and discrete ordinate Problem 3 solution with 2560 cells and 16 angles.

7. The plots for the problem time for discrete ordinates are shown in Figure 5. This result is surprising to me since I anticipated that if the time to solution scaled as N^3 for our matrix the log-log plot of solution time should be linearly.

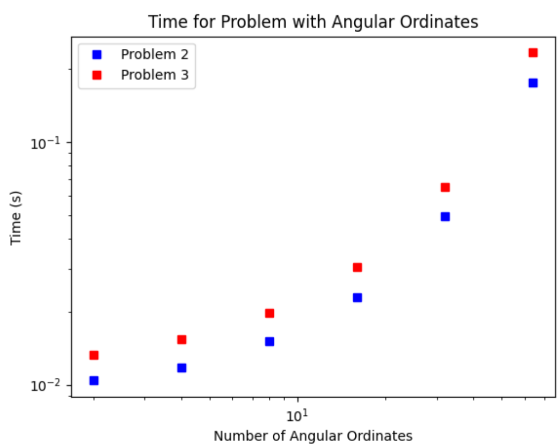

Figure 5: Time for Problems 2 and 3 as a function of the number of discrete ordinates.

8. Since we only computed P_2, we aren't able to test the angular convergence of our method for diffusion. However, in Figure 6 we plot the sum of absolute value of the sum of differences between the scalar flux for lower resolution simulations and the best resolution simulations (given time constraints). It's unclear why the convergence is the same for both problems. However, this shows that there is a steady decrease in error as the number of angular ordinates is increased.

3 References

I briefly discussed this homework with Max Hoffing and Harrison Reisinger. To solve the problems, I used code and documentation provided by the Scipy Linalg and Special library. Timing was

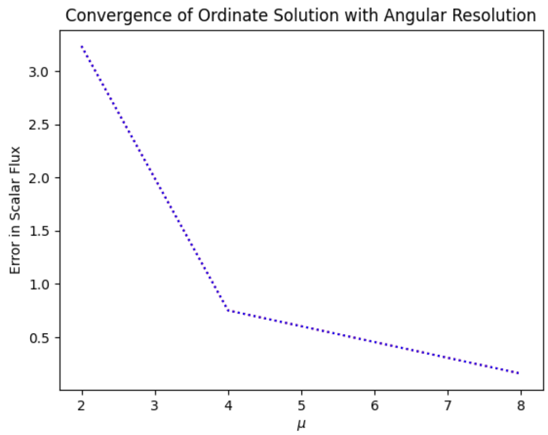

Figure 6: Decrease of total errors in scalar flux as number of angular ordinates is increased.

perfomed by the Python time module, code for which I obtained from Google AI. The equations used and reflecting and vacuum boundary conditions were obtained from the revised homework and the class slides on spherical harmonics and diffusion. Averaging the scalar and angular flux at nodes to obtain the scalar flux within a cell was based on diamond differences from class, finite volume methods, and a suggestion from Bell and Glasstone. The equations for Marshak boundary conditions that I used for the spherical harmonics were obtained from the Exnihilo manual page 22 (I changed the sign of ϕ_0 for the right boundary condition since we integrate from -1 to 0).