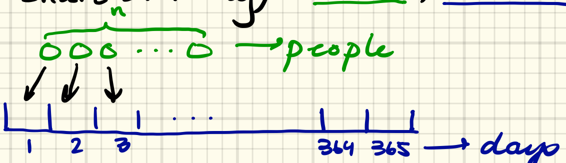
E.H. 7/22/18

Birthday Problem Approximation

Room of n people, 365 days (can use m days)

$\mathbb{P}(\text{any share birthday}) = 1 - \mathbb{P}(\text{none share birthday})$

$\mathbb{P}(\text{none share birthday})$: n balls, 365 bins



Each ball in different bin, no collisions

$\mathbb{P}(1^{st} \text{ ball no collisions}) = 1 \rightarrow$ will not collide (no balls in bins)

$\mathbb{P}(2^{nd} \text{ ball no collisions} \mid 1^{st} \text{ ball no collision}) = 1 - \frac{1}{365} \rightarrow$ any bin w/o 1^{st} ball

$\mathbb{P}(2^{nd} \,\&\, 1^{st} \text{ no collisions}) = \mathbb{P}(2^{nd} \text{ no collisions} \mid 1^{st} \text{ no collisions}) \cdot \mathbb{P}(1^{st} \text{ no collisions}) = 1 - \frac{1}{365}$

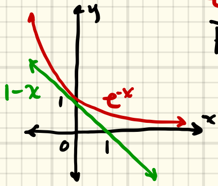
In general:

$\mathbb{P}(i^{th} \text{ ball no collisions} \mid 1^{st}, \dots, i-1^{th} \text{ no collision}) = 1 - \frac{i-1}{365}$

$\mathbb{P}(1, \dots, i^{th} \text{ ball no collisions}) = 1 \cdot \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{i-1}{365}\right)$

So $\mathbb{P}(\text{none share birthday})$ (of n people) $= \overbrace{1 \cdot \left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right)}^{\text{note: } n \text{ terms}}$

Observe: $e^{-x} \geq 1 - x$ or $e^{-x} \approx 1 - x$ when $x \approx 0$, and so:



$\mathbb{P}(\text{none share birthday}) = 1 \cdot \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdots \left(1 - \frac{n-1}{365}\right)$

$\approx e^0 \cdot e^{-1/365} \cdot e^{-2/365} \cdot e^{\frac{-(n-1)}{365}} = \exp\left\{\sum_{i=0}^{n-1} -\frac{i}{365}\right\}$

$= \exp\left\{-\frac{1}{365} \sum_{i=1}^{n-1} i\right\} \overset{*}{=} \exp\left\{\frac{-n(n-1)}{730}\right\} \underset{**}{\approx} \exp\left\{\frac{-n^2}{730}\right\}$

$\mathbb{P}(\text{any share birthday}) = 1 - \mathbb{P}(\text{none share birthday}) \approx 1 - \exp\left\{-n^2/730\right\}$

Calculate example $\mathbb{P}(\text{any share birthday}) = 1/2$ with approx.

$\mathbb{P}(\text{any share birthday}) = 1/2 \Rightarrow 1 - \exp\left\{\frac{-n^2}{730}\right\} = \frac{1}{2} \Rightarrow \exp\left\{n^2/730\right\} = 1/2$

$\Rightarrow \frac{-n^2}{730} = -\ln(2) \Rightarrow n = \sqrt{730 \cdot \ln(2)} \Rightarrow \boxed{n \approx 22.5}$

So 23 people until $\geq 50\%$ of sharing birthday.
(same result as w/o approx.)

$* \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

$**$ for large n