# Introduction to Geometric Algebra

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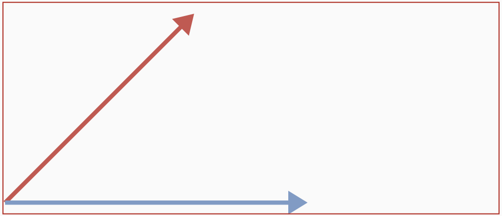
# Part 1

# Formal sum

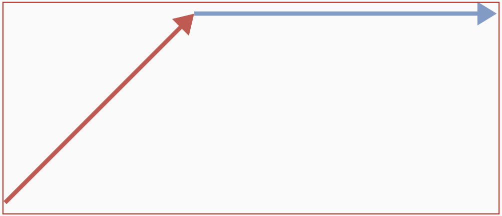
| Linear Algebra | Geometric Algebra |
|---|---|
| $\vec{\mathbf{u}} = \begin{pmatrix} a \\ b \\ c \\ \vdots \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \\ \vdots \end{pmatrix}$ | $\vec{\mathbf{u}} = a\,e_x + b\,e_y + c\,e_z + \cdots$ |

$$\vec{u} \wedge \vec{v}$$



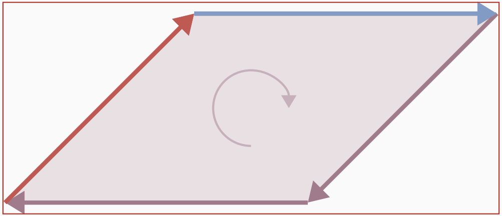
# Bivectors and Outer Product

$$\vec{u} \wedge \vec{v}$$

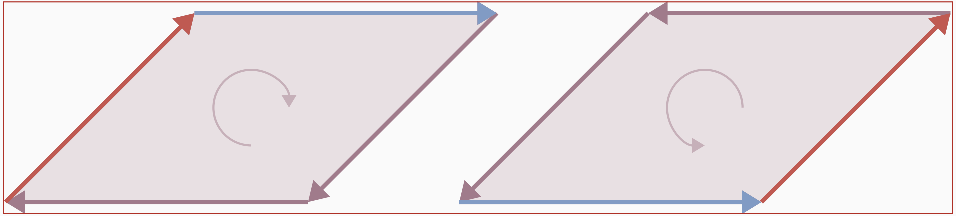


# Bivectors and Outer Product

$$\vec{u} \wedge \vec{v} = \vec{A}$$



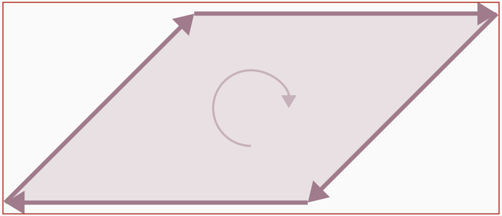
$$\vec{u} \wedge \vec{v} = -\vec{v} \wedge \vec{u}$$



Anti-Commutativity!

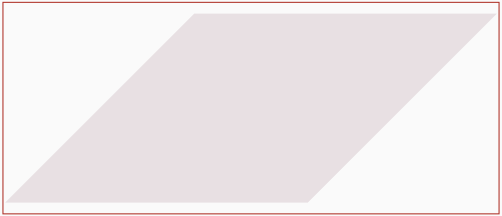
# Bivectors and Outer Product

$$\vec{A}$$



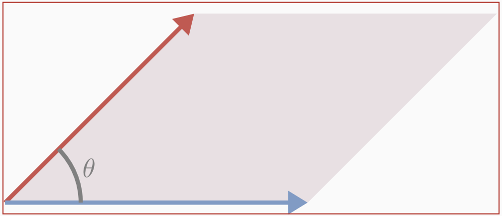
# Bivectors and Outer Product

$$\|\vec{A}\|$$

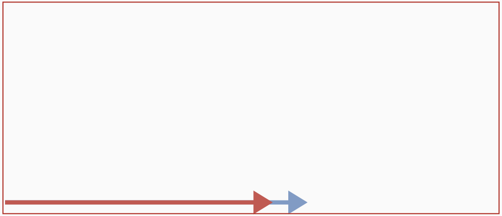


## Bivectors and Outer Product

$$\|\vec{A}\| = \sin(\theta)\|\vec{u}\|\|\vec{v}\|$$



$$\|\vec{A}\| = 0$$



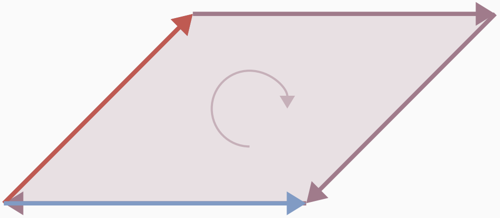
Contract parallel dimensions!

# Bivectors and Outer Product

$$\vec{u} \wedge \vec{v} = \vec{A}$$

$$\begin{aligned}
\lambda \wedge \vec{u} &= \lambda \vec{u} \\
(\vec{u} \wedge \vec{v}) \wedge \vec{w} &= \vec{u} \wedge (\vec{v} \wedge \vec{w}) \\
\vec{u} \wedge (\vec{v} + \vec{w}) &= (\vec{u} \wedge \vec{v}) + (\vec{u} \wedge \vec{w}) \\
\vec{u} \wedge (\lambda \vec{v}) &= \lambda (\vec{u} \wedge \vec{v})
\end{aligned}$$

$$\begin{aligned}
\vec{u} \wedge \vec{v} &= -\vec{v} \wedge \vec{u} \\
\vec{u} \wedge \lambda \vec{u} &= 0
\end{aligned}$$



# Bivectors and Outer Product

$$\vec{u} \wedge \vec{v} = \vec{A}$$

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\vec{u} \wedge (\lambda \vec{v}) &= \lambda (\vec{u} \wedge \vec{v})
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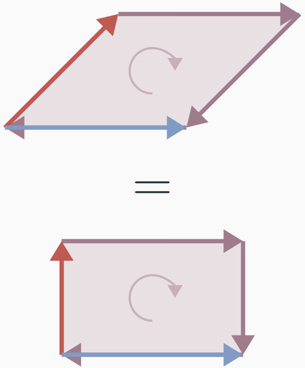



**Limitations (again)**
Not enough information about the vectors

# Bivector in 2D

$$\vec{u} = a_1\mathbf{e}_x + b_1\mathbf{e}_y \qquad \vec{v} = a_2\mathbf{e}_x + b_2\mathbf{e}_y$$

$$\begin{aligned}
\vec{u} \wedge \vec{v} &= (a_1\mathbf{e}_x + b_1\mathbf{e}_y) \wedge (a_2\mathbf{e}_x + b_2\mathbf{e}_y) \\
&= a_1 a_2(\mathbf{e}_x \wedge \mathbf{e}_x) + a_1 b_2(\mathbf{e}_x \wedge \mathbf{e}_y) \\
&\quad + b_1 a_2(\mathbf{e}_y \wedge \mathbf{e}_x) + b_1 b_2(\mathbf{e}_y \wedge \mathbf{e}_y) \\
&= (a_1 b_2 - b_1 a_2)(\mathbf{e}_x \wedge \mathbf{e}_y) \\
&= (a_1 b_2 - b_1 a_2)\mathbf{e}_{xy}
\end{aligned}$$

# Bivector in 3D

$$\vec{u} = a_1\mathbf{e}_x + b_1\mathbf{e}_y + c_1\mathbf{e}_z \qquad \vec{v} = a_2\mathbf{e}_x + b_2\mathbf{e}_y + c_2\mathbf{e}_z$$

$$\vec{u} \wedge \vec{v} = (a_1 b_2 - b_1 a_2)\,\mathbf{e}_{xy} + (c_1 a_2 - a_1 c_2)\,\mathbf{e}_{zx} + (b_1 c_2 - c_1 b_2)\,\mathbf{e}_{yz}$$

$$\vec{u} \times \vec{v} = (a_1 b_2 - b_1 a_2)\,\mathbf{e}_z + (c_1 a_2 - a_1 c_2)\,\mathbf{e}_y + (b_1 c_2 - c_1 b_2)\,\mathbf{e}_x$$

**Note**: looks like the cross product of $\mathbb{R}^3$ but:

- is actually defined in any dimension
- is associative: $(\vec{u} \wedge \vec{v}) \wedge \vec{w} = \vec{u} \wedge (\vec{v} \wedge \vec{w})$

# Trivectors

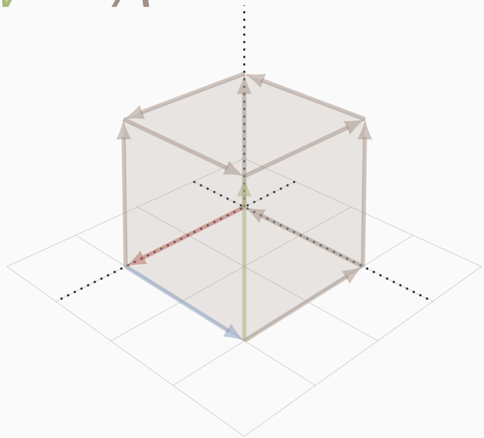
$$\vec{u} \wedge \vec{v} \wedge \vec{w} = A$$

**Properties**:

- Magnitude (Volume)
- Orientation

**Operations:**

- Addition
- Multiplication by a scalar
- ...



**Set of vector spaces in 3D:**

$$\left\{ \underbrace{1}_{\text{scalar}},\quad \underbrace{\mathbf{e}_x,\ \mathbf{e}_y,\ \mathbf{e}_z,}_{\text{vector space}}\quad \underbrace{\mathbf{e}_{xy},\ \mathbf{e}_{zx},\ \mathbf{e}_{yz},}_{\text{bivector space}}\quad \underbrace{\mathbf{e}_{xyz}}_{\text{trivector space}} \right\}$$

**Pseudo-scalar:** $\mathbf{I} = \mathbf{e}_x \wedge \mathbf{e}_y \wedge \mathbf{e}_z = \mathbf{e}_{xyz}$

# Part 2

**3D Projective geometric algebra:** $\mathbb{R}_{3,0,1}$

| | $\mathbf{e}_x$ | $\mathbf{e}_y$ | $\mathbf{e}_z$ | $\mathbf{e}_0$ |
|---|---|---|---|---|
| $\mathbf{e}_x$ | 1 | 0 | 0 | 0 |
| $\mathbf{e}_y$ | 0 | 1 | 0 | 0 |
| $\mathbf{e}_z$ | 0 | 0 | 1 | 0 |
| $\mathbf{e}_0$ | 0 | 0 | 0 | 0 |

$$\mathbf{p} = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z + w\,\mathbf{e}_0$$

$$\vec{\mathbf{u}} = a\,\mathbf{e}_x + b\,\mathbf{e}_y + c\,\mathbf{e}_z$$

# Line from points and vectors

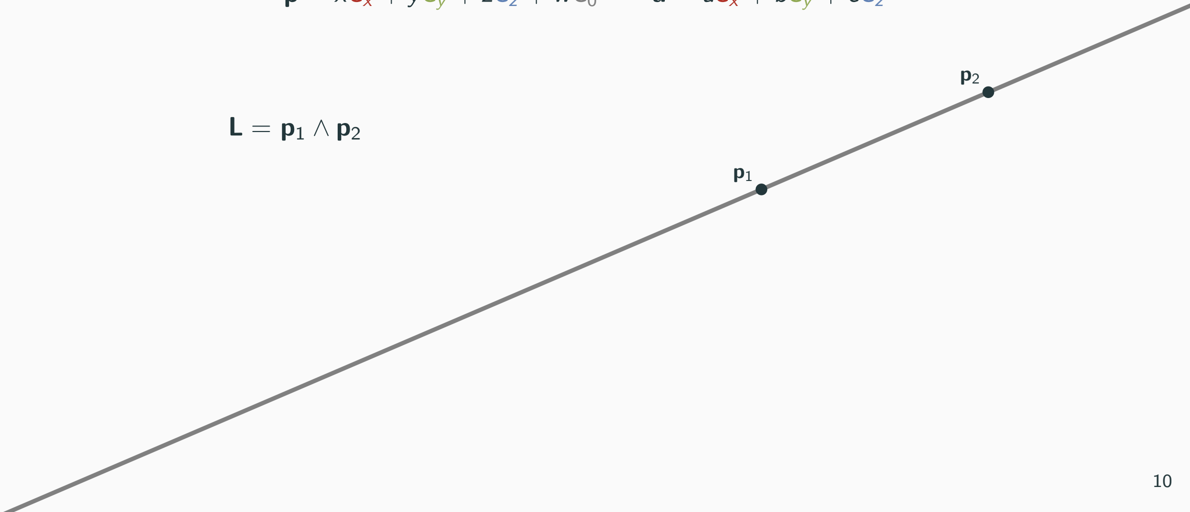
$$\mathbf{p} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z + w\mathbf{e}_0 \qquad \vec{\mathbf{u}} = a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z$$

$$\mathbf{p}_1 \wedge \mathbf{p}_2$$

# Line from points and vectors

$$\mathbf{p} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z + w\mathbf{e}_0 \qquad \vec{\mathbf{u}} = a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z$$

$$\mathbf{L} = \mathbf{p}_1 \wedge \mathbf{p}_2$$



# Line from points and vectors

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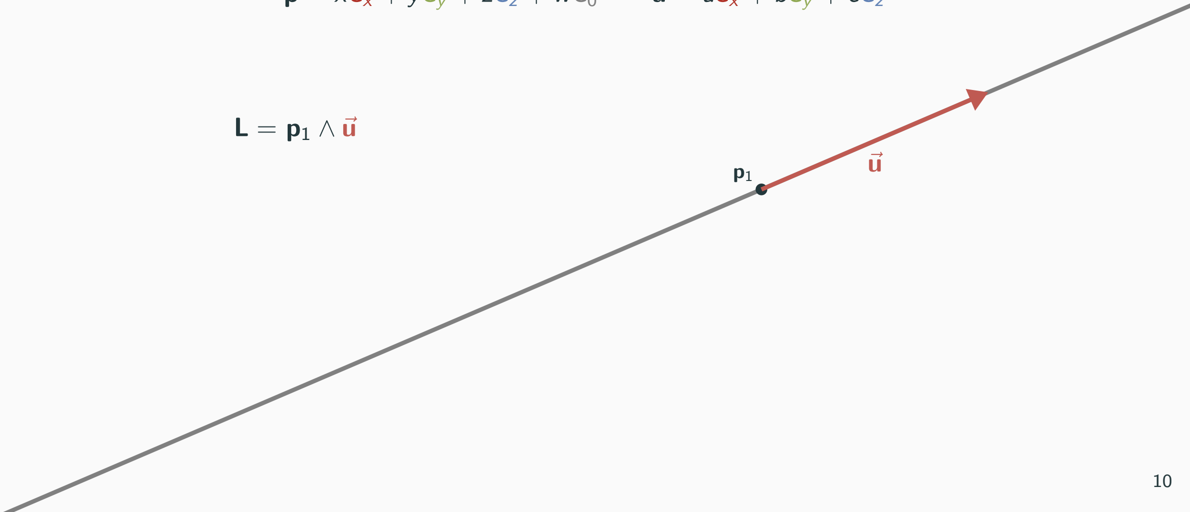

# Line from points and vectors

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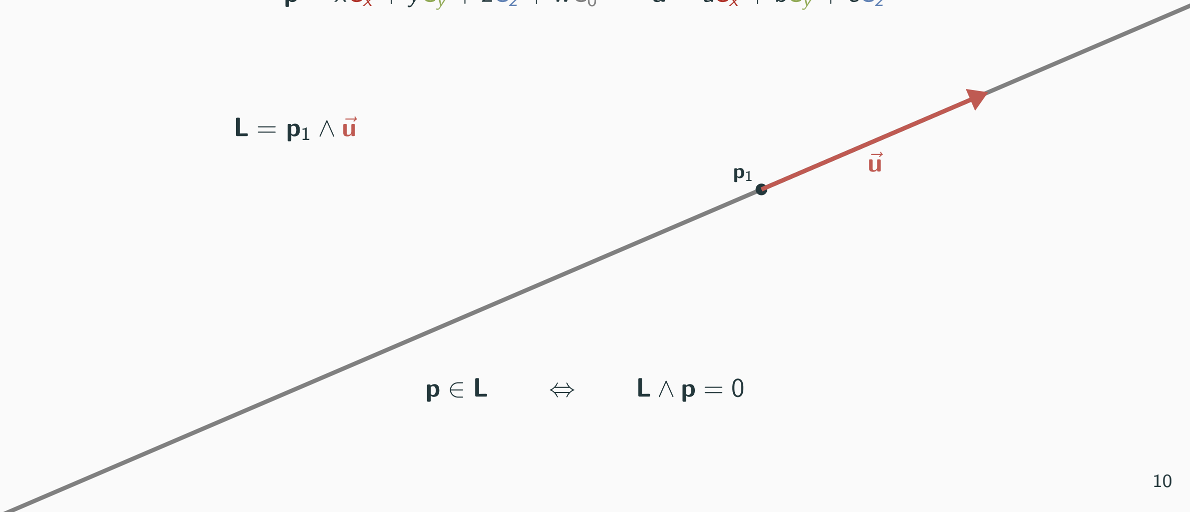



$$\mathbf{p} \in \mathbf{L} \qquad \Leftrightarrow \qquad \mathbf{L} \wedge \mathbf{p} = 0$$

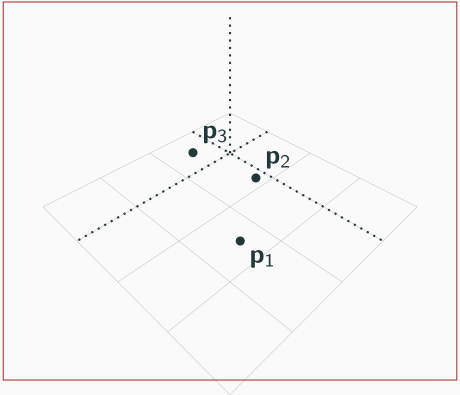
# Plane from points and vectors

$$\mathbf{p} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z + w\mathbf{e}_0 \qquad \vec{\mathbf{u}} = a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z$$

$$\mathbf{P} = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \mathbf{p}_3$$

$$\mathbf{P} = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \vec{\mathbf{u}}$$

$$\mathbf{P} = \mathbf{p}_1 \wedge \vec{\mathbf{u}} \wedge \vec{\mathbf{v}}$$



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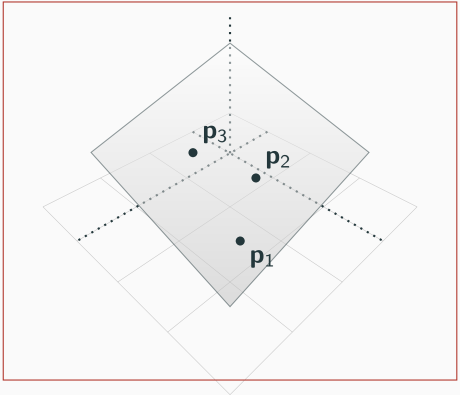

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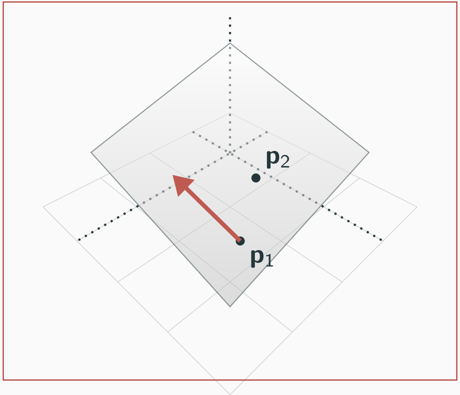

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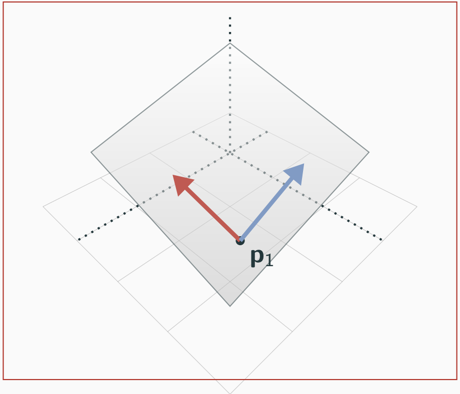

# Plane from points and vectors

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$$\mathbf{P} = \mathbf{p}_1 \wedge \mathbf{p}_2 \wedge \vec{\mathbf{u}}$$

$$\mathbf{P} = \mathbf{p}_1 \wedge \vec{\mathbf{u}} \wedge \vec{\mathbf{v}}$$

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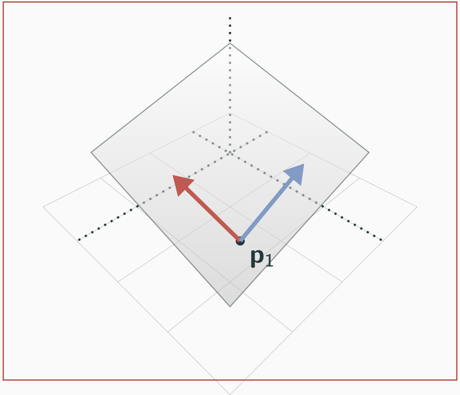
$$\mathbf{p} \in \mathbf{P} \quad \Leftrightarrow \quad \mathbf{P} \wedge \mathbf{p} = 0$$



# Plane line intersection