2-dimensional vacuum polarisation in presence of a delta-Kondo potential

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Convention The signature of our space-time is $(+,-)$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

1 Massless spin-1/2 particle, without extra electric field

With the same notation as in [1], we can write down the Dirac equation of the problem as following

$$i\partial\phi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i\partial\phi + \begin{pmatrix} v_3 & v_- \\ v_+ & -v_3 \end{pmatrix} \delta(x_1)\phi \tag{1}$$

By noting $\phi = \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix}$, eq. (1) leads to

$$\begin{cases} i\partial_0\phi_L = -i\partial_1\phi_L + (v_3\phi_L + v_-\phi_R)\delta(x_1) \\ i\partial_0\phi_R = i\partial_1\phi_R + (v_+\phi_L - v_3\phi_R)\delta(x_1) \end{cases} \tag{2}$$

By considering the right-hand side of eq. (1) as a Hamiltonian (verifying that it is self-adjoint),

$$\phi_{L,k} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{ikx^1} \quad \text{and} \quad \phi_{R,k} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ikx^1} \tag{3}$$

form a basis of solutions for eq. (1) for an eigenvalue of the Hamiltonian k. Let's consider now the matching conditions at $x^1 = 0$, given by

$$\begin{cases} -i(\phi_L(0^+) - \phi_L(0^-)) + \frac{1}{2}(v_3(\phi_L(0^+) + \phi_L(0^-)) + v_-(\phi_R(0+) + \phi_R(0^-))) = 0 \\ i(\phi_R(0^+) - \phi_R(0^-)) + \frac{1}{2}(v_+(\phi_L(0^+) + \phi_L(0^-)) - v_3(\phi_R(0^+) + \phi_R(0^-))) = 0 \end{cases} \tag{4}$$

After the operations $L_1 \leftarrow L_1 \times (i - \frac{1}{2}v_3) - L_2 \times \frac{1}{2}v_-$ and $L_2 \leftarrow L_2 \times (-i + \frac{1}{2}v_3) - L_1 \times \frac{1}{2}v_+$, we have

$$\begin{pmatrix} 1 - \frac{1}{4}\Sigma + iv_3 & 0 \\ 0 & 1 - \frac{1}{4}\Sigma + iv_3 \end{pmatrix} \begin{pmatrix} \phi_L(0^+) \\ \phi_R(0^+) \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{4}\Sigma & -iv_- \\ iv_+ & 1 + \frac{1}{4}\Sigma \end{pmatrix} \begin{pmatrix} \phi_L(0^-) \\ \phi_R(0^-) \end{pmatrix} \tag{5}$$

where $\Sigma = v_1^2 + v_2^2 + v_3^2$

One can notice that this system does not have unique solution for certain values of v_i, namely, when the matrix on the left-hand side vanishes, *i.e.,* when $1 - \frac{1}{4}\Sigma + iv_3 = 0$. This case will be excluded in the following.

Therefore, the matching condition eq. (4) implies

$$\begin{pmatrix} \phi_L(0^+) \\ \phi_R(0^+) \end{pmatrix} = \begin{pmatrix} \frac{A}{D} & \frac{C}{D} \\ \frac{C^*}{D} & \frac{A}{D} \end{pmatrix} \begin{pmatrix} \phi_L(0^-) \\ \phi_R(0^-) \end{pmatrix} \tag{6}$$

where $A = 1 + \frac{1}{4}\Sigma$, $C = -iv_-$, $D = 1 - \frac{1}{4}\Sigma + iv_3$.

1.1 Verification of the consistency of the solution

Let us justify the consistency of eq. (6). From eq. (3), we can consider $\phi_{L,k}$ ($\phi_{R,k}$) in the region $x^1 < 0$ ($x^1 > 0$) as an "in-coming" wave w.r.t. the origin and $\phi_{R,k}$ ($\phi_{L,k}$) in the region $x^1 > 0$ ($x^1 < 0$) as an "out-coming" wave w.r.t. the origin. In the stationary regime, the total density of probability of the in-coming waves should be equal to the total density of probability of the out-coming waves. One can re-write eq. (6) in the following way

$$\begin{pmatrix} \phi_L(0^+) \\ \phi_R(0^-) \end{pmatrix} = \begin{pmatrix} \frac{A}{D} - \frac{|C|^2}{DA} & \frac{C}{A} \\ -\frac{C^*}{A} & \frac{D}{A} \end{pmatrix} \begin{pmatrix} \phi_L(0^-) \\ \phi_R(0^+) \end{pmatrix} \tag{7}$$

Since $|A|^2 = |C|^2 + |D|^2$, the matrix on the r.h.s. is an unitary matrix. It follows that the norm of the vector on the l.h.s. is equal to the norm of the vector on the r.h.s. , which is indeed what we try to prove.

1.2 Two-point Hadamard form of the spatially bounded case

Consider now a system confined in $[-\frac{L}{2}, \frac{L}{2}]$. We apply the procedure given in [1] in order to calculate the vacuum polarisation. The boundary conditions are given by

$$i\gamma^1\psi\Big|_{\pm\frac{L}{2}} = \pm\psi\Big|_{\pm\frac{L}{2}}$$

Suppose that the the solution takes the form $\phi = \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix}$. In terms of $\phi = \gamma^0\psi$, the boundary conditions become

$$\begin{pmatrix} -i\phi_R \\ i\phi_L \end{pmatrix}\Bigg|_{\pm\frac{L}{2}} = \pm\begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix}\Bigg|_{\pm\frac{L}{2}}$$

For k an eigenvalue of the Hamiltonian, we can write down the solution for the region $x^1 < 0$ with $\phi_L = fe^{ikx}$ and $\phi_R = ge^{-ikx}$, where f and g are complex numbers that we have to determine. According to eq. (6), the components of the solution in the region $x > 0$ should be $\phi_L = \frac{1}{D}(Af + Cg)e^{ikx^1}$ and $\phi_R = \frac{1}{D}(C^*f + Ag)e^{-ikx^1}$. Note that the solution on the whole space $x^1 \in [-\frac{L}{2}, \frac{L}{2}] - \{0\}$ is totally determined by f and g due to the matching condition eq. (4). The boundary conditions imply

$$\begin{cases} -ie^{ik\frac{L}{2}}g = -fe^{-ik\frac{L}{2}} & \text{, at } x^1 = -\frac{L}{2} \\ \frac{A}{D}fe^{ik\frac{L}{2}} + \frac{C}{D}ge^{ik\frac{L}{2}} = -i(\frac{C^*}{D}fe^{-ik\frac{L}{2}} + \frac{A}{D}ge^{-ik\frac{L}{2}}) & \text{, at } x^1 = \frac{L}{2} \end{cases} \tag{8}$$

which can be re-arranged as [1]

$$\begin{cases} g = fe^{-i(kL + \frac{\pi}{2})} \\ g = \frac{A + iC^*e^{-ikL}}{-Ce^{ikL} - iA}fe^{ikL} \end{cases} \tag{9}$$

[1] We can verify that, as $|A|^2 - |C|^2 = |D|^2 > 0$ by assumption, $iA + C$ is always non-vanishing.

For a non-vanishing solution, this implies

$$e^{-i(kL+\frac{\pi}{2})} = \frac{A + iC^* e^{-ikL}}{(A + iC^* e^{-ikL})^*} e^{i(kL+\frac{\pi}{2})} \tag{10}$$

and

$$|f| = |g| \tag{11}$$

Thus, according to eq. (10), k has to take specific values such that

$$kL = \mathrm{Arg}(A - iCe^{ikL}) + \big(n + \frac{1}{2}\big)\pi \quad \text{for n} \in \mathbb{Z} \tag{12}$$

The case $|C| = 0$ is relatively easy to deal with. Let us focus on the cases where $|C| \neq 0$. We should consider separately eq. (12) for n odd and n even. The motivation of this distinction is due to the 2π-periodicity of the exponential term.

Let us start with the case where n is even. For $C = |C|e^{i\eta} \neq 0$, it follows[2]

$$\begin{aligned} &\mathrm{Arg}(A - iCe^{ikL}) \\ =&\mathrm{Arg}(A + |C|e^{i(kL-\frac{\pi}{2}+\eta)}) \\ =&\arctan\left(\frac{|C|\sin(kL - \frac{\pi}{2} + \eta)}{A + |C|\cos(kL - \frac{\pi}{2} + \eta)}\right) \end{aligned} \tag{13}$$

We want to find k such that $kL \in [0, \pi]$ in order to coincide with the allowed values of arctan. Therefore, by eq. (12), k must satisfy

$$\begin{aligned} &\frac{|C|\sin(kL - \frac{\pi}{2} + \eta)}{A + |C|\cos(kL - \frac{\pi}{2} + \eta)} = -\cot kL \\ \Leftrightarrow \quad & A\cot kL = |C|\cos(kL + \eta) - |C|\cot kL \sin(kL + \eta) \end{aligned} \tag{14}$$

As k should satisfy

$$A\cos kL + |C|\sin\eta = 0 \tag{15}$$

$kL = \arccos\big(-\frac{|C|}{A}\big)$. Other solutions for the case where n is even are equal to this value modulo 2π.

For the case where n is odd, the calculation is similar. We try to find k such that $kL - \pi \in [0, \pi]$, which gives $kL = 2\pi - \arccos\big(-\frac{|C|}{A}\big)$. Other solutions for the case where n is odd are equal to this value modulo 2π.

To sum up, the possible values of k are given by

$$k_n = \frac{1}{L}\big(\theta + (\pi - \theta)(1 - (-1)^n)\big) + \frac{\pi}{L}n \quad \text{where } \theta = \arccos\left(\frac{-|C|\sin\eta}{A}\right) \tag{16}$$

[2] For $\alpha, \beta, \theta \in \mathbb{R}$, assuming that $\alpha + \beta\cos\theta > 0$, $\alpha + \beta e^{i\theta} = \alpha + \beta\cos\theta + i\beta\sin\theta = (\alpha^2 + \beta^2 + 2\alpha\beta\cos\theta)e^{i\delta}$ with $\delta = \arctan\frac{\beta\sin\theta}{\alpha+\beta\cos\theta}$

The coefficients f_n and g_n for the mode k_n can be determined by using the normalisation condition $\int_{[-\frac{L}{2},\frac{L}{2}]} \phi^\dagger\phi = 1$. In the region $[-\frac{L}{2}, 0)$, $\phi^\dagger\phi = |f|^2 + |g|^2$. Whereas in the region $(0, \frac{L}{2}]$,

$$\begin{aligned}\phi^\dagger\phi &= \begin{pmatrix}\frac{1}{D^*}(Af^* + C^*g^*)e^{-ikx^1} & \frac{1}{D^*}(Cf^* + Ag^*)e^{ikx^1}\end{pmatrix}\begin{pmatrix}\frac{1}{D}(Af + Cg)e^{ikx^1} \\ \frac{1}{D}(C^*f + Ag)e^{-ikx^1}\end{pmatrix} \\ &= \frac{A^2 + |C|^2}{|D|^2}(|f|^2 + |g|^2) + 4\frac{A}{|D|^2}\Re\{Cf^*g\}\end{aligned} \tag{17}$$

By the first equation of eq. (9), the last term of the last expression is

$$4\frac{A|C|}{|D|^2}|f|^2\Re\{e^{-i(kL+\frac{\pi}{2}-\eta)}\} = -4\frac{A|C|}{|D|^2}|f|^2\sin(kL-\eta)$$

Hence, the normalisation condition and eq. (11) imply

$$|f_n| = \sqrt{\frac{1}{L(\alpha - \beta\sin(k_nL - \eta))}} \quad \text{where } \alpha = 1 + \frac{A^2 + |C|^2}{|D|^2}, \quad \beta = \frac{2A|C|}{|D|^2} \tag{18}$$

Therefore, for the mode $k_n > 0$, the solution space is spanned by

$$\begin{aligned}\phi_{k_n} =& \sqrt{\frac{1}{L(\alpha - \beta\sin(k_nL - \eta))}}\left(\begin{pmatrix}1 & 0 \\ 0 & e^{-i(kL+\frac{\pi}{2})}\end{pmatrix}\Theta(-x^1) + \right. \\ & \left.\begin{pmatrix}\frac{A}{D} + \frac{C}{D}e^{-i(kL+\frac{\pi}{2})} & 0 \\ 0 & \frac{C^*}{D} + \frac{A}{D}e^{-i(kL+\frac{\pi}{2})}\end{pmatrix}\Theta(x^1)\right)\begin{pmatrix}e^{ik_nx^1} \\ e^{-ik_nx^1}\end{pmatrix}\end{aligned} \tag{19}$$

where Θ is the Heaviside step function.

We take the Hadamard state defined in [1] to calculate the vacuum polarisation of this configuration, given by the two-point function

$$\omega(\psi^B(x)\bar{\psi}_A(y)) = \int_{E_k>0} \psi^B(x)\bar{\psi}_A(y)e^{-i(x^0-y^0)E_k}\mathrm{d}k \tag{20}$$

As we will multiply these two-point functions by γ^i to get observables (current and charge density), only the following terms should be considered in terms of ϕ [3]

$$\omega(\psi^B(x)\bar{\psi}_A(y)) = \gamma^B_A\omega(\phi^A(x)\phi^\dagger_A(y)) = \int_{E_k>0} \phi^A(x)\phi^\dagger_A(y)e^{-i(x^0-y^0)E_k}\mathrm{d}k \quad \text{for } A = 1, 2 \tag{21}$$

[3] The summation of Einstein is not applied here.

Firstly, consider $x^1, y^1 < 0$. The integral term for $A = 1$, with $z = x^0 - y^0 - x^1 + y^1$ becomes

$$\sum_{2p\geq 0} \frac{e^{-i(\theta+2p\pi)\frac{z}{L}}}{L(\alpha - \beta\sin(\theta-\eta))} + \sum_{2p+1\geq 0} \frac{e^{-i(-\theta+(2p+2)\pi)\frac{z}{L}}}{L(\alpha + \beta\sin(\theta+\eta))} \tag{22}$$

The above sum can be written as

$$\frac{1}{2iL\sin\frac{\pi}{L}z}\left(\frac{e^{i(-\theta+\pi)\frac{z}{L}}}{\alpha - \beta\sin(\theta-\eta)} + \frac{e^{i(\theta-\pi)\frac{z}{L}}}{\alpha + \beta\sin(\theta+\eta)}\right) \tag{23}$$

Developping the term in the parenthesis up to $\mathcal{O}(1)$, we get

$$\begin{aligned} & \frac{1}{\alpha - \beta\sin(\theta-\eta)} + \frac{1}{\alpha + \beta\sin(\theta+\eta)} \\ =& \frac{2(\alpha + \beta\sin\eta\cos\theta)}{(\alpha + \beta\sin\eta\cos\theta)^2 - \beta^2\sin^2\theta\cos^2\eta} \\ =& \frac{2(\alpha - \beta\frac{|C|}{A}\sin^2\eta)}{\alpha^2 - \beta^2 + \beta^2\sin^2\eta(1 + \frac{|C|^2}{A^2}) - 2\alpha\beta\frac{|C|}{A}\sin^2\eta} \end{aligned} \tag{24}$$

As

$$\begin{aligned} & \alpha^2 - \beta^2 = 2\alpha \\ & \beta^2\big(1 + \frac{|C|^2}{A^2}\big) - 2\alpha\beta\frac{|C|}{A} \\ =& \big(2\frac{A|C|}{|D|^2}\big)^2\big(1 + \frac{|C|^2}{A^2}\big) - 4\big(1 + \frac{|C|^2}{D^2})\big)\big(2\frac{A|C|}{|D|^2}\big)\frac{|C|}{A} \\ =& 4\frac{A^2|C|^2}{|D|^4} + 4\frac{|C|^4}{|D|^4} - 8\frac{|C|^2}{|D|^2} - 8\frac{|C|^4}{|D|^4} \\ =& -2\beta\frac{|C|}{A} \end{aligned}$$

we have

$$\frac{1}{\alpha - \beta\sin(\theta-\eta)} + \frac{1}{\alpha + \beta\sin(\theta+\eta)} = 1 \tag{25}$$

Therefore, the singularity of $\mathcal{O}(z^{-1})$ is the same as for the Hadamard parametrix of the vacuum case.

We calculate now the vacuum polarization in the region $[-\frac{L}{2}, 0)$. Since

$$\frac{1}{2i\sin\frac{\pi}{L}z} = \frac{-iL}{2\pi z} - \frac{i\pi z}{12L} + \mathcal{O}(z^3)$$

using eq. (25) and denoting

$$
\begin{aligned}
\xi(z) =&\Big(\frac{-i}{2\pi z}-\frac{i\pi z}{12L^2}+\mathcal{O}(z^3)\Big)\Big(1+\frac{i(-\theta+\pi)\frac{z}{L}}{\alpha-\beta\sin(\theta-\eta)}+\frac{i(\theta-\pi)\frac{z}{L}}{\alpha+\beta\sin(\theta+\eta)}\\
&-\frac{1}{2}\Big(\frac{(-\theta+\pi)^2}{\alpha-\beta\sin(\theta-\eta)}+\frac{(\theta-\pi)^2}{\alpha+\beta\sin(\theta+\eta)}\Big)\frac{z^2}{L^2}+\mathcal{O}(z^3)\Big)\\
=&\frac{-i}{2\pi z}+\frac{1}{2\pi L}\Big(\frac{-\theta+\pi}{\alpha-\beta\sin(\theta-\eta)}+\frac{\theta-\pi}{\alpha+\beta\sin(\theta+\eta)}\Big)+\frac{i\pi}{4L^2}z\Big(-\frac{1}{3}+\frac{(\theta-\pi)^2}{\pi^2}\Big)+\mathcal{O}(z^2)\\
=&\frac{-i}{2\pi z}+\frac{1}{2\pi L}\Big(\frac{\beta\sin\theta\cos\eta}{\alpha+\beta\sin\eta\cos\theta}\Big)(-\theta+\pi)+\frac{i\pi}{4L^2}\Big(-\frac{1}{3}+\frac{(\theta-\pi)^2}{\pi^2}\Big)z+\mathcal{O}(z^2)
\end{aligned} \tag{26}
$$

we thus have

$$
\begin{aligned}
\omega(\psi^2(x)\bar{\psi}_1(y)) &= \omega(\phi^1(x)\phi_1^\dagger(y)) = \xi(x^0-y^0-x^1+y^1)\\
\omega(\psi^1(x)\bar{\psi}_2(y)) &= \omega(\phi^2(x)\phi_2^\dagger(y)) = \xi(x^0-y^0+x^1-y^1)
\end{aligned}
$$

The off-diagonal components of the Hadamard parametrix of our problem are [1]

$$
\begin{aligned}
H^+(x,y)_1^2 &= \frac{-i}{2\pi(x^0-y^0-x^1+y^1-i\epsilon)} + \text{terms vanishing at coinciding point limit}\\
H^+(x,y)_2^1 &= \frac{-i}{2\pi(x^0-y^0+x^1-y^1-i\epsilon)} + \text{terms vanishing at coinciding point limit}
\end{aligned}
$$

Hence, in the region $[-\frac{L}{2},0)$, the charge density is

$$
\rho(x) = \frac{e}{\pi L}\Big(\frac{\beta\sin\theta\cos\eta}{\alpha+\beta\sin\eta\cos\theta}\Big)(-\theta+\pi) \tag{27}
$$

The same calculation allows us to get the two points functions and the Hadamard parametrix (which is the same) in the region $(0,\frac{L}{2}]$. By denoting

$$
\begin{aligned}
\chi(z) =&\omega(\phi^1(x)\phi_1^\dagger(y))\\
=&\Big(\frac{-i}{2\pi z}-\frac{i\pi z}{12L^2}+\mathcal{O}(z^3)\Big)\Big(1+\frac{i(-\theta+\pi)\frac{z}{L}}{\alpha+\beta\sin(\theta+\eta)}+\frac{i(\theta-\pi)\frac{z}{L}}{\alpha-\beta\sin(\theta-\eta)}\\
&-\frac{1}{2}\Big(\frac{(-\theta+\pi)^2}{\alpha+\beta\sin(\theta+\eta)}+\frac{(\theta-\pi)^2}{\alpha-\beta\sin(\theta-\eta)}\Big)\frac{z^2}{L^2}+\mathcal{O}(z^3)\Big)\\
=&\frac{-i}{2\pi z}-\frac{1}{2\pi L}\Big(\frac{\beta\sin\theta\cos\eta}{\alpha+\beta\sin\eta\cos\theta}\Big)(-\theta+\pi)+\frac{i\pi}{4L^2}\Big(-\frac{1}{3}+\frac{(\theta-\pi)^2}{\pi^2}\Big)z+\mathcal{O}(z^2)
\end{aligned}
$$

we find

$$
\omega(\psi^2(x)\bar{\psi}_1(y)) = \chi(x^0-y^0-x^1+y^1)
$$

$$\omega(\psi^1(x)\bar{\psi}_2(y)) = \chi(x^0 - y^0 + x^1 - y^1)$$

Hence, the charge density in the whole space $[-\frac{L}{2}, \frac{L}{2}] - \{0\}$ is

$$\rho(x) = \frac{e}{\pi L}\Big(\frac{\beta \sin\theta \cos\eta}{\alpha + \beta \sin\eta \cos\theta}\Big)(-\theta + \pi)\Big(\Theta(-x^1) - \Theta(x^1)\Big) \tag{28}$$

and the current density is zero.

1.3 Stress-energy tensor

It is also worthy to compute the stress-energy tensor of the system in order to see how it is related to the vacuum polarization that we have calculated. With its classical form, one can write down the components of stress-energy tensor as

$$\begin{aligned}
T_{00} &= \frac{i}{2}(\bar{\psi}\gamma_1\nabla_1\psi - \nabla_1\bar{\psi}\gamma_1\psi) \\
T_{11} &= \frac{i}{2}(\bar{\psi}\gamma_0\nabla_0\psi - \nabla_0\bar{\psi}\gamma_0\psi) \\
T_{01} &= \frac{i}{4}(\bar{\psi}\gamma_1\nabla_0\psi + \bar{\psi}\gamma_0\nabla_1\psi - \nabla_1\bar{\psi}\gamma_0\psi - \nabla_0\bar{\psi}\gamma_1\psi)
\end{aligned} \tag{29}$$

In the two-point function formulation, the expectation value of T_{ab} corresponds to the regular part of the state ω previously defined. For example[4],

$$T_{00}(x,y) = -\frac{i}{2}\big(\omega(\nabla_1\psi^B(x)\bar{\psi}_A(y))(\gamma_1)^A_B - \omega(\psi^B(x)\nabla_1\bar{\psi}_A(y))(\gamma_1)^A_B\big) - \text{singular part} \tag{30}$$

Practically, the singular part corresponds to the covariant derivatives of the Hadamard parametrix. In terms of $\phi = \gamma^0\psi$,

$$\bar{\psi}\gamma_1\nabla\psi = -\phi^\dagger\gamma^1\gamma^0\nabla\phi$$

As usual, we start with the region $[-\frac{L}{2}, 0)$. By denoting

$$\zeta = \gamma^1\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[4] Normal ordering is applied here when calculating the two-point function. The anti-commutation relation of the state is involved.

using the function ξ that we have introduced in eq. (26), the states evaluated at the components of the stress-energy tensor can be written as

$$\begin{aligned}
T_{00}(x,y) =& \frac{i}{2}\Big(\nabla_{x^1}\big(\omega(\phi^B(x)\phi^\dagger_A(y))\zeta^A_C - H^+(x,y)\big) - \nabla_{y^1}\big(\omega(\phi^B(x)\bar{\phi}_A(y))\zeta^A_C - H^+(x,y)\big)\Big)\delta^C_B \\
=& \frac{i}{2}\big((-\xi'(z) - \xi'(w)) - \xi'(z) - \xi'(w) + \frac{i}{\pi z} + \frac{i}{\pi w}\big) \\
T_{11}(x,y) =& -\frac{i}{2}\big(\xi'(z) + \xi'(w) + \xi'(z) + \xi'(w) - \frac{i}{\pi z} - \frac{i}{\pi w}\big) \\
T_{01}(x,y) =& \frac{i}{4}\Big(\nabla_{x^0}\big(\omega(\phi^B(x)\phi^\dagger_A(y))(\zeta_1)^A_C - H^+(x,y)\big) + \nabla_{x^1}\big(\omega(\phi^B(x)\phi^\dagger_A(y))\delta^A_C - H^+(x,y)\big) \\
& - \nabla_{y^0}\big(\omega(\phi^B(x)\bar{\phi}_A(y))\zeta^A_C - H^+(x,y)\big) - \nabla_{y^1}\big(\omega(\phi^B(x)\bar{\phi}_A(y))\delta^A_C - H^+(x,y)\big)\Big)\delta^C_B \\
=& \frac{i}{4}\Big(\big(\xi'(z) - \xi'(w)\big) + \big(-\xi'(z) + \xi'(w)\big) - \big(-\xi'(z) + \xi'(w)\big) - \big(\xi'(z) - \xi'(w)\big)\Big) \\
=& 0
\end{aligned} \tag{31}$$

where $z = x^0 - y^0 - x^1 + y^1$ and $w = x^0 - y^0 + x^1 - y^1$
Taking the coinciding point limit, we find

$$T_{ab} = \frac{\pi}{2L^2}\big(-\frac{1}{3} + \frac{(\theta-\pi)^2}{\pi^2}\big)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{32}$$

2 With static constant electric field

We introduce now a static constant electric field whose electromagnetic potential components are

$$(A_0, A_1) = (Ex^1, 0)$$

eq. (1) turns out to be

$$i\partial\phi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} i\partial\phi + eEx^1\phi + \begin{pmatrix} v_3 & v_- \\ v_+ & -v_3 \end{pmatrix} \delta(x_1)\phi \tag{33}$$

For eigenvalue k of the new Hamiltonian, the eigenvector in the region $[-\frac{L}{2}, 0)$ can be written as

$$\phi_k = \begin{pmatrix} \phi_{k,L} \\ \phi_{k,R} \end{pmatrix} \quad \text{where} \quad \begin{aligned} \phi_{k,L} &= f_k e^{-\frac{i}{2}eE(x^1)^2 + ikx^1} \\ \phi_{k,R} &= g_k e^{\frac{i}{2}eE(x^1)^2 - ikx^1} \end{aligned} \tag{34}$$

for $f, g \in \mathbb{C}$. The matching condition at $x^1 = 0$ is the same as eq. (6). With the same boundary conditions, we can derive a relation similar to eq. (9)

$$\begin{cases} g = f e^{i(kL + \frac{\pi}{2}) - i\frac{eE}{4}L^2} \\ g = \frac{A + iC^* e^{-ikL + i\frac{eE}{4}L^2}}{A - iCe^{ikL - i\frac{eE}{4}L^2}} f e^{ikL - i\frac{eE}{4}L^2 + i\frac{\pi}{2}} \end{cases} \tag{35}$$

which implies $|f| = |g|$ and

$$kL = \text{Arg}(A - iCe^{ikL - i\frac{eE}{4}L^2}) + (n + \frac{1}{2}) \quad n \in \mathbb{Z} \tag{36}$$

This equation can be solved in the same way which has been done for eq. (10) by replacing C by $Ce^{-i\frac{eE}{4}L^2}$.
Thus, by denoting $\eta = \text{Arg } C - \frac{eE}{4}L^2$, an eigenvalue k of the Hamiltonian can take the following values

$$k_n = \frac{(-1)^n}{L}\theta + \frac{2\pi}{L}n \quad \text{where } \theta = \arccos\left(\frac{-|C|\sin\eta}{A}\right) \tag{37}$$

The solution space is therefor spanned by

$$\begin{aligned} \phi_{k_n} = &\sqrt{\frac{1}{L(\alpha - \beta\sin(k_n L - \eta))}} \left(\begin{pmatrix} 1 & 0 \\ 0 & e^{-i(kL + \frac{\pi}{2}) + i\frac{eE}{4}L^2} \end{pmatrix} \Theta(-x^1) + \right. \\ &\left. \begin{pmatrix} \frac{A}{D} + \frac{C}{D}e^{-i(kL + \frac{\pi}{2}) + i\frac{eE}{4}L^2} & 0 \\ 0 & \frac{C^*}{D} + \frac{A}{D}e^{-i(kL + \frac{\pi}{2}) + i\frac{eE}{4}L^2} \end{pmatrix} \Theta(x^1) \right) \begin{pmatrix} e^{ik_n x^1 - i\frac{eE}{4}(x^1)^2} \\ e^{-ik_n x^1 + i\frac{eE}{4}(x^1)^2} \end{pmatrix} \end{aligned} \tag{38}$$

With $z = x^0 - y^0 - x^1 + y^1$, we define

$$
\begin{aligned}
\xi_E(z, x^1, y^1) =& \Big(\frac{-i}{2\pi z} - \frac{i\pi z}{12L^2} + \mathcal{O}(z^3)\Big)\Big(\frac{1}{\alpha - \beta\sin(\theta-\eta)} e^{i\frac{(-\theta+\pi)z}{L} + \frac{ieE}{2}((x^1)^2-(y^1)^2)} + \\
& \frac{1}{\alpha + \beta\sin(\theta+\eta)} e^{i\frac{(\theta-\pi)z}{L} + \frac{ieE}{2}((x^1)^2-(y^1)^2)}\Big) \\
=& \Big(\frac{-i}{2\pi z} - \frac{i\pi z}{12L^2} + \mathcal{O}(z^3)\Big)\Big(1 + \frac{i(-\theta+\pi)\frac{z}{L}}{\alpha - \beta\sin(\theta-\eta)} + \frac{i(\theta-\pi)\frac{z}{L}}{\alpha + \beta\sin(\theta+\eta)} \\
& + \frac{1}{2}\Big(\frac{(-\theta+\pi)^2}{\alpha - \beta\sin(\theta-\eta)} + \frac{(\theta-\pi)^2}{\alpha + \beta\sin(\theta+\eta)}\Big)\frac{z^2}{L^2} \\
& + \frac{ieE}{2}(x^1-y^1)(x^1+y^1) - \frac{e^2E^2}{8}(x^1-y^1)^2(x^1+y^1)^2 + \mathcal{O}(z^3)\Big) \\
=& \frac{-i}{2\pi z} + \frac{1}{2\pi L}\Big(\frac{\beta\sin\theta\cos\eta}{\alpha + \beta\sin\eta\cos\theta}\Big)(-\theta+\pi) + \frac{eE}{4\pi}\frac{(x^1-y^1)(x^1+y^1)}{z} \\
& + \frac{i\pi}{4L^2}\big(-\frac{1}{3} + \frac{(\theta-\pi)^2}{\pi^2}\big)z + \frac{ie^2E^2}{16\pi z}(x^1-y^1)^2(x^1+y^1)^2 + \mathcal{O}(z^2)
\end{aligned}
\tag{39}
$$

In the region $[-\frac{L}{2}, 0)$, the two-point functions defined in the last section become

$$
\begin{aligned}
\omega(\psi^2(x)\bar{\psi}_1(y)) = \omega(\phi^1(x)\phi_1^\dagger(y)) = \xi_E(x^0 - y^0 - x^1 + y^1, x^1, y^1) \\
\omega(\psi^1(x)\bar{\psi}_2(y)) = \omega(\phi^2(x)\phi_2^\dagger(y)) = \xi_E(x^0 - y^0 + x^1 - y^1, x^1, y^1)
\end{aligned}
$$

Similarly, by defining

$$
\begin{aligned}
\chi_E(z, x^1, y^1) =& \frac{-i}{2\pi z} + \frac{1}{2\pi L}\Big(\frac{\beta\sin\theta\cos\eta}{\alpha + \beta\sin\eta\cos\theta}\Big)(\theta-\pi) + \frac{eE}{4\pi}\frac{(x^1-y^1)(x^1+y^1)}{z} \\
& + \frac{i\pi}{4L^2}\big(-\frac{1}{3} + \frac{(\theta-\pi)^2}{\pi^2}\big)z + \frac{ie^2E^2}{16\pi z}(x^1-y^1)^2(x^1+y^1)^2 + \mathcal{O}(z^2)
\end{aligned}
\tag{40}
$$

we have the expressions for the states in the region $(0, \frac{L}{2}]$

$$
\begin{aligned}
\omega(\psi^2(x)\bar{\psi}_1(y)) = \omega(\phi^1(x)\phi_1^\dagger(y)) = \chi_E(x^0 - y^0 - x^1 + y^1, x^1, y^1) \\
\omega(\psi^1(x)\bar{\psi}_2(y)) = \omega(\phi^2(x)\phi_2^\dagger(y)) = \chi_E(x^0 - y^0 + x^1 - y^1, x^1, y^1)
\end{aligned}
$$

The coinciding point limit gives the charge density of the vacuum polarisation and the stress-tensor energy in the whole region (compared to the results of [1])

$$
\rho(x) = \frac{e}{\pi L}\Big(\frac{\beta\sin\theta\cos\eta}{\alpha + \beta\sin\eta\cos\theta}\Big)(-\theta+\pi)\Big(\Theta(-x^1) - \Theta(x^1)\Big) + \frac{e^2E}{\pi}x^1 \tag{41}
$$

$$
T_{ab}(x) = \left(\frac{\pi}{2L^2}\big(-\frac{1}{3} + \frac{(\theta-\pi)^2}{\pi^2}\big) + \frac{e^2E^2(x^1)^2}{2\pi}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{42}
$$

3 Without electric field and without boundary

In this section we examine the vacuum polarisation in the no-boundary case. The matching condition eq. (6) and the calculation in section 1 suggest that all eigenspaces for eigenvalue k of the Hamiltonian are 2-dimensional. We search therefore an orthonormal basis of the Hilbert space composed of eigenvectors (for eigenvalue k) of type

$$\begin{aligned}
|k_{(1)}\rangle &= \left(\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \Theta(-x^1) + \frac{1}{D} \begin{pmatrix} Af & 0 \\ 0 & C^* f \end{pmatrix} \Theta(x^1) \right) \begin{pmatrix} e^{ikx^1} \\ e^{-ikx^1} \end{pmatrix} \\
|k_{(2)}\rangle &= \left(\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \Theta(-x^1) + \frac{1}{D} \begin{pmatrix} Ag + Ch & 0 \\ 0 & C^* g + Ah \end{pmatrix} \Theta(x^1) \right) \begin{pmatrix} e^{ikx^1} \\ e^{-ikx^1} \end{pmatrix}
\end{aligned} \tag{43}$$

where $f, g, h \in \mathbb{C}$ The normalisation condition demands

$$\langle k'_{(m)} | k_{(n)} \rangle = \delta(k - k')\delta_{mn} \tag{44}$$

Since[5]

$$\begin{aligned}
\langle k'_{(1)} | k_{(1)} \rangle =& |f|^2 \Big(\int_{-\infty}^{0} e^{i(k-k')x} \mathrm{d}x + \int_{0}^{\infty} \frac{|A|^2}{|D|^2} e^{i(k-k')x} + \frac{|C|^2}{|D|^2} e^{-i(k-k')x} \mathrm{d}x \Big) \\
=& |f|^2 \Big(\frac{1}{i(k-k'-i\epsilon)} - \frac{A^2}{|D|^2} \frac{1}{i(k-k'+i\epsilon)} + \frac{|C|^2}{|D|^2} \frac{1}{i(k-k'-i\epsilon)} \Big) \\
=& \frac{A^2|f|^2}{i|D|^2} \Big(P\big(\frac{1}{k-k'}\big) + i\pi\delta(k-k') - P\big(\frac{1}{k-k'}\big) + i\pi\delta(k-k') \Big) \\
=& \frac{2\pi A^2 |f|^2}{|D|^2} \delta(k-k')
\end{aligned} \tag{45}$$

we choose $f = \frac{|D|}{\sqrt{2\pi}A}$ to fulfil the normalisation condition.

On the other hand,

$$\begin{aligned}
\langle k'_{(1)} | k_{(2)} \rangle =& f^* \Big(\int_{-\infty}^{0} g e^{i(k-k')x} \mathrm{d}x + \int_{0}^{\infty} \frac{A}{D^*} \big(\frac{A}{D} g + \frac{C}{D} h \big) e^{i(k-k')x} \mathrm{d}x + \int_{0}^{\infty} \frac{C}{D^*} \big(\frac{C^*}{D} g + \frac{A}{D} h \big) e^{-i(k-k')x} \mathrm{d}x \Big) \\
=& f^* \big(\frac{1}{i(k-k'-i\epsilon)} g - \frac{A}{D^*} \big(\frac{A}{D} g + \frac{C}{D} h \big) \frac{1}{i(k-k'+i\epsilon)} + \frac{C}{D^*} \big(\frac{C^*}{D} g + \frac{A}{D} h \big) \frac{1}{i(k-k'-i\epsilon)} \big) \\
=& f^* \pi \delta(k-k') \Big(\big(1 + \frac{A^2}{|D|^2} + \frac{|C|^2}{|D|^2}\big) g + \frac{2AC}{|D|^2} h \Big)
\end{aligned} \tag{46}$$

[5] The distributional formula $\frac{1}{x \pm i\epsilon} = P\big(\frac{1}{x}\big) \mp i\pi\delta(x)$, where $P\big(\frac{1}{x}\big)$ is the principal value of $\frac{1}{x}$, is used.

gives the following condition for the orthogonality

$$g = -\frac{\beta e^{i\eta}}{\alpha} h \quad \text{where } \alpha = 1 + \frac{A^2 + |C|^2}{|D|^2} \quad \beta = \frac{2A|C|}{|D|^2} \quad \eta = \mathrm{Arg}\ C \tag{47}$$

Finally, the normalisation of $|k_{(2)}\rangle$ imposes

$$\begin{aligned}
\delta(k-k') =& \langle k'_{(2)}|k_{(2)}\rangle \\
=& |h|^2 \Bigg(\int_{\infty}^{0} \Big(\frac{\beta^2}{\alpha^2} e^{i(k-k')x} + e^{-i(k-k')x} \Big) \mathrm{d}x + \int_0^{\infty} \underbrace{\Big| -\frac{A\beta}{D\alpha} e^{i\eta} + \frac{C}{D} \Big|^2}_{E} e^{i(k-k')x} \mathrm{d}x \\
&+ \int_0^{\infty} \underbrace{\Big| -\frac{C^*\beta}{D\alpha} e^{i\eta} + \frac{A}{D} \Big|^2}_{F} e^{-i(k-k')x} \mathrm{d}x \Bigg) \\
=& |h|^2 \Bigg(\frac{\beta^2}{i\alpha^2} \frac{1}{k-k'-i\epsilon} - \frac{1}{i(k-k'+i\epsilon)} - E \frac{1}{i(k-k'+i\epsilon)} + F \frac{1}{i(k-k'-i\epsilon)} \Bigg) \\
=& |h|^2 \Bigg(\Big(\frac{\beta^2}{\alpha^2} - 1 - E + F \Big) P\big(\frac{1}{x}\big) + i\pi\delta(x) \Big(\frac{\beta^2}{\alpha^2} + 1 + E + F \Big) P\big(\frac{1}{x}\big) \Bigg) \\
=& |h|^2 \pi \Bigg(\alpha \Big(\frac{\beta^2}{\alpha^2} + 1 \Big) - 2\frac{\beta^2}{\alpha} \Bigg) \\
\underset{\alpha^2-\beta^2=2\alpha}{=}& 2\pi |h|^2
\end{aligned} \tag{48}$$

we can therefore choose

$$h = \frac{1}{\sqrt{2\pi}}$$

We have thus found an orthonormal basis for our Hilbert space

$$\begin{aligned}
|k_{(1)}\rangle &= \frac{|D|}{\sqrt{2\pi}A} \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Theta(-x^1) + \frac{1}{D} \begin{pmatrix} A & 0 \\ 0 & C^* \end{pmatrix} \Theta(x^1) \right) \begin{pmatrix} e^{ikx^1} \\ e^{-ikx^1} \end{pmatrix} \\
|k_{(2)}\rangle &= \frac{1}{2\pi} \left(\begin{pmatrix} -\frac{\beta e^{i\eta}}{\alpha} & 0 \\ 0 & 1 \end{pmatrix} \Theta(-x^1) + \frac{1}{D} \begin{pmatrix} -A\frac{\beta e^{i\eta}}{\alpha} + C & 0 \\ 0 & -C^*\frac{\beta e^{i\eta}}{\alpha} + A \end{pmatrix} \Theta(x^1) \right) \begin{pmatrix} e^{ikx^1} \\ e^{-ikx^1} \end{pmatrix}
\end{aligned} \tag{49}$$

4 Self-adjointness of the Hamiltonian

In this section, we study the self-adjointness of the Hamiltonian in which the Kondo potential is involved. The Hilbert space in which we are going to work is $L^2(I_+ \cup I_-) \times L^2(I_+ \cup I_-)$, where $I_- = [-\frac{L}{2}, 0)$ and $I_+ = (0, \frac{L}{2}]$, with $I = [-\frac{L}{2}, \frac{L}{2}]$. The Hamiltonian as an operator on this space is defined as

$$H\phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \phi + \begin{pmatrix} v_3 & v_- \\ v_+ & -v_3 \end{pmatrix} \delta(x)\phi \tag{50}$$

To start, we choose for the domain of H as $\mathrm{Dom}(H) = \Big\{ \phi \mid \bar{\phi}_{|I_-} \in \mathcal{C}^1(I_-) \times \mathcal{C}^1(I_-),\ \ \bar{\phi}_{|I_+} \mathcal{C}^1(I_+) \times \mathcal{C}^1(I_+),\ \ -i\gamma^1 \phi|_{\pm \frac{L}{2}} = \pm \phi|_{\pm \frac{L}{2}} \Big\}$ where $\bar{\phi}$ is the extension of ϕ by continuity at 0 from the given interval. H is symmetric because of the boundary conditions and the fact that the elements in its domain possess right and left limits at $x = 0$.

A basic criterion of self-adjointness is given in [2]

Theorem 1 *Let T be a symmetric operator on a Hilbert space $\mathcal{H}$. The 3 following statement are equivalent*

1. *T is self-adjoint*
2. *T is closed and* $\ker(T^* \pm i) = \{0\}$
3. $\mathrm{Ran}(T \pm i) = \mathcal{H}$

We denote $\mathcal{K}_\pm = \ker(i \mp H^*)$ for the deficiency subspaces of H. The corollary of the Theorem X.2 of [3] states that $\dim \mathcal{K}_+ = \dim \mathcal{K}_-$ is a necessary and sufficient condition such that H possesses an self-adjoint extension (all closed extension of H is self-adjoint if this two numbers are equal to zero).
We start by calculate $\mathcal{K}_+$. Let $\phi \in \mathcal{K}_+$. Then $i\phi = H^*\phi$. As H is symmetric, this implies, for $\phi = \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix}$,

$$i \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} = i \begin{pmatrix} v_3 & v_- \\ v_+ & -v_3 \end{pmatrix} \delta(x) \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} \tag{51}$$

Thus, ϕ could be written as

$$\begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix} = \Theta(-x) \begin{pmatrix} f_- e^{x} \\ g_- e^{-x} \end{pmatrix} + \Theta(x) \begin{pmatrix} f_+ e^{x} \\ g_+ e^{-x} \end{pmatrix} \tag{52}$$

The boundary conditions give

$$\begin{cases} -ig_- e^{\frac{L}{2}} = -f_- e^{-\frac{L}{2}} \\ -ig_+ e^{-\frac{L}{2}} = f_+ e^{\frac{L}{2}} \end{cases} \Leftrightarrow \begin{cases} g_- = -if_- e^{-L} \\ g_+ = if_+ e^{L} \end{cases} \tag{53}$$

We have found that the matching condition at $x = 0$ gives a linear transformation, namely, $\phi(0^+) = T\phi(0^-)$ with T a certain matrix depending on v_i. With the boundary conditions, this implies

$$f_+ \begin{pmatrix} 1 \\ ie^L \end{pmatrix} = f_- T \begin{pmatrix} 1 \\ -ie^L \end{pmatrix} = f_- \begin{pmatrix} \frac{A}{D} - i\frac{C}{D}e^{-L} \\ \frac{C^*}{D} - i\frac{A}{D}e^{-L} \end{pmatrix} \tag{54}$$

f_+ and f_- are non-vanishing if and only if

$$\begin{aligned} ie^L &= \frac{C^* - iAe^{-L}}{A - iCe^{-L}} \\ \Leftrightarrow \quad 1 &= \frac{C^* e^L - iA}{iA + Ce^{-L}} \\ \Leftrightarrow \quad 2iA &= -Ce^{-L} + C^* e^L \end{aligned} \tag{55}$$

For $C = -iv_- = -iv_1 - v_2$ and $A = 1 + \frac{1}{4}(v_1^2 + v_2^2 + v_3^2)$, this requires

$$\begin{cases} v_2 = 0 \\ v_1 \cosh L = 1 + \frac{1}{4}(v_1^2 + v_3^2) \end{cases} \tag{56}$$

This condition is not verified in general, as the v_i do not depend on L, in which case we get $\dim \mathcal{K}_+ = 0$.
For $\mathcal{K}_-$, it suffices to replace L by $-L$ in the above calculation. Therefore, besides certain specific potentials, H is essentially self-adjoint.

References

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