Homework 5. CS 511.

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Problem 1

a $\phi_1(x) := x \times 0$

b $\phi_2(x) := (x \times 0) + 1$

c $\phi_3(x) :=< (x \times y), (x \times y) + 1 >$

d $\phi_4(x) :=< x, x + y + 1 >$

Problem 2

1. $(\forall x.\phi(x, f(x))) \vdash (\forall x.\exists y.\phi(x, y))$

	1	$(\forall x.\phi(x, f(x)))$	premise
x_0	2		fresh x_0
	3	$\phi(x_0, f(x_0))$	$\forall E$ 1
	4	$\exists y.\phi(x_0, y)$	$\exists I$ 3
	5	$\forall x.\exists y.(phi(x, y))$	I 2, 4

2. Let $(\forall x.\phi(x, f(x)))$ and $(\forall x.\exists y.\phi(x, y))$ fit under the model $M := \{\mathbb{Z}, <, -, f(x), \phi(x, y)\}$ where $\phi(x, y) := x < y$, and $f(x) := -x$.

   This holds true for $(\forall x.\exists y.\phi(x, y))$ since for all integers, there does exist one smaller than itself, but does not hold true for $(\forall x.\phi(x, f(x)))$, since the negative of an integer is not always smaller than itself, therefore $(\forall x.\exists y.\phi(x, y)) \not\models (\forall x.\phi(x, f(x)))$.

3. By completeness, we know that $(\forall x.\exists y.\phi(x, y)) \models (\forall x.\phi(x, f(x))) \to (\forall x.\exists y.\phi(x, y)) \vdash (\forall x.\phi(x, f(x)))$ and by soundness that $(\forall x.\exists y.\phi(x, y)) \vdash (\forall x.\phi(x, f(x))) \to (\forall x.\exists y.\phi(x, y)) \models (\forall x.\phi(x, f(x)))$, so $(\forall x.\exists y.\phi(x, y)) \vdash (\forall x.\phi(x, f(x))) \leftrightarrow (\forall x.\exists y.\phi(x, y)) \models (\forall x.\phi(x, f(x)))$.

Therefore, because we showed that $(\forall x.\exists y.\phi(x,y)) \not\models (\forall x.\phi(x, f(x)))$, then we also know that $(\forall x.\exists y.\phi(x,y)) \nvdash (\forall x.\phi(x, f(x)))$.

Problem 3

1. Let the model $M := \{\mathbb{N}, P(m,n)\}$ where

   $P(m,n) := m \geq n$.

   This satisfies that P is reflexive because arbitrary natural number is equal to itself, and P is transitive, since for any given numbers x, y, z, if x and $y \geq z$, then we know that $x geq z$. It does not satisfy that P is symmetric because if for 2 numbers x, y, $x \geq y$, then it is not necessarily true that $y \geq x$, in fact it is only true if $x = y$.

2. Let the model $M := \{\mathbb{R}, P(m,n)\}$ where

   $P(m,n) := m \times n > 0$.

   This does not satisfy that P is reflexive because 0 multiplied by itself is 0, which is not > 0. P is transitive, since for any given numbers x, y, z, if $x \times y > 0$ and $y \times z > 0$, then we know that x, and z must have the same sign, therefore $xz > 0$. P is symmetric because if for 2 numbers x, y, $x \times y > 0$, then it is necessarily true that $yx > 0$, since multiplication is commutative.

3. Let the model $M := \{\mathbb{N}, P(m,n)\}$ where

   $P(m,n) := (|m - n| \leq 1$.

   This satisfies that P is reflexive because the difference between a number and itself is 0 which is ≤ 1. It is symmetric since for any 2 numbers x, y, $x - y = -(y - x)$, which have the same absolute value. It is not transitive because it is not the case that for numbers x, y, z, where $|x - y| \leq 1$, and $|y - z| \leq 1$, that $|x - z| \leq 1$. Take $x = 2, y = 3, z = 4$. $|x - y| \leq 1$ and $|y - z| \leq 1$, but $|x - z| = 2$ which is not ≤ 1.

   Therefore, symmetry, reflexivity, and transitivity are each respectively not semantically entailed by the other 2.

Problem 4

Problem 5 code uploaded to this github repository: `https://github.com/ehchao88/CS511_HW5`

The multiplication table of G, as found by mace4 is the following:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	3	2	5	4
2	2	4	0	5	1	3
3	3	5	1	4	0	2
4	4	2	5	0	3	1
5	5	3	4	1	2	0