Singular value decomposition

Any $p \times q$ matrix A of rank r can be expressed as

$$A = L \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} M^T$$

where

(i) $L_{p\times p}$ and $M_{q\times q}$ are orthogonal matrices

(ii) $\Delta^2 = \Delta\,\Delta$ contains the positive (non-zero) eigenvalues of $A^T A$ and $A\,A^T$ and

$$L'AA'L = \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}, \quad M'A'AM = \begin{bmatrix} \Delta^2 & 0 \\ 0 & 0 \end{bmatrix}_{q\times q},$$

that is, L and M are orthogonal matrices having eigenvectors of AA' and $A'A$ as columns.

Singular value decomposition

Assume A be a symmetric & positive definite matrix.

- $A'A = AA'$ because $A = A'$ and thus $L = M$
- By the definition of spectral decomposition,

  $$A = \sum \lambda_i \mathbf{u}_i \mathbf{u}_i', \quad \lambda_1 \geq \lambda_1 \geq \cdots \geq \lambda_{p=q} > 0.$$

  and

  $$AA' = A'A = \sum \lambda_i^2 \mathbf{u}_i \mathbf{u}_i'.$$

  That is, eigenvalues of A is Δ.
- Thus, singular value decomposition is equivalent to spectral decomposition