Chapter 2. Transformations and Expectations

"We want something more than mere theory and preaching now, though." [Sherlock Homes, A Study in Scarlet]

Often, if we are able to model a phenomenon in terms of a random variable X with cdf $F_X(x)$, we will also be concerned with the behavior of **functions of** X. In this chapter we study techniques that allow us to gain information about functions of X that may be of interest.

2.1. Distribution of functions of a RV

- Probability, Random Variable (RV)

$$(S, \mathcal{B}, P) \xrightarrow{X} (R, \mathcal{B}^1, P_X)$$

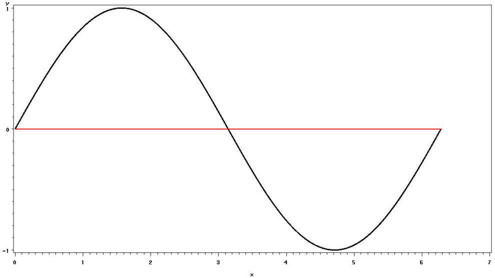
Q: What is the distribution of $g(X)$? That is,
We want to find $F_Y(y) = P[Y \leq y]$ given $X \sim F_X(x)$, where $Y = g(X)$.
$\triangleright$ Example: let X be the number thrown on a fair dice. Then the support of X is $\mathcal{X} = \{1, 2, 3, 4, 5, 6\}$. Let $Y = g(X) = 2X$ then support of Y is

$$\mathcal{Y} = \{2, 4, 6, 8, 10, 12\} = \{y : g(x) = y, x \in \mathcal{X}\},$$

where $g(\cdot)$ is an one-to-one and monotone increasing.

2.1. Distribution of functions of a RV

▷ Example: let $X \sim Unif(0, 2\pi)$. $Y = g(X) = \sin(X)$. Then the support of X is $\mathcal{X} = \{x : 0 < x < 2\pi\}$ and the support of Y is $\mathcal{Y} = \{y : -1 < y < 1\}$.



$g(\cdot)$ is not an one-to-one and not monotone increasing.

2.1. Distribution of functions of a RV

▷ Example: Toss a 6-sided die with $Y = g(X) = 2X$.

$$F_Y(y) = P[Y \leq y] = P[2X \leq y] = P[X \leq y/2].$$

$$F_Y(y) = \begin{cases} 0 & , y < 2, \\ 1/6 & , 2 \leq y < 4, \\ 2/6 & , 4 \leq y < 6, \\ 3/6 & , 6 \leq y < 8, \\ 4/6 & , 8 \leq y < 10, \\ 5/6 & , 10 \leq y < 12, \\ 1 & , 12 \leq y. \end{cases}$$

2.1. Distribution of functions of a RV

Lemma

If X is a continuous random variable with pdf $f_X(x)$ then the cdf of $Y = g(X)$ is

$$F_Y(y) = P[Y \leq y] = P[g(X) \leq y] = \int_{\{x \in \mathcal{X} : g(x) \leq y\}} f_X(x)dx.$$

▷ Example: Let $X \sim Unif(0,1)$. $Y_1 = g_1(X) = X^2$.

$$\mathcal{X} = \{x : 0 < x < 1\}\ ,\ \mathcal{Y} = \{y : 0 < y < 1\}.$$

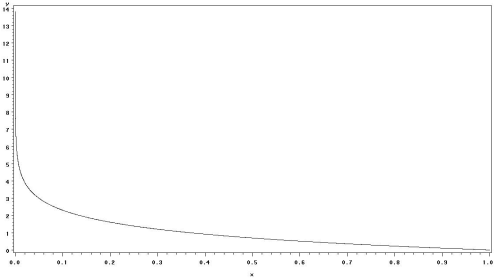
$$F_{Y_1}(y) = P[Y_1 \leq y] = P[X^2 \leq y] = P[X \leq \sqrt{y}]$$

$$F_{Y_1}(y) = \begin{cases} 0 & , y < 0, \\ & , 0 \leq y < 1, \\ 1 & , 1 \leq y. \end{cases}$$

2.1. Distribution of functions of a RV

▷ Example-Continued: $Y_2 = g_2(X) = -\ln(X)$.

$$\mathcal{X} = \{x : 0 < x < 1\}, \quad \mathcal{Y} = \{y : 0 < y < \infty\}.$$



$$\begin{aligned} F_{Y_2}(y) =& P[Y_2 \le y] = P[-\ln(X) \le y] = P[X \ge e^{-y}] \\ =& 1 - P[X \le e^{-y}] = \qquad\qquad . \end{aligned}$$

2.1. Distribution of functions of a RV

Using the cdf of X, we could get the cdf of $Y = g(X)$. If the function $g(\cdot)$ is a monotone function, cdf of Y can be derived systematically.

Definition

- Monotone Decreasing

$$u < \nu \Longrightarrow g(u) > g(\nu) \quad , \quad \text{for all} \quad u, \nu \in \mathcal{X}$$

- Monotone Increasing

$$u < \nu \Longrightarrow g(u) < g(\nu) \quad , \quad \text{for all} \quad u, \nu \in \mathcal{X}$$

Are monotone functions one-to-one function on $\mathcal{X}$?

2.1. Distribution of functions of a RV

Theorem

(Theorem 2.1.3) $X \sim F_X(x)$, $Y = g(X)$. $\mathcal{X}$ *and* $\mathcal{Y}$ *are supports of RV* X *and* Y*.*

- *If* $g(\cdot)$ *is monotone increasing*

$$F_Y(y) = F_X\left[g^{-1}(y)\right] \quad , \quad \text{for all } \; y \in \mathcal{Y}$$

- *If* $g(\cdot)$ *is monotone decreasing and* X *is* **continuous**

$$F_Y(y) = 1 - F_X\left[g^{-1}(y)\right] \quad , \quad \text{for all } \; y \in \mathcal{Y}$$

Proof: (For decreasing $g(\cdot)$, see what happen if $X = 0$ or 1 with equal probability.)

2.1. Distribution of functions of a RV

Theorem

(Theorem 2.1.5) Consider a continuous RV X with the pdf $f_X(x)$. $Y = g(X)$, where $g(\cdot)$is a continuous function. $\mathcal{X}$ and $\mathcal{Y}$ denote the supports of RV X and Y, respectively. If $f_X(x)$ is continuous on $\mathcal{X}$ and $g^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$, then

$$f_Y(y) = \begin{cases} f_X\left[g^{-1}(y)\right]\left|\frac{d}{dy}g^{-1}(y)\right| & , \quad y \in \mathcal{Y} \\ 0 & , \quad \textit{elsewhere} \end{cases}$$

$$\text{Jacobian} = J = \frac{d}{dy}g^{-1}(y)$$

2.1. Distribution of functions of a RV

See Example 2.1.4, and Example 2.1.7.

▷ Example: Let $X \sim Unif(0,1)$. $Y_1 = g_1(X) = X^2$. $Y_2 = g_2(X) = -\ln(X)$.

$$g_1^{-1}(y) = \qquad\qquad , J = \frac{d}{dy} g_1^{-1}(y) =$$

$$f_{Y_1}(y) =$$

$$g_2^{-1}(y) = \qquad\qquad , J = \frac{d}{dy} g_2^{-1}(y) =$$

$$f_{Y_2}(y) =$$

2.1. Distribution of functions of a RV

▷ Example:

$$X \sim f_X(x) = \begin{cases} 4x^3 & , \quad \mathcal{X} = \{x : 0 < x < 1\}, \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

$$Y = g(X) = e^X \quad , \quad \mathcal{Y} = \{y : \qquad\qquad\qquad \} \, .$$

$$J =$$

$$f_Y(y) =$$

▷ Example: X is a continuous RV with cdf $F_X(x)$. Let $Y = g(X) = X^2$. Then, for $y > 0$,

$$\begin{aligned} F_Y(y) =& P[Y \leq y] \\ =& p[-\sqrt{y} \leq X \leq \sqrt{y}] \\ =& \qquad\qquad\qquad\qquad . \end{aligned}$$

Thus, the pdf of Y is

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \quad .$$

Note that

$$\left| \frac{d}{dy} g^{-1}(y) \right| =$$

Theorem

Let X be a continuous RV with pdf $f_X(x)$. Partition of the support $\mathcal{X}$, $\{A_0, A_1, \cdots, A_k\}$, satisfies ($P[X \in A_0] = 0$) and $f_X(x)$ is continuous on each A_i. If $g_1(x), \cdots, g_k(x)$ defined on $A_1, \cdots, A_k$ satisfies

1. $g(x) = g_i(x) \ , \ \ x \in A_i$
2. *$g_i(x)$ is monotone on A_i*
3. *$\mathcal{Y} = \{y : y = g_i(x), x \in A_i\}$ is the same for each i*
4. *$g_i^{-1}(y)$ has a continuous derivative on $\mathcal{Y}$ for each i*

then

$$f_Y(y) = \sum_{i=1}^{k} f_X\left[g_i^{-1}(y)\right] \left|\frac{d}{dy} g_i^{-1}(y)\right| \quad , \quad y \in \mathcal{Y}$$

2.1. Distribution of functions of a RV

▷ Example

$$X \sim f_X(x) = \begin{cases} 4x^3 & , \quad \mathcal{X} = \{x : 0 < x < 1\}, \\ 0 & , \quad \text{elsewhere.} \end{cases}$$

$$Y = g(X) = (X - 0.5)^2 \quad , \quad \mathcal{Y} = \{y : \qquad\qquad\qquad \} \; .$$

$$A_0 = \{0.5\}, \; A_1 = \{ \qquad\qquad \}, \; A_2 = \{ \qquad\qquad \}$$

$$f_Y(y) =$$

Theorem

Let X be a continuous RV with cdf $F_X(x)$. Define a function of X as $Y = F_X(x)$. Then Y has an uniform distribution between 0 and 1. That is,

$$F_Y(y) = y \quad , \quad 0 < y < 1 \quad .$$

Proof)

(Read some detailed explanation in page 55.)

2.2. Expected Value

- Averaging according to the distribution of X
- Expected Value/ Average Value/ Mean Value

Definition

The *expected value* of a random variable $g(X)$, denoted $Eg(X)$, is

$$Eg(X) = \begin{cases} \int_{-\infty}^{\infty} g(x) f_X(x) dx & \text{if } X \text{ is continuous ,} \\ \sum_x g(x) f_X(x) & \text{if } X \text{ is discrete ,} \end{cases}$$

provided that the integral and sum exists.

Note: Existence of expected value $E|g(X)| < \infty$ should be checked first.

2.2. Expected Value

Example 2.2.2 and Example 2.2.4

- Example (Exponential mean)

$$f_X(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & x \geq 0, \lambda > 0 \\ 0, & \text{elsewhere} \end{cases}$$

$$Eg(X) = EX =$$

- Example (Cauchy mean)

$$f_X(x) = \frac{1}{\pi(1 + x^2)}, \quad -\infty < x < \infty.$$

$$Eg(X) = EX =$$

2.2. Expected Value

▷ Example: Let Y be the number of a fair coin tosses necessary to obtain the first tail. $p =$probability of tail.

$$f_Y(y) = \begin{cases} \qquad\qquad\qquad\qquad , & y = 1, 2, 3, \cdots, \\ 0 \qquad\qquad\qquad\quad , & \text{elsewhere} \end{cases}$$

$EY =$

2.2. Expected Value

Theorem

Let X be a RV and a, b, and c be constants. Then for any functions $g_1(x)$ and $g_2(x)$ whose expectations exist,

a. $E[ag_1(X) + bg_2(X) + c] = aEg_1(X) + bEg_2(X) + c$
b. *If* $g_1(x) \geq 0$ *for all* x, *then* $Eg_1(X) \geq 0$
c. *If* $g_1(x) \geq g_2(x)$ *for all* x, *then* $Eg_1(X) \geq Eg_2(x)$
d. *If* $a \leq g_1(x) \leq b$ *for all* x, *then* $a \leq Eg_1(X) \leq b$

2.3. Moments & mgf

Definition

For each integer n, the nth *moment* of X, μ'_n is

$$\mu'_n = EX^n \quad .$$

The nth *central moment* of X, μ_n, is

$$\mu_n = E(X - \mu)^n \quad ,$$

where $\mu = \mu'_1 = EX$.

Note:

1. If rth moment exist for $r > 0$, then the sth moment exists for $0 \le s \le r$
2. If rth moment fails to exist for $r > 0$, then the sth moment fails to exists for $s \ge r$

2.3. Moments & mgf

▷ Example

$$X \sim f_X(x) = \begin{cases} \frac{1}{x^2}, & x > 1, \\ 0, & \text{elsewhere}, \end{cases}$$

Is this a pdf ?

Does $\mu = EX$ exist ?

2.3. Moments & mgf

▷ Example

$$X \sim f_X(x) = \begin{cases} \frac{2}{x^3}, & x > 1, \\ 0, & \text{elsewhere}, \end{cases}$$

Is this a pdf ?

Does $\mu = EX$ exist ?

Does $\mu'_2 = EX^2$ exist ?

2.3. Moments & mgf

Definition

The *variance* of a random variable X is its second central moment,

$$Var(X) = E(X - EX)^2 = EX^2 - (EX)^2 \ .$$

The positive square root of $Var(X)$ is the standard deviation of X.

Theorem

If X is a random variable with finite variance, then for any constants a and b,

$$Var(aX + b) = a^2 Var(X)$$

Proof: (See Theorem 2.3.4)

2.3. Moments & mgf

Definition

Skewness of a random variable X is

$$\alpha_3 = \frac{\mu_3}{\mu_2^{3/2}} .$$

Note:

1. Measure of symmetry of the distribution.
2. $\alpha_3 = 0 \to$ Symmetric
3. $\alpha_3 < 0 \to$ Skewed to left
4. $\alpha_3 > 0 \to$ Skewed to right

2.3. Moments & mgf

Definition

Kurtosis of a random variable X is

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} .$$

Note:

1. Measure of peakedness or flatness of the distribution.
2. $\alpha_4 = 3 \rightarrow$ Normal distribution

$\triangleright$ Example

$$X \sim f_X(x) = 3x^2 \quad , \quad 0 < x < 1$$

$EX, \; EX^2, \; Var(X)$

2.3. Moments & mgf

▷ Example: Poisson(λ)

$$X \sim f_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, \cdots$$

EX, EX^2, $Var(X)$? Note that

$$e^y = \sum_{i=0}^{\infty} \frac{y^i}{i!} \quad .$$

2.3. Moments & mgf

Definition

Let X be a RV with cdf $F_X(x)$. The *moment generating function*(mgf) of X is

$$M_X(t) = Ee^{tX} = \begin{cases} \sum_x e^{tx} f_X(x), & \text{discrete}, \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)dx, & \text{continuous}, \end{cases}$$

provided that the expectation exists for t in some neighborhood of 0.
[Ee^{tX} exist for all t in $-h < t < h$, for some $h > 0$]

Theorem

If X has mgf $M_X(t)$, then

$$EX^n = M_X^{(n)}(0) \quad ,$$

where

$$M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)\Big|_{t=0} \quad .$$

Proof:

2.3. Moments & mgf

▷ Example: $X \sim \text{Poisson}(\lambda)$
$M_X(t) = Ee^{tX}$

EX, EX^2, $Var(X)$

▷ Example: $X \sim \text{Gamma}(\alpha, \beta)$, $\alpha > 0$, $\beta > 0$.

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \quad , \quad x > 0$$

Note:

1. $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$
2. $\Gamma(n) = (n-1)!$ for integer $n \geq 1$.
3. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$

$M_X(t) = Ee^{tX}$

EX, EX^2, $Var(X)$

Theorem

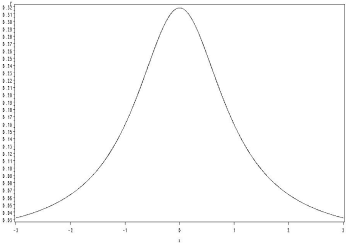
Let $F_X(x)$ and $F_Y(y)$ be two cdfs all of whose moments exist.

1. *If X and Y have bounded support, then $F_X(u) = F_Y(u)$ for all u, if and only if $E(X^r) = E(Y^r)$, for all integers $r = 0, 1, 2, \cdots$.*
2. *If the mgfs exist and $M_X(t) = M_Y(t)$ for all t in some neighborhood of 0, then $F_X(u) = F_Y(u)$ for all u.*

2.3. Moments & mgf

▷ Example: Cauchy distribution - No finite moments

$$f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \quad -\infty < x < \infty \quad .$$



$E|X| = \infty$

Theorem

Consider a sequence of RV's $\{X_i,\ i = 1, 2, \cdots\}$ *with mgf of each* $M_{X_i}(t)$. *Furthermore, suppose that*

$$\lim_{i\to\infty} M_{X_i}(t) = M_X(t), \quad \textit{for all,} \quad -h < t < h,\ \ h > 0$$

and $M_X(t)$ *is a mgf. Then there is an unique cdf* F_X *whose mgf is* $M_X(t)$ *and for all* x *where* $F_X(x)$ *is continuous, we have*

$$\lim_{x\to\infty} F_{X_i}(x) = F_X(x) \quad .$$

▷ Example: Binomial to Poisson

Let $X \sim \text{poisson}(\lambda)$ and $Y \sim \text{binomial}(n, \lambda/n)$. Note that the mgfs of X and Y are

$$M_X(t) = \exp[\lambda(e^t - 1)] \quad , \quad M_Y(t) = \left(\frac{\lambda}{n}e^t + 1 - \frac{\lambda}{n}\right)^n$$

$\lim_{n\to\infty} M_Y(t) =$

2.3. Moments & mgf

Theorem

(Theorem 2.3.15) Let $M_X(t)$ be a mgf of the RV X. Then the mgf of $Y = g(X) = aX + b$, for constant a and b, is

$$M_Y(t) = Ee^{tg(X)} = e^{tb}M_X(at) \quad .$$

Theorem

Let $X_1, \cdots X_n$ be independent RVs with mgf's $M_{X_i}(t)$, then mgf of $Y = \sum X_i$ is

$$M_Y(t) = M_{X_1}(t) \cdots M_{X_n}(t) \quad .$$

If X_i has the same distribution of X,

$$M_Y(t) = [M_X(t)]^n \quad .$$

2.3. Moments & mgf

▷ Example

$$X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

$$M_X(t) = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right] \quad .$$

Let $Y = \sum_{i=1}^n X_i$.
$M_Y(t) =$

Other generating function

Miscellanies(2.6.2)

1. Factorial moment generating functions (Fmgf)

$$G_X(t) = E\left(t^X\right)$$

$$\frac{d^r}{dt^r} E\left(t^X\right)\Big|_{t=1} = E[X(X-1)(X-2)\cdots(X-r+1)]$$

   Called r-th factorial moment.

2. Characteristic function

$$C_X(t) = E\left(e^{itX}\right) = E\left[\cos(tX) + i\sin(tX)\right]$$

   - Always exist
   - Unique
   - Moments can be generated: $C_X^{(r)}(0) = i^r EX^r$