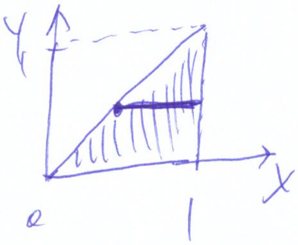
Chapeer 4. < slide #8 > ①

* Support ⇒

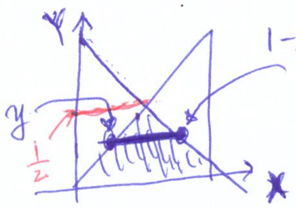


$$*\iint f(x,y)\,dxdy = \int_0^1\int_y^1 \frac{1}{x}\,dxdy = \int_0^1 \log x\Big|_y^1 dy = -\int_0^1 \log y\,dy = -(y\log y - y)\Big|_0^1$$

$$= -(1\cdot 0 - 1 - 0) = 1. \therefore \text{pdf}.$$

$$* \begin{cases} f_X(x) = \int_0^x f(x,y)\,dy = \int_0^x \frac{1}{x}\,dy = \frac{1}{x}\cdot y\Big|_0^x = 1, & 0<x<1 \\ f_Y(y) = \int_y^1 f(x,y)\,dx = \int_y^1 \frac{1}{x}\,dx = \log x\Big|_y^1 = -\log y, & 0<y<1 \end{cases}$$

* $P(X+Y\le 1) = ?$ Consider the support.



$$\int_0^{\frac{1}{2}}\int_y^{1-y} \frac{1}{x}\,dx\,dy = \int_0^{\frac{1}{2}} \log(1-y) - \log(y)\,dy$$

$$= -y - (1-y)\log(1-y) - (y\log y - y)\Big|_0^{\frac{1}{2}} = \log 2.$$

$$* E(XY) = \iint f(x,y)\,xy\,dxdy = \int_0^1\int_y^1 y\,dxdy = \int_0^1 y(1-y)\,dy$$

$$= \left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]\Big|_0^1 = \left(\frac{1}{2}-\frac{1}{3}\right) - 0 = \frac{1}{6}.$$

※ Try $P(X+Y\le 1)$ by

$$\int_0^{\frac{1}{2}}\int_0^x \frac{1}{x}\,dy\,dx + \int_{\frac{1}{2}}^1\int_0^{1-x} \frac{1}{x}\,dy\,dx$$

This will give the same result as

$$\int_0^{\frac{1}{2}}\int_y^{1-y} \frac{1}{x}\,dx\,dy.$$

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Support. $\{(x,y) : x=0,1,\cdots,y, \quad y=0,1,2,\cdots\}$.

$$f_X(x) = \sum_{y=0}^{\infty} f(x,y) = \sum_{y=0}^{\infty} \binom{y}{x} p^x (1-p)^{y-x} \frac{e^{-\lambda}\lambda^y}{y!}$$

$$= p^x e^{-\lambda} \sum_{y=0}^{\infty} \frac{\cancel{y!}}{x!(y-x)!} \frac{(1-p)^{y-x}\lambda^y}{\cancel{y!}}$$

$$= \frac{p^x e^{-\lambda}}{x!} \sum_{y=0}^{\infty} \frac{(\lambda(1-p))^{y-x} e^{-\lambda(1-p)}}{(y-x)!} \cdot e^{\lambda(1-p)} \lambda^x$$

$$= \frac{(\lambda p)^x e^{-\lambda p}}{x!} \sum_{y=x}^{\infty} \frac{(\lambda(1-p))^{y-x} e^{-\lambda(1-p)}}{(y-x)!}.$$

Note that the maximum of x is y.
(or, equivalently the minimum of y is x).

$$= \frac{(\lambda p)^x e^{-\lambda p}}{x!}, \quad x = 0,1,2,\cdots$$

$$f_Y(y) = \frac{e^{-\lambda}\lambda^y}{y!} \sum_{x=0}^{y} \underbrace{\binom{y}{x} p^x (1-p)^{y-x}}_{\text{pmf of Binomial}(y,p)} = \frac{e^{-\lambda}\lambda^y}{y!}, \quad y=0,1,\cdots$$

$$E(XY) = \sum_{y=0}^{\infty} \frac{y e^{-\lambda}\lambda^y}{y!} \underbrace{\sum_{x=0}^{y} x \binom{y}{x} p^x (1-p)^{y-x}}_{= E[X] \text{ when } X \sim B(y,p)}$$

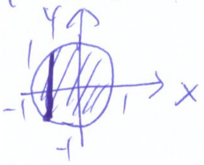
$$= \sum_{y=0}^{\infty} \frac{y e^{-\lambda}\lambda^y}{y!} yp = p\,E[Y^2] = p\left(Var(Y) + E(Y)^2\right)$$

$$= p(\lambda^2 + \lambda).$$

⟨Exercise 4.14-(a)⟩

$X \sim N(0,1)$, $Y \sim N(0,1)$. X and Y are indep.

$P(X^2+Y^2<1)=?$



Let $x = r\cos\theta$, $y = r\sin\theta$. then $x^2+y^2=r^2$, $dxdr = r\,dr\,d\theta$.

$$P(X^2+Y^2<1) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2\pi} e^{-(x^2+y^2)/2}\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 \frac{1}{2\pi} e^{-r^2/2}\cdot r\,dr\,d\theta$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} e^{-r^2/2}\Big|_0^1\,d\theta = -\frac{1}{2\pi}\int_0^{2\pi}\left(e^{-1/2}-1\right)d\theta$$

$$= \frac{1}{2\pi}\left(1-e^{-1/2}\right)2\pi = 1-\frac{1}{\sqrt{e}} \approx 0.39347$$

⟨Exercise 4.16-(a)⟩

$X \sim Geo(p)$, $Y \sim Geo(p)$, $X \perp Y$. $U = \min(X,Y)$. $V = X - Y$.

Show that U and V are indep (i.e. $U \perp V$).

$$U = \begin{cases} X & \text{if } X<Y \quad \text{- case 1} \\ Y & \text{if } X \ge Y \quad \text{- case 2.} \end{cases}$$

1) $X<Y$.

$P(U=u, V=v)$

$= P(X=u, V=X-Y=v)$

$= P(X=u, Y=u-v)$

$= P(X=u)\,P(Y=u-v)$

$= p(1-p)^{u-1}\cdot p(1-p)^{u-v-1}$

Here $u = 1,2,3,\cdots$

$v = -1, -2, -3, \cdots$

2) $X \ge Y$

$P(U=u, V=v)$

$= P(Y=u, X-Y=v)$

$= P(Y=u, X=Y+v=u+v)$

$= P(Y=u)\,P(X=u+v)$

$= p(1-p)^{u-1}\cdot p(1-p)^{u+v-1}$

Here $u=1,2,3,\cdots$

$v = 0,1,2,3,\cdots$

Combine case 1) and case 2)

$\Rightarrow P(U=u, V=v)$

$= p(1-p)^{u-1}\cdot p(1-p)^{u+|v|-1}$

$= P(U=u)\cdot P(V=v)$

where $u=1,2,3\cdots$

$v = 0, \pm1, \pm2, \cdots$

Thus U and V are independent.

$$f_{X,Y}(x,y) = \binom{y}{x} p^x (1-p)^{y-x} \frac{e^{-\lambda}\lambda^y}{y!}, \quad x=0,1,\cdots,y, \; y=0,1,\cdots$$

$*$ $Y \sim Poisson(\lambda)$, $X \sim Poisson(\lambda p)$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{\binom{y}{x} p^x (1-p)^{y-x} \frac{e^{-\lambda}\lambda^y}{y!}}{e^{-\lambda p}(\lambda p)^x / x!} = \frac{\frac{\cancel{y}!}{x!(y-x)!} \frac{p^x (1-p)^{y-x} e^{-\lambda}\lambda^y}{y!}}{e^{-\lambda p}(\lambda p)^x / x!}$$

$$= \frac{1}{(y-x)!} e^{-\lambda(1-p)} (\lambda(1-p))^{y-x}, \quad y-x = 0,1,2,\cdots$$

$\therefore$ $Y|X \sim Poisson(\lambda(1-p))$

$$f_{X|Y}(x|y) = f_{X,Y}(x,y) / f_Y(y)$$

$$= \frac{\binom{y}{x} p^x (1-p)^{y-x} \frac{e^{-\lambda}\lambda^y}{y!}}{e^{-\lambda}\lambda^y / y!}$$

$$= \binom{y}{x} p^x (1-p)^{y-x}$$

$\therefore$ $X|Y \sim Binomial(y, p)$

$$f_U(u) = \sum_{v \in Z} f_{U,V}(u,v) = p^2(1-p)^{2u-2} \sum_{v \in Z} (1-p)^{|v|} \quad , v \in Z = \{0, \pm 1, \pm 2, \ldots\}, \; u = 1,2,\cdots$$

$$= p^2(1-p)^{2u-2}\left(1 + 2\frac{1-p}{1-(1-p)}\right)$$

$$= p(1-p)^{2u-2}(2-p)$$

$$f_V(v) = \sum_{u=1}^{\infty} f_{U,V}(u,v) = p^2(1-p)^{|v|-2} \sum_{u=1}^{\infty} (1-p)^{2u} \quad , \; v = 0, \pm 1, \pm 2, \cdots$$

$$= p^2(1-p)^{|v|-2}\left(\frac{(1-p)^2}{1-(1-p)^2}\right)$$

$$= \frac{p(1-p)^{|v|}}{2-p}$$

Because $f_{U,V}(u,v) = f_U(u) \bullet f_V(v)$, U and V are independent.

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$f_{Y|X}(y|x) = \frac{1}{x} I(0<y<x<1) / I(0<x<1) = \frac{1}{x} I(0<y<x<1)$.

$f_{X|Y}(x|y) = \frac{1}{x} I(0<y<x<1) / ((-\log y) I(0<y<1))$

$= -\frac{1}{x \log y} I(0<y<x<1)$

$P(X<\frac{3}{4}|Y=\frac{1}{2}) = \int_{\frac{1}{2}}^{3/4} -\frac{1}{x \log \frac{1}{2}} dx = \frac{1}{\log 2} \int_{\frac{1}{2}}^{3/4} \frac{1}{x} dx = \frac{\log(\frac{3}{4}) - \log(\frac{1}{2})}{\log 2}$.

< Slide #21 >

$f(x,y) = e^{-y} I(0<x<y<\infty)$, $f_X(x) = e^{-x} I(0<x<\infty)$, $f_Y(y) = y e^{-y} I(0<y<\infty)$

① $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{e^{-y} I(0<x<y<\infty)}{e^{-x} I(0<x<\infty)} = e^{-(y-x)} I(0<x<y<\infty)$.

② $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-y} I(0<x<y<\infty)}{y e^{-y} I(0<y<\infty)} = \frac{1}{y} I(0<x<y<\infty)$

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$E[Y] = \iint y f(x,y) dy dx = \iint y \frac{f_{X,Y}(x,y)}{f_X(x)} f_X(x) dy dx$

$= \int f_X(x) \int y f_{Y|X}(y|x) dy dx = \int f_X(x) E[Y|X] dx = E[E[Y|X]]$

Let $Var(Y) = \mu_Y$.

$Var(Y) = E[(Y-\mu_Y)^2] = E[(Y - E[Y|X] + E[Y|X] - \mu_Y)^2]$

$= E[(Y-E(Y|X))^2] + E[(E(Y|X) - \mu_Y)^2]$

$+ 2E[(Y - E(Y|X))(E(Y|X) - \mu_Y)]$

$= Var(E(Y|X)) + E[Var(Y|X)] + 0$, where the last term is zero because

$\iint (y - E[Y|x])(E[Y|x] - \mu_Y) f_{X,Y}(x,y) dy dx$

$= \int (E[Y|x] - \mu_Y) \left\{ \int (y - E[Y|X]) \frac{f_{X,Y}(x,y)}{f_X(x)} dy \right\} f_X(x) dx$ is zero as the term in $\{ \}$ is zero

<Slide #36>.

$X \sim N(0,1)$, $Y \sim N(0,1)$. $X \perp\!\!\!\perp Y$.

$U = \dfrac{X+Y}{\sqrt{2}}$, $V = \dfrac{X-Y}{\sqrt{2}}$.

Find the joint and marginal distribution of U and V.

$\circledast$ $X = \dfrac{U+V}{\sqrt{2}}$, $Y = \dfrac{U-V}{\sqrt{2}}$.

$$f_{U,V}(u,v) = f_{X,Y}(x,y)\,|J|$$

$$\overset{\text{ind}}{=} f_X\left(\frac{u+v}{\sqrt{2}}\right) f_Y\left(\frac{u-v}{\sqrt{2}}\right)|J|, \quad J = \begin{vmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{vmatrix} = -1$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(u+v)^2}{4}\right\} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(u-v)^2}{4}\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) \cdot \frac{1}{\sqrt{2\pi}} \exp(-v^2/2) = f_U(u) \cdot f_V(v), \text{ where}$$

$-\infty < u < \infty$, $-\infty < v < \infty$.

<Slide #38>.

$X \sim \text{Gamma}(\alpha, 1)$, $Y \sim \text{Gamma}(\beta, 1)$, $X \perp\!\!\!\perp Y$, $U = \dfrac{X}{X+Y}$, $V = X+Y$.

$f_U(u) = ?$

Note $X = UV$, $Y = V(1-U)$.

$$f_{U,V}(u,v) = f_{X,Y}(uv, v(1-u))\,|J|, \quad J = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v$$

$$= \frac{(uv)^{\alpha-1}}{\Gamma(\alpha)} e^{-uv} \cdot \frac{\{v(1-u)\}^{\beta-1}}{\Gamma(\beta)} e^{-v(1-u)} \cdot v$$

$$= u^{\alpha-1}(1-u)^{\beta-1} v^{\alpha+\beta-1} e^{-v} / \Gamma(\alpha)\Gamma(\beta)$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \cdot \underbrace{\frac{1}{\Gamma(\alpha+\beta)} v^{\alpha+\beta-1} e^{-v}}_{\sim \text{Gamma}(\alpha+\beta,\,1)}, \quad 0 < u < 1,\ 0 < v < \infty.$$

$$f_U(u) = \int_0^\infty f_{U,V}(u,v)\,dv = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1}. \quad \therefore U \sim \text{Beta}(\alpha, \beta).$$

Extra Example

$X \sim N(0,1)$, $Y \sim \chi^2(r)$. $X \perp\!\!\!\perp Y$, $U = X\sqrt{\frac{r}{Y}}$.

$f_U(u) = ?$

Let $V = \sqrt{\frac{Y}{r}}$, then $X = UV$, $Y = rV^2$.

$$J = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\ \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 2rv \end{vmatrix} = 2rv^2.$$

$$f_{U,V}(u,v) = f_{X,Y}(uv, rv^2)|J| = f_X(uv) f_Y(rv^2) \cdot 2rv^2$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2v^2}{2}\right) \cdot \frac{1}{\Gamma(\frac{r}{2}) 2^{r/2}} (rv^2)^{\frac{r}{2}-1} e^{-rv^2/2} \cdot 2rv^2$$

$$= \underbrace{\frac{2r \cdot r^{\frac{r}{2}-1}}{\sqrt{2\pi}\, \Gamma(\frac{r}{2}) 2^{r/2}}}_{\text{Let } C} \exp\left(-\frac{v^2}{2}(u^2+r)\right) v^r, \quad 0 < v < \infty, -\infty < u < \infty$$

$$f_U(u) = \int_0^\infty f_{U,V}(u,v)\,dv = C\int_0^\infty \exp\left(-\frac{v^2}{2}(u^2+r)\right) v^r\,dv$$

Let $t = \frac{v^2}{2}(u^2+r)$, then $dt = v\,dv(u^2+r)$. Or $v\,dv = (u^2+r)^{-1}dt$.

And we have $\frac{2t}{u^2+r} = v^2 \Leftrightarrow v^{r-1} = \left[\frac{2t}{u^2+r}\right]^{\frac{r-1}{2}}$

Then $f_U(u) = C\int_0^\infty e^{-t} \left[\frac{2t}{u^2+r}\right]^{\frac{r-1}{2}} \frac{1}{u^2+r}\,dt$

$$= \frac{C\, 2^{\frac{r-1}{2}}}{(u^2+r)^{\frac{r+1}{2}}} \cdot \Gamma\left(\frac{r+1}{2}\right) \int_0^\infty \underbrace{\frac{e^{-t} t^{\frac{r+1}{2}-1}}{\Gamma(\frac{r+1}{2})}}_{\text{pdf of Gamma}(\frac{r+1}{2}, 1)} dt$$

$$= \frac{2r \cdot r^{\frac{r}{2}-1}\, 2^{\frac{r-1}{2}}\, \Gamma(\frac{r+1}{2})}{\sqrt{2\pi}\, \Gamma(\frac{r}{2}) 2^{r/2} (u^2+r)^{\frac{r+1}{2}}}$$

$$= \frac{1}{\sqrt{\pi r}} \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r}{2})} \left[\frac{u^2}{r}+1\right]^{-\frac{r+1}{2}}, \quad -\infty < u < \infty \quad \therefore U \sim t(r).$$

t-dist. w/ r degrees of freedom.

$X \sim N(0,1)$, $Y \sim N(0,1)$, $X \perp\!\!\!\perp Y$, $Z = \begin{cases} X & \text{if } XY > 0 \\ -X & \text{if } XY < 0. \end{cases}$

$f_Z(z) = ?$

We consider case 1) $X > 0$ and case 2) $X < 0$.

case 1). $P(Z \le z) = P(X \le z, Y > 0) + P(-X \le z, Y < 0)$

$\overset{\text{indep}}{=} P(X \le z)\, P(Y > 0) + P(X \le -z)\, P(Y < 0)$

$= \Phi(z)\frac{1}{2} + \{\underbrace{1 - \Phi(-z)}_{= \Phi(z)}\}\frac{1}{2}$, where $\Phi(\cdot)$ is CDF of standard normal.

$= \Phi(z)$

case 2). $P(Z \le z) = P(X \le z, Y < 0) + P(-X \le z, Y > 0)$

$= \Phi(z) \cdot \frac{1}{2} + \{1 - \Phi(-z)\}\frac{1}{2} = \Phi(z)$.

Thus, from case 1 and 2, we see $P(Z \le z) = \Phi(z)$

$\Rightarrow P(Z = z) = \phi(z)$. $\therefore Z \sim N(0, 1)$.

First note that $YZ > 0$ always!

Now, consider (1) $X>0, Y>0$, (2) $X>0, Y<0$, (3) $X<0, Y>0$, (4) $X<0, Y<0$.

$P(Y \le y, Z \le z) = P(Y \le y, X \le z, Y > 0) + P(Y \le y, X \ge -z, Y < 0) + P(Y \le y, X \ge -z, Y > 0) + P(Y \le y, X \le z, Y < 0)$

if $y > 0$

$\Rightarrow P(0 < Y \le y)\Phi(z) + P(Y < 0)\Phi(z) + P(0 < Y \le y)\Phi(z) + P(Y < 0)\Phi(z)$

$= \{\Phi(y) - \frac{1}{2}\}\Phi(z) + \frac{1}{2}\Phi(z) + \{\Phi(y) - \frac{1}{2}\}\Phi(z) + \frac{1}{2}\Phi(z) = 2\Phi(y)\Phi(z)$

if $y < 0$

$\Rightarrow \Phi(y)\Phi(z) + \Phi(y)\Phi(z) = 2\Phi(y)\Phi(z)$

Thus $P(Y = y, Z = z) = \begin{cases} 2\phi(y)\phi(z) & \text{if } yz > 0 \\ 0 & \text{if } yz < 0 \end{cases}$

(This is not a bivariate normal).