Ch 6. Comparisons of Several Multivariate Means

- Paired Comparisons

  - Measurements are often recorded under different sets of experimental conditions to see whether the responses differ significantly over these sets.

  - Example. Efficacy of a new drug is measured before and after the "treatment."

  - To compare two treatments, or the presence and absence of a single treatment, we can assign both treatments to the *same* or *identical* units.

  - The paired responses are analyzed by computing their differences, thereby eliminating much of the influence of extraneous unit-to-unit variation.

Ch 6. Comparisons of Several Multivariate Means

- Univariate case

  - X_{j1}: the response to treatment 1 (or the response before treatment) for jth trial

    X_{j2} : the response to treatment 2 (or the response after treatment) for jth trial

    (X_{j1}, X_{j2}) are measurements recorded on the jth unit or jth pair of like units.

  - $D_j = X_{j1} - X_{j2}$, $j = 1, 2, .., n$, reflect the differential effects of the treatments.

  - Given that $D_j \sim N(\delta, \sigma^2_d)$,

  $$t = \frac{\overline{D} - \delta}{s_d / \sqrt{n}} \sim t_{n-1},$$

  where $\overline{D} = \dfrac{1}{n}\sum_{j=1}^{n} D_j$ and $s_d^2 = \dfrac{1}{n-1}\sum_{j=1}^{n}\left(D_j - \overline{D}\right)^2.$

  - To test H_0: $\delta = 0$ vs. H_1: $\delta \neq 0$, compare $|t|$ with $t_{n-1}(\alpha/2)$.

  - A $100(1 - \alpha)$% confidence interval for the mean difference $\delta = E(X_{j1} - X_{j2})$ is

  $$\overline{d} - t_{n-1}(\alpha/2)\frac{s_d}{\sqrt{n}} \leq \delta \leq \overline{d} + t_{n-1}(\alpha/2)\frac{s_d}{\sqrt{n}}.$$

Paired Comparisons

- Multivariate extension of the paired comparison

  - p responses within the jth unit

    X_{1j1} = variable 1 under treatment 1

    X_{1j2} = variable 2 under treatment 1

    $\vdots$

    X_{1jp} = variable p under treatment 1

    X_{2j1} = variable 1 under treatment 2

    X_{2j2} = variable 2 under treatment 2

    $\vdots$

    X_{2jp} = variable p under treatment 2

  - The p paired-difference random variables become

    $D_{j1} = X_{1j1} - X_{2j1}$

    $D_{j2} = X_{1j2} - X_{2j2}$

    $\vdots$

    $D_{jp} = X_{1jp} - X_{2jp}$

  - Let $D_j{}' = [D_{j1}, D_{j2}, \ldots D_{jp}]$.

Paired Comparisons

- Assume, for $j = 1, 2, ..., n$, that

$$E(D_j) = \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_p \end{bmatrix} \quad \text{and} \quad Cov(D_j) = \Sigma_d.$$

  - If $D_1, D_2, \ldots, D_n$ are independent $N_p(\delta, \Sigma_d)$ random vectors, inferences about the vector of mean differences δ can be based upon a T^2-statistic:

  $$T^2 = n(\overline{D} - \delta)' S_d^{-1}(\overline{D} - \delta),$$

  where $\overline{D} = \dfrac{1}{n}\sum_{j=1}^{n} D_j$ and $S_d = \dfrac{1}{n-1}\sum_{j=1}^{n}(D_j - \overline{D})(D_j - \overline{D})'$.

- <u>Result 6.1</u> Let the differences $D_1, D_2, \ldots, D_n$ be a random sample from an $N_p(\delta, \Sigma_d)$ population. Then,

  $$T^2 = n(\overline{D} - \delta)' S_d^{-1}(\overline{D} - \delta),$$

  is distributed as an $[(n-1)p/(n-p)]F_{p, n-p}$ random variable, whatever the true δ and Σ_d.

  If n and $n - p$ are both large, T^2 is approximately distributed as a χ^2_p random variable, regardless of the form of the underlying population of differences.

Paired Comparisons

- The condition $\delta = 0$ is equivalent to "no average difference between the two treatments." For the ith variable, $\delta_i > 0$ implies that treatment 1 is larger, on average, than treatment 2. In general, inferences about δ can be made using Result 6.1.

- Given the observed differences $d_j' = [d_{j1}, d_{j2},\ldots, d_{jp}], j = 1,2,\ldots, n$, corresponding to the random variables, $D_{j1}, D_{j2},\ldots D_{jp}$, an **$\alpha$-level test** of H_0: $\delta = 0$ vs. H_1: $\delta \neq 0$ for an $N_p(\delta, \Sigma_d)$ population rejects H_0 if the observed

$$T^2 = n\bar{d}' S_d^{-1} \bar{d} > \frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha),$$

where $F_{p, n-p}(\alpha)$ is the upper (100α)th percentile of an F-distribution with p and $n - p$ d.f.

Paired Comparisons

- A $100(1 - \alpha)\%$ **confidence region** for δ consists of all δ such that

$$n\left(\bar{d} - \delta\right)' S_d^{-1}\left(\bar{d} - \delta\right) \leq \frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha).$$

Also, $100(1 - \alpha)\%$ **simultaneous confidence intervals** for the *individual mean differences* δ_i are given by

$$\delta_i : \quad \bar{d}_i \pm \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{d_i}^2}{n}},$$

where $\bar{d}_i$ is the *i*th elements of $\bar{d}$ and $s_{d_i}^2$ is the *i*th diagonal elements of S_d. For $n - p$ large, $[(n - 1)p/(n - p)]F_{p,n\text{-}p}(\alpha) \cong \chi^2_p(\alpha)$ and normality needs not be assumed.

The **Bonferroni $100(1 - \alpha)\%$ simultaneous confidence intervals** for the individual mean differences are

$$\delta_i : \quad \bar{d}_i \pm t_{n-1}\left(\frac{\alpha}{2p}\right) \sqrt{\frac{s_{d_i}^2}{n}},$$

where $t_{n\text{-}1}(\alpha/2p)$ is the upper $100(\alpha/2p)$th percentile of a *t*-distribution with $n - 1$ d.f.

6.3. Comparing Mean Vectors from Two Populations

- A T^2-statistic for testing the equality of vector means from two multivariate populations can be developed by analogy with the univariate procedure.

- Consider a random sample of size n_1 from population 1 and a sample of size n_2 from population 2.

  - Observations on p variables can be arranged as follows:

Sample	Summary statistics	
(Population 1) $x_{11}, x_{12}, ..., x_{1n_1}$	$\bar{x}_1 = \dfrac{1}{n_1}\displaystyle\sum_{j=1}^{n_1} x_{1j}$	$S_1 = \dfrac{1}{n_1 - 1}\displaystyle\sum_{j=1}^{n_1} \left(x_{1j} - \bar{x}_1\right)\left(x_{1j} - \bar{x}_1\right)'$
(Population 2) $x_{21}, x_{22}, ..., x_{2n_2}$	$\bar{x}_2 = \dfrac{1}{n_2}\displaystyle\sum_{j=1}^{n_2} x_{2j}$	$S_2 = \dfrac{1}{n_2 - 1}\displaystyle\sum_{j=1}^{n_2} \left(x_{2j} - \bar{x}_2\right)\left(x_{2j} - \bar{x}_2\right)'$

  - Want to make inferences about

    (mean vector of population 1) - (mean vector of population 2) = $\mu_1 - \mu_2$.

6.3. Comparing Mean Vectors from Two Populations

- *Assumptions Concerning the Structure of the Data*

  1. The sample $X_{11}, X_{12}, ..., X_{1n_1}$ is a random sample of size n_1 from a p-variate population with mean vector μ_1 and covariance matrix Σ_1.

  2. The sample $X_{21}, X_{22}, ..., X_{2n_2}$ is a random sample of size n_2 from a p-variate population with mean vector μ_2 and covariance matrix Σ_2.

  3. $X_{11}, X_{12}, ..., X_{1n_1}$ are independent of $X_{21}, X_{22}, ..., X_{2n_2}$.

- *Further Assumptions When n_1 and n_2 Are Small*

  1. Both populations are multivariate normal.

  2. $\Sigma_1 = \Sigma_2$ (same covariance matrix).

  - The second assumption is much stronger than its univariate counterpart.

6.3. Comparing Mean Vectors from Two Populations

- When $\Sigma_1 = \Sigma_2 = \Sigma$, $\sum_{j=1}^{n_1} (x_{1j} - \bar{x}_1)(x_{1j} - \bar{x}_1)'$ is an estimate of $(n_1 - 1)\Sigma$ and

  $\sum_{j=1}^{n_2} (x_{2j} - \bar{x}_2)(x_{2j} - \bar{x}_2)'$ is an estimate of $(n_2 - 1)\Sigma$. Consequently, we can pool

  the information in both samples in order to estimate the common
  covariance Σ:

$$S_{pooled} = \frac{\sum_{j=1}^{n_1} (x_{1j} - \bar{x}_1)(x_{1j} - \bar{x}_1)' + \sum_{j=1}^{n_2} (x_{2j} - \bar{x}_2)(x_{2j} - \bar{x}_2)'}{n_1 + n_2 - 2} = \frac{n_1 - 1}{n_1 + n_2 - 2} S_1 + \frac{n_2 - 1}{n_1 + n_2 - 2} S_2.$$

- Since $\sum_{j=1}^{n_1} (x_{1j} - \bar{x}_1)(x_{1j} - \bar{x}_1)'$ has $(n_1 - 1)$ d.f. and $\sum_{j=1}^{n_2} (x_{2j} - \bar{x}_2)(x_{2j} - \bar{x}_2)'$ has

  $(n_2 - 1)$ d.f., the divisor $(n_1 - 1) + (n_2 - 1)$ is obtained by combining the two
  component degrees of freedom.

6.3. Comparing Mean Vectors from Two Populations

- Note that

$$E(\bar{X}_1 - \bar{X}_2) = E(\bar{X}_1) - E(\bar{X}_2) = \mu_1 - \mu_2$$

and since the independence assumption implies that $\bar{X}_1$ and $\bar{X}_2$ are independent and thus $Cov(\bar{X}_1, \bar{X}_2) = 0$,

$$Cov(\bar{X}_1 - \bar{X}_2) = Cov(\bar{X}_1) + Cov(\bar{X}_2) = \frac{1}{n_1}\Sigma + \frac{1}{n_2}\Sigma = \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\Sigma.$$

- Since S_{pooled} estimates Σ, $\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)S_{pooled}$ is an estimator of $Cov(\bar{X}_1 - \bar{X}_2)$.

- The likelihood ratio test of $H_0: \mu_1 - \mu_2 = \delta_0$

- Reject H_0 if

$$T^2 = (\bar{x}_1 - \bar{x}_2 - \delta_0)'\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)S_{pooled}\right]^{-1}(\bar{x}_1 - \bar{x}_2 - \delta_0) > c^2,$$

where the critical distance c^2 is determined from the distribution of the two-sample T^2-statistic.

6.3. Comparing Mean Vectors from Two Populations

- <u>Result 6.2</u> If $X_{11}, X_{12}, ..., X_{1n_1}$ is a random sample of size n_1 from $N_p(\mu_1, \Sigma)$ and $X_{21}, X_{22}, ..., X_{2n_2}$ is an independent random sample of size n_2 from $N_p(\mu_2, \Sigma)$, then

$$T^2 = \left(\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)\right)' \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled}\right]^{-1} \left(\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)\right)$$

is distributed as

$$\frac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} F_{p, n_1 + n_2 - p - 1}.$$

Consequently,

$$P\left[\left(\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)\right)' \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled}\right]^{-1} \left(\overline{X}_1 - \overline{X}_2 - (\mu_1 - \mu_2)\right) \leq c^2\right] = 1 - \alpha,$$

where $c^2 = \dfrac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} F_{p, n_1 + n_2 - p - 1}(\alpha).$

6.3. Comparing Mean Vectors from Two Populations

- Result 6.2 *Proof.*

  First note that $\bar{X}_1 - \bar{X}_2 = \frac{1}{n_1}X_{11} + \frac{1}{n_1}X_{12} + ... + \frac{1}{n_1}X_{1n_1} - \frac{1}{n_2}X_{21} + \frac{1}{n_2}X_{22} + ... + \frac{1}{n_2}X_{2n_2}$

  is distributed as $N_p\left(\mu_1 - \mu_2, \left(\frac{1}{n_1} + \frac{1}{n_2}\right)\Sigma\right)$

  by Result 4.8 with $c_1 = c_2 = \cdots = c_{n_1} = \frac{1}{n_1}$ and $c_{n_1+1} = c_{n_1+2} = \cdots = c_{n_1+n_2} = -\frac{1}{n_2}$.

  Second, $(n_1 - 1)S_1$ is distributed as $W_{n_1-1}(\Sigma)$ and $(n_2 - 1)S_2$ as $W_{n_2-1}(\Sigma)$.

  By assumption, the X_{1j}'s and the X_{2j}'s are independent, so $(n_1 - 1)S_1$ and $(n_2 - 1)S_2$ are also independent. Then, $(n_1 - 1)S_1 + (n_2 - 1)S_2$ is distributed as $W_{n_1+n_2-2}(\Sigma)$.

  Therefore,

  $$T^2 = \left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-\frac{1}{2}}\left(\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)\right)' S_{pooled}^{-1}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)^{-\frac{1}{2}}\left(\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)\right)$$

  $$= \left(\begin{array}{c}\text{multivariate normal} \\ \text{random vector}\end{array}\right)'\left(\frac{\text{Wishart random matrix}}{\text{d.f.}}\right)^{-1}\left(\begin{array}{c}\text{multivariate normal} \\ \text{random vector}\end{array}\right)$$

  $$= N_p(0,\Sigma)'\left[\frac{W_{n_1+n_2-2}(\Sigma)}{n_1 + n_2 - 2}\right]^{-1} N_p(0,\Sigma),$$

  which is the T^2-distribution, with n replaced by $n_1 + n_2 - 1$.

6.3. Comparing Mean Vectors from Two Populations

- When the primary interest is in confidence regions for $\mu_1 - \mu_2$, all $\mu_1 - \mu_2$ within squared statistical distance c^2 of $\bar{x}_1 - \bar{x}_2$ constitute the confidence region.

  - This region is an ellipsoid centered at the observed difference $\bar{x}_1 - \bar{x}_2$ and whose axes are determined by the eigenvalues and eigenvectors of S_{pooled} $\left(\text{or } S_{pooled}^{-1} \right)$.

- Example 6.3 Constructing a confidence region for the difference of two mean vectors
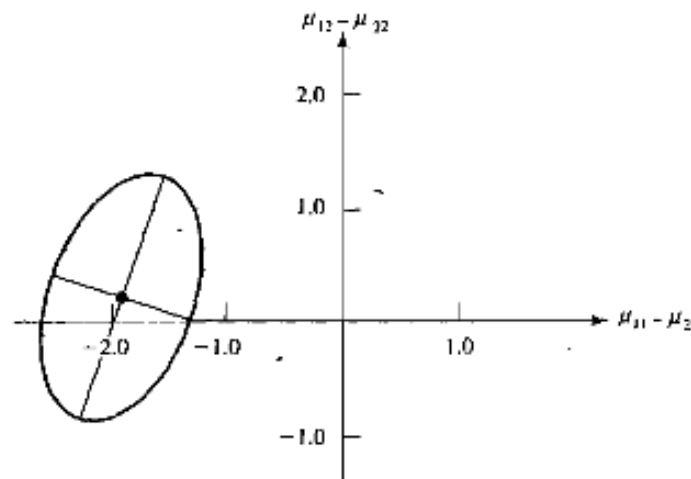

Figure 6.1 95% confidence ellipse for $\mu_1 - \mu_2$.

Simultaneous Confidence Intervals

- It is possible to derive simultaneous confidence intervals for the components of the vector $\mu_1 - \mu_2$.
  - These confidence intervals are developed from a consideration of all possible linear combinations of the differences in the mean vectors.
  - Assume that the parent multivariate populations are normal with a common covariance Σ.

- Result 6.3 Let $c^2 = \left[\dfrac{(n_1 + n_2 - 2)p}{(n_1 + n_2 - p - 1)} \right] F_{p, n_1 + n_2 - p - 1}(\alpha)$. With probability $1 - \alpha$,

$$a'(\overline{X}_1 - \overline{X}_2) \pm c \sqrt{a'\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled} \, a}$$

will cover $a'(\mu_1 - \mu_2)$ for all a. In particular $\mu_{1i} - \mu_{2i}$ will be covered by

$$(\overline{X}_{1i} - \overline{X}_{2i}) \pm c \sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{ii,pooled}} \qquad \text{for } i = 1, 2, \ldots, p.$$

Simultaneous Confidence Intervals

- The **Bonferroni $100(1 - \alpha)\%$ simultaneous confidence intervals** for the p population mean differences are

$$\mu_{1i} - \mu_{2i}: \quad \left(\bar{x}_{1i} - \bar{x}_{2i}\right) \pm t_{n_1+n_2-2}\left(\frac{\alpha}{2p}\right)\sqrt{\left(\frac{1}{n_1} + \frac{1}{n_2}\right)S_{ii,pooled}},$$

where $t_{n_1+n_2-2}\left(\dfrac{\alpha}{2p}\right)$ is the upper $100(\alpha/2p)$th percentile of a t-distribution

with $n_0 + n_2 - 2$ d.f.

The Two-Sample Situation When $\Sigma_1 \neq \Sigma_2$

- When $\Sigma_1 \neq \Sigma_2$, we are unable to find a distance measure like T^2, whose distribution does not depend on the unknowns Σ_1 and Σ_2.

- Bartlett's test is used to test the equality of Σ_1 and Σ_2 in terms of generalized variances.

  - The conclusions based on the Bartlett's test can be seriously misleading when the populations are nonnormal.

  - Nonnormality and unequal covariances cannot be separated with Bartlett's test.

  - In the univariate case, any discrepancy of the order $\sigma_{1,ii} = 4\sigma_{2,ii}$, or vice versa, is probably serious.

  - The size of the discrepancies that are critical in the multivariate situation probably depends, to a large extent, on the number of variables p.

  - A transformation may improve things when the marginal variances are quite different.

- For n_1 and n_2 large, we can avoid the complexities due to unequal covariance matrices.

The Two-Sample Situation When $\Sigma_1 \neq \Sigma_2$

- <u>Result 6.4</u> Let the sample sizes be such that $n_1 - p$ and $n_2 - p$ are large. Then, an approximate $100(1 - \alpha)\%$ confidence ellipsoid for $\mu_1 - \mu_2$ is given by

$$\left(\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)\right)'\left[\frac{1}{n_1}S_1 + \frac{1}{n_2}S_2\right]^{-1}\left(\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)\right) \leq \chi_p^2(\alpha),$$

where $\chi_p^2(\alpha)$ is the upper (100α)th percentile of a chi-square distribution with p d.f. Also, $100(1 - \alpha)\%$ simultaneous confidence intervals for all linear combinations $a'(\mu_1 - \mu_2)$ are provided by

$$a'(\mu_1 - \mu_2) \text{ belongs to } a'(\bar{x}_1 - \bar{x}_2) \pm \sqrt{\chi_p^2(\alpha)}\sqrt{a'\left(\frac{1}{n_1}S_1 + \frac{1}{n_2}S_2\right)a}.$$

- **Remark.** If $n_1 = n_2 = n$, then $\dfrac{(n-1)}{(n+n-2)} = \dfrac{1}{2}$, so

$$\frac{1}{n_1}S_1 + \frac{1}{n_2}S_2 = \frac{1}{n}(S_1 + S_2) = \frac{(n-1)S_1 + (n-1)S_2}{n+n-2}\left(\frac{1}{n} + \frac{1}{n}\right) = S_{pooled}\left(\frac{1}{n} + \frac{1}{n}\right).$$

  - With equal sample sizes, the large sample procedure is essentially the same as the procedure based on the pooled covariance matrix.

An Approximation to the Distribution of T^2 for Normal Populations When Sample Sizes Are Not Large

- H_0: $\mu_1 - \mu_2 = 0$ can be tested when the population covariance matrices are unequal even if the two sample sizes are not large, provided the two populations are multivariate normal.

  - Called the multivariate Behrens-Fisher problem.

  - It is required that both sample sizes n_1 and n_2 are greater than p.

  - This approach depends on an approximation to the distribution of the statistic

$$T^2 = \left(\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)\right)' \left[\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2\right]^{-1} \left(\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)\right)$$

  which is identical to the large sample statistic in Result 6.4.

An Approximation to the Distribution of T^2 for Normal Populations When Sample Sizes Are Not Large

- Instead of using the chi-square approximation to obtain the critical value for testing H_0, the recommended approximation for smaller samples is given by

$$T^2 = \frac{vp}{v-p+1} F_{p,v-p+1},$$

  where the degrees of freedom v is estimated from the sample covariance matrices using the relation

$$v = \frac{p+p^2}{\sum_{i=1}^{2} \frac{1}{n_i} \left\{ tr\left[\left(\frac{1}{n_i} S_i \left(\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2 \right)^{-1} \right)^2 \right] + \left(tr\left[\frac{1}{n_i} S_i \left(\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2 \right)^{-1} \right] \right)^2 \right\}},$$

  where $\min(n_1, n_2) \le v \le n_1 + n_2$.

- With moderate sample sizes and two normal populations, the approximate level α test for equality of means rejects H_0: $\mu_1 - \mu_2 = 0$ if

$$\left(\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2) \right)' \left[\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2 \right]^{-1} \left(\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2) \right) > \frac{vp}{v-p+1} F_{p,v-p+1}(\alpha).$$

An Approximation to the Distribution of T^2 for Normal Populations When Sample Sizes Are Not Large

- This procedure is consistent with the large sample procedures in Result 6.4 except that the critical value $\chi^2_p(\alpha)$ is replaced by the larger constant

$$\frac{vp}{v-p+1} F_{p,v-p+1}(\alpha).$$

- The approximate $100(1 - \alpha)\%$ confidence region is given by all $\mu_1 - \mu_2$ such that

$$\left(\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)\right)'\left[\frac{1}{n_1}S_1 + \frac{1}{n_2}S_2\right]^{-1}\left(\bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2)\right) < \frac{vp}{v+p-1} F_{p,v+p-1}(\alpha).$$

- For normal populations, the approximation to the distribution of T^2 usually gives reasonable results.

6.4. Comparing Several Multivariate Population Means (One-Way MANOVA)

- Often, more than two populations need to be compared. Random samples, collected from each of g populations, are arranged as

$$\text{Population } 1: \quad X_{11}, X_{12}, ..., X_{1n_1}$$
$$\text{Population } 2: \quad X_{21}, X_{22}, ..., X_{2n_2}$$
$$\vdots$$
$$\text{Population } g: \quad X_{g1}, X_{g2}, ..., X_{gn_g}$$

  - MANOVA is used (1) to investigate whether the population mean vectors are the same, and (2) if not, which mean components differ significantly.

- ***Assumptions about the structure of the data for one-way MANOVA***

  1. $X_{l1}, X_{l2}, ..., X_{ln_l}$ is a random sample of size n_l from a population with mean μ_l, $l = 1, 2, ..., g$. The random samples from different populations are independent.

  2. All populations have a common covariance matrix Σ.

  3. Each population is multivariate normal.

  - Condition 3 can be relaxed by appealing to the central limit theorem when the sample sizes n_l are large.

A Summary of Univariate ANOVA

- In the univariate situation, the assumptions are

(1) $X_{l1}, X_{l2}, ..., X_{ln_l}$ is a random sample from an $N(\mu_l, \sigma^2)$ population, $l = 1, 2, ..., g$.

(2) The random samples are independent.

- It is customary to regard μ_l as the sum of an overall mean component, such as μ, and a component due to the specific population:

$$\underset{\substack{\text{(lth population} \\ \text{mean)}}}{\mu_l} = \underset{\substack{\text{(overall} \\ \text{mean)}}}{\mu} + \underset{\substack{\text{(lth population} \\ \text{(treatment) effect)}}}{\tau_l} ,$$

where $\tau_l = \mu_l - \mu$.

- The response X_{lj}, distributed as $N(\mu + \tau_l, \sigma^2)$, can be expressed as

$$X_{lj} = \underset{\substack{\text{(overall} \\ \text{mean)}}}{\mu} + \underset{\substack{\text{(treatment} \\ \text{effect)}}}{\tau_l} + \underset{\substack{\text{(random} \\ \text{error)}}}{e_{lj}} ,$$

where e_{lj} are independent $N(0, \sigma^2)$ random variables.

- It is customary to impose the constraint $\sum_{l=1}^{g} n_l \tau_l = 0$.

A Summary of Univariate ANOVA

- To test H_0: $\tau_1 = \tau_2 = \ldots = \tau_g = 0$, the analysis of variance is based upon a decomposition of the observations

$$x_{lj} = \underbrace{\overline{x}}_{\substack{\text{overall} \\ \text{sample mean}}} + \underbrace{(\overline{x}_l - \overline{x})}_{\substack{\text{estimated} \\ \text{treatment effect}}} + \underbrace{(x_{lj} - \overline{x}_l)}_{\text{(residual)}},$$

where $\overline{x}$ is an estimate of μ, $\hat{\tau}_l = (\overline{x}_l - \overline{x})$ is an estimate of τ_l, and $(x_{lj} - \overline{x}_l)$ is an estimate of the error e_{lj}.

- $(x_{lj} - \overline{x})^2 = (\overline{x}_l - \overline{x})^2 + (x_{lj} - \overline{x}_l)^2 + 2(\overline{x}_l - \overline{x})(x_{lj} - \overline{x}_l).$

- Summing both sides over j results in

$$\sum_{j=1}^{n_l}(x_{lj} - \overline{x})^2 = n_l(\overline{x}_l - \overline{x})^2 + \sum_{j=1}^{n_l}(x_{lj} - \overline{x}_l)^2.$$

- Summing both sides over l gives

$$\underbrace{\sum_{l=1}^{g}\sum_{j=1}^{n_l}(x_{lj} - \overline{x})^2}_{\substack{SS_{cor} \\ \text{total (corrected) SS}}} = \underbrace{\sum_{l=1}^{g}n_l(\overline{x}_l - \overline{x})^2}_{\substack{SS_{tr} \\ \text{between (samples) SS}}} + \underbrace{\sum_{l=1}^{g}\sum_{j=1}^{n_l}(x_{lj} - \overline{x}_l)^2}_{\substack{SS_{res} \\ \text{within (samples) SS}}}$$

or $\underbrace{\sum_{l=1}^{g}\sum_{j=1}^{n_l}x_{lj}^2}_{(SS_{obs})} = \underbrace{(n_1 + n_2 + \cdots + n_g)\overline{x}^2}_{(SS_{mean})} + \underbrace{\sum_{l=1}^{g}n_l(\overline{x}_l - \overline{x})^2}_{(SS_{tr})} + \underbrace{\sum_{l=1}^{g}\sum_{j=1}^{n_l}(x_{lj} - \overline{x}_l)^2}_{(SS_{res})}.$

A Summary of Univariate ANOVA

- ANOVA Table for Comparing Univariate Population Means

ANOVA Table for Comparing Univariate Population Means

Source of variation	Sum of squares (SS)	Degrees of freedom (d.f.)
Treatments	$SS_{tr} = \sum_{\ell=1}^{g} n_\ell (\bar{x}_\ell - \bar{x})^2$	$g - 1$
Residual (error)	$SS_{res} = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x}_\ell)^2$	$\sum_{\ell=1}^{g} n_\ell - g$
Total (corrected for the mean)	$SS_{cor} = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (x_{\ell j} - \bar{x})^2$	$\sum_{\ell=1}^{g} n_\ell - 1$

A Summary of Univariate ANOVA

- The usual F-test rejects H_0: $\tau_1 = \tau_2 = \ldots = \tau_g = 0$ at level α if

$$F = \frac{SS_{tr}/(g-1)}{SS_{res}\bigg/\left(\displaystyle\sum_{l=1}^{g} n_l - g\right)} > F_{g-1,\Sigma n_l - g}(\alpha),$$

where $F_{g-1,\Sigma n_l - g}(\alpha)$ is the upper (100α)th percentile of the F-distribution with g - 1 and Σn_l - g degrees of freedom.

- This is equivalent to reject H_0 for large values of SS_{tr}/SS_{res} or $1 + (SS_{tr}/SS_{res})$.

- The statistic appropriate for a multivariate generalization rejects H_0 for *small* values of the reciprocal

$$\frac{1}{1 + SS_{tr}/SS_{res}} = \frac{SS_{res}}{SS_{res} + SS_{tr}}.$$

Multivariate Analysis of Variance (MANOVA)

- **MANOVA Model for Comparing g Population Mean Vectors**

$$X_{lj} = \mu + \tau_l + e_{lj}, \quad j = 1, 2, \ldots, n_l \text{ and } l = 1, 2, \ldots, g,$$

where e_{lj} are independent $N_p(0, \Sigma)$ variables.

- The parameter vector μ is an overall mean (level), and τ_l represents the lth treatment effect with $\sum_{l=1}^{g} n_l \tau_l = 0$.

- *Each component* of the observation vector X_{lj} satisfies the univariate model.

- The errors for the components of X_{lj} are correlated, but the covariance matrix Σ is the same for all populations.

- A vector of observations may be decomposed as

$$\underset{(\text{observation})}{x_{lj}} = \underset{\left(\substack{\text{overall sample} \\ \text{mean } \hat{\mu}}\right)}{\overline{x}} + \underset{\left(\substack{\text{estimated treatment} \\ \text{effect } \hat{\tau}_l}\right)}{\left(\overline{x}_l - \overline{x}\right)} + \underset{\left(\substack{\text{residual} \\ \hat{e}_{lj}}\right)}{\left(x_{lj} - \overline{x}_l\right)}.$$

Multivariate Analysis of Variance (MANOVA)

- Noting that

$$\left(x_{lj} - \bar{x}\right)\left(x_{lj} - \bar{x}\right)' = \left[\left(x_{lj} - \bar{x}_l\right) + \left(\bar{x}_l - \bar{x}\right)\right]\left[\left(x_{lj} - \bar{x}_l\right) + \left(\bar{x}_l - \bar{x}\right)\right]'$$

$$= \left(x_{lj} - \bar{x}_l\right)\left(x_{lj} - \bar{x}_l\right)' + \left(x_{lj} - \bar{x}_l\right)\left(\bar{x}_l - \bar{x}\right)' + \left(\bar{x}_l - \bar{x}\right)\left(x_{lj} - \bar{x}_l\right)' + \left(\bar{x}_l - \bar{x}\right)\left(\bar{x}_l - \bar{x}\right)'.$$

  - The sum over j of the middle two expressions is the zero matrix because

$$\sum_{j=1}^{n_l} \left(x_{lj} - \bar{x}_l\right) = 0.$$

- Summing the cross product over l and j yields

$$\sum_{l=1}^{g}\sum_{j=1}^{n_l}\left(x_{lj} - \bar{x}\right)\left(x_{lj} - \bar{x}\right)' = \sum_{l=1}^{g} n_l\left(\bar{x}_l - \bar{x}\right)\left(\bar{x}_l - \bar{x}\right)' + \sum_{l=1}^{g}\sum_{j=1}^{n_l}\left(x_{lj} - \bar{x}_l\right)\left(x_{lj} - \bar{x}_l\right)'.$$

$\begin{pmatrix} \text{total (corrected) sum} \\ \text{of squares and cross} \\ \text{products} \end{pmatrix}$ $\begin{pmatrix} \text{treatment (Between)} \\ \text{sum of squares and} \\ \text{cross products} \end{pmatrix}$ $\begin{pmatrix} \text{residual (Within) sum} \\ \text{of squares and cross} \\ \text{products} \end{pmatrix}$

Multivariate Analysis of Variance (MANOVA)

- Note that the **within** sum of squares and cross products matrix can be expressed as

$$W = \sum_{l=1}^{g}\sum_{j=1}^{n_l}\left(x_{lj}-\bar{x}_l\right)\left(x_{lj}-\bar{x}_l\right)' = (n_1-1)S_1 + (n_2-1)S_2 + \cdots + \left(n_g-1\right)S_g,$$

  where S_l is the sample covariance matrix for the lth sample.

  - This matrix is a generalization of the $(n_1 + n_2 - 2)\,S_{pooled}$ matrix in the two sample case.

- A test of no treatment effects,

$$H_0:\ \tau_1 = \tau_2 = \ldots = \tau_g = 0$$

  is conducted by considering the relative sizes of the treatment and residual sums of squares and cross products.

Multivariate Analysis of Variance (MANOVA)

- MANOVA Table for Comparing Multivariate Population Means

MANOVA Table for Comparing Population Mean Vectors

Source of variation	Matrix of sum of squares and cross products (SSP)	Degrees of freedom (d.f.)
Treatment	$\mathbf{B} = \sum_{\ell=1}^{g} n_\ell (\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})(\bar{\mathbf{x}}_\ell - \bar{\mathbf{x}})'$	$g - 1$
Residual (Error)	$\mathbf{W} = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)(\mathbf{x}_{\ell j} - \bar{\mathbf{x}}_\ell)'$	$\sum_{\ell=1}^{g} n_\ell - g$
Total (corrected for the mean)	$\mathbf{B} + \mathbf{W} = \sum_{\ell=1}^{g} \sum_{j=1}^{n_\ell} (\mathbf{x}_{\ell j} - \bar{\mathbf{x}})(\mathbf{x}_{\ell j} - \bar{\mathbf{x}})'$	$\sum_{\ell=1}^{g} n_\ell - 1$

Multivariate Analysis of Variance (MANOVA)

- One test of no treatment effects,

$$H_0: \tau_1 = \tau_2 = \ldots = \tau_g = 0$$

  involves generalized variances.

  - Reject H_0 if the ratio of generalized variances

$$\Lambda^* = \frac{|W|}{|B + W|} = \frac{\left| \sum\limits_{l=1}^{g} \sum\limits_{j=1}^{n_l} (x_{lj} - \bar{x}_l)(x_{lj} - \bar{x}_l)' \right|}{\left| \sum\limits_{l=1}^{g} \sum\limits_{j=1}^{n_l} (x_{lj} - \bar{x})(x_{lj} - \bar{x})' \right|}$$

  is too small.

  - The quantity $\Lambda^* = \dfrac{|W|}{|B + W|}$, called **Wilks' lambda**, corresponds to the

  equivalent form $\dfrac{SS_{res}}{SS_{res} + SS_{tr}}$ of the F-test of H_0: no treatment effects

  in the univariate case.

Multivariate Analysis of Variance (MANOVA)

- Table 6.3 provides exact distributions of Λ^* for the special cases.

Table 6.3 Distribution of Wilks' Lambda, $\Lambda^* = |\mathbf{W}|/|\mathbf{B}+\mathbf{W}|$

No. of variables	No. of groups	Sampling distribution for multivariate normal data
$p = 1$	$g \geq 2$	$\left(\dfrac{\Sigma n_\ell - g}{g-1}\right)\left(\dfrac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{g-1,\,\Sigma n_\ell - g}$
$p = 2$	$g \geq 2$	$\left(\dfrac{\Sigma n_\ell - g - 1}{g-1}\right)\left(\dfrac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2(g-1),\,2(\Sigma n_\ell - g - 1)}$
$p \geq 1$	$g = 2$	$\left(\dfrac{\Sigma n_\ell - p - 1}{p}\right)\left(\dfrac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{p,\,\Sigma n_\ell - p - 1}$
$p \geq 1$	$g = 3$	$\left(\dfrac{\Sigma n_\ell - p - 2}{p}\right)\left(\dfrac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2p,\,2(\Sigma n_\ell - p - 2)}$

Multivariate Analysis of Variance (MANOVA)

- For large samples, if H_0 is true and $\Sigma\, n_l = n$ is large,

$$-\left(n-1-\frac{(p+g)}{2}\right)\ln\Lambda^* = -\left(n-1-\frac{(p+g)}{2}\right)\ln\left(\frac{|W|}{|B+W|}\right)$$

has approximately a chi-square distribution with $p(g-1)$ d.f.

- Reject H_0 at significance level α if

$$-\left(n-1-\frac{(p+g)}{2}\right)\ln\left(\frac{|W|}{|B+W|}\right) > \chi^2_{p(g-1)}(\alpha),$$

where $\chi^2_{p(g-1)}(\alpha)$ is the upper (100α)th percentile of a chi-square distribution with $p(g-1)$ d.f.

6.5. Simultaneous Confidence Intervals for Treatment Effects

- When the hypothesis of equal treatment effects is rejected, those effects that led to the rejection of the hypothesis are of interest.
  - For pairwise comparisons, the Bonferroni approach can be used to construct simultaneous confidence intervals for the components of the differences $\tau_k - \tau_l$ (or $\mu_k - \mu_l$).

- Let τ_{ki} be the ith component of τ_k.
  - Since τ_k is estimated by $\hat{\tau}_k = \bar{x}_k - \bar{x}$, $\hat{\tau}_{ki} = \bar{x}_{ki} - \bar{x}_i$.
  - $\hat{\tau}_{ki} - \hat{\tau}_{li} = \bar{x}_{ki} - \bar{x}_{li}$ is the difference between two independent sample means.
  - $Var(\hat{\tau}_{ki} - \hat{\tau}_{li}) = Var(\bar{X}_{ki} - \bar{X}_{li}) = \left(\frac{1}{n_k} + \frac{1}{n_l} \right) \sigma_{ii}$,

    where σ_{ii} is the ith diagonal element of Σ.
  - $Var(\bar{X}_{ki} - \bar{X}_{li})$ is estimated by dividing the corresponding element of W by its degrees of freedom:

$$\hat{Var}(\bar{X}_{ki} - \bar{X}_{li}) = \left(\frac{1}{n_k} + \frac{1}{n_l} \right) \frac{w_{ii}}{n - g},$$

    where w_{ii} is the ith diagonal element of W and $n = n_1 + \ldots + n_g$.

6.5. Simultaneous Confidence Intervals for Treatment Effects

- There are p variables and $g(g-1)/2$ pairwise differences, so each two-sample t-interval employs the critical value $t_{n-g}(\alpha/2m)$, where $m = pg(g-1)/2.$

- Result 6.5 Let $n = \sum_{k=1}^{g} n_k$. With confidence at least $(1-\alpha)$,

$$\tau_{ki} - \tau_{li} \text{ belongs to } \bar{x}_{ki} - \bar{x}_{li} \pm t_{n-g}\left(\frac{\alpha}{pg(g-1)}\right)\sqrt{\frac{w_{ii}}{n-g}\left(\frac{1}{n_k}+\frac{1}{n_l}\right)}$$

for all components $i = 1, \ldots, p$ and all differences $l < k = 1, \ldots, g$. Here w_{ii} is the ith diagonal element of W.

6.6. Testing for Equality of Covariance Matrices

- One of the assumptions made when comparing two or more multivariate mean vectors is that the covariance matrices of the potentially different populations are the same.

  - Before pooling the variation across samples to form a pooled covariance matrix when comparing mean vectors, it is worthwhile to test the equality of the population covariance matrices.

  - One commonly employed test for equal covariance matrices is Box's M-test.

- With g populations, the null hypothesis is

  $$H_0: \Sigma_1 = \Sigma_2 = \ldots = \Sigma_g = \Sigma,$$

  where Σ_l is the covariance matrix for the lth population, $l = 1, 2, \ldots, g$, and Σ is the presumed common covariance matrix.

  - The alternative hypothesis is that at least two of the covariance matrices are not equal.

6.6. Testing for Equality of Covariance Matrices

- Assuming multivariate normal populations, a likelihood ratio statistic for testing H_0 is

$$\Lambda = \prod_l \left(\frac{|S_l|}{|S_{pooled}|} \right)^{\frac{(n_l - 1)}{2}},$$

where n_l is the sample size for the lth population,

$\quad S_l$ is the lth group sample covariance matrix,

and S_{pooled} is the pooled sample covariance matrix given by

$$S_{pooled} = \frac{1}{\sum_l (n_l - 1)} \left\{ (n_1 - 1)S_1 + (n_2 - 1)S_2 + \cdots + (n_g - 1)S_g \right\}.$$

6.6. Testing for Equality of Covariance Matrices

- Box's test is based on chi-square approximation to the sampling distribution of $-2 \ln\Lambda$.
  - Setting $-2 \ln\Lambda = M$ (Box's M statistic) gives

$$M = \left[\sum_l (n_l - 1) \right] \ln|S_{pooled}| - \sum_l \left[(n_l - 1) \ln|S_l| \right]$$

  - If H_0 is true, the individual sample covariance matrices are not expected to differ too much, and consequently, do not differ too much from the pooled covariance matrix. In this case, the ratio of the determinant in Λ will all be close to 1, Λ will be near 1 and Box's M statistic will be small.
  - If H_0 is false, the sample covariance matrices can differ more and the differences in their determinants will be more pronounced. In this case, Λ will be small and M statistic will be relatively large.

6.6. Testing for Equality of Covariance Matrices

- Box's test for Equality of Covariance Matrices

  Set

$$u = \left[\sum_l \frac{1}{(n_l - 1)} - \frac{1}{\sum_l (n_l - 1)} \right] \left[\frac{2p^2 + 3p - 1}{6(p+1)(g-1)} \right],$$

  where p is the number of variables and g is the number of groups. Then

$$C = (1-u)M = (1-u)\left\{ \left[\sum_l (n_l - 1) \right] \ln|S_{pooled}| - \sum_l \left[(n_l - 1) \ln|S_l| \right] \right\}$$

  has an approximate χ^2 distribution with

$$v = g\frac{1}{2}p(p+1) - \frac{1}{2}p(p+1) = \frac{1}{2}p(p+1)(g-1)$$

  degrees of freedom. At significance level α, reject H_0 if $C > \chi^2_{p(p+1)(g-1)/2}(\alpha)$.

  - Box's χ^2 approximation works well if each n_l exceed 20 and if p and g do not exceed 5.

6.6. Testing for Equality of Covariance Matrices

- Box's M-test is known to be sensitive to some forms of non-normality.

  - In the presence of non-normality, normal theory tests on covariances are influenced by the kurtosis of the parent populations.

  - With reasonably large samples, the MANOVA tests of means or treatment effects are rather robust to nonnormality.

  - With equal sample sizes, some differences in covariance matrices have little effect on the MANOVA tests.

  - May decide to continue with the usual MANOVA tests even though the M-test leads to rejection of H_0.

6.7. Two-way Multivariate Analysis of Variance

- **Univariate Two-way Fixed-Effects *Model with Interaction***
  - Assume that measurements are recorded at various levels of two factors.
  - Assume that observations at different combinations of experimental conditions are independent of one another.
  - Suppose there are g levels of factor 1 and b levels of factor 2.
  - n independent observations can be observed at each of the gb combinations of levels.
  - X_{lkr}: rth observation at level l of factor 1 and level k of factor 2
  - The univariate two-way model is

  $$X_{lkr} = \mu + \tau_l + \beta_k + \gamma_{lk} + e_{lkr}, \quad l = 1, 2, \ldots, g; \ k = 1, 2, \ldots, b; \ r = 1, 2, \ldots, n,$$

  where $\displaystyle\sum_{l=1}^{g} \tau_l = \sum_{k=1}^{b} \beta_k = \sum_{l=1}^{g} \gamma_{lk} = \sum_{k=1}^{b} \gamma_{lk} = 0,$

  and e_{lkr} are independent $N(0, \sigma^2)$ random variables.
  - Note that μ represents an overall level,

    τ_l represents the fixed effect of factor 1,

    β_k represents the fixed effect of factor 2,

    and γ_{lk} is the interaction between factor 1 and factor 2.

Univariate Two-Way Fixed-Effects Model with Interaction

- $$E\left(X_{lkr}\right) = \underset{\binom{\text{mean}}{\text{response}}}{\mu} + \underset{\binom{\text{overall}}{\text{level}}}{\mu} + \underset{\binom{\text{effect of}}{\text{factor 1}}}{\tau_l} + \underset{\binom{\text{effect of}}{\text{factor 2}}}{\beta_k} + \underset{\binom{\text{factor 1-factor 2}}{\text{interaction}}}{\gamma_{lk}},$$

$$l = 1, 2, \ldots, g, \quad k = 1, 2, \ldots, b.$$

- The presence of interaction, γ_{lk}, implies that the factor effects are not additive (see p. 313, Figure 6.3) .

- Each observation can be decomposed as

$$x_{lkr} = \bar{x} + \left(\bar{x}_{l\cdot} - \bar{x}\right) + \left(\bar{x}_{\cdot k} - \bar{x}\right) + \left(\bar{x}_{lk} - \bar{x}_{l\cdot} - \bar{x}_{\cdot k} + \bar{x}\right) + \left(x_{lkr} - \bar{x}_{lk}\right),$$

where $\bar{x}$ is the overall average,

$\bar{x}_{l\cdot}$ is the average for the lth level of factor 1,

$\bar{x}_{\cdot k}$ is the average for the kth level of factor 2,

and $\bar{x}_{lk}$ is the average for the lth level of factor 1 and the kth level

of factor 2.

Univariate Two-Way Fixed -Effects Model with Interaction

- Squaring and summing the deviations $(x_{lkr} - \bar{x})$ gives

$$\sum_{l=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(x_{lkr} - \bar{x})^2 = \sum_{l=1}^{g}bn(\bar{x}_{l.} - \bar{x})^2 + \sum_{k=1}^{b}gn(\bar{x}_{.k} - \bar{x})^2$$

$$+ \sum_{l=1}^{g}\sum_{k=1}^{b}n(\bar{x}_{lk} - \bar{x}_{l.} - \bar{x}_{.k} - \bar{x})^2 + \sum_{l=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}(x_{lkr} - \bar{x}_{lk})^2$$

or

$$SS_{cor} = SS_{fac1} + SS_{fac2} + SS_{int} + SS_{res}.$$

- The corresponding degrees of freedom associated with the sums of squares are $gbn - 1 = (g - 1) + (b - 1) + (g - 1)(b - 1) + gb(n - 1)$.

- See ANOVA Table in p. 314.

- The F-ratios of the mean squares, $SS_{fac1}/(g - 1)$, $SS_{fac2}/(b - 1)$, and $SS_{int}/(g - 1)(b - 1)$ to the mean square, $SS_{res}/(gb(n - 1))$, can be used to test for the effects of factor 1, factor 2, and factor 1-factor 2 interaction, respectively.

Univariate Two-Way Fixed -Effects Model with Interaction

ANOVA Table for Comparing Effects of Two Factors and Their Interaction

Source of variation	Sum of squares (SS)	Degrees of freedom (d.f.)
Factor 1	$SS_{fac1} = \sum_{\ell=1}^{g} bn(\bar{x}_{\ell\cdot} - \bar{x})^2$	$g - 1$
Factor 2	$SS_{fac2} = \sum_{k=1}^{b} gn(\bar{x}_{\cdot k} - \bar{x})^2$	$b - 1$
Interaction	$SS_{int} = \sum_{\ell=1}^{g} \sum_{k=1}^{b} n(\bar{x}_{\ell k} - \bar{x}_{\ell\cdot} - \bar{x}_{\cdot k} + \bar{x})^2$	$(g - 1)(b - 1)$
Residual (Error)	$SS_{res} = \sum_{\ell=1}^{g} \sum_{k=1}^{b} \sum_{r=1}^{n} (x_{\ell k r} - \bar{x}_{\ell k})^2$	$gb(n - 1)$
Total (corrected)	$SS_{cor} = \sum_{\ell=1}^{g} \sum_{k=1}^{b} \sum_{r=1}^{n} (x_{\ell k r} - \bar{x})^2$	$gbn - 1$

Multivariate Two-Way Fixed-Effects Model with Interaction

- Two–way fixed effects model for a **vector** response consisting of p components is

$$X_{lkr} = \mu + \tau_l + \beta_k + \gamma_{lk} + e_{lkr}$$

$$l = 1, 2, \ldots, g,$$
$$k = 1, 2, \ldots, b,$$
$$r = 1, 2, \ldots, n,$$

where $\sum_{l=1}^{g} \tau_l = \sum_{k=1}^{b} \beta_k = \sum_{l=1}^{g} \gamma_{lk} = \sum_{k=1}^{b} \gamma_{lk} = 0$,

and e_{lkr} are independent $N_p(0, \Sigma)$ random vectors.

- Note that the responses consist of p measurements replicated n times at each of the possible combinations of levels of factors 1 and 2.

- Decompose the observation vectors x_{lkr} as

$$x_{lkr} = \bar{x} + \left(\bar{x}_{l\cdot} - \bar{x} \right) + \left(\bar{x}_{\cdot k} - \bar{x} \right) + \left(\bar{x}_{lk} - \bar{x}_{l\cdot} - \bar{x}_{\cdot k} + \bar{x} \right) + \left(x_{lkr} - \bar{x}_{lk} \right),$$

where $\bar{x}$ is the overall average of the observation vectors,

$\bar{x}_{l\cdot}$ is the average of the observation vectors at the lth level of factor 1,

$\bar{x}_{\cdot k}$ is the average of the observation vectors at the kth level of factor 2,

and $\bar{x}_{lk}$ is the average of the observation vectors at the lth level of factor 1 and the kth level of factor 2.

Multivariate Two-Way Fixed-Effects Model with Interaction

- Straightforward generalizations of the univariate case give

$$\sum_{l=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}\left(x_{lkr}-\bar{x}\right)\left(x_{lkr}-\bar{x}\right)' = \sum_{l=1}^{g}bn\left(\bar{x}_{l\cdot}-\bar{x}\right)\left(\bar{x}_{l\cdot}-\bar{x}\right)'$$

$$+\sum_{k=1}^{b}gn\left(\bar{x}_{\cdot k}-\bar{x}\right)\left(\bar{x}_{\cdot k}-\bar{x}\right)'$$

$$+\sum_{l=1}^{g}\sum_{k=1}^{b}n\left(\bar{x}_{lk}-\bar{x}_{l\cdot}-\bar{x}_{\cdot k}-\bar{x}\right)\left(\bar{x}_{lk}-\bar{x}_{l\cdot}-\bar{x}_{\cdot k}-\bar{x}\right)'$$

$$+\sum_{l=1}^{g}\sum_{k=1}^{b}\sum_{r=1}^{n}\left(x_{lkr}-\bar{x}_{lk}\right)\left(x_{lkr}-\bar{x}_{lk}\right)'$$

  - The corresponding degrees of freedom are

    $gbn - 1 = (g - 1) + (b - 1) + (g - 1)(b - 1) + gb(n - 1).$

  - The generalization from the univariate to the multivariate analysis consists simply of replacing a scalar such as $\left(\bar{x}_{l\cdot}-\bar{x}\right)^{2}$ with the corresponding matrix $\left(\bar{x}_{l\cdot}-\bar{x}\right)\left(\bar{x}_{l\cdot}-\bar{x}\right)'$.

- See MANOVA Table in p. 316.

Multivariate Two-Way Fixed-Effects Model with Interaction

The MANOVA table is the following:

MANOVA Table for Comparing Factors and Their Interaction

Source of variation	Matrix of sum of squares and cross products (SSP)	Degrees of freedom (d.f.)
Factor 1	$\mathrm{SSP}_{fac1} = \sum\limits_{\ell=1}^{g} bn(\bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}})'$	$g - 1$
Factor 2	$\mathrm{SSP}_{fac2} = \sum\limits_{k=1}^{b} gn(\bar{\mathbf{x}}_{\cdot k} - \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\cdot k} - \bar{\mathbf{x}})'$	$b - 1$
Interaction	$\mathrm{SSP}_{int} = \sum\limits_{\ell=1}^{g}\sum\limits_{k=1}^{b} n(\bar{\mathbf{x}}_{\ell k} - \bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}}_{\cdot k} + \bar{\mathbf{x}})(\bar{\mathbf{x}}_{\ell k} - \bar{\mathbf{x}}_{\ell\cdot} - \bar{\mathbf{x}}_{\cdot k} + \bar{\mathbf{x}})'$	$(g-1)(b-1)$
Residual (Error)	$\mathrm{SSP}_{res} = \sum\limits_{\ell=1}^{g}\sum\limits_{k=1}^{b}\sum\limits_{r=1}^{n} (\mathbf{x}_{\ell kr} - \bar{\mathbf{x}}_{\ell k})(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}}_{\ell k})'$	$gb(n-1)$
Total (corrected)	$\mathrm{SSP}_{cor} = \sum\limits_{\ell=1}^{g}\sum\limits_{k=1}^{b}\sum\limits_{r=1}^{n} (\mathbf{x}_{\ell kr} - \bar{\mathbf{x}})(\mathbf{x}_{\ell kr} - \bar{\mathbf{x}})'$	$gbn - 1$

Multivariate Two-Way Fixed-Effects Model with Interaction

- A test (the likelihood ratio test) of $H_0: \gamma_{11} = \gamma_{12} = \ldots = \gamma_{gb} = 0$ (no interaction effect) versus H_1: At least one $\gamma_{lk} \neq 0$

  - Reject H_0 for small values of the ratio

  $$\Lambda^* = \frac{\left| SSP_{res} \right|}{\left| SSP_{int} + SSP_{res} \right|}.$$

  - For large samples, Wilk's lambda, Λ^*, can be referred to a chi-square percentile.

  - Using Bartlett's multiplier to improve the chi-square approximation, reject $H_0: \gamma_{11} = \gamma_{12} = \ldots = \gamma_{gb} = 0$ at the α level if

  $$-\left[gb(n-1) - \frac{p+1-(g-1)(b-1)}{2} \right] \ln \Lambda^* > \chi^2_{(g-1)(b-1)p}(\alpha),$$

  where $\chi^2_{(g-1)(b-1)p}(\alpha)$ is the upper (100α)th percentile of a chi-square distribution with $(g-1)(b-1)p$ d.f.

Multivariate Two-Way Fixed-Effects Model with Interaction

● Ordinarily, the test for interaction is carried out before the tests for main factor effects.

- If interaction effects exist, the factor effects do not have a clear interpretation.

- p univariate two-way analyses of variance (one for each variable) are often conducted to see whether the interaction appears in some response but not others.

- Responses without interaction may be interpreted in terms of additive factor 1 and 2 effects.

Multivariate Two-Way Fixed-Effects Model with Interaction

- A test (the likelihood ratio test) of H_0: $\tau_1 = \tau_2 = \ldots = \tau_g = 0$ versus H_1: At least one $\tau_l \neq 0$

  - These hypotheses specify *no* factor 1 effects vs. *some* factor 1 effects, respectively.

  - Reject H_0 for small values of the ratio

  $$\Lambda^* = \frac{\left| SSP_{res} \right|}{\left| SSP_{fac1} + SSP_{res} \right|}.$$

  - Using Bartlett's correction, the likelihood ratio test rejects H_0: $\tau_1 = \tau_2 = \ldots = \tau_g = 0$ (no factor 1 effects) at the α level if

  $$-\left[gb(n-1) - \frac{p+1-(g-1)}{2} \right] \ln \Lambda^* > \chi^2_{(g-1)p}(\alpha),$$

  where $\chi^2_{(g-1)p}(\alpha)$ is the upper (100α)th percentile of a chi-square distribution with $(g-1)p$ d.f.

Multivariate Two-Way Fixed-Effects Model with Interaction

- A test (the likelihood ratio test) of $H_0: \beta_1 = \beta_2 = \ldots = \beta_b = 0$ versus H_1: At least one $\beta_k \neq 0$

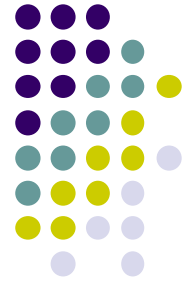 - Reject H_0 for small values of the ratio

  $$\Lambda^* = \frac{|SSP_{res}|}{|SSP_{fac2} + SSP_{res}|}.$$

  - For large samples, using Bartlett's correction, the likelihood ratio test rejects $H_0: \beta_1 = \beta_2 = \ldots = \beta_b = 0$ (no factor 2 effects) at the α level if

  $$-\left[gb(n-1) - \frac{p+1-(b-1)}{2} \right] \ln \Lambda^* > \chi^2_{(b-1)p}(\alpha),$$

  where $\chi^2_{(b-1)p}(\alpha)$ is the upper (100α)th percentile of a chi-square distribution with $(b-1)p$ d.f.

Multivariate Two-Way Fixed-Effects Model with Interaction

- Simultaneous confidence intervals for contrasts in the model parameters can provide insights into the nature of the factor effects.
  - When interaction effects are negligible, may concentrate on contrasts in the factor 1 and factor 2 main effects.

- Using Bonferroni approach, the $100(1 - \alpha)\%$ simultaneous confidence intervals for $\tau_{li} - \tau_{mi}$ are

$$\tau_{li} - \tau_{mi} \text{ belongs to } (\bar{x}_{l\cdot i} - \bar{x}_{m\cdot i}) \pm t_v\left(\frac{\alpha}{pg(g-1)}\right)\sqrt{\frac{E_{ii}}{v}\frac{2}{bn}},$$

  where $v = gb(n - 1)$, E_{ii} is the ith diagonal element of $E = SSP_{res}$, and $\bar{x}_{l\cdot i} - \bar{x}_{m\cdot i}$ is the ith component of $\bar{x}_{l\cdot} - \bar{x}_{m\cdot}$.

  - Similarly, the $100(1 - \alpha)\%$ simultaneous confidence interval for $\beta_{ki} - \beta_{qi}$ are

$$\beta_{ki} - \beta_{qi} \text{ belongs to } (\bar{x}_{\cdot ki} - \bar{x}_{\cdot qi}) \pm t_v\left(\frac{\alpha}{pb(b-1)}\right)\sqrt{\frac{E_{ii}}{v}\frac{2}{gn}},$$

  where $\bar{x}_{\cdot ki} - \bar{x}_{\cdot qi}$ is the ith component of $\bar{x}_{\cdot k} - \bar{x}_{\cdot q}$.

6.10. Perspectives and a Strategy for Analyzing Multivariate Models

- With several characteristics, it is important to control the overall probability of making any incorrect decision.

  - A single multivariate test, with its associated single p-value, is preferable to performing a large number of univariate tests.

  - The outcome of the multivariate test may tell us whether or not it is worthwhile to look closer on a variable by variable and group by group analysis.

6.10. Perspectives and a Strategy for Analyzing Multivariate Models

- In the random samples from several populations, multivariate tests are based on the matrices

$$W = \sum_{l=1}^{g} \sum_{j=1}^{n_l} \left(x_{lj} - \bar{x}_l \right) \left(x_{lj} - \bar{x}_l \right)' \quad \text{and} \quad B = \sum_{l=1}^{g} n_l \left(x_l - \bar{x} \right) \left(x_l - \bar{x} \right)'.$$

- Wilks' lambda statistic $\Lambda^* = \dfrac{|W|}{|B+W|}$.

- The test based on Wilks' lambda statistic is equivalent to the likelihood ratio test.

- Three other multivariate statistics

(1) Lawley-Hotelling trace = $tr[BW^{-1}]$

(2) Pillai trace = $tr[B(B+W)^{-1}]$

(3) Roy's largest root = maximum eigenvalue of $W(B+W)^{-1}$.

6.10. Perspectives and a Strategy for Analyzing Multivariate Models

- All four of these tests appear to be nearly equivalent for extremely large samples.

- For moderate sample sizes, all comparisons are based on simulation.

  - The first three tests have similar power, while the last, Roy's test, behaves differently.

  - The power of the Roy's test is best only when there is a single nonzero eigenvalue and the power is large. (This may approximate situations where a large difference exists in just one characteristic and it is between one group and all of the others.)

  - Pillai's trace is slightly more robust against nonnormality.

- All four statistics apply in the two-way setting and in even more complicated MANOVA.

6.10. Perspectives and a Strategy for Analyzing Multivariate Models

- Strategy for the multivariate comparison of treatments

  1. *Try to identify outliers.*

     - Check the data group by group.

     - Check the collection of residual vectors from any fitted model.

  2. *Perform a multivariate test of hypothesis.*

     - Perform the likelihood ratio test (equivalent to Wilks' lambda).

  3. *Calculate the Bonferroni simultaneous confidence intervals.*

     - If the multivariate test reveals a difference, proceed to calculate the Bonferroni confidence intervals for all pairs of groups or treatments, and all characteristics.

     - If no differences are significant, try looking at Bonferroni intervals for the larger set of responses that includes the differences and sums of pairs of responses.