X_i's $\overset{iid}{\sim} N(\mu, \sigma^2)$, $i=1,\cdots,n$. Show $E[S^2] = \sigma^2$.

The standardization says $\frac{X_i - \mu}{\sigma} \sim N(0,1)$ $\forall i = 1,\cdots,n$

Then, $\frac{\sum_{i=1}^n (X_i - \mu)^2}{\sigma^2} \sim \chi^2_n$, and $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$ $\left[\text{or, } \frac{n(\bar{X}-\mu)^2}{\sigma^2} \sim \chi^2_1\right]$

$$\Rightarrow \sum_{i=1}^n (X_i - \mu)^2/\sigma^2 = \Sigma(X_i - \bar{X} + \bar{X} - \mu)^2/\sigma^2 = \underbrace{\Sigma(X_i - \bar{X})^2/\sigma^2}_{①} + \underbrace{n(\bar{X} - \mu)^2/\sigma^2}_{②}$$

From the above equality, LHS follows χ^2_n, and the 2^{nd} quantity follows χ^2_1. As the $\underbrace{\text{rank}}_{=d.f.}$ of $\frac{\Sigma(X_i - \bar{X})^2}{\sigma^2}$ is $n-1$, we apply Cochran's theorem.

Another result: ① and ② are indep.

$$\therefore \frac{\Sigma(X_i - \bar{X})^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}.$$

$$\therefore E\left[\frac{(n-1)S^2}{\sigma^2}\right] = (n-1) \Leftrightarrow E[S^2] = \sigma^2.$$

<Exercise 5.12>

$Y_1 = \left|\frac{1}{n}\sum_{i=1}^n X_i\right|$, $Y_2 = \frac{1}{n}\sum_{i=1}^n |X_i|$, where X_i's $\overset{iid}{\sim} N(0,1)$, $i=1,\cdots,n$.

$E[Y_1] = ?$, $E[Y_2] = ?$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$. Then $\bar{X} \sim N(0, \frac{1}{n})$.

$$EY_1 = \int_{-\infty}^{\infty} \frac{|t|}{\sqrt{2\pi/n}} e^{-\frac{t^2}{2/n}} dt = \frac{2\sqrt{n}}{\sqrt{2\pi}} \int_0^\infty t e^{-nt^2/2} dt = \frac{2\sqrt{n}}{\sqrt{2\pi}} \left[-\frac{1}{n} e^{-nt^2/2}\Big|_0^\infty\right]$$

$$= \sqrt{\frac{2}{n\pi}}.$$

For EY_2, note that $EY_2 = E|X_1|$ since X_i's are iid.

Then, $EY_2 = 2\int_0^\infty t \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \sqrt{\frac{2}{\pi}}$.

Thus $EY_1 \le EY_2$ where equality holds when $n=1$.

(Or, the inequality holds by the triangle inequality).

1) $X \sim F(a,b)$ then $1/X \sim F(b,a)$.

$$f_X(x) = \frac{\Gamma(\frac{a+b}{2})}{\Gamma(\frac{a}{2})\Gamma(\frac{b}{2})}\left(\frac{a}{b}\right)^{a/2}\frac{x^{\frac{a}{2}-1}}{[1+\frac{a}{b}x]^{\frac{a+b}{2}}}, \quad 0<x<\infty.$$

sol) Let $Y = \frac{1}{X}$. $\Rightarrow$ $X = \frac{1}{Y}$. $J = \frac{\partial X}{\partial Y} = -y^{-2}$.

$$f_Y(y) = f_X(\tfrac{1}{y})\,|-y^{-2}| = \underbrace{\frac{\Gamma(\frac{a+b}{2})}{\Gamma(\frac{b}{2})\Gamma(\frac{a}{2})}}_{\text{let } C_1}\left(\frac{a}{b}\right)^{a/2}\frac{\left(\frac{1}{y}\right)^{\frac{a}{2}-1}}{[1+\frac{a}{b}\cdot\frac{1}{y}]^{\frac{a+b}{2}}}\cdot\left(\frac{1}{y}\right)^2$$

multiply/divide by $\left(\frac{b}{a}\right)^{\frac{a+b}{2}}$

$$\overset{\downarrow}{=} C_1\cdot\frac{\left(\frac{a}{b}\right)^{\frac{a}{2}}\cdot\left(\frac{1}{y}\right)^{\frac{a}{2}+1}\cdot\left(\frac{b}{a}\right)^{\frac{a+b}{2}}}{[\frac{b}{a}+\frac{1}{y}]^{\frac{a+b}{2}}} = C_1\frac{\left(\frac{b}{a}\right)^{b/2}\cdot\left(\frac{1}{y}\right)^{\frac{a}{2}+1}}{[\frac{1}{y}+\frac{b}{a}]^{\frac{a+b}{2}}}$$

multiply/divide by $y^{\frac{a+b}{2}}$

$$= C_1\cdot\frac{\left(\frac{b}{a}\right)^{b/2}\cdot y^{\frac{b}{2}-1}}{[1+\frac{b}{a}\cdot y]^{\frac{a+b}{2}}} = \frac{\Gamma(\frac{b+a}{2})}{\Gamma(\frac{b}{2})\Gamma(\frac{a}{2})}\left(\frac{b}{a}\right)^{b/2}\frac{y^{\frac{b}{2}-1}}{[1+\frac{b}{a}y]^{\frac{b+a}{2}}}, \quad 0<y<\infty.$$

---

2) $X \sim t(a)$, then $X^2 \sim F(1,a)$.

Let $Y = X^2$. Then $X = \pm\sqrt{Y}$. $|J| = \left|\frac{\partial}{\partial y}\pm\sqrt{y}\right| = \frac{1}{2\sqrt{y}}$.

$$f_Y(y) = \{f_X(\sqrt{y}) + f_X(-\sqrt{y})\}\frac{1}{2\sqrt{y}} \qquad * f_X(\sqrt{y}) = f_X(-\sqrt{y}).$$

$$= f_X(\sqrt{y})\frac{1}{\sqrt{y}}$$

$$= \frac{\Gamma(\frac{a+1}{2})}{\Gamma(a/2)}\frac{1}{\sqrt{\pi a}}\left(1+\frac{y}{a}\right)^{-\frac{a+1}{2}}\cdot\frac{1}{\sqrt{y}}$$

$$= \frac{\Gamma(\frac{1}{2}+\frac{a}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{a}{2})}\left(\frac{1}{a}\right)^{\frac{1}{2}}\frac{y^{\frac{1}{2}-1}}{[1+\frac{1}{a}y]^{\frac{1+a}{2}}}, \quad 0<y<\infty$$

$*\ \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$\therefore Y \sim F(1,a)$.

---

3). Set $Y = \frac{(a/b)X}{1+(a/b)X}$. Then conduct transformation.

<Slide #11>.

$X \sim F(\nu_1, \nu_2)$. Let $X = \dfrac{W/\nu_1}{V/\nu_2}$, where $V \perp\!\!\!\perp W$ &

$V \sim \chi^2(\nu_2)$

$W \sim \chi^2(\nu_1)$.

Then $E[X^2] = E\left[\dfrac{W^2}{\nu_1}\right] \cdot E\left[\dfrac{\nu_2^2}{V^2}\right]$.

$$E\left[\frac{1}{V^2}\right] = \int_0^\infty \frac{1}{v^2} \frac{v^{\nu_2/2-1} e^{-v/2}}{\Gamma(\nu_2/2)\, 2^{\nu_2/2}}\, dv$$

$$= \frac{\Gamma(\frac{\nu_2}{2}-2)\, 2^{-2}}{\Gamma(\frac{\nu_2}{2})} \cdot \int_0^\infty \underbrace{\frac{v^{(\frac{\nu_2}{2}-2)-1} e^{-v/2}}{\Gamma(\frac{\nu_2}{2}-2) \cdot 2^{\nu_2/2-2}}}_{\text{Gamma}(\frac{\nu_2}{2}-2,\ 2)}\, dv$$

$$= \frac{\Gamma(\frac{\nu_2}{2}-2)}{\Gamma(\frac{\nu_2}{2}) \cdot 4}.$$

* see page ④ for more about $\Gamma(\frac{\nu}{2})$

Assume ν_2 is even number and large.

Then $E\left[\dfrac{1}{V^2}\right] = \dfrac{1}{(\frac{\nu_2}{2}-1)(\frac{\nu_2}{2}-2) \cdot 4} = \dfrac{1}{(\nu_2-2)(\nu_2-4)}$.

Assume ν_2 is a large odd number.

$$E\left[\frac{1}{V^2}\right] = \frac{(\frac{\nu_2}{2}-3)(\frac{\nu_2}{2}-4)\cdots\frac{1}{2}\sqrt{\pi}}{(\frac{\nu_2}{2}-1)(\frac{\nu_2}{2}-2)\cdots\frac{1}{2}\sqrt{\pi} \cdot 4} = \frac{1}{(\nu_2-2)(\nu_2-4)}$$

<Slide #14>.

$X_1, \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, $\sigma^2 < \infty$. Then, $\bar{X}_n \xrightarrow{P} \mu$. WLLN.

$$P(|\bar{X}_n - \mu| \geq \varepsilon) = P((\bar{X}_n-\mu)^2 \geq \varepsilon^2) \underset{\uparrow}{\leq} \frac{E(\bar{X}_n-\mu)^2}{\varepsilon^2}$$

Chebychev's inequality

$$= \frac{Var(\bar{X}_n)}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2} \longrightarrow 0 \text{ as } n \to \infty.$$

$\therefore \lim_{n\to\infty} P(|\bar{X}_n - \mu| \geq \varepsilon) = 0$. $\therefore \bar{X}_n \xrightarrow{P} \mu$. $[\bar{X}_n - \mu = O_p(n^{-1/2})]$

<More on Convergence in probability>. (4)

Consider $X_n = \begin{cases} n & w/p \ \ 1/n \\ 0 & w/p \ \ 1-1/n. \end{cases}$

Then, $P(|X_n - 0| \le \varepsilon) = 1 - \frac{1}{n} \to 1$. Thus $X_n \xrightarrow{P} 0$.

But, $E[X_n] = n \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n}) = 1$.

$E[X_n^2] = n^2 \cdot \frac{1}{n} + 0\,(1 - \frac{1}{n}) = n$. $\therefore Var(X_n) = n - 1$.

Thus $X_n \xrightarrow{P} c$ does NOT imply $E X_n \to c$ nor $Var(X_n) \to 0$.

Let $S^2 = \frac{\Sigma(X_i - \bar{X})^2}{(n-1)}$, X_i's $\overset{iid}{\sim} N(\mu, \sigma^2)$, $\sigma^2 < \infty$, We know $E[S^2] = \sigma^2$

Then ① $P(|S^2 - \sigma^2| \ge \varepsilon) \le \frac{E(S^2 - \sigma^2)}{\varepsilon^2} = \frac{Var(S^2)}{\varepsilon^2} = \frac{2\sigma^4}{(n-1)\varepsilon^2} \xrightarrow{as\ n \to \infty} 0$

$\therefore S^2 \xrightarrow{P} \sigma^2$.

② $S^2 = \underbrace{\frac{n}{n-1}}_{\to 1} \Big[\underbrace{\frac{1}{n} \Sigma X_i^2}_{\xrightarrow{WLLN} E[X^2] = \mu^2 + \sigma^2} - \bar{X}^2 \Big]$

ⓐ $\bar{X} \xrightarrow{P} \mu$ by WLLN.

ⓑ $(\bar{X})^2 \xrightarrow{P} \mu^2$ by Thm 5.5.4.

as $g(x) = x^2$ is a continuous function.

$\therefore S^2 \xrightarrow{P} \sigma^2$. by Slutsky's Thm.

③ $S \xrightarrow{P} \sigma$. as $g(x) = \sqrt{x}$ is a continuous function.

$$\frac{\Gamma(J + \frac{1}{2})}{\Gamma(J)} = \sqrt{J}\left(1 - \frac{1}{8J} + o\left(\frac{1}{J}\right)\right)$$

$$\therefore \frac{\Gamma(\frac{\nu}{2} + \frac{1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\nu\pi}} = \frac{1}{\sqrt{\nu\pi}} \sqrt{\frac{\nu}{2}} \left(1 - \frac{1}{4\nu} + o\left(\frac{1}{\nu}\right)\right) \xrightarrow{as\ \nu \to \infty} \frac{1}{\sqrt{2\pi}}.$$

This justifies $T_\nu \to N(0,1)$ in <Slide #8>.

<Slide #14, #17>.

Do we need $\sigma^2 < \infty$ for WLLN & SLLN?

Consider $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ where $\mu < \infty$.

Then $\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \varepsilon) = \lim_{n\to\infty} P\left(\frac{|\bar{X}_n - \mu|}{\sigma/\sqrt{n}} > \frac{\varepsilon\sqrt{n}}{\sigma}\right)$

$= \lim_{n\to\infty} P\left(|Z| < \frac{\varepsilon\sqrt{n}}{\sigma}\right) = \lim_{n\to\infty}\left[\Phi\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right) - \Phi\left(-\frac{\varepsilon\sqrt{n}}{\sigma}\right)\right]$

Note that $\frac{\sqrt{n}}{\sigma} \to \infty$ and $-\frac{\sqrt{n}}{\sigma} \to -\infty$ even if $\sigma = o(\sqrt{n})$.

(or, $\sigma = C \cdot n^{\alpha}, 0<\alpha<\frac{1}{2}$)

But, we still need $\mu < \infty$. Otherwize $E[\bar{X}_n]$ does not exist.

So, $\sigma^2 < \infty$ is a stronger condition than needed. (Sufficient but not necessary).

<Slide #18>.

X_i's $\overset{iid}{\sim} U(0,1)$. $X_{(n)} = \max_{1 \le i \le n} X_i$.

Show that $n(1 - X_{(n)}) \xrightarrow{D} Exp(1)$.

$P(X_{(n)} \le t) = P(X_1 \le t) P(X_2 \le t) \cdots P(X_n \le t) = t^n$.

Then $P(n(1 - X_{(n)}) \le t) = P(X_{(n)} \ge 1 - t/n) = 1 - P(X_{(n)} \le 1 - t/n)$

$= 1 - (1 - t/n)^n$.

$\xrightarrow{n\to\infty} 1 - e^{-t}, \ 0<t<1$.

Because $\int_0^t e^{-x}dx = -e^{-x}\Big|_0^t = 1 - e^{-t}$, $n(1 - X_{(n)}) \xrightarrow{D} Exp(1)$

Exercise 5.42.

(a) X_i's $\overset{iid}{\sim}$ Beta$(1,\beta)$. Find ν s.t. $n^{\nu}(1-X_{(n)})$ converges in distr.

where $X_{(n)} = \max_{1\le i\le n} X_i$ and Y is a random variable.

$$P(X_i \le t) = \int_0^t \frac{\Gamma(1+\beta)}{\Gamma(1)\Gamma(\beta)} x^{1-1}(1-x)^{\beta-1}dx = \int_0^t \beta(1-x)^{\beta-1}dx = 1-(1-t)^{\beta}, \quad 0\le t\le 1.$$

$$\Rightarrow P(X_{(n)} \le t) = [P(X_1\le t)]^n = [1-(1-t)^{\beta}]^n.$$

Thus
$$P[n^{\nu}(1-X_{(n)}) \le t] = P[X_{(n)} \ge 1 - t/n^{\nu}]$$
$$= 1-P(X_{(n)} < 1-t/n^{\nu})$$
$$= 1-[1-(1-(1-t/n^{\nu}))^{\beta}]^n$$
$$= 1-\left[1-\frac{t^{\beta}}{n^{\beta\nu}}\right]^n, \quad 0\le t<\infty.$$

Note that $\left(1-\frac{t^{\beta}}{n^{\beta\nu}}\right)^n \xrightarrow{n\to\infty} e^{-t^{\beta}}$ if $\nu=\frac{1}{\beta}$.

Then $P[n^{1/\beta}(1-X_{(n)}) \le t] \xrightarrow{n\to\infty} 1-e^{-t^{\beta}}, \quad 0\le t<\infty$

This is a cdf of Weibull distribution w/ scale parameter $=1$,
shape " $=\beta$.

$\boxed{\therefore\ \nu = \frac{1}{\beta}.}$

(b) Similar to (a), $P(X_{(n)}\le t) = [1-e^{-t}]^n$. Thus

$$P(X_{(n)} - a_n \le t) = \left(1-e^{-(a_n+t)}\right)^n \quad \text{Let } a_n = \log n.$$
$$= \left(1-e^{-\log n}\cdot e^{-t}\right)^n = \left(1-\frac{e^{-t}}{n}\right)^n \xrightarrow{n\to\infty} e^{-e^{-t}}, \quad -\infty<t<\infty$$

Standard Gumbel distribution is the limiting distribution.

<Slide #20>. C.L.T.

Consider the central moment $m(t) = Ee^{t(X-\mu)}$.

Then, there exist $0 \le \xi \le t$ such that

$m(t) = m(0) + m'(0)t + m''(\xi)t^2/2$ by Taylor expansion. ⓐ

We have $m(0)=1$, $m'(0)=0$, $m''(0) = \sigma^2$. $\Rightarrow m(t) = 1 + \frac{\sigma^2 t^2}{2} + \frac{(m''(\xi)-\sigma^2)t^2}{2}$.

Let $Y_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$. Then $m_{Y_n}(t) = Ee^{tY_n} = Ee^{t\left(\frac{\sum X_i - n\mu}{\sigma\sqrt{n}}\right)}$

$$\overset{iid}{=} \left[Ee^{t/\sigma\sqrt{n}\,(X_i-\mu)}\right]^n =$$

By ⓐ $= \left[1 + \frac{\sigma^2}{2}\left(\frac{t}{\sigma\sqrt{n}}\right)^2 + \frac{(m''(\xi)-\sigma^2)}{2}\cdot\left(\frac{t}{\sigma\sqrt{n}}\right)^2\right]^n$, where $0 \le \xi \le \frac{t}{\sigma\sqrt{n}}$.

Then, we have $\xi \to 0$ as $n \to \infty$, and therefore $m''(\xi) - \sigma^2 \to 0$ as $n \to \infty$

Thus $m_{Y_n}(t) = \left[1 + \frac{t^2}{2n} + \underbrace{\frac{(m''(\xi)-\sigma^2)t^2}{2n\sigma^2}}_{\text{This term goes to zero quickly}}\right]^n \to e^{t^2/2}$

Noting that $e^{t^2/2}$ is a mgf of $N(0,1)$, we have

mgf of $Y_n \to$ mgf of $N(0,1)$.

Then the convergence in distribution is verified by the continuity theorem (in Slide #20).

<Slide #22>. (8)

We have $\sqrt{n}(Y_n - \theta) \xrightarrow{D} N(0, \sigma^2)$. $\exists g'(\theta) \neq 0$:

By Taylor expansion of $g(Y_n)$ around θ provides

$$g(Y_n) = g(\theta) + g'(\theta)(Y_n - \theta) + g''(\xi)\frac{(Y_n-\theta)^2}{2}, \quad \xi \in (Y_n, \theta) \quad (a)$$

Note that $P(: |Y_n - \theta| < \varepsilon) = P\left(\frac{\sqrt{n}|Y_n-\theta|}{\sigma} < \sqrt{n}\,\varepsilon/\sigma\right)$

$$= \Phi\left(\frac{\varepsilon\sqrt{n}}{\sigma}\right) - \Phi\left(-\frac{\varepsilon\sqrt{n}}{\sigma}\right) \xrightarrow{n\to\infty} 1.$$

$\therefore Y_n \xrightarrow{P} \theta$. Further, we have $Y_n - \theta = O_p(n^{-1/2}) \Rightarrow (Y_n-\theta)^2 = O_p(n^{-1})$.

Note: the 3rd term in the RHS of (a) goes to zero quickly.

Now, $\sqrt{n}(g(Y_n) - g(\theta)) = g'(\theta)\cdot\sqrt{n}(Y_n - \theta) + O_p(n^{-1/2})$.

As $Y_n \xrightarrow{P} \theta$ and $\sqrt{n}(Y_n - \theta) \xrightarrow{D} N(0, \sigma^2)$, by the Slutsky's theorem, $g'(\theta)\cdot\sqrt{n}(Y_n - \theta) \xrightarrow{D} N(0, \{g'(\theta)\}^2\sigma^2)$.

$g_1(\mu) = e^{\mu}$, $g_2(\mu) = 1/\mu$. $\Rightarrow g_1'(\mu) = e^{\mu}$. $g_2'(\mu) = -\mu^{-2}$.

By the Central Limit Theorem, we have

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2).$$

By Delta method,

* $\sqrt{n}(e^{\bar{X}_n} - e^{\mu}) \xrightarrow{D} N(0, \sigma^2 e^{2\mu})$
* $\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\mu}\right) \xrightarrow{D} N(0, \sigma^2/\mu^4)$.

If we don't know μ and σ^2, we can estimate them by $\bar{X}_n$, S_n^2.

$f_{X_{(1)},\cdots,X_{(n)}}(y_1,\cdots,y_n) = n!\, f_X(y_1)\cdots f_X(y_n), \quad -\infty < y_1 < \cdots < y_n < \infty$

Then $f_{X_{(j)}}(y_j) = \int_{-\infty}^{y_j}\int_{-\infty}^{y_{j-1}}\cdots\int_{-\infty}^{y_2}\int_{y_j}^{\infty}\int_{y_{j+1}}^{\infty}\cdots\int_{y_{n-1}}^{\infty} n!\, f(y_1)\cdots f(y_j)\cdots f(y_n)\,dy_n\cdots dy_{j+2}\,dy_{j+1}\,dy_1\cdots dy_{j-1}$

Consider the integral one-by-one.

① st $\int_{y_{n-1}}^{\infty} f(y_n)\,dy_n = 1-F(y_{n-1})$, ② nd $\int_{y_{n-2}}^{\infty}[1-F(y_{n-1})]f(y_{n-1})\,dy_{n-1} = \frac{\{1-F(y_{n-2})\}^2}{2}$

③ rd $\int_{y_{n-3}}^{\infty}\frac{\{1-F(y_{n-2})\}^2}{2}f(y_{n-2})\,dy_{n-2} = \frac{\{1-F(y_{n-3})\}^3}{3!}$, $\cdots$ (n−j) th $\frac{\{1-F(y_j)\}^{n-j}}{(n-j)!}$.

Now, we integrate w.r.t. $y_1, y_2, \cdots, y_{j-1}$ in this order.

① st $\int_{-\infty}^{y_2} f(y_1)\,dy_1 = F(y_2)$. ② nd $\int_{-\infty}^{y_3} F(y_2)f(y_2)\,dy_2 = F(y_3)^2/2$

③ rd. $\int_{-\infty}^{y_4} F(y_3)^2/2 \cdot f(y_3)\,dy_3 = \frac{F(y_4)^3}{3!}$ $\cdots$ (j−1) th $\frac{\{F(y_j)\}^{j-1}}{(j-1)!}$

We combine the above results, then

$$f_{X_{(j)}}(y_j) = \frac{n!}{(j-1)!(n-j)!}\{F(y_j)\}^{j-1}\{1-F(y_j)\}^{n-j}\cdot f(y_j), \quad -\infty<y_j<\infty$$

$\therefore X_{(j)} \sim Beta(j, n-j+1)$

Similarly $f_{X_{(j)},X_{(k)}}(y_j,y_k)$ [Slide #26] can be found.

Example) X_i's $\overset{iid}{\sim} U(0,1)$, $i=1,\cdots,n$. Find the distn. of $R = X_{(n)} - X_{(1)}$

Let $V = X_{(n)} + X_{(1)}$.

Then, $X_{(n)} = \frac{R+V}{2}$, $X_{(1)} = \frac{V-R}{2}$. $J = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\end{vmatrix} = -\frac{1}{2}$.

$f_{R,V}(r,v) = f_{X_{(1)},X_{(n)}}\left(\frac{v-r}{2}, \frac{v+r}{2}\right)|J|$,

$= \frac{1}{2}\cdot\frac{n!}{(1-1)!\,(n-2)!\,(n-n)!}\left\{F_X\left(\frac{v+r}{2}\right)-F_X\left(\frac{v-r}{2}\right)\right\}^{n-2} f_X\left(\frac{v+r}{2}\right)f_X\left(\frac{v-r}{2}\right)$,

$0<r<v<2-r$ → continue.

$\circledast$ (As $0 < X_{(1)} < X_{(n)} < 1$, we have $0 < \frac{v-r}{2} < \frac{v+r}{2} < 1$
or, equivalently $0 < r < v < 2-r$.)

$$f_{R,V}(r,v) = \frac{1}{2} n(n-1) \left\{ \frac{v+r}{2} - \frac{v-r}{2} \right\}^{n-2} = \frac{n(n-1)}{2} r^{n-2}, \quad 0 < r < v < 2-r.$$

$$f_R(r) = \int_r^{2-r} \frac{1}{2} n(n-1) r^{n-2} dv = n(n-1) r^{n-2} (1-r), \quad 0 < r < 1.$$

$\Rightarrow \quad R \sim Beta(n-1, 2)$ //

Now, let's find $f_V(v)$.

From $\circledast$, we see $0 < r < v$ if $0 < v < 1$, $0 < r < 2-v$ if $1 < v < 2$.

Then, $f_V(v) = \begin{cases} \int_0^v \frac{1}{2} n(n-1) r^{n-2} dr = \frac{n}{2} v^{n-1}, & 0 < v < 1 \\ \int_0^{2-v} \frac{1}{2} n(n-1) r^{n-2} dr = \frac{n}{2} (2-v)^{n-1}, & 1 < v < 2 \end{cases}$

Check $f_V(v)$ is a proper pdf. What is $E(R)$, $E(V) = ?$

$E[R] = \frac{n-1}{n+1}$ // < 1, but $E[R] \longrightarrow 1$ as $n \to \infty$.

$$E[V] = \int_0^1 \frac{n}{2} v^n dv + \int_1^2 \frac{n}{2} (2-v)^{n-1} v \, dv$$

$$= \frac{n}{2(n+1)} + \int_0^1 \frac{n}{2} (2-t) t^{n-1} dt \quad (\text{by setting } t = 2-v)$$

$$= \frac{n}{2(n+1)} + \int_0^1 n t^{n-1} dt - \int_0^1 \frac{n}{2} t^n dt = 1 \text{ //}$$

$\rightarrow$ On average, the distance between 0 and $X_{(1)}$ is the same as the distance between $X_{(n)}$ and 1. $\therefore E[V] = 1$