Ch 9. Interval Estimation

Intro

Definition

Let $X_1, \cdots X_n$ have joint pdf/pmf $f(\mathbf{x} : \theta)$, $\theta \in \Theta$. Let $L(\mathbf{X})$ and $U(\mathbf{X})$ be two statistics such that $L(\mathbf{X}) \leq U(\mathbf{X})$ with probability 1.

1. The random interval $I(\mathbf{X}) = [L(\mathbf{X})\ ,\ U(\mathbf{X})]$ is called an interval estimator for θ.
2. $I(\mathbf{X}) = (-\infty\ ,\ U(\mathbf{X})]$ is said to be a one-sided upper interval estimator for θ.
3. $I(\mathbf{X}) = [L(\mathbf{X})\ ,\ \infty)$ is said to be a one-sided lower interval estimator for θ.
4. The **coverage probability** of an interval estimator $I(\mathbf{X})$ is defined as $P_\theta[\theta \in I(\mathbf{X})]$.
5. The **confidence coefficient** of $I(\mathbf{X})$ is defined as $\inf_{\theta \in \Theta} P_\theta[\theta \in I(\mathbf{X})]$.

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim}$ Uniform$(0, \theta)$. Consider the following three interval estimators of θ.

$I_1(\mathbf{X}) = [aX_{(n)}\ ,\ bX_{(n)}],\ 1 \leq a < b$
$I_2(\mathbf{X}) = [X_{(n)} + c\ ,\ \infty)$
$I_3(\mathbf{X}) = [X_{(n)} + a\ ,\ X_{(n)} + b]$

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, 1)$

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta \neq \theta_0$$

The LRT of size α is

$$\phi(\mathbf{x}) = \begin{cases} 1, & |\sqrt{n}(\bar{x} - \theta_0)| \geq z_{\alpha/2}, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\Longrightarrow P_{\theta_0}\{\bar{X} - z_{\alpha/2}/\sqrt{n} \leq \theta_0 \leq \bar{X} + z_{\alpha/2}/\sqrt{n}\} = 1 - \alpha$$

Theorem

Let $X_1, \cdots, X_n$ have joint pdf/pmf $f(\mathbf{x} : \theta)$, $\theta \in \Theta$. For each $\theta_0 \in \Theta$, let $A(\theta_0)$ denote the acceptance region of a size α simple test for testing

$$H_0 : \theta = \theta_0 \quad vs \quad H_1 : \theta \neq \theta_0$$

Define a set $C(\mathbf{x}) = \{\theta_0 \in \Theta : \mathbf{x} \in A(\theta_0)\}$. Then $C(\mathbf{X})$ is a **confidence set** *with confidence coefficient $1 - \alpha$.*

$\triangleleft$ Note:

1. $C(\mathbf{X})$ is not necessarily an interval.
2. One may need to consider one-sided test for one-sided confidence interval.

$\triangleright$ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. μ and σ^2 are unknown. Find a $1-\alpha$ two-sided CI and one-sided lower CI for μ.

$\triangleright$ Example: $X_1, \cdots, X_n \overset{iid}{\sim} f(x|\theta) = \theta^2 x e^{-\theta x}, x > 0, \theta > 0$. Find an approximate (and exact) $1-\alpha$ confidence set of θ.

Definition

Let $X_1, \cdots, X_n$ have joint pdf/pmf $f(\mathbf{x} : \theta)$, $\theta \in \Theta$. A random variable $Y = Q(\mathbf{X} : \theta)$ is called a *pivotal quantity (PQ)* if the distribution of $Y = Q(\mathbf{X} : \theta)$ does not depend on θ.

$\triangleright$ Example: $X_1, \cdots, X_n \overset{iid}{\sim} f(x : \theta)$. Consider the following families of distributions and statistics.

1. $f(x : \theta) = f_0(x - \theta)$
2. $f(x : \theta) = \frac{1}{\theta} f_0(x)$
3. $f(x : \theta) = \frac{1}{\theta_2} f_0[(x - \theta_1)/\theta_2]$

$$\bar{X}_n - \theta, \quad \bar{X}_n/\theta, \quad (\bar{X}_n - \theta_1)/\theta_2$$

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim} \exp(\lambda)$.

$$T = \sum X_i \sim \text{Gamma}(n, \lambda)$$

($Gamma(n, \lambda)$ is a scale family)
Find a $(1 - \alpha)\%$ confidence interval of λ.

▷ Example 9.2.15: $X_1, \cdots, X_n \overset{iid}{\sim} Poisson(\lambda)$. Find a $(1 - \alpha)\%$ confidence interval of λ. (Hint: Use Gamma-Poisson relationship $P(U \leq u) = P(V \geq \alpha), U \sim Gamma(\alpha, \beta), V \sim Poisson(u/\beta)$.)

Theorem (See the theorem 2.1.10 for reference)

Suppose $T = T(\mathbf{X})$ is a statistic calculated from $X_1, \cdots, X_n$. Assume T has a continuous distribution with cdf

$$F(t:\theta) = P_\theta(T \leq t).$$

Then

$$Q(T:\theta) = F(T:\theta)$$

is a PQ.

◁ Note: In order for $Q(T:\theta) = F(T:\theta)$ to result in a confidence interval, we want $F(T:\theta)$ to be monotone in θ. A cdf $F(T:\theta)$ that is increasing or decreasing in θ for all t is said to be stochastically increasing or decreasing.

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 is known. Let $T(\mathbf{X}) = \bar{X}$. Then

$$Q(T:\mu) = F(T:\mu) = \Phi\left(\frac{T-\mu}{\sigma/\sqrt{n}}\right).$$

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim} f(x:\mu) = e^{-(x-\mu)}, \quad x \geq \mu$. Let $T(\mathbf{X}) = X_{(1)} = \min_{1\leq i\leq n} X_i$. Derive $(1-\alpha)100\%$ confidence interval using the cdf of T.

Definition

$[L(\mathbf{x}), U(\mathbf{x})]$ is called a $(1-\alpha)100\%$ *credible set (or Bayesian interval)* if

$$
\begin{aligned}
1-\alpha &= P[L(\mathbf{x}) < \theta < U(\mathbf{x}) | \mathbf{X} = \mathbf{x}] \\
&= \begin{cases} \sum_{\theta} \pi(\theta|\mathbf{x}) & \text{discrete} \\ \int_{\theta} \pi(\theta|\mathbf{x}) d\theta & \text{continuous} \end{cases}
\end{aligned}
$$

$\triangleright$ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$. $\theta \sim N(\mu, \tau^2)$. Find a $(1-\alpha)$ credible set.

CI: Length of CI *vs* Coverage probability

Definition

$f(x)$ is a unimodal pdf if $f(x)$ is nondecreasing for $x \leq x^*$ and nonincreasing for $x \geq x^*$ in which case x^* is the mode of the distribution.

Theorem (Theorem 9.3.2. Shortest CI for unimodal pdf.)

Let $f(x)$ be a unimodal pdf. If the interval $[a, b]$ satisfies

i. $\int_a^b f(x)dx = 1 - \alpha$
ii. $f(a) = f(b) > 0$
iii. $a \leq x^* \leq b$, *when x^* is a mode of $f(x)$*

Then no other interval estimator satisfying (i) is shorter than $[a, b]$.

▷ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. σ^2 is known.

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Definition (Probability of false coverage)

For $\theta' \neq \theta$, $P_\theta[L(\mathbf{X}) \leq \theta' \leq U(\mathbf{X})]$

For $\theta' < \theta$, $P_\theta[L(\mathbf{X}) \leq \theta']$

For $\theta' > \theta$, $P_\theta[\theta' \leq U(\mathbf{X})]$

Definition

A $1-\alpha$ confidence interval with minimum probability of false coverage is called a *Uniformly Most Accurate (UMA)* $1-\alpha$ confidence interval.

Ch 9. Interval Estimation

Optimal theory for CI

Theorem (UMA CI based on UMP test)

Let $X_1, \cdots, X_n$ have a joint pdf/pmf $f(\mathbf{x} : \theta)$. Suppose that a UMP test of size α for testing $H_0 : \theta \leq \theta_0 \quad vs \quad H_1 : \theta > \theta_0$ exists and given as

$$\phi(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \notin A^*(\theta_0), \\ 0, & \mathbf{x} \in A^*(\theta_0). \end{cases}$$

Let $C^*(\mathbf{X})$ be the confidence interval obtained by inverting the UMP acceptance region. Then, for any other $1 - \alpha$ confidence region(set, interval),

$$P_\theta[\theta' \in C^*(\mathbf{X})] \leq P_\theta[\theta' \in C^*(\mathbf{X})],$$

for all $\theta' < \theta$. That is, inverting UMP test yields a UMA confidence region(set, interval).

$\triangleleft$ Note: UMP unbiased test can be inverted to obtain UMA unbiased confidence region(set, interval).

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$\triangleright$ Example: $X_1, \cdots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$. σ^2 is known.