1. The first two parts of this problem were just exercises in applying the formulas to compute moments of quadratic forms. Here we make use of S-Plus to perform the calculations.

(a) $E(\mathbf{Y}^T A\mathbf{Y}) = \mu^{\mathbf{T}}\mathbf{A}\mu + \mathbf{tr}(\mathbf{A\Sigma})$. **(Result 4.6)**

```
> mu <- c(-1,0,-3);
> sigma <- matrix(c(.75,.00,.25,.00,1.00,.00,.25,.00,.75),3,3)
> A <- matrix(c(1.5,0.0,-0.5,0.0,1.0,0.0,-0.5,0.0,1.5),3,3)
> E <- t(mu)%*%A%*%mu+sum(diag(A%*%sigma))
> E
     [,1]
[1,]   15
```

(b) $var(\mathbf{Y}^T A\mathbf{Y}) = \mathbf{4}\mu^{\mathbf{T}}\mathbf{A\Sigma A}\mu + \mathbf{2tr}(\mathbf{A\Sigma A\Sigma})$.

```
> Var <- 4*t(mu)%*%A%*%sigma%*%A%*%mu+2*sum(diag(A%*%sigma%*%A%*%sigma))
> Var
     [,1]
[1,]   54
```

(c) We can see that A is symmetric, and Σ is symmetric and positive definite. Next we need to check if $A\Sigma$ is idempotent.

```
> V1 <- A%*%sigma%*%A%*%sigma
> V1
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
> V2 <- A%*%sigma
> V2
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
```

It turns out that $A\Sigma A\Sigma = A\Sigma$,so we conclude $\mathbf{y^T A y}$ has a noncentral chi-square distribution with 3 degrees of freedom and noncentrality parameter $\mu^T A\mu = 12$.

(d) Now we need to check if $(A/\sigma^2)\sigma^2 I = A$ is idempotent. It turns out $AA \neq A$. So we conclude $\mathbf{y^T A y}/\sigma^\mathbf{2}$ does not have a chi-square distribution.

```
> A%*%A
     [,1] [,2] [,3]
[1,]  2.5    0 -1.5
[2,]  0.0    1  0.0
[3,] -1.5    0  2.5
> A
     [,1] [,2] [,3]
[1,]  1.5    0 -0.5
[2,]  0.0    1  0.0
[3,] -0.5    0  1.5
```

2. In this case, $\underset{\sim}{Y} \sim N(\mu\underset{\sim}{1}, \sigma^2 I)$, where $\underset{\sim}{Y} = (Y_1, \ldots, Y_n)^T$.

(a) We know that $(I - P_{\underset{\sim}{1}})\underset{\sim}{Y} = \underset{\sim}{Y} - \overline{Y}\underset{\sim}{1}$. Since $I - P_{\underset{\sim}{1}}$ is a projection matrix, it is symmetric and idempotent. Then,

$$\frac{1}{n-1}\underset{\sim}{Y}^T(I - P_{\underset{\sim}{1}})\underset{\sim}{Y} = \frac{1}{n-1}\underset{\sim}{Y}^T(I - P_{\underset{\sim}{1}})^T(I - P_{\underset{\sim}{1}})\underset{\sim}{Y} = \frac{1}{n-1}((I - P_{\underset{\sim}{1}})\underset{\sim}{Y})^T((I - P_{\underset{\sim}{1}})\underset{\sim}{Y}).$$

Consequently,

$$\frac{1}{n-1}((I - P_{\underset{\sim}{1}})\underset{\sim}{Y})^T((I - P_{\underset{\sim}{1}})\underset{\sim}{Y}) = \frac{1}{n-1}(\underset{\sim}{Y} - \overline{y}\underset{\sim}{1})^T(\underset{\sim}{Y} - \overline{y}\underset{\sim}{1}) = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y})^2 = S^2.$$

$E(\underset{\sim}{Y}^T A \underset{\sim}{Y}) = \mu^T A \mu + tr(A\Sigma)$. **(Result 4.6)**

$$\begin{aligned}
E(S^2) &= \frac{1}{n-1}[(\mu\underset{\sim}{1})^T(I - P_{\underset{\sim}{1}})(\mu\underset{\sim}{1}) + tr((I - P_{\underset{\sim}{1}})\sigma^2 I)] \\
&= \frac{1}{n-1}[\mu^2(\underset{\sim}{1}^T\underset{\sim}{1} - \underset{\sim}{1}^T\underset{\sim}{1}(\underset{\sim}{1}^T\underset{\sim}{1})^{-1}\underset{\sim}{1}^T\underset{\sim}{1}) \;+\; \sigma^2 tr(I - P_{\underset{\sim}{1}})] \\
&= \frac{1}{n-1}[\,0 + \sigma^2 rank(I - P_{\underset{\sim}{1}})] \\
&= \frac{1}{n-1}[\sigma^2(n-1)] \\
&= \sigma^2
\end{aligned}$$

$$
\begin{aligned}
Var(S^2) &= 4\mu^T A\Sigma A\mu + 2tr(A\Sigma A\Sigma) \\
&= 4((\mu \underset{\sim}{1})^T \frac{I - P_{\underset{\sim}{1}}}{n-1}\sigma^2 I \frac{I - P_{\underset{\sim}{1}}}{n-1}\mu \underset{\sim}{1}) + 2tr(\frac{I - P_{\underset{\sim}{1}}}{n-1}\sigma^2 I \frac{I - P_{\underset{\sim}{1}}}{n-1}\sigma^2 I) \\
&= \frac{4}{(n-1)^2}\sigma^2\mu^2(\underset{\sim}{1})^T(I - P_{\underset{\sim}{1}})\underset{\sim}{1} + \frac{2}{(n-1)^2}\sigma^4 tr(I - P_{\underset{\sim}{1}}) \\
&= \frac{2\sigma^4}{n-1}
\end{aligned}
$$

(b) $\underset{\sim}{Y} \sim N(\mu \underset{\sim}{1}, \sigma^2 I)$, Let $A = \frac{1}{\sigma^2}(I - P_{\underset{\sim}{1}})$, then $A\Sigma = I - P_{\underset{\sim}{1}}$ is idempotent from part (a), and $(I - P_{\underset{\sim}{1}})$ is symmetric and Σ is symmetric and positive definite. Hence, the conditions of **Result 4.7** are satisfied and we have $\underset{\sim}{Y}^T A \underset{\sim}{Y} = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}(\delta^2)$, where the degrees of freedom are rank$(I - P_{\underset{\sim}{1}})$=n-1. Here, the non-centrality parameter is $\delta^2 = \mu^T A \mu = (\mu \underset{\sim}{1})^T (I - P_{\underset{\sim}{1}})(\mu \underset{\sim}{1}) = 0$, and $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$. Then, from the moments of a central chi-square random variable, we have

$$
n - 1 = E(\frac{(n-1)S^2}{\sigma^2}) = \frac{(n-1)}{\sigma^2}E(S^2)
$$

and it follows that $E(S^2) = \sigma^2$.

From the variance of a central chi-square random variable, we have

$$
2(n-1) = Var(\frac{(n-1)S^2}{\sigma^2}) = \frac{(n-1)^2}{\sigma^4}Var(S^2)
$$

and it follows that $Var(S^2) = \frac{2\sigma^4}{(n-1)}$. These match the results from part (a).

(c) Note that $n\overline{Y}^2 = \frac{1}{n}[\sum_{i=1}^{n} Y_i]^2 = \underset{\sim}{Y}^T P_1 \underset{\sim}{Y}$. To apply **Result 4.8**, check that

$A_1 \Sigma A_2 = P_1(\sigma^2 I)\frac{1}{(n-1)}(I - P_1) = \frac{\sigma^2}{(n-1)}P_{\underset{\sim}{1}}(I - P_{\underset{\sim}{1}}) = 0$. Consequently, since P_1 and $I - P_1$ are both symmetric, the conditions of **Result 4.8** are satisfied and we have established that $n\overline{Y}^2$ and S^2 are independent.

(d) From the results above we can write

$$
n\overline{Y}^2 = \underset{\sim}{Y}^T P_{\underset{\sim}{1}} \underset{\sim}{Y} \quad \text{and} \quad (n-1)S^2 = \underset{\sim}{Y}^T (I - P_{\underset{\sim}{1}})\underset{\sim}{Y}
$$

Since $I = P_1 + (I - P_1)$, $n = rank(P_1) + rank(I - P_1)$, and P_1 and $I - P_1$ are symmetric, by Cochran's Theorem

$$
\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}
$$

and

$$
\frac{1}{\sigma^2}\underset{\sim}{Y}^T P_{\underset{\sim}{1}} \underset{\sim}{Y} \sim \chi^2_1(\delta^2) \quad \text{where } \delta^2 = \frac{1}{\sigma^2}(\mu \underset{\sim}{1})^T P_{\underset{\sim}{1}}(\mu \underset{\sim}{1}) = \frac{n\mu^2}{\sigma^2}
$$

and these random quantities are distributed independently. It follows that

$$F = \frac{[\frac{1}{\sigma^2}\underset{\sim}{Y}^T P_1 \underset{\sim}{Y}]/1}{[\frac{1}{\sigma^2}\underset{\sim}{Y}^T(I-P_1)\underset{\sim}{Y}]/(n-1)} = \frac{n\overline{Y}^2}{S^2} \sim F_{(1,n-1)}(\frac{n\mu^2}{\sigma^2}).$$

(e) $\delta^2 = \frac{n\mu^2}{\sigma^2}$. Since $\sigma^2 > 0$ and $n > 0$, then $\delta^2 = 0$ if and only if $\mu = 0$. Consequently, we can test $H_o : \mu = 0$ against $H_a : \mu \neq 0$, and H_o is rejected if $F \geq F(\alpha; 1, n-1)$.

3.(a) To show H_0 is testable, we need show (1) the linear combinations of parameters that are restricted by the null hypothesis ($C\beta$) is estimable and (2) the rank of the C matrix is equal to the number of rows in C. To show $C\beta$ is estimable you can use the definition of estimability or result 3.8 or 3.9 from the course notes. For example, $C\beta$ is estimable if and only if for any $X\underline{d} = 0$, we have $C\underline{d} = 0$. In this case, the only solution to $X\underline{d} = 0$ is $\underline{d}^T = k[-1\ 1\ 1\ 1]$, where k is a constant, and we have to check if $C[-1\ 1\ 1\ 1]^T = 0$.

	function	C	estimable	rank(C)=rows	testable
(i)	$\alpha_1 = \alpha_2$	[0 1 -1 0]	Yes	Yes	Yes
(ii)	$\alpha_1 - 2\alpha_2 + 3\alpha_3 = 0$	[0 1 -2 3]	No		No
(iii)	$\alpha_3 = 0$	[0 0 0 1]	No		No
(iv)	$\mu = 0$	[1 0 0 0]	No		No
(v)	$\alpha_1 = \alpha_3$ and $\alpha_1 - 2\alpha_2 + \alpha_3 = 0$	$\begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix}$	Yes	Yes	Yes
(vi)	$\alpha_1 = \alpha_2 = \alpha_3 \text{and} 2\alpha_1 - \alpha_2 - \alpha_3 = 0$	$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & -1 & -1 \end{bmatrix}$	Yes	No	No

In this case, the last row of C is the sum of the first two rows of C.

(b) First note that the null hypothesis $H_0 : \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is testable.

(i) Note that $Var(\underset{\sim}{Y})=\sigma^2 I$ is positive definite, $\frac{I-P_x}{\sigma^2}$ is symmetric, $\frac{I-P_x}{\sigma^2}\sigma^2 I = I - P_x$ is idempotent, and $\delta^2 = (X\beta)^T \frac{I-P_x}{\sigma^2} X\beta = 0$. Consequently, the conditions of Result 4.7 are satisfied and SSE has a central chi-square distribution with 5 d.f. (rank($I - P_x$)=rank(I-rank(X)=8-3=5).

(ii) Define SS$H_0 = (C\underline{b})^T[C(X^TX)^- C^T]^{-1} C\underline{b}$, where $C = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix}$.

Note that, $Cov(C\underline{b},\ (I-P_x)\underset{\sim}{Y}) = Cov(C(X^TX)^-X^T\underset{\sim}{Y},\ (I-P_x)\underset{\sim}{Y}) = C(X^TX)^-X^T\sigma^2 I(I-P_x)^T$
$= \sigma^2 C(X^TX)^-X^T(I-P_x)^T = 0$, since $X^T(I-P_x)^T = 0$. Furthermore, the joint distribution of

$$\begin{bmatrix} C\underline{b} \\ (I-P_x)\underset{\sim}{Y} \end{bmatrix} = \begin{bmatrix} C(X^TX)^-X^T\underset{\sim}{Y} \\ (I-P_x)\underset{\sim}{Y} \end{bmatrix} = \begin{bmatrix} C(X^TX)^-X^T \\ (I-P_x) \end{bmatrix}\underset{\sim}{Y}$$

is multivariate normal (by result 4.1) because it is a linear function of a multivariate normal random vector $\underset{\sim}{Y}$. Consequently, $C(X^TX)^-X^T\underset{\sim}{Y}$ and $(I-P_x)\underset{\sim}{Y}$ are independent random vectors. Since SSE is a function only of $(I-P_x)\underset{\sim}{Y}$ while SSH_0 is a function only of $C\underline{b}$, it follows that they are independently distributed.

Or we can use result 4.8 from the notes. First define

$$SSH_0 = (C\underline{b})^T[C(X^TX)^-C^T]^{-1}C\underline{b} = \underset{\sim}{Y}^T X(X^TX)^-C^T[C(X^TX)^-C^T]^{-1}C(X^TX)^-X^T\underline{y} = \underset{\sim}{Y}^T A_0 \underset{\sim}{Y}\,.$$

The condition of Result 4.8 is satisfied because

$$(I-P_x)\sigma^2 I A_0 = (I-P_x)X(X^TX)^-C^T[C(X^TX)^-C^T]^{-1}C(X^TX)^-X^T = 0,$$

since $(I-P_x)X = 0$. Therefore, SSE and SSH_0 are independently distributed.

(iii) We know that $C\underline{b} \sim N(C\underline{\beta},\ \sigma^2 C(X^TX)^-C^T)$ and $\frac{1}{\sigma^2}[C(X^TX)^-C^T]^{-1}Var(C\underline{b}) = I$ is symmetric and idempotent. It follows from Result 4.7 that $\frac{SSH_0}{\sigma^2} \sim \chi^2_2(\delta^2)$, where 2=rank(C) provides the degrees of freedom. The non-centrality parameter is

$$\delta^2 = \frac{1}{\sigma^2}\underline{\beta}^T C^T(C(X^TX)^-C^T)^{-1}C\underline{\beta}\,.$$

(iv) It follows from the definition of the F-distribution that

$$F = \frac{\frac{SSH_0}{2\sigma^2}}{\frac{SSE}{5\sigma^2}} = \frac{SSH_0/2}{SSE/5} \sim F_{(2,5)}(\delta^2)$$

where $\delta^2 = \frac{1}{\sigma^2}\underline{\beta}^T C^T(C(X^TX)^-C^T)^{-1}C\underline{\beta} = \frac{1}{\sigma^2}(1.5\alpha_1^2 + 2\alpha_2^2 + 1.5\alpha_3^2 - 2\alpha_1\alpha_2 - \alpha_1\alpha_3 - 2\alpha_2\alpha_3)$. This was obtained by using S-Plus to evaluate $C^T(C(X^TX)^-C^T)^{-1}C$.

```
> library(MASS)
> X<-matrix(c(rep(1,10),rep(0,8),rep(1,4),rep(0,8),1,1),ncol=4,byrow=F)
> XTX<-t(X)%*%X
> c<-matrix(c(0,1,0,-1,0,1,-2,1),ncol=4,byrow=T)
> t(c)%*%solve(c%*%ginv(XTX)%*%t(c))%*%c
     [,1] [,2] [,3] [,4]
```

```
[1,]    0  0.0    0  0.0
[2,]    0  1.5   -1 -0.5
[3,]    0 -1.0    2 -1.0
[4,]    0 -0.5   -1  1.5
```

(v) Obviously, when the null hypothesis is true we have $C\underline{\beta} = 0$, and $\delta^2 = \frac{1}{\sigma^2}(C\underline{\beta})^T(C(X^TX)^-C^T)^{-1}C\underline{\beta} = 0$. Consequently, the test statistic has a central F-distribution.

(c) The null hypothesis is $H_0 : \alpha_1 = \alpha_2 = \alpha_3$. It was incorrectly printed on the first version of the assignment that was posted on the course web page. Here, $C^* = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, whose rows span the same space as the rows of C, i.e.,

$$a_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ -a_1 - a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ (a_1 + a_2/2) - a_2/2 \\ a_2 \\ (-a_1 - a_2/2) - a_2/2 \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 + a_2/2 \\ 0 \\ -(a_1 + a_2/2) \end{bmatrix} + \begin{bmatrix} 0 \\ -a_2/2 \\ a_2 \\ -a_2/2 \end{bmatrix} =$$

$$(a_1 + a_2/2) \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} - (a_2/2) \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

Therefore the null hypothesis is the same in parts (b) and (c), and the F-test in part (c) is the same as the F-test in (b).

4.(a) Define $C = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then, $H_0 : C\underline{\beta} = 0 \Leftrightarrow \alpha_1 - \alpha_2 = \beta = 0$.

(b) From the previous assignment we know that, $C\underline{\beta}$ is estimable if $C\mathbf{d} = \mathbf{0}$, where $d^T = w[-1\ 1\ 1\ 0]$. So, here $C\underline{\beta}$ is estimable. Also, it is easy to see that rank(C)=number of rows in C = 2. Therefore, the null hypothesis is testable.

(c) Use $SSH_0 = (C\underline{b})^T[C(X^TX)^-C^T]^{-1}C\underline{b}$. It follows from Result 4.7 in the notes, that $\frac{1}{\sigma^2}SSH_0 \sim \chi^2_2(\delta^2)$, where $\delta^2 = \frac{1}{\sigma^2}(C\underline{\beta})^T[C(X^TX)^-C^T]^{-1}C\underline{\beta}$. Since $C\underline{\beta} = 0$, we have $\delta^2 = 0$ and $\frac{1}{\sigma^2}SSH_0 \sim \chi^2_2$. See the solution to question 1.(b) for details.

(d) Check the conditions of Result 4.7 in the notes. Note that $\frac{1}{\sigma^2}(I - P_x)\sigma^2 I = (I - P_x)$ is idempotent and symmetric, $\delta^2 = (X\underline{\beta})^T \frac{(I-P_x)}{\sigma^2} X\underline{\beta} = 0$, and rank$(I - P_x)$=rank(I)-rank$(P_x)$=10-3=7. Therefore, $\frac{1}{\sigma^2}SSE \sim \chi^2_7$.

(e) You can check the conditions of Result 4.8 in the notes. See the solution to question 1.(b).(ii).

(f) From the results in parts (c), (d), and (e) we have that $\frac{SSH_0/(2\times\sigma^2)}{SSE/(7\sigma^2)} = \frac{SSH_0/2}{SSE/7} \sim F_{2,7}(\delta^2)$, where $\delta^2 = \frac{5}{2\sigma^2}(\alpha_1-\alpha_2)^2 + \frac{500}{\sigma^2}\beta^2$. Therefore, $\delta^2 = 0$ if and only if $\alpha_1 = \alpha_2$ and $\beta = 0$. (H_0 is true.) The test is performed by rejecting the null hypothesis if $F = \frac{SSH_0/2}{SSE/7} > F_{(2,7),\alpha}$.

(g)
```
> y <- c(20,24,27,33,38,25,29,32,37,41)
> x <- matrix(c(rep(1,15),rep(0,10),rep(1,5),rep(c(-10,-5,0,5,10),2)),10,4)
> C <- matrix(c(0,0,1,0,-1,0,0,1),2,4)
> SSH0 <- t(C%*%ginv(t(x)%*%x)%*%t(x)%*%y)%*%solve(C%*%ginv(t(x)%*%x)%*%t(C))%*%C
            %*%ginv(t(x)%*%x)%*%t(x)%*%y
> SSH0
       [,1]
[1,] 409.65
> SSE <- crossprod(y-x%*%ginv(t(x)%*%x)%*%t(x)%*%y,y-x%*%ginv(t(x)%*%x)%*%t(x)%*%y)
> SSE
     [,1]
[1,] 4.75
> V <- (SSH0/2)/(SSE/7)
> V
         [,1]
[1,] 301.8474
> pvalue <- 1-pf(V,2,7)
> pvalue
             [,1]
[1,] 1.612347e-07
```

Since the p-value is less than 0.0001, we reject the null hypothesis at the 0.0001 significance level. The mean yields do not appear to be the same for all combinations of catalysts and temperatures used in the experiment.

5.(a) Let

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad Q = \begin{bmatrix} 1 & .5 & .5 & 0 \\ 0 & .5 & -.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then, W=XR and X=WQ. Therefore, these two models are reparameterizations of each other.

(b) Since these parameters are estimable, each has an unambiguous interpretation:

γ_1 is the mean yield of the process, averaging across the results for catalyst A and catalyst B, when the process is run at 100^oC.

γ_2 is half the difference between the mean yield of the process using catalyst A and the mean yield of the process using catalyst B when both processes are run at the same temperature.

γ_3 represents the change in the mean yield of the process when the temperature is increased 1^oC and the same catalyst (either catalyst A or catalyst B) is used.

(c) Calculations can be done in different ways. S-Plus code is show for each calculation.

(i) $\hat{\gamma} = (W^T W)^{-1} W^T \underset{\sim}{Y}$.

```
> w <- matrix(c(rep(1,15),rep(-1,5),-10,-5,0,5,10,-10,-5,0,5,10),ncol=3,byrow=F)
> y <- c(20,24,27,33,38,25,29,32,37,41)
> gamma.hat <- solve(t(w)%*%w)%*%t(w)%*%y
> gamma.hat
       [,1]
[1,]  30.60
[2,]  -2.20
[3,]   0.85
```

(ii) $\hat{y} = W\hat{\gamma}$.

```
> yhat <- w%*%gamma.hat
> yhat
       [,1]
 [1,] 19.90
 [2,] 24.15
 [3,] 28.40
 [4,] 32.65
 [5,] 36.90
 [6,] 24.30
 [7,] 28.55
 [8,] 32.80
 [9,] 37.05
[10,] 41.30
```

(iii) $e = y - \hat{y}$.

```
> e <- y-yhat
> e
       [,1]
 [1,]  0.10
 [2,] -0.15
 [3,] -1.40
 [4,]  0.35
 [5,]  1.10
 [6,]  0.70
 [7,]  0.45
 [8,] -0.80
 [9,] -0.05
[10,] -0.30
```

(iv)-(viii)

	SS	$d.f.$	MS	F	$p-value$
$R(\gamma_1)$	9363.6	1	9363.6	13798	0.00
$R(\gamma_2\|\gamma_1)$	48.4	1	48.4	71.32	0.00
$R(\gamma_3\|\gamma_1\ \gamma_2)$	361.25	1	361.25	532.37	0.00
SSE	4.75	7	.6786		

```
>  # function to compute projection matrices
>  ject <- function(w){w%*%ginv(t(w)%*%w)%*%t(w)}
>
>  # Identity matrix
>  one <- diag(rep(1,10))
>  SSE <- t(y)%*%(one-ject(w))%*%y
>  SSE
     [,1]
[1,] 4.75
>
>  w1 <- w[,1]
>  r1 <- t(y)%*%ject(w1)%*%y
>  r1
       [,1]
[1,] 9363.6
>
>  w2 <- w[,1:2]
>  r2 <- t(y)%*%(ject(w2)-ject(w1))%*%y
>  r2
     [,1]
[1,] 48.4
```

```
>
> r3 <- t(y)%*%(ject(w)-ject(w2))%*%y
> r3
       [,1]
[1,] 361.25
```

(d) Check the conditions of Cochran's Theorem.

1) $(I - P_w) + P_{w1} + (P_{w2} - P_{w1}) + (P_w - P_{w2}) = I$

2) $rank(I-P_w)+rank(P_{w1})+rank(P_{w2}-P_{w1})+rank(P_w-P_{w2}) = (10-3)+1+(2-1)+(3-2) = 10 = n$

3) $I - P_w$, P_{w1}, $P_{w2} - P_{w1}$, $P_w - P_{w2}$ are all symmetric

Therefore, by Cochran's theorem, the sums of squares in (vi)-(vii) are independently distributed as chi-square random variables multiplied by σ^2, with d.f. 7, 1, 1, 1 respectively.

(e)

$$\delta^2 = \frac{1}{\sigma^2}\beta^T W^T (P_w - P_{w2}) W\beta = \frac{500}{\sigma^2}\gamma_3^2$$

Therefore, $\delta^2 = 0$ if and only if $\gamma_3 = 0$. You can use S-Plus to evaluate $W^T(P_w - P_{w2})W$:

```
> round(t(w)%*%(ject(w)-ject(w2))%*%w,4)
     [,1] [,2] [,3]
[1,]    0    0    0
[2,]    0    0    0
[3,]    0    0  500
```

(f) $\delta^2 = \frac{1}{\sigma^2}\beta^T W^T (P_{w1} - P_{w2}) W\beta = \frac{10}{\sigma^2}\gamma_2^2$. Therefore, $\delta^2 = 0$ if and only if $\gamma_2 = 0$.

(g)(i) Use R to evaluate $cov(\hat{\gamma}) = (\hat{\sigma}^2)(W^T W)^{-1}$

```
> a <- c(SSE/7)
> cov <- a*(solve(t(w)%*%w))
> cov
           [,1]       [,2]        [,3]
[1,] 0.06785714 0.00000000 0.000000000
[2,] 0.00000000 0.06785714 0.000000000
[3,] 0.00000000 0.00000000 0.001357143
```

Then, a 95% confidence interval forγ_2 is: $\hat{\gamma}_2 \pm t_{.975,7}\sqrt{.06785714} = [-2.816, -1.584]$

(g)(ii) The least squares estimator of the mean yield is $\hat{Y} = c^T * \hat{\gamma}$, where $c^T = (1, 1, 20)$. A 95% confidence interval is $\hat{Y} \pm t_{.975,7}\sqrt{c^T * cov * c} = [43.45, 47.35]$

(h) A 95% CI for error variance is: $(\frac{SSE}{\chi^2_{7,.975}}, \frac{SSE}{\chi^2_{7,.025}}) = (0.297, 2.811)$

```
> SSE/(qchisq(.975,7))
[1] 0.2966384
> SSE/(qchisq(.025,7))
[1] 2.810868
```

6.(a) By letting $W\gamma = X\beta$, we can get

$$\begin{aligned}\gamma_1 &= \mu + \tfrac{\alpha_1+\alpha_2}{2} + 100\gamma,\\ \gamma_2 &= \tfrac{\alpha_1-\alpha_2}{2},\\ \gamma_3 &= \gamma.\end{aligned}$$

Therefore, $\gamma = \begin{bmatrix} 1 & 1/2 & 1/2 & 100 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \beta.$

Now consider how four parameters can be expressed in terms of three parameters. Then,

$$\beta = \begin{bmatrix} 1 & 0 & -100 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \gamma$$

.

Here we see that $\gamma_2 = \alpha_1$ and $\gamma_2 = -\alpha_2$. Consequently, the restriction that is being imposed is $\alpha_1 + \alpha_2 = 0$. Note that $\alpha_1 + \alpha_2$ is not an estimable quantity.

(b) Define $\alpha_1 + \alpha_2 = [0\ 1\ 1\ 0]\beta = \underline{c}^T\beta$. In this case, $\underline{c}^T\beta$ is estimable if and only if $\underline{c}^T\underline{d} = 0$ where $\underline{d}^T = w[-1110]$ for $w \neq 0$. However, $\underline{c}^T\underline{d} = 2w \neq 0$. Therefore $\alpha_1 + \alpha_2$ is not estimable.

7. This is not a reparameterization of the models in problem 4 and 5. Denote this model matrix as X, and denote the model matrix in problem 4 as W. Each column of X can be written as a linear combination of the columns of W (note they share the same first 3 columns, and the 4th column of X is the sum of the last 2 columns of W). However, not all columns of W can be written as a linear combinations of the columns of X because rank(W) is larger than rank(X) (check the 4th and 5th columns of W). Hence, the columns of X do not span the same model space as the columns of W and these models are not reparameterizations of each other.

8. $H_0 : C\beta = (0, 1, -1, 0) \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta \end{bmatrix} = 0$

$H_a : C\beta = 0.5\sigma$, and the power of the test is:
$power = Pr(F_{1,10n-3}(\delta^2) > F_{1,10n-3,.95})$
where $\delta^2 = \frac{1}{\sigma^2} 0.5\sigma [C(X^T X)^- C^T]^{-1} 0.5\sigma$
In this case,

$$X^T X = \begin{bmatrix} 10n & 5n & 5n & 500n \\ 5n & 5n & 0 & 250n \\ 5n & 0 & 5n & 250n \\ 500n & 250n & 250n & 500n \end{bmatrix}$$

$C(X^T X)^- C^T = 0.4/n$ and $\delta^2 = 0.5 * (1/0.4) * 0.5 = 5n/8$

Choose n such that
$.90 = power = Pr(F_{1,10n-3}(5n/8) > F_{1,10n-3,.95})$
n=18

```
> n<-1
> powerfun<-function(n){
1-pf(qf(.95,1,10*n-3),1,10*n-3,5*n/8)
}
> while (powerfun(n)<.90){
        n<-n+1
        powerfun(n)
}
> n
   [,1]
[1,] 18
```