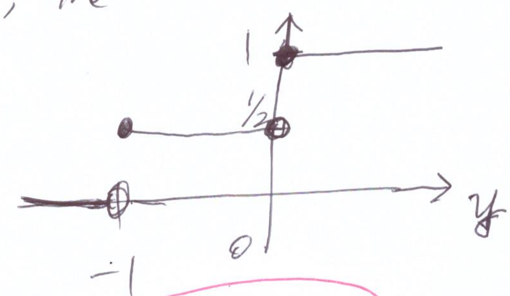
(Thm 2.1.3). If $g(\cdot)$ is monotone decreasing but X is "NOT continuous".

Consider $Y = g(X) = -X$, $X = \begin{cases} 0 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2. \end{cases}$

Then, we know $Y = \begin{cases} -1 & \text{w.p. } 1/2 \\ 0 & \text{w.p. } 1/2. \end{cases}$

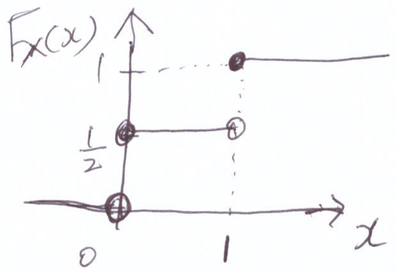
Thus, the cdf of Y is a step function.



We see $F_Y(0) = 1$ and $F_Y(-1) = \frac{1}{2}$.

However, $F_Y(y) = 1 - F_X(g^{-1}(y))$ provides

$$F_Y(0) = 1 - F_X(0) = 1 - \frac{1}{2} = \frac{1}{2},$$ which is wrong.



This phenomenon does not happen when $g(\cdot)$ is monotone increasing. It is because cdf is right continuous, but NOT left continuous.

Slide # 14.

$$X \sim f_X(x) = \begin{cases} 4x^3, & 0 < x < 1 \\ 0, & o/w. \end{cases}$$

$Y = (X-0.5)^2$. $\quad \mathcal{Y} = \{y;\ 0 \le y < \frac{1}{4}\}$

$A_0 = \{0.5\}$. $\quad A_1 = \{x:\ 0 < x < \frac{1}{2}\}$, $A_2 = \{x:\ \frac{1}{2} < x < 1\}$

$f_Y(y) = ?$

$$g^{-1}(y) = X = \begin{cases} 0.5 + \sqrt{y} & \text{if } x \in A_2 \\ 0.5 - \sqrt{y} & \text{if } x \in A_1. \end{cases}$$

* Check four conditions in Theorem 2.1.8.

$$\Rightarrow f_Y(y) = \sum_{i=1}^{2} f(g_i^{-1}(y)) \left| \frac{\partial}{\partial y} g_i^{-1}(y) \right|$$

$$= 4(0.5+\sqrt{y})^3 \left| \frac{1}{2\sqrt{y}} \right| + 4(0.5-\sqrt{y})^3 \left| -\frac{1}{2\sqrt{y}} \right|$$

$$= \begin{cases} \{2(0.5+\sqrt{y})^3 + 2(0.5-\sqrt{y})^3\} / \sqrt{y}, & 0 \le y < \frac{1}{4} \\ 0, & o/w. \end{cases}$$

<slide #18>

$$f_Y(y) = \begin{cases} (1-p)^{y-1} p & , y = 1, 2, 3, \ldots \\ 0 & , o/w. \end{cases}$$

$$E[Y] = \sum_{y=1}^{\infty} y (1-p)^{y-1} p$$

$$= p \left\{ \sum_{y=1}^{\infty} (1-p)^{y-1} + \sum_{y=2}^{\infty} (1-p)^{y-1} + \sum_{y=3}^{\infty} (1-p)^{y-1} + \cdots \right\}$$

$$= p \left\{ \frac{1}{1-(1-p)} + \frac{(1-p)}{1-(1-p)} + \frac{(1-p)^2}{1-(1-p)} + \cdots \right\}$$

$$= 1 + (1-p) + (1-p)^2 + \cdots$$

$$= \frac{1}{1-(1-p)} = \frac{1}{p}.$$

<slide #21>

$$f_X(x) = \frac{1}{x^2} I(x > 1).$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_1^{\infty} \frac{1}{x^2}\, dx = -x^{-1}\Big|_1^{\infty} = -[0-1] = 1 \Rightarrow \text{pdf.}$$

$$E|X| = \int_1^{\infty} |x| \frac{1}{x^2}\, dx = \int_1^{\infty} \frac{1}{x}\, dx = \log(x)\Big|_1^{\infty} = \infty.$$

As $E|X| = \infty$, EX does not exist.

<Slide #22>

$f_X(x) = 2/x^3 \, I(x>1)$.

$\int_{-\infty}^{\infty} f_X(x)dx = \int_1^\infty 2x^{-3}dx = -x^{-2}\Big|_1^\infty = -[0-1] = 1 \Rightarrow$ pdf.

$E|X| = EX = \int_1^\infty x\frac{2}{x^3}dx = \int_1^\infty \frac{2}{x^2}dx = -2x^{-1}\Big|_1^\infty = -2[0-1] = 2$.

$EX^2 = \int_1^\infty \frac{2}{x}dx = 2\log(X)\Big|_1^\infty = \infty$. does not exist.

<Slide #26>

$$f_X(x) = \begin{cases} e^{-\lambda}\lambda^x/x!, & x=0,1,\cdots \\ 0 & \text{o/w}. \end{cases}$$

Let $t = x-1$.

$$E[X] = \sum_{x=0}^{\infty} x\frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty}\frac{e^{-\lambda}\lambda^x}{(x-1)!} = \lambda\underbrace{\boxed{\sum_{t=0}^{\infty}\frac{e^{-\lambda}\lambda^t}{t!}}}_{=1} = \lambda.$$

$$E[X^2] = \sum_{x=0}^{\infty}\frac{x^2e^{-\lambda}\lambda^x}{x!} = \sum_{x=1}^{\infty}\frac{xe^{-\lambda}\lambda^x}{(x-1)!} = \sum_{x=1}^{\infty}\frac{(x-1+1)\cdot e^{-\lambda}\lambda^x}{(x-1)!}$$

$$= \sum_{x=1}^{\infty}\frac{(x-1)e^{-\lambda}\lambda^x}{(x-1)!} + \lambda\sum_{x=1}^{\infty}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}$$

$$= \lambda^2\sum_{x=2}^{\infty}\frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} + \lambda\underbrace{\boxed{\sum_{t=0}^{\infty}\frac{e^{-\lambda}\lambda^t}{t!}}}_{=1}$$

Let $u = x-2$

$$= \lambda^2\sum_{u=0}^{\infty}\frac{e^{-\lambda}\lambda^u}{u!} + \lambda = \lambda^2+\lambda.$$

$\therefore Var(X) = E(X^2) - E(X)^2 = \lambda^2+\lambda-\lambda^2 = \lambda$.

<Slide #29>.

$$M_X(t) = Ee^{tX} = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-(\lambda e^t)}(\lambda e^t)^x}{x!} (e^{-\lambda} \cdot e^{\lambda e^t}) = e^{\lambda(e^t-1)}$$ //

$$E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = \underbrace{\lambda e^t \cdot e^{\lambda(e^t-1)}}\Big|_{t=0} = \lambda.$$

$$E[X^2] = \frac{\partial}{\partial t^2} M_X(t)\Big|_{t=0} = \frac{\partial}{\partial t} \lambda e^t e^{\lambda(e^t-1)}\Big|_{t=0}$$

$$= \left[\lambda e^t e^{\lambda(e^t-1)} + \lambda^2 e^{2t} e^{\lambda(e^t-1)}\right]\Big|_{t=0}$$

$$= \lambda + \lambda^2.$$

$\therefore Var(X) = \lambda.$

<Slide #34>.

$$\lim_{n\to\infty} M_Y(t) = \lim_{n\to\infty} \left(1 + \frac{\lambda e^t - \lambda}{n}\right)^n$$

$$= \exp\{\lambda(e^t-1)\}.$$

$\rightarrow$ This is mgf of Poisson(λ) distribution.

$$* \lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a.$$