< Chapter 3 > Slide #2.

We have $\sum_{y=1}^{n} y = \frac{n(n+1)}{2}$, $\sum_{y=1}^{n} y^2 = \frac{n(n+1)(2n+1)}{6}$.

$Y \sim$ Discrete Uniform (a, b), $f_Y(y) = \frac{1}{b-a+1}$, $y = a, \cdots, b$.

$$E[Y] = \sum_{y=a}^{b} \frac{y}{b-a+1} = \frac{1}{b-a+1}\left(\sum_{y=1}^{b} y - \sum_{y=1}^{a-1} y\right)$$

$$= \frac{1}{b-a+1}\left[\frac{b(b+1)}{2} - \frac{a(a-1)}{2}\right] = \frac{b^2 - a^2 + b + a}{2(b-a+1)}$$

$$= \frac{(b-a)(b+a)+(b+a)}{2(b-a+1)} = \frac{(b-a+1)(a+b)}{2(a-b+1)} = \frac{a+b}{2}.$$

$$E[Y^2] = \sum_{y=a}^{b} \frac{y^2}{b-a+1} = \frac{1}{b-a+1}\left[\sum_{y=1}^{b} y^2 - \sum_{y=1}^{a-1} y^2\right]$$

$$= \frac{1}{6(b-a+1)}\left(b(b+1)(2b+1) - (a-1)a(2a-1)\right)$$

$$= \{2(b^3-a^3) + 3(b^2+a^2) + (b-a)\}/6(b-a+1).$$

$$\therefore Var(Y) = E[Y^2] - E[Y]^2$$

$$= \frac{4(b^3-a^3) + 6(b^2+a^2) + 2(b-a) - 3(b-a+1)(a+b)^2}{12(b-a+1)}$$

$$= \frac{b^3 + 3b^2(1-a) + b(3a^2-6a+2) - a(a-1)(a-2)}{12(b-a+1)}$$

plug-in $a, (a-1), (a-2)$ into b will give zero.

$$= (b-a)(b-a+1)(b-a+2)/12(b-a+1)$$

$$= \frac{(b-a)(b-a+2)}{12}$$

$X \sim \text{Bernoulli}(p)$.

$$Ee^{tX} = \sum_{x=0}^{1} e^{tx} p^x (1-p)^{1-x} = (1-p) + pe^t.$$

$X_i \overset{iid}{\sim} \text{Bernoulli}(p)$

$Y_1 = \sum_{i=1}^{n} X_i$.

$$Ee^{tY} = Ee^{t\sum X_i} \overset{\text{indep}}{=} Ee^{tX_1} Ee^{tX_2} \cdots Ee^{tX_n}$$

$$\overset{\text{identical}}{=} [(1-p) + pe^t]^n.$$

Let $Y_2 \sim \text{Binomial}(p)$. $f_{Y_2}(y) = \binom{n}{y} p^y (1-p)^{n-y}$.

$$M_{Y_2}(t) = \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y (1-p)^{n-y}$$

Binomial Theorem

$$= (pe^t + (1-p))^n.$$

Thus Y_1 is a Binomial (n, p).

$$E[Y_1] = \frac{\partial}{\partial t} M_{Y_1}(t)\Big|_{t=0} = n(pe^t + (1-p))^{n-1} \cdot pe^t \Big|_{t=0} = np.$$

$$E[Y_1^2] = \frac{\partial^2}{\partial t^2} M_{Y_1}(t)\Big|_{t=0} = \frac{\partial}{\partial t} n(pe^t + (1-p))^{n-1} pe^t \Big|_{t=0}$$

$$= n(n-1)(pe^t + (1-p))^{n-2} p^2 e^{2t} + n(pe^t + (1-p))^{n-1} pe^t \Big|_{t=0}$$

$$= n(n-1)p^2 + np$$

$$\therefore \text{Var}(Y_1) = E[Y_1^2] - E[Y_1]^2 = np(1-p).$$

<Slide #11>

$*\ X \sim \text{Geometry}(p)$

$\Rightarrow f_X(x) = p(1-p)^{x-1},\ x=1,2,\cdots$

"x is positive"

$$P(X = x_0 + x \mid X > x_0)$$

$$= \frac{P(X = x_0 + x \text{ and } X > x_0)}{P(X > x_0)}$$

$$= \frac{P(X = x_0 + x)}{1 - P(X \le x_0)} = \frac{p(1-p)^{x_0+x-1}}{1 - \sum_{x=1}^{x_0} p(1-p)^{x-1}}$$

$$= \frac{p(1-p)^{x_0+x-1}}{1 - p\frac{1-(1-p)^{x_0}}{1-(1-p)}} = \frac{p(1-p)^{x_0+x-1}}{(1-p)^{x_0}} = p(1-p)^{x-1}$$

$= P(X = x)$ $\Rightarrow$ memoryless property.

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(1-p)^{x-1}$$

$$= \frac{p}{1-p} \sum_{x=1}^{\infty} \{(1-p)e^t\}^x$$

$$= \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1-(1-p)e^t} = pe^t[1-(1-p)e^t]^{-1}$$

<Slide #12>

$$P(X > 50) = 1 - P(X \le 50) = 1 - \sum_{x=1}^{50} p(1-p)^{x-1} = (1-p)^{50} = 0.99^{50}.$$

< Slide #13 >

$X =$ # of trial when the r^{th} success occur.

$\Rightarrow X \sim NB(r, p)$, $f_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$, $x = r, r+1, \cdots$

Let $Y =$ # of failure before r^{th} success.

Note : $Y = X - r$.

This is an alternative form of NB.

So, $Y \sim NB(r, p)$.

$$f_Y(y) = f_X(y+r) \left| \frac{\partial}{\partial y}(y+r) \right|$$

$$= \binom{y+r-1}{r-1} p^r (1-p)^y, \; y = 0, 1, 2, \cdots \quad \left(\text{or } \binom{y+r-1}{y} p^r (1-p)^y \right).$$

$$E[Y] = \sum_{y=0}^{\infty} y \binom{y+r-1}{r-1} p^r (1-p)^y$$

$$= \sum_{y=0}^{\infty} y \frac{(y+r-1)!}{(r-1)!\, y!} p^r (1-p)^y$$

$$= \sum_{y=1}^{\infty} \frac{(y+r-1)!}{(r-1)!\,(y-1)!} p^{r+1-1} (1-p)^{y-1+1}$$

Let $t = y-1$.

$$= \sum_{t=0}^{\infty} r \frac{(t+r)!}{r!\, t!} p^{(r+1)-1} (1-p)^{t+1}$$

$$= \frac{r(1-p)}{p} \underbrace{\sum_{t=0}^{\infty} \binom{t+r}{r} p^{r+1} (1-p)^t}_{\text{p.m.f of } Y \text{ with } (r+1) \text{ success}}$$

$$= \frac{r(1-p)}{p} \quad \left(\text{Or, } E[X] = E[Y] + r = \frac{r}{p} \right).$$

$$E[Y(Y-1)] = \sum_{y=0}^{\infty} y(y-1)\binom{y+r-1}{r-1} p^r (1-p)^y$$

$$= \sum_{y=2}^{\infty} \frac{(y+r-1)!}{(r-1)!\,(y-2)!} p^r (1-p)^{y-2+2}$$

Let $V = y-2$
$$= \sum_{V=0}^{\infty} r(r+1)\frac{(V+r+1)!}{(r+1)!\,V!} p^{r+2-2} (1-p)^{V+2}$$

$$= r(r+1)\frac{(1-p)^2}{p^2} \sum_{V=0}^{\infty} \underbrace{\binom{V+r+1}{r+1} p^{r+2} (1-p)^V}_{\text{p.m.f. of } NB(r+2,p)}$$

$$= \frac{(1-p)^2 r(r+1)}{p^2}$$

Thus $Var(Y) = E(Y(Y-1)) + E(Y) - E(Y)^2$

$$= \frac{(1-p)^2 r(r+1)}{p^2} + \frac{r(1-p)}{p} - \frac{r^2(1-p)^2}{p^2}$$

$$= \frac{(1-p)^2 r(r+1) + rp(1-p) - r^2(1-p)^2}{p^2} = \frac{r(1-p)}{p^2} = Var(X)$$

$$Ee^{tY} = \sum_{y=0}^{\infty} e^{ty}\binom{y+r-1}{r-1} p^r (1-p)^y$$

$$= \sum_{y=0}^{\infty} \binom{y+r-1}{r-1} p^r (\underbrace{e^t(1-p)}_{\text{Let } 1-p^*})^y \quad p^* = 1 - e^t(1-p).$$

$$= \sum_{y=0}^{\infty} \binom{y+r-1}{r-1} p^{*r} (1-p^*)^y \cdot p^r \, p^{*-r} = \left[\frac{p}{1-e^t(1-p)}\right]^r,$$

for $0 < p^* < 1 \Leftrightarrow -\infty < t < -\log(1-p)$.

$\therefore Ee^{tY} = \left[\frac{p}{1-(1-p)e^t}\right]^r$ for $t < -\log(1-p)$.

$Ee^{tX} = e^{tr} Ee^{tY} = \left[e^t p/(1-(1-p)e^t)\right]^r$ for $t < -\log(1-p)$

$X \sim Poisson(\lambda)$

$M_X(t) = Ee^{tX} = \exp(\lambda(e^t - 1))$

$E(X) = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = \lambda e^t \exp(\lambda(e^t-1))\Big|_{t=0} = \lambda.$

$E(X^2) = \frac{\partial^2}{\partial t^2} M_X(t)\Big|_{t=0} = \lambda e^t \exp(\lambda(e^t-1)) + \lambda^2 e^{2t} \exp(\lambda(e^t-1))\Big|_{t=0}$

$= \lambda + \lambda^2.$

$\therefore Var(X) = \lambda.$

$Y = X_1 + X_2$, $X_1 \sim Poisson(\lambda_1)$, $X_2 \sim Poisson(\lambda_2)$, X_1, X_2 are indep.

$M_Y(t) = Ee^{tY} = Ee^{tX_1 + tX_2} = Ee^{tX_1} \cdot Ee^{tX_2}$

$= \exp(\lambda_1(e^t - 1)) \cdot \exp(\lambda_2(e^t - 1))$

$= \exp((\lambda_1 + \lambda_2)(e^t - 1)).$

$\Rightarrow$ M.G.F. of $Poisson(\lambda_1 + \lambda_2)$.

$\therefore Y \sim Poisson(\lambda_1 + \lambda_2).$

* Recursive property.

$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{\lambda}{x} \cdot \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} = \frac{\lambda}{x} P(X=x-1)$

$= \frac{\lambda^2}{x(x-1)} \cdot \frac{e^{-\lambda}\lambda^{x-2}}{(x-2)!} = \frac{\lambda^2}{x(x-1)} P(X=x-2)$

$= \cdots = \frac{\lambda^x}{x!} P(X=0).$

<Slide #16>

$X \sim \mathrm{Unif}(a,b)$.

$$M_X(t) = \int_a^b e^{tx}/(b-a)\,dx = \frac{1}{(b-a)}\ \frac{1}{t} e^{tx}\Big|_a^b$$

$$= \frac{e^{tb}-e^{ta}}{t(b-a)} \quad \text{if } t \neq 0.$$

When $t=0$, by L'Hopital's rule

$$\lim_{t\to 0} \frac{e^{tb}-e^{ta}}{t(b-a)} = \lim_{t\to 0} \frac{\frac{\partial}{\partial t}(e^{tb}-e^{ta})}{\frac{\partial}{\partial t} t(b-a)} = \lim_{t\to 0} \frac{be^{tb}-ae^{ta}}{(b-a)} = 1.$$

Thus
$$M_X(t) = \begin{cases} \dfrac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0. \end{cases}$$

$$\frac{\partial}{\partial t} M_X(t) = \frac{1}{b-a}\left[\frac{be^{tb}-ae^{ta}}{t} - \frac{(e^{tb}-e^{ta})}{t^2}\right] = \frac{1}{b-a}\,\frac{t(be^{tb}-ae^{ta})-(e^{tb}-e^{ta})}{t^2}$$

$\Rightarrow$ MGF is not differentiable at zero.

But $E(X)$ can be calculated by taking $\lim_{t\to 0}$.

$$\lim_{t\to 0}\frac{\partial}{\partial t} M_X(t) = \lim_{t\to 0} \frac{1}{b-a}\,\frac{t(be^{tb}-ae^{ta})-(e^{tb}-e^{ta})}{t^2}$$

L'Hopital's rule

$$= \frac{1}{b-a}\lim_{t\to 0} \frac{(be^{tb}-ae^{ta})+(be^{tb}-ae^{ta})-(be^{tb}-ae^{ta})}{2t}$$

$$= \frac{1}{b-a}\lim_{t\to 0} \frac{(b^2e^{tb}-a^2e^{ta})+(b^2e^{tb}-a^2e^{ta})-(b^2e^{tb}-a^2e^{ta})}{2}$$

$$= \frac{1}{b-a}\,\frac{b^2-a^2}{2} = \frac{a+b}{2} \,/\!/ = E(X).$$

<Slide #18>

$X \sim \text{Gamma}(\alpha, \beta)$, $Y \sim \text{Poisson}(x/\beta)$.

Show $P(X \le x \mid \alpha, \beta) = P(Y \ge \alpha \mid \beta)$

$$P(X \le x) = \int_0^x \frac{v^{\alpha-1} e^{-v/\beta}}{\Gamma(\alpha)\beta^\alpha} dv$$

$$= -\beta \frac{v^{\alpha-1} e^{-v/\beta}}{\Gamma(\alpha)\beta^\alpha}\Big|_0^x + \beta(\alpha-1)\int_0^x \frac{v^{\alpha-2} e^{-v/\beta}}{\Gamma(\alpha)\beta^\alpha} dv$$

$$= -\frac{(x/\beta)^{\alpha-1} e^{-x/\beta}}{(\alpha-1)!} + \int_0^x \frac{v^{\alpha-2} e^{-v/\beta}}{\Gamma(\alpha-1)\beta^{\alpha-1}} dv$$

$$= -P(Y=\alpha-1) - \beta \frac{v^{\alpha-2} e^{-v/\beta}}{\Gamma(\alpha-1)\beta^{\alpha-1}}\Big|_0^x + \int_0^x \frac{v^{\alpha-3} e^{-v/\beta}}{\Gamma(\alpha-2)\beta^{\alpha-2}} dv$$

$$= -P(Y=\alpha-1) - P(Y=\alpha-2) - \cdots - P(Y=1) + \int_0^x \frac{v^0 e^{-v/\beta}}{\Gamma(1)\beta} dv$$

$$= -\sum_{y=1}^{\alpha-1} P(Y=y) - e^{-v/\beta}\Big|_0^x$$

$$= -\sum_{y=1}^{\alpha-1} P(Y=y) - \left[e^{-x/\beta} - 1\right]$$

$$= 1 - \sum_{y=0}^{\alpha-1} P(Y=y) = 1 - P(Y \le \alpha-1)$$

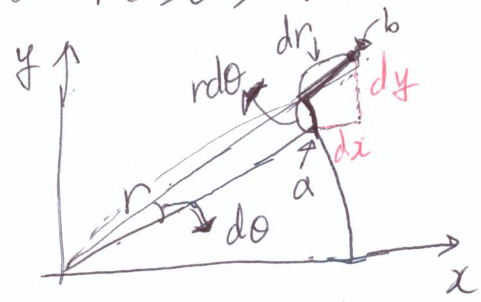
$$= P(Y \ge \alpha)$$

Verify $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz = 1$.

Let $A = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)\, dz$.

Then $A^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp(-(x^2+y^2)/2)\, dx\,dy$.

Let $x = r\cos\theta$, $y = r\sin\theta$. $\Rightarrow x^2+y^2 = r^2$. & $dxdy = rdrd\theta$.



by moving "a" point to "b", we see the area is increased $dx \cdot dy$ or $rdrd\theta$.

By this polar coordinate

$$A^2 = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \exp(-r^2/2)\, rdrd\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left[-\exp(-r^2/2) \Big|_0^{\infty} \right] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 1\, d\theta = 1.$$ Since $A \neq -1$, $A = 1$.

<Slide #17>

$\Gamma(\frac{1}{2}) = \int_0^{\infty} t^{-1/2} e^{-t} dt \overset{\text{verify}}{=} \sqrt{\pi}$.

Let $v = \sqrt{t}$. then $dv = \frac{1}{2\sqrt{t}} dt$.

Thus $\Gamma(\frac{1}{2}) = \int_0^{\infty} 2e^{-v^2} dv \underset{\text{symmetry}}{=} \int_{-\infty}^{\infty} e^{-v^2} dv = \sqrt{\pi} \int \overbrace{\frac{1}{\sqrt{2\pi \cdot \frac{1}{2}}} e^{-v^2/(2\cdot\frac{1}{2})}}^{\text{p.d.f. of } N(0,\frac{1}{2})} dv$

$= \sqrt{\pi}$.

<Slide#30, #31>

Ch3.4. Exponential Family

ex1) $X \sim Binomial(n, p)$] → see page 112, page 113.

ex2) $X \sim N(\mu, \sigma^2)$

ex3) $X \sim Geometric(p)$

$$f_X(x|p) = p(1-p)^{x-1}$$

$$= p \cdot \exp((x-1)\log(1-p))$$

$c(p) = p, \quad 0 \le p < 1$ $\quad h(x) = \begin{cases} 1 & \text{if } x = 1, 2, \cdots \\ 0 & o/w \end{cases}$

$t_1(x) = (x-1)$ $\quad w_1(p) = \log(1-p), \quad 0 \le p < 1$

ex4) $X \sim Gamma(\alpha, \beta)$

$$f_X(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad 0 < x < \infty, \quad \alpha > 0, \beta > 0$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \exp\left\{(\alpha-1)\log x - \frac{1}{\beta} \cdot x\right\}$$

$c(\theta) = c(\alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}, \quad \alpha > 0, \beta > 0$

$h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & o/w \end{cases}$

$t_1(x) = \log x, \quad x > 0$

$w_1(\alpha, \beta) = \alpha - 1$

$t_2(x) = x$

$w_2(\alpha, \beta) = -\frac{1}{\beta}$

<Slide #43>

$X \sim Normal(\mu, \sigma^2)$, $E|g'(X)| < \infty$

$E[g(X)(X-\mu)] = \sigma^2 E[g'(X)]$

$$\Rightarrow E[g(X)(X-\mu)] = \int_{-\infty}^{\infty} \underbrace{g(x)}_{u} \underbrace{(x-\mu)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}_{dv} dx$$

$v = -\sigma^2 \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$

$$= -g(x)\sigma^2\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]\Bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} g'(x)\sigma^2\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx$$

$$= \sigma^2\int_{-\infty}^{\infty} g'(x)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx = \sigma^2 E[g'(X)]$$

<Extra equalities.>

$X \sim Gamma(\alpha, \beta)$. $E[g(X)(X-\alpha\beta)] = \beta E[Xg'(X)]$

$$\int_0^{\infty} \underbrace{g(x)}_{u} \underbrace{(x-\alpha\beta)\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}}_{dv} dx, \quad v = -\frac{\beta x^{\alpha} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}$$

$$= -g(x)\frac{\beta x^{\alpha}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}\Bigg|_0^{\infty} + \int_0^{\infty} g'(x)\beta\frac{x^{\alpha}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}dx$$

$$= 0 + \beta\int_0^{\infty} g'(x)\,x\,\frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}dx$$

$$= \beta E[Xg'(X)].$$

$X \sim \text{Beta}(\alpha, \beta)$. $E|g'(X)| < \infty$.

$$E\left[g(X)\left\{\beta - (\alpha-1)\frac{(1-X)}{X}\right\}\right] = E\left[(1-X)g'(X)\right]$$

$$\Rightarrow \int_0^1 \underbrace{g(x)}_{u} \underbrace{\left\{\beta - (\alpha-1)\frac{(1-x)}{x}\right\} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx}_{dv}$$

$$v = -\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta}$$

$$\Rightarrow -g(x)\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta}\Big|_0^1 + \int_0^1 g'(x)\cdot(1-x)\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx$$

$$= 0 - 0 + E\left[g'(X)(1-X)\right].$$