$\boxed{\text{Slide 10.}}$ Another Example that $P(S)=1$ for continuous case.

X is a R.V. w/ pdf

$$f_X(x) = \begin{cases} (e+e^{-1})^{-1}(x+1)e^{x}, & -1<x<1 \\ 0 & o/w. \end{cases}$$

Show that $f_X(x)$ is a p.d.f.

sol). $\int_{-1}^{1} f_X(x)\,dx = \int_{-1}^{1} (e+e^{-1})^{-1}(x+1)e^{x}\,dx$

$= (e+e^{-1})^{-1}\underbrace{\int_{-1}^{1} xe^{x}\,dx}_{\downarrow} + (e+e^{-1})^{-1}\int_{-1}^{1} e^{x}\,dx.$

* Use Integration by part.

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$$

$$\int_{-1}^{1} xe^{x}\,dx = xe^{x}\Big|_{-1}^{1} - \int_{-1}^{1} e^{x}\,dx$$

$$= (e+e^{-1}) - e^{x}\Big|_{-1}^{1} = 2e^{-1}.$$

$= (e+e^{-1})^{-1}\cdot 2e^{-1} + (e+e^{-1})^{-1}(e-e^{-1})$

$= (e+e^{-1})^{-1}\{2e^{-1} + (e-e^{-1})\}$

$= 1.$

Slide 14 Binomial Coefficients. Exercise 1.27-(b)

Show that $\sum_{k=1}^{n} k \binom{n}{k} = n 2^{n-1}$.

sol) Let $r = k-1$, then $k = r+1$.

Then $\sum_{r=0}^{n-1} (r+1) \binom{n}{r+1} = \sum_{r=0}^{n-1} (r+1) \frac{n!}{(n-r-1)!(r+1)!}$

$= n \sum_{r=0}^{n-1} \frac{(n-1)!}{(n-1-r)!\, r!}$

$= n 2^{n-1} \underbrace{\sum_{r=0}^{n-1} \frac{(n-1)!}{(n-1-r)!\, r!} \left(\frac{1}{2}\right)^{r} \left(\frac{1}{2}\right)^{n-1-r}}_{\text{p.m.f. of Binomial}((n-1),\ \frac{1}{2})}$

$= n 2^{n-1}$.

Exercise 1.28

Show that $\displaystyle\lim_{n\to\infty} \frac{n!}{n^{n+(1/2)} e^{-n}} = C$, C is a constant.

sol). Let $a_n = \log\left[\dfrac{n!}{n^{n+(1/2)} e^{-n}}\right]$

Then $a_n = \log n! - \{(n+\tfrac12)\log n - n\}$

$$- \;\; a_{n+1} = \log(n+1)! - \{(n+\tfrac32)\log(n+1) - (n+1)\}$$

$$\Rightarrow a_n - a_{n+1} = -\log(n+1) - n(\log n - \log(n+1)) + \tfrac32\log(n+1) - \tfrac12\log n - 1$$

$$= (\tfrac12+n)\log(n+1) - (\tfrac12+n)\log n - 1 = (\tfrac12+n)\log(1+\tfrac1n) - 1.$$

$a_n - a_{n+1}$ decreases as n increases, and $a_1 - a_2 = \frac32\log 2 - 1 > 0$ — 1

we will show $0 < a_n - a_{n+1}$.

Now, consider $\displaystyle\sum_{k=1}^{\infty}(a_k - a_{k+1}) = \lim_{N\to\infty}\{(a_1-a_2)+(a_2-a_3)+\cdots+(a_N-a_{N+1})\}$

$$= a_1 - \lim_{N\to\infty} a_{N+1} = 1 - \lim_{N\to\infty} a_{N+1} \quad — 2$$

From the Taylor expansion of $\log(1+x)$, we have

$$^{*} f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \frac{f''''(a)(x-a)^4}{4!} + \cdots$$

here

$f(x) = \log(1+x)$ and $a = 0$

$\Rightarrow f'(x) = \frac{1}{1+x}$, $f'(a) = 1$ // $f''(x) = -(1+x)^{-2}$, $f''(a) = -1$

$f'''(x) = 2(1+x)^{-3}$, $f'''(a) = 2$ // $f''''(x) = -3\cdot2(1+x)^{-4}$, $f''''(a) = -3!$

$$\Rightarrow \log(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \cdots$$

$$\Rightarrow \log(1+x) > x - x^2/2 + x^3/3 - x^4/4$$

$$a_n - a_{n+1} = (\tfrac12+n)\log(1+\tfrac1n) - 1 \;\cancel{> (\tfrac12+n)(\tfrac1n - \tfrac{1}{2n^2} + \tfrac{1}{3n^3} - \tfrac{1}{4n^4}) - 1}$$

~~lower bound~~

(continued)

$$=\left(\frac{1}{2}+n\right)\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots\right)-1$$

$$=\left(\frac{1}{2n}+1-\frac{1}{4n^2}-\frac{1}{2n}+\frac{1}{6n^3}+\frac{1}{3n^2}-\frac{1}{8n^4}-\frac{1}{4n^5}+\cdots\right)-1$$

$$=\frac{1}{12n^2}+o\left(\frac{1}{n^2}\right)$$ little o. "smaller than $\frac{1}{n^2}$".

Then $\sum_{n=1}^{\infty}(a_n-a_{n-1})=\sum_{n=1}^{\infty}\frac{1}{12n^2}+o\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)$

$\rightarrow$ converges to a constant C_1.

Thus $a_1-\lim_{n\to\infty}a_n=C_1$

$\Rightarrow \lim_{n\to\infty}a_n=a_1-C_1=1-C_1$. Converges to a constant.

Then $\lim_{n\to\infty}\frac{n!}{n^{n+(1/2)}e^{-n}}=C_2$, where $C_2=e^{1-C_1}$.

* $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ (Basel Problem)

* $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots$

$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$

$=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$

$\rightarrow$ diverges.

As the lower bound diverges, $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Exercise 1.55. T has pdf $f_T(t) = \frac{1}{1.5} e^{-t/(1.5)}$, $t > 0$.

$V = 5$ if $T < 3$ and $V = 2T$ if o/w.

Find cdf of V.

sol) $P(V=5) = P(T<3) = \int_0^3 \frac{1}{1.5} e^{-t/(1.5)} dt$

$$= -e^{-t/(1.5)} \Big|_0^3$$

$$= 1 - e^{-2}.$$

Note that the above quantity is, in fact, held for $5 \le v < 6$.

For $v \ge 6$, $V = 2T$, therefore

$$P(V \le v) = P(2T \le v) = P(T \le v/2)$$

$$= \int_0^{v/2} \frac{1}{1.5} e^{-t/(1.5)} dt$$

$$= -e^{-t/(1.5)} \Big|_0^{v/2} = 1 - e^{-v/3}, \quad 6 \le v.$$

Thus, $$P(V \le v) = \begin{cases} 0 & \text{for } -\infty < v < 5 \\ 1 - e^{-2} & \text{for } 5 \le v < 6 \\ 1 - e^{-v/3} & \text{for } 6 \le v. \end{cases}$$