Two-way crossed classification

Example 7.1 Days to first germination of three varieties of carrot seed grown in two types of potting soil.

Soil Type	Variety 1	Variety 2	Variety 3
1	$Y_{111} = 6$ $Y_{112} = 10$ $Y_{113} = 11$	$Y_{121} = 13$ $Y_{122} = 15$	$Y_{131} = 14$ $Y_{132} = 22$
2	$Y_{211} = 12$ $Y_{212} = 15$ $Y_{213} = 19$ $Y_{214} = 18$	$Y_{221} = 31$	$Y_{231} = 18$ $Y_{232} = 9$ $Y_{233} = 12$

This might be called an *unbalanced factorial experiment*.

Sample sizes:

Soil type	Variety 1	Variety 2	Variety 3
1	$n_{11} = 3$	$n_{12} = 2$	$n_{13} = 2$
2	$n_{21} = 4$	$n_{22} = 1$	$n_{23} = 3$

In general we have

$i = 1, 2, \ldots, a$: levels for the first factor

$j = 1, 2, \ldots, b$: levels for the second factor

$n_{ij} > 0$: observations at the i-th level of the first factor and the j-th level of the second factor

We will restrict our attention to normal-theory Gauss-Markov models.

Cell means model:

$$Y_{ijk} = \mu_{ij} + \epsilon_{ijk}$$

where

$$\epsilon_{ijk} \sim NID(0, \sigma^2) \quad \begin{cases} i = 1, \ldots, a \\ j = 1, \ldots, b \\ k = 1, \ldots, n_{ij} \end{cases}$$

Clearly, $E(Y_{ijk}) = \mu_{ij}$ is estimable if $n_{ij} > 0$.

Overall mean response:

$$\bar{\mu}_{..} = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \mu_{ij}$$

Mean response at i-th level of factor 1, averaging across the levels of factor 2.

$$\bar{\mu}_{i.} = \frac{1}{b} \sum_{j=1}^{b} \mu_{ij}$$

Mean response at j-th level of factor 2, averaging across the levels of factor 1

$$\bar{\mu}_{.j} = \frac{1}{a} \sum_{i=1}^{a} \mu_{ij}$$

Contrasts of interest:

Main effects for factor 1:

$$\bar{\mu}_{i.} - \bar{\mu}_{..} \qquad i = 1, 2, \ldots, a$$

$$\bar{\mu}_{i.} - \bar{\mu}_{k.} \qquad i \neq k$$

Main effects for factor 2:

$$\bar{\mu}_{.j} - \bar{\mu}_{..} \qquad j = 1, 2, \ldots, b$$

$$\bar{\mu}_{.j} - \bar{\mu}_{.\ell} \qquad j \neq \ell$$

Conditional effects:

$$\mu_{ij} - \mu_{kj} \quad \begin{cases} i \neq k \\ j = 1, 2, \ldots, b \end{cases}$$

$$\mu_{ij} - \mu_{i\ell} \quad \begin{cases} j \neq \ell \\ i = 1, 2, \ldots, a \end{cases}$$

Interaction contrasts:

$$\begin{aligned}(\mu_{ij} - \mu_{kj}) - (\mu_{i\ell} - \mu_{k\ell}) =& (\mu_{ij} - \mu_{i\ell}) - (\mu_{kj} - \mu_{k\ell}) \\ =& \mu_{ij} - \mu_{kj} - \mu_{i\ell} + \mu_{k\ell}\end{aligned}$$

All of these contrasts are estimable when

$$n_{ij} > 0 \quad \text{for all } (i,j)$$

because

- $E(\bar{Y}_{ij.}) = \mu_{ij}$
- Any linear function of estimable functions is estimable

An *effects* model

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

where

$$\epsilon_{ijk} \sim NID(0, \sigma^2)$$

$$i = 1, 2, \ldots, a$$

$$j = 1, 2, \ldots, b$$

$$k = 1, 2, \ldots, n_{ij} > 0$$

$$
\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{113} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{213} \\ Y_{214} \\ Y_{221} \\ Y_{231} \\ Y_{232} \\ Y_{233} \end{bmatrix}
=
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix}
+
\begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{113} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{213} \\ \epsilon_{214} \\ \epsilon_{221} \\ \epsilon_{231} \\ \epsilon_{232} \\ \epsilon_{233} \end{bmatrix}
$$

The resulting restricted model is

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

where

$$\epsilon_{ijk} \sim NID(0, \sigma^2) \quad \begin{cases} i = 1, \ldots, a \\ j = 1, \ldots, b \\ k = 1, \ldots, n_{ij} \end{cases}$$

and

$$\begin{aligned} \alpha_a &= 0 \\ \beta_b &= 0 \\ \gamma_{ib} &= 0 \quad \text{for all} \quad i = 1, \ldots, a \\ \gamma_{aj} &= 0 \quad \text{for all} \quad j = 1, \ldots, b \end{aligned}$$

We will call these the "baseline" restrictions.

Soil Type	Variety 1	Variety 2	Variety 3	Soil Type Means
1	$\mu_{11} = \mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu_{12} = \mu + \alpha_1 + \beta_2 + \gamma_{12}$	$\mu_{13} = \mu + \alpha_1$	$\mu + \alpha_1 + \frac{\beta_1+\beta_2}{3} + \frac{\gamma_{11}+\gamma_{12}}{3}$
2	$\mu_{21} = \mu + \beta_1$	$\mu_{22} = \mu + \beta_2$	$\mu_{23} = \mu$	$\mu + \frac{\beta_1+\beta_2}{3}$
Var. means	$\mu + \frac{\alpha_1}{2} + \beta_1 + \frac{\gamma_{11}}{2}$	$\mu + \frac{\alpha_1}{2} + \beta_2 + \frac{\gamma_{12}}{2}$	$\mu + \frac{\alpha_1}{2}$	

Interpretation:

$$\mu = \mu_{ab} = E(Y_{abk})$$

the mean response when factor 1 is at level a and factor 2 is at level b.

$$\alpha_i = \mu_{ib} - \mu_{ab} = E(Y_{ibk}) - E(Y_{abk})$$

is a difference in mean responses between levels i and a of factor 1 when factor 2 is at its highest level.

Soil Type	Variety 1	Variety 2	Variety 3	Soil Type Means
1	$\mu_{11} = \mu + \alpha_1$ $+\beta_1 + \gamma_{11}$	$\mu_{12} = \mu + \alpha_1$ $+\beta_2 + \gamma_{12}$	$\mu_{13} = \mu + \alpha_1$	$\mu + \alpha_1$ $+\frac{\beta_1+\beta_2}{3}$ $+\frac{\gamma_{11}+\gamma_{12}}{3}$
2	$\mu_{21} = \mu + \beta_1$	$\mu_{22} = \mu + \beta_2$	$\mu_{23} = \mu$	$\mu + \frac{\beta_1+\beta_2}{3}$
Var. means	$\mu + \frac{\alpha_1}{2} + \beta_1 + \frac{\gamma_{11}}{2}$	$\mu + \frac{\alpha_1}{2} + \beta_2 + \frac{\gamma_{12}}{2}$	$\mu + \frac{\alpha_1}{2}$	

Interpretation:

$$\beta_j = \mu_{aj} - \mu_{ab} = E(Y_{ajk}) - E(Y_{abk}) \quad \text{for} j = 1, 2, \ldots, b$$

is the difference in the mean responses for levels j and b of factor 2 when factor 1 is at its highest level.

Soil Type	Variety 1	Variety 2	Variety 3	Soil Type Means
1	$\mu_{11} = \mu + \alpha_1 + \beta_1 + \gamma_{11}$	$\mu_{12} = \mu + \alpha_1 + \beta_2 + \gamma_{12}$	$\mu_{13} = \mu + \alpha_1$	$\mu + \alpha_1 + \frac{\beta_1+\beta_2}{3} + \frac{\gamma_{11}+\gamma_{12}}{3}$
2	$\mu_{21} = \mu + \beta_1$	$\mu_{22} = \mu + \beta_2$	$\mu_{23} = \mu$	$\mu + \frac{\beta_1+\beta_2}{3}$
Var. means	$\mu + \frac{\alpha_1}{2} + \beta_1 + \frac{\gamma_{11}}{2}$	$\mu + \frac{\alpha_1}{2} + \beta_2 + \frac{\gamma_{12}}{2}$	$\mu + \frac{\alpha_1}{2}$	

Interaction:

$$
\begin{aligned}
\gamma_{ij} &= (\mu_{ij} - \mu_{ib}) - (\mu_{aj} - \mu_{ab}) \\
&= (\mu_{ij} - \mu_{aj}) - (\mu_{ib} - \mu_{ab})
\end{aligned}
$$

Note that

$$\gamma_{ij} - \gamma_{i\ell} - \gamma_{kj} + \gamma_{k\ell} = \mu_{ij} - \mu_{i\ell} - \mu_{kj} + \mu_{k\ell}$$

for any (i, j) and (k, ℓ)

Matrix formulation:

$$\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{113} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{213} \\ Y_{214} \\ Y_{221} \\ Y_{231} \\ Y_{232} \\ Y_{233} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \beta_1 \\ \beta_2 \\ \gamma_{11} \\ \gamma_{12} \end{bmatrix} + \boldsymbol{\epsilon},$$

where $\mathbf{Y} \sim N(X\beta, \sigma^2 I)$.

Least squares estimation: $\mathbf{b} = (X^T X)^{-1} X^T \mathbf{Y}$

$$\mathbf{b} = \begin{bmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\gamma}_{11} \\ \hat{\gamma}_{12} \end{bmatrix} = \begin{bmatrix} n_{\bullet\bullet} & n_{1\bullet} & n_{\bullet 1} & n_{\bullet 2} & n_{11} & n_{12} \\ n_{1\bullet} & n_{1\bullet} & n_{11} & n_{12} & n_{11} & n_{12} \\ n_{\bullet 1} & n_{11} & n_{\bullet 1} & 0 & n_{11} & 0 \\ n_{\bullet 2} & n_{12} & 0 & n_{\bullet 2} & 0 & n_{12} \\ n_{11} & n_{11} & n_{11} & 0 & n_{11} & 0 \\ n_{12} & n_{12} & 0 & n_{12} & 0 & n_{12} \end{bmatrix}^{-1} \begin{bmatrix} Y_{\bullet\bullet\bullet} \\ Y_{1\bullet\bullet} \\ Y_{\bullet 1\bullet} \\ Y_{\bullet 2\bullet} \\ Y_{11\bullet} \\ Y_{12\bullet} \end{bmatrix}$$

$$= \begin{bmatrix} \bar{Y}_{23\bullet} \\ \bar{Y}_{13\bullet} - \bar{Y}_{23\bullet} \\ \bar{Y}_{21\bullet} - \bar{Y}_{23\bullet} \\ \bar{Y}_{22\bullet} - \bar{Y}_{23\bullet} \\ \bar{Y}_{11\bullet} - \bar{Y}_{13\bullet} - \bar{Y}_{21\bullet} + \bar{Y}_{23\bullet} \\ \bar{Y}_{12\bullet} - \bar{Y}_{13\bullet} - \bar{Y}_{22\bullet} + \bar{Y}_{23\bullet} \end{bmatrix}$$

Comments:

Imposing a set of restrictions on the parameters in the *effects* model

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

to obtain a model matrix with full column rank.

(i) Avoids the use of a generalized inverse in least squares estimation.

(ii) Is equivalent to choosing a generalized inverse for

$$\mathbf{b} = (X^T X)^- X^T \mathbf{Y}$$

in the unrestricted *effects* model.

(iii) Restrictions must involve *non-estimable* quantities for the unrestricted *effects* model.

Baseline restrictions: (SAS)

$$\begin{aligned} \alpha_a &= 0 \\ \beta_b &= 0 \\ \gamma_{ib} &= 0 \quad \text{for all } i = 1, \ldots, a \\ \gamma_{aj} &= 0 \quad \text{for all } j = 1, \ldots, b \end{aligned}$$

Baseline restrictions: (R)

$$\begin{aligned} \alpha_1 &= 0 \\ \beta_1 &= 0 \\ \gamma_{i1} &= 0 \quad \text{for all } i = 1, \ldots, a \\ \gamma_{1j} &= 0 \quad \text{for all } j = 1, \ldots, b \end{aligned}$$

Σ-restrictions:

$$Y_{ijk} = \underline{\omega + \gamma_i + \delta_j + \eta_{ij}} + \epsilon_{ijk}$$

$$\mu_{ij} = E(Y_{ijk})$$

where

$$\epsilon_{ijk} \sim NID(0, \sigma^2), \quad \sum_{i=1}^{a} \gamma_i = 0, \quad \sum_{j=1}^{b} \delta_j = 0$$

$$\sum_{i=1}^{a} \eta_{ij} = 0 \qquad \text{for each } j = 1, \ldots, b$$

$$\sum_{j=1}^{b} \eta_{ij} = 0 \qquad \text{for each } i = 1, \ldots, a$$

	Variety 1	Variety 2	Variety 3	Means
Soil type 1	$\mu_{11} = \omega + \gamma_1 + \delta_1 + \eta_{11}$	$\mu_{12} = \omega + \gamma_1 + \delta_2 + \eta_{12}$	$\mu_{13} = \omega + \gamma_1 + \delta_3 + \eta_{13}$	$\bar{\mu}_{1.} = \omega + \gamma_1$
Soil type 2	$\mu_{21} = \omega + \gamma_2 + \delta_1 + \eta_{21}$	$\mu_{22} = \omega + \gamma_2 + \delta_2 + \eta_{22}$	$\mu_{23} = \omega + \gamma_2 + \delta_3 + \eta_{23}$	$\bar{\mu}_{2.} = \omega + \gamma_2$
means	$\bar{\mu}_{.1} = \omega + \delta_1$	$\bar{\mu}_{.2} = \omega + \delta_2$	$\bar{\mu}_{.3} = \omega + \delta_3$	

Interpretation:

$$\omega = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \mu_{ij}$$

is the overall mean germination time, averaging across all soil types and all varieties used in this study.

	Variety 1	Variety 2	Variety 3	Means
Soil type 1	$\mu_{11} = \omega + \gamma_1 + \delta_1 + \eta_{11}$	$\mu_{12} = \omega + \gamma_1 + \delta_2 + \eta_{12}$	$\mu_{13} = \omega + \gamma_1 + \delta_3 + \eta_{13}$	$\bar{\mu}_{1.} = \omega + \gamma_1$
Soil type 2	$\mu_{21} = \omega + \gamma_2 + \delta_1 + \eta_{21}$	$\mu_{22} = \omega + \gamma_2 + \delta_2 + \eta_{22}$	$\mu_{23} = \omega + \gamma_2 + \delta_3 + \eta_{23}$	$\bar{\mu}_{2.} = \omega + \gamma_2$
means	$\bar{\mu}_{.1} = \omega + \delta_1$	$\bar{\mu}_{.2} = \omega + \delta_2$	$\bar{\mu}_{.3} = \omega + \delta_3$	

Interpretation:

$$\omega + \delta_j = \bar{\mu}_{.j}, \quad \delta_j = \bar{\mu}_{.j} - \bar{\mu}_{..}$$

and

$$\begin{aligned} \delta_j - \delta_k &= (\bar{\mu}_{.j} - \bar{\mu}_{..}) - (\bar{\mu}_{.k} - \bar{\mu}_{..}) \\ &= \bar{\mu}_{.j} - \bar{\mu}_{.k} \end{aligned}$$

is the difference between mean germination times for varieties j and k, averaging across soil types.

	Variety 1	Variety 2	Variety 3	Means
Soil type 1	$\mu_{11} = \omega + \gamma_1 + \delta_1 + \eta_{11}$	$\mu_{12} = \omega + \gamma_1 + \delta_2 + \eta_{12}$	$\mu_{13} = \omega + \gamma_1 + \delta_3 + \eta_{13}$	$\bar{\mu}_{1.} = \omega + \gamma_1$
Soil type 2	$\mu_{21} = \omega + \gamma_2 + \delta_1 + \eta_{21}$	$\mu_{22} = \omega + \gamma_2 + \delta_2 + \eta_{22}$	$\mu_{23} = \omega + \gamma_2 + \delta_3 + \eta_{23}$	$\bar{\mu}_{2.} = \omega + \gamma_2$
means	$\bar{\mu}_{.1} = \omega + \delta_1$	$\bar{\mu}_{.2} = \omega + \delta_2$	$\bar{\mu}_{.3} = \omega + \delta_3$	

Similarly,

$$\gamma_1 - \gamma_2 = \bar{\mu}_{1.} - \bar{\mu}_{2.}$$

is the difference in the mean germination times for different soil types, averaging across varieties.

	Variety 1	Variety 2	Variety 3	Means
Soil type 1	$\mu_{11} = \omega + \gamma_1 + \delta_1 + \eta_{11}$	$\mu_{12} = \omega + \gamma_1 + \delta_2 + \eta_{12}$	$\mu_{13} = \omega + \gamma_1 + \delta_3 + \eta_{13}$	$\bar{\mu}_{1.} = \omega + \gamma_1$
Soil type 2	$\mu_{21} = \omega + \gamma_2 + \delta_1 + \eta_{21}$	$\mu_{22} = \omega + \gamma_2 + \delta_2 + \eta_{22}$	$\mu_{23} = \omega + \gamma_2 + \delta_3 + \eta_{23}$	$\bar{\mu}_{2.} = \omega + \gamma_2$
means	$\bar{\mu}_{.1} = \omega + \delta_1$	$\bar{\mu}_{.2} = \omega + \delta_2$	$\bar{\mu}_{.3} = \omega + \delta_3$	

For a model that includes the Σ-restrictions:

$$\eta_{ij} = \mu_{ij} - (\omega + \gamma_i + \delta_j)$$

is a deviation from an additive model. Then,

$$\begin{aligned} & \eta_{ij} - \eta_{kj} - \eta_{i\ell} + \eta_{k\ell} \\ = \; & \mu_{ij} - \mu_{kj} - \mu_{i\ell} + \mu_{k\ell} \end{aligned}$$

Matrix formulation:

$$
\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{113} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{213} \\ Y_{214} \\ Y_{221} \\ Y_{231} \\ Y_{232} \\ Y_{233} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \omega \\ \gamma_1 \\ \delta_1 \\ \delta_2 \\ \eta_{11} \\ \eta_{12} \end{bmatrix} + \boldsymbol{\epsilon}
$$

This uses the Σ-restrictions to obtain

$$
\begin{aligned}
\gamma_2 &= -\gamma_1 & \delta_3 &= -\delta_1 - \delta_2 \\
\eta_{21} &= -\eta_{11} & \eta_{13} &= -\eta_{11} - \eta_{12} \\
\eta_{22} &= -\eta_{12} & \eta_{23} &= -\eta_{13} = \eta_{11} + \eta_{12}
\end{aligned}
$$

Least squares estimation:

$$
\begin{aligned}
\mathbf{b} &= (X^T X)^{-1} X^T \mathbf{Y} \\
&= \begin{bmatrix} X_\omega^T X_\omega & X_\omega^T X_\gamma & X_\omega^T X_\delta & X_\omega^T X_\eta \\ X_\gamma^T X_\omega & X_\gamma^T X_\gamma & X_\gamma^T X_\delta & X_\gamma^T X_\eta \\ X_\gamma^T X_\omega & X_\gamma^T X_\gamma & X_\gamma^T X_\delta & X_\gamma^T X_\eta \\ X_\eta^T X_\omega & X_\eta^T X_\gamma & X_\eta^T X_\delta & X_\eta^T X_\eta \end{bmatrix}^{-1} \begin{bmatrix} X_\omega^T \mathbf{Y} \\ X_\gamma^T \mathbf{Y} \\ X_\delta^T \mathbf{Y} \\ \mathbf{X}_\eta^T \mathbf{Y} \end{bmatrix} \\
&= \begin{bmatrix} 15 & -1 & 2 & -2 & 0 & 2 \\ -1 & 15 & 0 & -2 & 2 & 2 \\ 2 & 0 & 12 & 5 & -2 & 2 \\ -2 & -2 & 5 & 8 & -1 & 0 \\ 0 & 2 & -2 & -1 & 12 & 5 \\ 2 & 2 & -1 & 0 & 5 & 8 \end{bmatrix}^{-1} \begin{bmatrix} Y_{...} \\ Y_{1..} - Y_{2..} \\ Y_{.1.} - Y_{.3.} \\ Y_{.2.} - Y_{.3.} \\ Y_{11.} - Y_{13.} - Y_{21.} + Y_{23.} \\ Y_{12.} - Y_{13.} - Y_{22.} + Y_{23.} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{6}\sum_i \sum_j \bar{Y}_{ij.} \\ \frac{1}{3}\sum_j \bar{Y}_{1j.} - \frac{1}{6}\sum\sum \bar{Y}_{ij.} \\ \frac{1}{2}\sum_i \bar{Y}_{i1.} - \frac{1}{6}\sum\sum \bar{Y}_{ij.} \\ \frac{1}{2}\sum_i \bar{Y}_{i2.} - \frac{1}{6}\sum\sum \bar{Y}_{ij.} \\ \bar{Y}_{11.} - \hat{\omega} - \hat{\gamma}_1 - \hat{\delta}_1 \\ \bar{Y}_{12.} - \hat{\omega} - \hat{\gamma}_1 - \hat{\delta}_2 \end{bmatrix} = \begin{bmatrix} \hat{\omega} \\ \hat{\gamma} \\ \hat{\delta}_1 \\ \hat{\delta}_2 \\ \hat{\eta}_{11} \\ \hat{\eta}_{12} \end{bmatrix} = \begin{bmatrix} 16.83 \\ -3.17 \\ -4.33 \\ 5.67 \\ -0.33 \\ -5.33 \end{bmatrix}
\end{aligned}
$$

If restrictions are placed on *non-estimable* functions of parameters in the unrestricted *effects* model, then

- The resulting models are reparameterizations of each other.
- $\hat{\mathbf{Y}} = P_X \mathbf{Y}$

  $\mathbf{e} = \mathbf{Y} - \hat{\mathbf{Y}} = (I - P_X)\mathbf{Y}$

  $SSE = \mathbf{e}^T \mathbf{e} = \mathbf{Y}^T (I - P_X)\mathbf{Y}$

  $\hat{\mathbf{Y}}^T \hat{\mathbf{Y}} = \mathbf{Y}^T P_X \mathbf{Y}$

  $SS_{\text{model}} = \mathbf{Y}^T (P_X - P_{\mathbf{1}})\mathbf{Y}$

  are the same for any set of restrictions.

- The solution to the normal equations

$$\mathbf{b} = (X^T X)^{-1} X^T \mathbf{Y}$$

  and interpretations of the corresponding parameters will not be the same for all such sets of restrictions.
  If you were to place restrictions on estimable functions of parameters in

$$Y_{ijk} = \mu + \alpha_1 + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

  then you would change
- rank(X)
- space spanned by the columns of X
- $\hat{\mathbf{Y}} = X(X^T X)^- X^T \mathbf{Y}$ and OLS estimators of other estimable quantities.

Normal Theory Gauss-Markov Model

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$$

Analysis of variance:

$$\begin{aligned}\mathbf{Y}^T\mathbf{Y} =& \mathbf{Y}^T P_\mu \mathbf{Y} + \mathbf{Y}^T(P_{\mu,\alpha} - P_\mu)\mathbf{Y} + \mathbf{Y}^T(P_{\mu,\alpha,\beta} - P_{\mu,\alpha})\mathbf{Y} \\ &+ \mathbf{Y}^T(P_X - P_{\mu,\alpha,\beta})\mathbf{Y} + \mathbf{Y}^T(I - P_X)\mathbf{Y} \\ =& R(\mu) + R(\boldsymbol{\alpha}|\mu) + R(\boldsymbol{\beta}|\mu,\alpha) + R(\boldsymbol{\gamma}|\mu,\boldsymbol{\alpha},\boldsymbol{\beta}) + SSE\end{aligned}$$

By Cochran's Theorem, these quadratic forms (or sums of squares) have independent chi-square distributions with 1, $a-1$, $b-1$, $(a-1)(b-1)$, and $n_{\bullet\bullet} - ab$ degrees of freedom, respectively, (if $n_{ij} > 0$ for all (i,j))

$$
\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{113} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{213} \\ Y_{214} \\ Y_{221} \\ Y_{231} \\ Y_{232} \\ Y_{233} \end{bmatrix}
=
\left[\begin{array}{c|cc|ccc|cccccc}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
\begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix}
+ \boldsymbol{\epsilon}
$$

call this X_μ call this X_α call this X_β call this X_γ

Define:

$$X_\mu = X_\mu, \qquad P_\mu = X_\mu (X_\mu^T X_\mu)^{-1} X_\mu^T$$

$$X_{\mu,\alpha} = [X_\mu | X_\alpha], \qquad P_{\mu,\alpha} = X_{\mu,\alpha}(X_{\mu,\alpha}^T X_{\mu,\alpha})^- X_{\mu,\alpha}^T$$

$$X_{\mu,\alpha,\beta} = [X_\mu | X_\alpha | X_\beta], \qquad P_{\mu,\alpha,\beta} = X_{\mu,\alpha,\beta}\left(X_{\mu,\alpha,\beta}^T X_{\mu,\alpha,\beta}\right)^- X_{\mu,\alpha,\beta}^T$$

$$X = [X_\mu | X_\alpha | X_\beta | X_\gamma], \qquad P_X = X(X^T X)^- X^T$$

The following three model matrices correspond to reparameterizations of the same model:

$$
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23}
\end{bmatrix}
$$

$$
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\qquad
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1
\end{array}\right]
$$

$R(\mu) = \mathbf{Y}^T P_\mu \mathbf{Y}$ is the same for all three models

$$
\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23}
\end{bmatrix}
$$

$$
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\qquad
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1
\end{array}\right]
$$

$R(\mu, \boldsymbol{\alpha}) = \mathbf{Y}^T P_{\mu,\alpha} \mathbf{Y}$ is the same for all three models and so is
$R(\alpha|\mu) = R(\mu, \boldsymbol{\alpha}) - R(\mu)$

$$\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23}
\end{bmatrix}$$

$$\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\qquad
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1
\end{array}\right]$$

$R(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbf{Y}^T P_{\mu,\alpha,\beta} \mathbf{Y}$ is the same for all three models and so is $R(\beta|\boldsymbol{\alpha}) = R(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) - R(\mu, \boldsymbol{\alpha})$

$$\begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23}
\end{bmatrix}$$

$$\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\qquad
\left[\begin{array}{c|ccccc}
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 1 & -1 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & -1 & 1 & 1
\end{array}\right]$$

$R(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = \mathbf{Y}^T P_X \mathbf{Y}$ is the same for all three models and so is $R(\gamma|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) = R(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) - R(\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \gamma_{11} \\ \gamma_{12} \\ \gamma_{13} \\ \gamma_{21} \\ \gamma_{22} \\ \gamma_{23} \end{bmatrix}$$

$$\left[\begin{array}{c|ccccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{c|ccccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{array}\right]$$

Consequently, the partition

$$\begin{aligned}
\mathbf{Y}^T\mathbf{Y} &= \mathbf{Y}^T P_\mu \mathbf{Y} + \mathbf{Y}^T(P_{\mu,\beta} - P_\mu)\mathbf{Y} \\
&\quad + \mathbf{Y}^T(P_{\mu,\alpha,\beta} - P_{\mu,\beta})\mathbf{Y} \\
&\quad + \mathbf{Y}^T(P_X - P_{\mu,\alpha,\beta})\mathbf{Y} \\
&\quad + \mathbf{Y}^T(I - P_X)\mathbf{Y} \\
&= R(\mu) + R(\boldsymbol{\beta}|\mu) + R(\boldsymbol{\alpha}|\mu,\boldsymbol{\beta}) + R(\gamma|\mu,\alpha,\beta) + SSE
\end{aligned}$$

is the same for all three models.

By Cochran's Theorem, these quadratic forms (or sums of squares) have independent chi-square distributions with 1, $b-1$, $a-1$, $(a-1)(b-1)$, and $n_{\bullet\bullet} - ab$ degrees of freedom, respectively, when $n_{ij} > 0$ for all (i, j).

Using result 4.7, we have also shown earlier that

$$
\begin{aligned}
SSE &= \mathbf{Y}^T(I - P_X)\mathbf{Y} \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n_{ij}}(Y_{ijk} - \bar{Y}_{ij\bullet})^2 \\
&\sim \chi^2_{n_{\bullet\bullet}-ab}
\end{aligned}
$$

What null hypotheses are tested by F-tests derived from such ANOVA tables (Type I sums of squares in SAS)?

$$\begin{aligned} R(\mu) &= \mathbf{Y}^T P_{\mathbf{1}} \mathbf{Y} \\ &= \mathbf{Y}^T P_{\mathbf{1}} P_{\mathbf{1}} \mathbf{Y} \\ &= (P_{\mathbf{1}} \mathbf{Y})^T (P_{\mathbf{1}} \mathbf{Y}) \\ &= (\bar{Y}_{\dots} \mathbf{1})^T (\bar{Y}_{\dots} \mathbf{1}) = n_{..} \bar{Y}_{\dots}^2 \end{aligned}$$

$\frac{1}{\sigma^2} R(\boldsymbol{\mu}) \sim \chi_1^2(\delta^2)$ and

$$F = \frac{R(\mu)}{SSE/(n_{..} - ab)} \sim F_{(1, n_{..} - ab)}(\delta^2)$$

where

$$\begin{aligned} \delta^2 &= \frac{1}{\sigma^2} \boldsymbol{\beta}^T X^T P_{\mathbf{1}} X \boldsymbol{\beta} \\ &= \frac{1}{\sigma^2} (\boldsymbol{\beta}^T X^T P_{\mathbf{1}})(P_{\mathbf{1}} X \boldsymbol{\beta}) \\ &= \frac{1}{\sigma^2} (P_{\mathbf{1}} X \boldsymbol{\beta})^T (P_{\mathbf{1}} X \boldsymbol{\beta}) \end{aligned}$$

For the carrot seed germination study:

$$\begin{aligned} P_{\mathbf{1}}X\boldsymbol{\beta} &= \frac{1}{n_{..}}\mathbf{1}\,\mathbf{1}^T X\boldsymbol{\beta} \\ &= \frac{1}{n_{..}}\,\mathbf{1}[n_{..}, n_{1.}, n_{2.}, n_{.1}, n_{.2}, n_{.3}, n_{11}, n_{12}, n_{13}, n_{21}, n_{22}, n_{23}]\ \boldsymbol{\beta} \\ &= \frac{1}{n_{..}}\,\mathbf{1}\left(n_{..}\mu + \sum_{i=1}^{a} n_{i.}\alpha_i + \sum_{j=1}^{b} n_{.j}\beta_j + \sum_{i=1}^{a}\sum_{j=1}^{b}\gamma_{ij}\right) \end{aligned}$$

The null hypothesis is

$$H_0 : 0 = n_{..}\,\mu + \sum_{i=1}^{a} n_{i.}\,\alpha_i + \sum_{j=1}^{b} n_{.j}\,\beta_j + \sum_{i}\sum_{j} n_{ij}\gamma_{ij}$$

With respect to the cell means $E(Y_{ijk}) = \mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$, this null hypothesis is

$$H_0 : 0 = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{n_{ij}}{n_{..}}\mu_{ij}$$

Consider

$$R(\boldsymbol{\alpha}|\mu) = \mathbf{Y}^T(P_{\mu,\alpha} - P_\mu)\mathbf{Y}$$

and

$$F = \frac{R(\boldsymbol{\alpha}|\mu)/(a-1)}{MSE} \sim F_{(a-1,n_{\cdot\cdot}-ab)}(\delta^2)$$

Here,

$$\frac{1}{\sigma^2} R(\boldsymbol{\alpha}|\boldsymbol{\mu}) \sim \chi^2_{a-1}(\delta^2)$$

where $a-1 = \text{rank}(X_{\mu,\boldsymbol{\alpha}}) -$ rank(X_μ) and

$$\begin{aligned}
\delta^2 &= \frac{1}{\sigma^2}\boldsymbol{\beta}^T X^T (P_{\mu,\alpha} - P_\mu) X\boldsymbol{\beta} \\
&= \frac{1}{\sigma^2}\left[(P_{\mu,\alpha} - P_\mu)X\boldsymbol{\beta}\right]^T \left[(P_{\mu,\alpha} - P_\mu)X\boldsymbol{\beta}\right]
\end{aligned}$$

For the general effects model for the carrot seed germination study:

$$P_{\mu,\alpha} X = X_{\mu,\alpha}(X_{\mu,\alpha}^T X_{\mu,\alpha})^- X_{\mu,\alpha}^T X = X_{\mu,\alpha} \begin{bmatrix} n_{..} & n_{1.} & n_{2.} \\ n_{1.} & n_{1.} & 0 \\ n_{2.} & 0 & n_{2.} \end{bmatrix}^-$$

$$\times \begin{bmatrix} n_{..} & n_{1.} & n_{2.} & n_{.1} & n_{.2} & n_{.3} & n_{11} & n_{12} & n_{13} & n_{21} & n_{22} & n_{23} \\ n_{1.} & n_{1.} & 0 & n_{11} & n_{12} & n_{13} & n_{11} & n_{12} & n_{13} & 0 & 0 & 0 \\ n_{2.} & 0 & n_{2.} & n_{21} & n_{22} & n_{23} & 0 & 0 & 0 & n_{21} & n_{22} & n_{23} \end{bmatrix}$$

$$\downarrow$$

$$= \quad X_{\mu,\alpha} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{n_{1.}} & 0 \\ 0 & 0 & \frac{1}{n_{2.}} \end{bmatrix} \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & \frac{n_{11}}{n_{1.}} & \frac{n_{12}}{n_{1.}} & \frac{n_{13}}{n_{1.}} & \frac{n_{11}}{n_{..}} & \frac{n_{12}}{n_{..}} & \frac{n_{13}}{n_{..}} & 0 & 0 & 0 \\ 1 & 0 & 1 & \frac{n_{11}}{n_{1.}} & \frac{n_{22}}{n_{2.}} & \frac{n_{23}}{n_{2.}} & 0 & 0 & 0 & \frac{n_{21}}{n_{1.}} & \frac{n_{22}}{n_{2.}} & \frac{n_{23}}{n_{1.}} \end{bmatrix}$$

Then, the first seven rows of $\underline{(P_{\mu,\alpha} - P_\mu)X\boldsymbol{\beta}}$ are

$$\left[\mu + \alpha_1 + \textstyle\sum_{j=1}^{b} \frac{n_{1j}}{n_{1.}} \left(\beta_j + \gamma_{1j}\right)\right]$$

$$-\left[\mu + \sum_{i=1}^{a} \frac{n_{i.}}{n_{..}} \alpha_i + \sum_{j=1}^{b} \frac{n_{.j}}{n_{..}} \beta_j + \sum_i \sum_j \frac{n_{ij}}{n_{..}} \gamma_{ij}\right]$$

The last eight rows of $(P_{\mu,\alpha} - P_\mu)X\boldsymbol{\beta}$ are

$$\left[\mu + \alpha_2 + \textstyle\sum_{j=1}^{b} \frac{n_{2j}}{n_{2.}} \left(\beta_j + \gamma_{2j}\right)\right]$$

$$-\left[\mu + \sum_{i=1}^{a} \frac{n_{i.}}{n_{..}} \alpha_i + \sum_{j=1}^{b} \frac{n_{.j}}{n_{..}} \beta_j + \sum_i \sum_j \frac{n_{ij}}{n_{..}} \gamma_{ij}\right]$$

The null hypothesis is

$$H_0 : \alpha_i + \sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} (\beta_j + \gamma_{ij}) \text{ are all equal } (i = 1, \ldots, a).$$

with respect to the cell means model,

$$\mu_{ij} = E(Y_{ijk}) = \mu + \alpha_i + \beta_j + \gamma_{ij} \ ,$$

this null hypothesis is

$$H_0 : \sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} \mu_{ij} \text{ are all equal } (i = 1, \ldots, a).$$

Comments:

For $R(\boldsymbol{\alpha}|\mu)$,

(i) $\displaystyle\sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} \mu_{ij}$ may not be equal for all $i = 1, \ldots, a$, even though $\displaystyle\frac{1}{b} \sum_{j=1}^{b} \mu_{ij}$ are equal for all $i = 1, \ldots, a$.

(ii) $\displaystyle\sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} \mu_{ij}$ may be equal for all $i = 1, \ldots, a$, even though $\displaystyle\frac{1}{b} \sum_{j=1}^{b} \mu_{ij}$ are not equal for some $i = 1, \ldots, a$.

Consider $R(\boldsymbol{\beta}|\boldsymbol{\mu},\boldsymbol{\alpha}) = \mathbf{Y}^T(P_{\mu,\alpha,\beta} - P_{\mu,\alpha})\mathbf{Y}$ and the corresponding F-statistic

$$F = \frac{R(\boldsymbol{\beta}|\mu,\boldsymbol{\alpha})/(b-1)}{MSE} \sim F_{(b-1,n_{..}-ab)}(\delta^2)$$

Here,

$$\frac{1}{\sigma^2}R(\boldsymbol{\beta}|\boldsymbol{\mu},\boldsymbol{\alpha}) \quad \sim \quad \chi^2_{\text{rank}(\mathrm{X}_{\mu,\alpha,\beta})-\text{rank}(\mathrm{X}_{\mu,\alpha})}(\delta^2)$$

$$\nearrow \qquad\qquad \nwarrow$$

$$[1+(a-1)+(b-1)] - [1+(a-1)]$$
$$= b-1 \text{ degrees of freedom}$$

and

$$\delta^2 = \frac{1}{2\sigma^2}\Big[(P_{\mu,\alpha,\beta} - P_{\mu,\alpha})X\boldsymbol{\beta}\Big]^T\Big[(P_{\mu,\alpha,\beta} - P_{\mu,\alpha})X\boldsymbol{\beta}\Big]$$

$$
\begin{aligned}
P_{\mu,\alpha,\beta}X &= X_{\mu,\alpha,\beta}\left[X^T_{\mu,\alpha,\beta}X_{\mu,\alpha,\beta}\right]^{-}X^T_{\mu,\alpha,\beta}X \\
&= X_{\mu,\alpha,\beta}\left[\begin{array}{c|cc|ccc}
n_{..} & n_{1.} & n_{2.} & n_{.1} & n_{.2} & n_{.3} \\ \hline
n_{1.} & n_{1.} & 0 & n_{11} & n_{12} & n_{13} \\
n_{2.} & 0 & n_{2.} & n_{21} & n_{22} & n_{23} \\ \hline
n_{.l} & n_{11} & n_{21} & n_{.1} & 0 & 0 \\
n_{.2} & n_{12} & n_{22} & 0 & n_{.2} & 0 \\
n_{.3} & n_{13} & n_{23} & 0 & 0 & n_{.3}
\end{array}\right]^{-}X^T_{\mu,\alpha,\beta}X
\end{aligned}
$$

$\nearrow$

call this $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$

$$
\begin{aligned}
\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} &= \begin{bmatrix} A^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A^{-1}B \\ I \end{bmatrix}[C - B^TA^{-1}B]^{-1}[-B^TA^{-1} \mid I] \\
&= \begin{bmatrix} 0 & 0 \\ 0 & C^{-1} \end{bmatrix} + \begin{bmatrix} I \\ -C^{-1}B^T \end{bmatrix}[A - BC^{-1}B^T]^{-1}[I \mid -BC^{-1}] \\
&= \begin{bmatrix} W & -WBC^{-1} \\ -C^{-1}B^TW & C^{-1} + C^{-1}B^TWBC^{-1} \end{bmatrix}
\end{aligned}
$$

where $W = [A - BC^{-1}B^T]^{-1}$

The null hypothesis is

$$H_0: \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} (\beta_j + \gamma_{ij}) - \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} \left(\sum_{k=1}^{b} \frac{n_{ik}}{n_{i.}} (\beta_k + \gamma_{ik}) \right) = 0$$

for all $j = 1, \ldots, b$

With respect to the cell means,

$$E(Y_{ijk}) = \mu_{ij},$$

this null hypothesis is

$$H_0: \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} \mu_{ij} - \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} \left(\sum_{k=1}^{b} \frac{n_{ik}}{n_{i.}} \mu_{ik} \right) = 0$$

for all $j = 1, 2, \ldots, b$.

Consider

$$R(\boldsymbol{\gamma}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathbf{Y}^T[P_X - P_{\mu,\alpha,\beta}]\mathbf{Y}$$

and the associated F-statistic

$$\begin{aligned} F &= \frac{R(\boldsymbol{\gamma}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta})/[(a-1)(b-1)]}{MSE} \\ &\sim F_{(a-1)(b-1), n_{..}-ab}(\delta^2) \end{aligned}$$

The null hypothesis is:

$$H_0: \ (\mu_{ij} - \mu_{i\ell} - \mu_{kj} + \mu_{k\ell}) = (\gamma_{ij} - \gamma_{i\ell} - \gamma_{kj} + \gamma_{k\ell}) = 0$$

for all (i, j) and (k, ℓ) .

Type I sums of squares

Source of variat.	d.f.	sums of squares	Mean square	F	p-value
Soil types	$a-1=1$	$R(\boldsymbol{\alpha}\|\boldsymbol{\mu})=52.5$	52.5	3.94	.0785
Var.	$b-1=2$	$R(\boldsymbol{\beta}\|\mu,\boldsymbol{\alpha})=124.73$	62.4	4.68	.0405
Inter-action	(a-1)(b-1)=2	$R(\boldsymbol{\gamma}\|\mu,\boldsymbol{\alpha},\boldsymbol{\beta})=222.76$	111.38	8.35	.0089
Resid.	$n_{\bullet\bullet}-ab=9$	$\mathbf{Y}^T(I-P_X)\mathbf{Y}=120$	13.33		
Corr. total	$n_{\bullet\bullet}-1=14$	$\mathbf{Y}^T(I-P_\mathbf{1})\mathbf{Y}=520$			
Corr. for the mean	1	$R(\mu)=3375$			

ANOVA Summary:

Sums of Squares	Associated null hypothesis
$R(\mu)$	$H_0 : \mu + \sum_{i=1}^{a} \frac{n_{i.}}{n_{..}} \alpha_i + \sum_{j=1}^{b} \frac{n_{.j}}{n_{..}} \beta_j + \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{n_{ij}}{n_{..}} \gamma_{ij} = 0 \left(\text{or } H_0 : \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{n_{ij}}{n_{..}} \mu_{ij} = 0 \right)$
$R(\boldsymbol{\alpha} \mid \mu)$	$H_0 : \alpha_i + \sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} (\beta_j + \gamma_{ij})$ are equal $\left(\text{or } H_0 : \sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} \mu_{ij} \text{ are equal} \right)$
$R(\boldsymbol{\beta} \mid \mu, \boldsymbol{\alpha})$	$H_0 : \beta_j + \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} \gamma_{ij} = \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} \sum_{k=1}^{b} \frac{n_{ik}}{n_{k.}} (\beta_k + \gamma_{ik})$ for all $j = 1, \ldots, b$ $\left(\text{or } H_0 : \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} \mu_{ij} = \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} \sum_{k=1}^{b} \frac{n_{ik}}{n_{i.}} \mu_{ik} \text{ for all } j = 1, \ldots, b \right)$
$R(\boldsymbol{\gamma} \mid \mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$	$H_0 : \gamma_{ij} - \gamma_{kj} - \gamma_{i\ell} + \gamma_{k\ell} = 0$ for all (i, j) and (k, ℓ) (or $H_0 : \mu_{ij} - \mu_{kj} - \mu_{i\ell} + \mu_{k\ell} = 0$ for all $(i, j)$and (k, ℓ)

Sums of Squares	Associated null hypothesis
$R(\mu)$	$H_0 : \mu + \sum_{i=1}^{a} \frac{n_{i.}}{n_{..}} \alpha_i + \sum_{j=1}^{b} \frac{n_{.j}}{n_{..}} \beta_j + \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{n_{ij}}{n_{..}} \gamma_{ij} = 0$ $\left(\text{or } H_0 : \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{n_{ij}}{n_{..}} \mu_{ij} = 0 \right)$
$R(\boldsymbol{\beta} \| \mu)$	$H_0 : \beta_j + \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} (\alpha_j + \gamma_{ij})$ are equal for all $j = 1, \ldots, b$ $\left(\text{or } H_0 : \sum_{i=1}^{a} \frac{n_{ij}}{n_{.j}} \mu_{ij} \text{ are equal for all } j = 1, \ldots, b \right)$
$R(\boldsymbol{\alpha} \| \mu, \boldsymbol{\beta})$	$H_0 : \sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} (\alpha_{ij} + \gamma_{ij}) = \sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} \sum_{k=1}^{a} \frac{n_{kj}}{n_{.j}} (\alpha_k + \gamma_{kj})$ for all $i = 1, \ldots, a$ $\left(\text{or } H_0 : \sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} \mu_{ij} = \sum_{j=1}^{b} \frac{n_{ij}}{n_{i.}} \left[\sum_{k=1}^{a} \frac{n_{kj}}{n_{.j}} \mu_{kj} \right] \text{ for all } i = 1, \ldots, a \right)$
$R(\boldsymbol{\gamma} \| \mu, \boldsymbol{\alpha}, \boldsymbol{\beta})$	$H_0 : \gamma_{ij} - \gamma_{kj} - \gamma_{i\ell} + \gamma_{k\ell} = 0$ for all (i, j) and (k, ℓ) $\left(\text{or } H_0 : \mu_{ij} - \mu_{kj} - \mu_{i\ell} + \mu_{k\ell} = 0 \text{ for all } (i, j) \text{ and } (k, \ell) \right)$

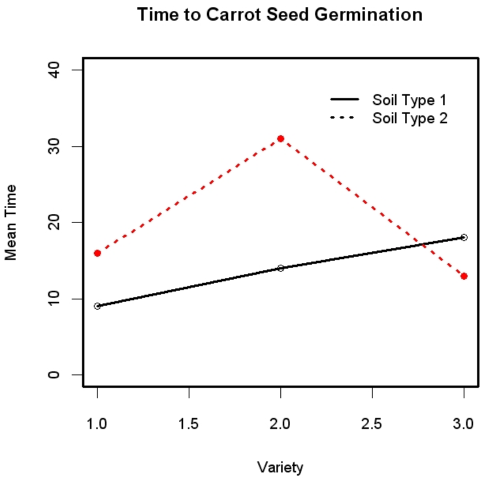
Type I sums of squares

Source of variat.	d.f.	sums of squares	Mean square	F	p-value
"Soils"	$a-1=1$	$R(\boldsymbol{\alpha}\|\boldsymbol{\mu})=52.50$	52.5	3.94	.0785
"Var."	$b-1=2$	$R(\boldsymbol{\beta}\|\mu,\boldsymbol{\alpha})=124.73$	62.4	4.68	.0405
Inter-action	(a-1)(b-1)=2	$R(\boldsymbol{\gamma}\|\mu,\boldsymbol{\alpha},\boldsymbol{\beta})=222.76$	111.38	8.35	.0089
''Res."	$\Sigma\Sigma(n_{ij}-1)$=9	$\mathbf{Y}^T(I-P_X)\mathbf{Y}=120.00$	13.33		
Corr. total	$n_{\cdot\cdot}-1=14$	$\mathbf{Y}^T(I-P_\mathbf{1})\mathbf{Y}=520.00$			

Source of variat.	d.f.	sums of squares	Mean square	F	p-value
"Var."	$b-1=2$	$R(\boldsymbol{\beta}\|\mu)=93.33$	46.67	3.50	.0751
"Soils"	$a-1=1$	$R(\alpha\|\boldsymbol{\mu},\boldsymbol{\beta})=83.90$	83.90	6.29	.0334
Inter-action	(a-1)(b-1)=2	$R(\boldsymbol{\gamma}\|\mu,\boldsymbol{\alpha},\boldsymbol{\beta})=222.76$	111.38	8.35	.0089
"Res."	$\Sigma\Sigma(n_{ij}-1)=9$	$\mathbf{Y}^T(I-P_X)\mathbf{Y}=120.00$	13.33		
Corr.total	$n_{\cdot\cdot}-1=14$	$\mathbf{Y}^T(I-P_\mathbf{1})\mathbf{Y}=520.00$			

Type II sums of squares: (SAS)

Source of variat.	d.f.	sums of squares	Mean square	F	p-value
"Soils"	$a-1=1$	$R(\boldsymbol{\alpha}\vert\boldsymbol{\mu},\boldsymbol{\beta}) = 83.90$	83.90	6.3	.0339
"Var."	$b-1=2$	$R(\boldsymbol{\beta}\vert\mu,\boldsymbol{\alpha}) = 124.73$	62.37	4.7	.0405
Inter-action	(a-1)(b-1)=2	$R(\boldsymbol{\gamma}\vert\mu,\boldsymbol{\alpha},\boldsymbol{\beta}) = 222.76$	111.38	8.4	.0089
"Res."	$n_{\bullet\bullet} - ab = 9$	$\mathbf{Y}^T(I-P_X)\mathbf{Y} = 120$	13.33		
Corr. total	$n_{\bullet\bullet} - 1$	$\mathbf{Y}^T(I-P_{\mathbf{1}})\mathbf{Y} = 520$			


Time to Carrot Seed Germination
Soil Type 1
Soil Type 2
Mean Time
0
10
20
30
40
1.0
1.5
2.0
2.5
3.0
Variety

Examine the soil type effect on time to germination for each variety:

Time to Germination

	Soil Type 1		Soil Type 2			
Variety	$\bar{Y}_{ij.}$	$S_{\bar{Y}_{ij.}}$	$\bar{Y}_{2j.}$	$S_{\bar{Y}_{2j.}}$	t	p-value
$j = 1$	9.0	2.11	16.0	1.83	-2.51	.0333
$j = 2$	14.0	2.58	31.0	3.65	-3.80	.0042
$j = 3$	18.0	2.58	13.0	2.11	1.50	.1679

- Time to germination for variety 2 is shorter in soil type 1.
- Time to germination for variety 1 may also be shorter in soil type 1.
- For variety 3 there is no significant difference in average germination times for the two soil types.

In the previous analysis:

$$\bar{Y}_{ij.} = \hat{\mu}_{ij} = \hat{\mu} + \hat{\alpha}_i + \hat{\beta}_j + \hat{\gamma}_{ij}$$

is the OLS estimator (b.l.u.e.) for

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij},$$

where

$$S_{\bar{Y}_{ij.}} = \sqrt{\frac{MSE}{n_{ij}}}$$

for $i = 1, \ldots, a,$ and $j = 1, \ldots, b$ and

$$t = \frac{\bar{Y}_{1j.} - \bar{Y}_{2j.}}{\sqrt{MSE(\frac{1}{n_{1j}} + \frac{1}{n_{2j}})}}$$

for $j = 1, \ldots, b$

Method of Unweighted Means

(Type III sums of squares in SAS when $n_{ij} > 0$ for all (i, j)).

Use the cell means reparameterization of the model:

$$
\begin{aligned}
Y_{ijk} &= \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk} \\
&= \mu_{ij} + \epsilon_{ijk}
\end{aligned}
$$

$$\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{113} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{213} \\ Y_{214} \\ Y_{221} \\ Y_{231} \\ Y_{232} \\ Y_{233} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{bmatrix} + \boldsymbol{\epsilon}$$

The model is

$$\mathbf{Y} = D\boldsymbol{\mu} + \boldsymbol{\epsilon}$$

The least squares estimator (b.l.u.e.) for $\boldsymbol{\mu}$ is

$$\begin{aligned}
\hat{\boldsymbol{\mu}} &= (D^T D)^{-1} D^T \mathbf{Y} \\
&= \begin{bmatrix} n_{11}^{-1} & & & & & \\ & n_{12}^{-1} & & & & \\ & & n_{13}^{-1} & & & \\ & & & n_{21}^{-1} & & \\ & & & & n_{22}^{-1} & \\ & & & & & n_{23}^{-1} \end{bmatrix} \begin{bmatrix} Y_{11.} \\ Y_{12.} \\ Y_{12.} \\ Y_{21.} \\ Y_{22.} \\ Y_{23.} \end{bmatrix} \\
&= \begin{bmatrix} \bar{Y}_{11.} \\ \bar{Y}_{12.} \\ \bar{Y}_{13.} \\ \bar{Y}_{21.} \\ \bar{Y}_{22.} \\ \bar{Y}_{23.} \end{bmatrix}
\end{aligned}$$

Test the null hypothesis

$$H_0 : \frac{1}{b}\sum_{j=1}^{b} \mu_{ij} = \frac{1}{b}\sum_{j=1}^{b} \mu_{2j} = \cdots = \frac{1}{b}\sum_{j=1}^{b} \mu_{aj}$$

vs.

$$H_A : \frac{1}{b}\sum_{j=1}^{b} \mu_{ij} \neq \frac{1}{b}\sum_{j=1}^{b} \mu_{kj} \text{ for some } i \neq k$$

The OLS estimator (b.l.u.e.) for $\frac{1}{b}\sum_{j=1}^{b} \mu_{ij}$ is

$$\tilde{Y}_{i..} = \frac{1}{b}\sum_{j=1}^{b} \bar{Y}_{ij.}$$

with

$$Var(\tilde{Y}_{i..}) = \frac{1}{b^2}\sum_{j=1}^{b} \frac{\sigma^2}{n_{ij}} = \sigma^2 \left(\frac{1}{b^2}\sum_{j=1}^{b} \frac{1}{n_{ij}} \right)$$

Express the null hypothesis in matrix form: $H_0 : C_1\boldsymbol{\mu} = \mathbf{0}$, where

$$
\begin{aligned}
C_1\boldsymbol{\mu} &= \left(\left[I_{a-1} \middle| -\mathbf{1}_{a-1} \right] \otimes \mathbf{1}_b^T \right) \boldsymbol{\mu} \\
&= \begin{bmatrix} \mathbf{1}_b^T & & & -\mathbf{1}_b^T \\ & \mathbf{1}_b^T & & -\mathbf{1}_b^T \\ & & \ddots & \vdots \\ & & & \mathbf{1}_b^T \; -\mathbf{1}_b^T \end{bmatrix} \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{1b} \\ \mu_{21} \\ \vdots \\ \mu_{2b} \\ \vdots \\ \mu_{ab} \end{bmatrix} \\
&= \begin{bmatrix} \sum_j \mu_{1j} - \sum_j \mu_{aj} \\ \vdots \\ \sum_j \mu_{a-1,j} - \sum_j \mu_{aj} \end{bmatrix}
\end{aligned}
$$

Then

$$C_1\mathbf{b} = C_1(D^TD)^{-1}D^T\mathbf{Y} = \begin{bmatrix} \sum_j \bar{Y}_{1j.} - \sum_j \bar{Y}_{aj.} \\ \vdots \\ \vdots \\ \sum_j \bar{Y}_{a-1,j.} - \sum_j \bar{Y}_{aj.} \end{bmatrix}$$

is the OLS estimator (b.l.u.e.) of $C_1\boldsymbol{\mu}$, and

$$\begin{aligned} Var(C_1\mathbf{b}) &= Var(C_1(D^TD)^{-1}D^T\mathbf{Y}) \\ &= C_1(D^TD)^{-1}D^T(\sigma^2 I)D(D^TD)^{-1}C_1^T \\ &= \sigma^2 C_1(D^TD)^{-1}D^TD(D^TD)^{-1}C_1^T \\ &= \sigma^2 C_1(D^TD)^{-1}C_1^T \end{aligned}$$

Compute

$$
\begin{aligned}
SS_{H_0} &= (C_1\mathbf{b}-\mathbf{0})^T[C_1(D^TD)^{-1}C_1^T]^{-1}(C_1\mathbf{b}-\mathbf{0}) \\
&= \mathbf{Y}^TD(D^TD)^{-1}C_1^T[C_1(D^TD)^{-1}C_1^T]^{-1}C_1(D^TD)^{-1}D^T\mathbf{Y}
\end{aligned}
$$

Use result 4.7 to show

$$\frac{1}{\sigma^2}SS_{H_0} \sim \chi^2_{(a-1)}(\delta^2)$$

Check that

$$
\begin{aligned}
A\Sigma = \frac{1}{\sigma^2}D(D^TD)^{-1}C_1^T\left[C_1(D^TD)^{-1}C_1^T\right]^{-1} \\
C_1(D^TD)^{-1}D^T(\sigma^2I)
\end{aligned}
$$

is idempotent and that

$$a-1 = \text{rank}(C_1(D^TD)^{-1}C_1^T)$$

Compute:

$$SSE = \mathbf{Y}^T(I - P_D)\mathbf{Y}$$

where

$$P_D = D(D^TD)^{-1}D^T$$

Use result **4.7** to show

$$\frac{1}{\sigma^2}SSE \sim \chi^2_{\Sigma\Sigma(n_{ij}-1)}$$

Use result **4.8** to show that

$$SSE = \mathbf{Y}^T\underbrace{(I - P_D)}_{\text{call this } A_1}\mathbf{Y}$$

is distributed independently of

$$SS_{H_0} = \mathbf{Y}^T\underbrace{D(D^TD)^{-1}C_1^T[C_1(D^TD)^{-1}C_1^T]^{-1}C_1(D^TD)^{-1}}_{\text{call this } A_2}D^T\mathbf{Y}$$

Check that

$$
\begin{aligned}
A_1 \Sigma A_2 &= A_1(\sigma^2 I) A_2 \\
&= \sigma^2 A_1 A_2 \\
&= \sigma^2 (I - P_D)(D(D^T D)^{-1} C_1^T (C_1 (D^T D)^{-1} C_1^T)^{-1} C_1 (D^T D)^{-1} D^T \\
&= 0
\end{aligned}
$$

This is true because $(I - P_D)D = 0$.

Then

$$
F = \frac{SS_{H_0}/(a-1)}{SSE/_{(\Sigma\Sigma(n_{ij}-1))}} \sim F_{(a-1, \Sigma\Sigma(n_{ij}-1))}(\delta^2)
$$

where

$$
\delta^2 = \frac{1}{\sigma^2} \boldsymbol{\mu}^T C_1^T \left[C_1 (D^T D)^{-1} C_1^T \right]^{-1} C_1 \boldsymbol{\mu}
$$

Reject

$$H_0 : \frac{1}{b}\sum_{j=1}^{b}\mu_{1j} = \frac{1}{b}\sum_{j=1}^{b}\mu_{2j} = \cdots = \frac{1}{b}\sum_{j=1}^{b}\mu_{aj}$$

if

$$F = \frac{SS_{H_0}/(a-1)}{SSE/_{(\Sigma\Sigma(n_{ij}-1))}} \;>\; F_{(a-1,\Sigma\Sigma(n_{ij}-1))}\,,\alpha$$

or if

$$\begin{aligned} p-\text{value} \quad &= \quad Pr\left\{F_{(a-1,\Sigma\Sigma(n_{ij}-1))} > F\right\} \\ &< \quad \alpha \end{aligned}$$

Test

$$H_0 : \frac{1}{a}\sum_{i=1}^{a}\mu_{i1} = \frac{1}{a}\sum_{i=1}^{a}\mu_{i2} = \cdots = \frac{1}{a}\sum_{i=1}^{a}\mu_{ib}$$

vs.

$$H_A : \frac{1}{a}\sum_{i=1}^{a}\mu_{ij} \neq \frac{1}{a}\sum_{i=1}^{a}\mu_{ik} \quad \text{for some } j \neq k$$

Write the null hypothesis in matrix form as $H_0 : C_2\boldsymbol{\mu} = \mathbf{0}$, where

$$\begin{aligned} C_2 &= \mathbf{1}_a^T \otimes \left[I_{b-1} \middle| -\mathbf{1}_{b-1} \right] \\ &= \left[\begin{array}{ccccc|cc|ccccc} 1 & & & & -1 & \cdots & \cdots & 1 & & & & -1 \\ & 1 & & & -1 & & & & 1 & & & -1 \\ & & \ddots & & \vdots & & & & & \ddots & & \vdots \\ & & & 1 & -1 & & & & & & 1 & -1 \end{array}\right] \end{aligned}$$

then

$$C_2\boldsymbol{\mu} = C_2 \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{1b} \\ \mu_{21} \\ \vdots \\ \mu_{ab} \end{bmatrix} = \begin{bmatrix} \frac{1}{a}\sum_{i=1}^{a}\mu_{i1} - \frac{1}{a}\sum_{i=1}^{a}\mu_{ib} \\ \vdots \\ \frac{1}{a}\sum_{i=1}^{a}\mu_{i,b-1} - \frac{1}{a}\sum_{i=1}^{a}\mu_{ib} \end{bmatrix}$$

Compute

$$SS_{H_{0,2}} = \mathbf{Y}^T D(D^T D)^{-1} C_2^T [C_2(D^T D)^{-1} C_2^T]^{-1} C_2(D^T D)^{-1} D^T \mathbf{Y}$$

and reject H_0 if

$$F = \frac{SS_{H_{0,2}}/(b-1)}{SSE/_{(\Sigma\Sigma(n_{ij}-1))}} > F_{(b-1,\Sigma\Sigma(n_{ij}-1)),\alpha}$$

Test for Interaction:

Test

$$H_0 : \mu_{ij} - \mu_{i\ell} - \mu_{kj} + \mu_{k\ell} = 0$$

for all (i, j) and (k, ℓ)

vs.

$$H_A : \mu_{ij} - \mu_{i\ell} - \mu_{kj} + \mu_{k\ell} \neq 0$$

for all (i, k) and $(j \neq \ell)$.

Write the null hypothesis in matrix form as

$$H_0 : C_3\, \boldsymbol{\mu} = \mathbf{0}$$

where

$$C_3 = \left[I_{a-1} | - \mathbf{1}_{a-1} \right] \otimes \left[I_{b-1} | - 1_{b-1} \right]$$

Compute

$$\mathbf{b} = (D^T D)^{-1} D^T \mathbf{Y} = \begin{bmatrix} \bar{Y}_{11.} \\ \vdots \\ \bar{Y}_{ab.} \end{bmatrix}$$

$$\begin{aligned} SS_{H_{0,3}} &= (C_3\mathbf{b} - \mathbf{0})^T [C_3(D^T D)^{-1} C_3^T]^{-1} (C_3\mathbf{b} - \mathbf{0}) \\ &= \mathbf{Y}^T D(D^T D)^{-1} C_3^T [C_3(D^T D)^{-1} C_3^T]^{-1} C_3 (D^T D)^{-1} D^T \mathbf{Y} \end{aligned}$$

and reject H_0 if

$$F = \frac{SS_{H_{0,3}}/((a-1)(b-1))}{SSE/_{(\Sigma\Sigma(n_{ij}-1))}}$$

$$> F_{((a-1)(b-1),\Sigma\Sigma(n_{ij}-1)),\alpha}$$

PROC GLM is SAS reports this as Type III sums of squares.

Source of variation	Sum of d.f.	Mean Squares	Square	F	p-value
Soils	a-1=1	$SS_{H_0} = 123.77$	123.77	9.28	.0139
Var.	b-1=2	$SS_{H_{0,2}} = 192.13$	96.06	7.20	.0135
Inter.	(a-1)(b-1)=2	$SS_{H_{0,3}} = 222.76$	111.38	8.35	.0089

Note that

$$
\begin{aligned}
\mathbf{Y}^T P_{\mathbf{1}} \mathbf{Y} \quad &+ \quad \mathbf{Y}^T D(D^T D)^{-1}[C_1(D^T D)^{-1} C_1^T]^{-1} C_1(D^T D)^{-1} D^T \mathbf{Y} \\
&+ \quad \mathbf{Y}^T D(D^T D)^{-1} C_2^T [C_2(D^T D)^{-1} C_2^T]^{-1} C_2(D^T D)^{-1} D^T \mathbf{Y} \\
&+ \quad \mathbf{Y}^T D(D^T D)^{-1} C_3^T [C_3(D^T D)^{-1} C_3^T]^{-1} C_3(D^T D)^{-1} D^T \mathbf{Y} \\
&+ \quad \mathbf{Y}^T (I - P_D) \mathbf{Y}
\end{aligned}
$$

do not necessarily sum to $\mathbf{Y}^T \mathbf{Y}$, nor do the middle three terms $(SS_{H_0}\,,\ SS_{H_0,2}\,,\ SS_{H_0,3})$ necessarily sum to

$$
SS_{\text{model,corrected}} = \mathbf{Y}^T (P_D - P_{\mathbf{1}}) \mathbf{Y} \ ,
$$

nor are $(SS_{H_0}\,,\ SS_{H_0,2}\,,\ SS_{H_0,3})$ necessarily independent of each other.

Note that

$$SS_{H_0} = \sum_{i=1}^{a} w_i \left[\tilde{Y}_{i.} - \frac{\sum_{k=1}^{a} w_k \tilde{Y}_{k.}}{\sum_{k=1}^{a} w_k} \right]^2$$

where

$$\tilde{Y}_{i.} = \frac{1}{b} \sum_{j=1}^{b} \bar{Y}_{ij.}, \quad w_i = \left[\frac{1}{b^2} \sum_{j=1}^{b} \frac{a}{n_{ij}} \right]^{-1} = \sigma^2 \left[Var(\tilde{Y}_{i.}) \right]^{-1}$$

and $\tilde{Y}_{i.}$ is not necessarily equal to

$$\bar{Y}_{i.} = \frac{\sum_{j=1}^{b} \sum_{k=1}^{n_{ij}} Y_{ijk}}{\sum_{j=1}^{b} n_{ij}} = \frac{\sum_{j=1}^{b} n_{ij} \bar{Y}_{ij.}}{\sum_{j=1}^{b} n_{ij}}$$

Furthermore,

$$SS_{H_0,2} = \sum_{j=1}^{b} w_j \left[\tilde{Y}_{.j} - \frac{\sum_{\ell=1}^{a} w_\ell \tilde{Y}_{.\ell}}{\sum_{\ell=1}^{a} w_\ell} \right]^2$$

where

$$\tilde{Y}_{.j} = \frac{1}{a} \sum_{i=1}^{a} \bar{Y}_{ij.} w_j = \left[\frac{1}{a^2} \sum_{i=1}^{a} \frac{a}{n_{ij}} \right]^{-1} = \sigma^2 \left[Var(\tilde{Y}_{.j}) \right]^{-1}$$

and $\tilde{Y}_{.j}$ is not necessarily equal to

$$\bar{Y}_{.j} = \frac{\sum_{i=1}^{a} \sum_{k=1}^{n_{ij}} Y_{ijk}}{\sum_{i=1}^{a} n_{ij}} = \frac{\sum_{i=1}^{a} n_{ij} \bar{Y}_{ij.}}{\sum_{i=1}^{a} n_{ij}}$$

Balanced factorial experiments

$$n_{ij} = n \quad \text{for} \quad i = 1, \ldots, a, \quad j = 1, \ldots, b$$

Example 8.2: Sugar Cane Yields (from Snedecor and Cochran)

Nitrogem Level

	150 lb/acre	210 lb/acre	270 lb/acre
Variety 1	$Y_{111} = 70.5$	$Y_{121} = 67.3$	$Y_{131} = 79.9$
	$Y_{112} = 67.5$	$Y_{122} = 75.9$	$Y_{132} = 72.8$
	$Y_{113} = 63.9$	$Y_{123} = 72.2$	$Y_{133} = 64.8$
	$Y_{114} = 64.2$	$Y_{124} = 60.5$	$Y_{134} = 86.3$
Variety 2	$Y_{211} = 58.6$	$Y_{221} = 64.3$	$Y_{231} = 64.4$
	$Y_{212} = 65.2$	$Y_{222} = 48.3$	$Y_{232} = 67.3$
	$Y_{213} = 70.2$	$Y_{223} = 74.0$	$Y_{233} = 78.0$
	$Y_{214} = 51.8$	$Y_{224} = 63.6$	$Y_{234} = 72.0$
Variety 3	$Y_{311} = 65.8$	$Y_{321} = 64.1$	$Y_{331} = 56.3$
	$Y_{312} = 68.3$	$Y_{322} = 64.8$	$Y_{332} = 54.7$
	$Y_{313} = 72.7$	$Y_{323} = 70.9$	$Y_{331} = 66.2$
	$Y_{314} = 67.6$	$Y_{324} = 58.3$	$Y_{334} = 54.4$

For a balanced experiment ($n_{ij} = n$), Type I, Type II, and Type III sums of squares are the same:

$$\begin{aligned} R(\boldsymbol{\alpha}|\mu) &= R(\boldsymbol{\alpha}|\mu, \boldsymbol{\beta}) = SS_{H_0} \\ &= n\, b \sum_{i=1}^{a} (\bar{Y}_{i..} - \bar{Y}_{...})^2 \end{aligned}$$

$$\begin{aligned} R(\boldsymbol{\beta}|\mu) &= R(\boldsymbol{\beta}|\mu, \boldsymbol{\alpha}) = SS_{H_{0,2}} \\ &= n\, a \sum_{j=1}^{b} (\bar{Y}_{.j.} - \bar{Y}_{...})^2 \end{aligned}$$

$$\begin{aligned} R(\boldsymbol{\gamma}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) &= SS_{H_{0,3}} \\ &= n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2 \end{aligned}$$

ANOVA

Sum of Squares	Associated null hypothesis
$R(\mu) = \mathbf{Y}^T P_{\mathbf{1}} \mathbf{Y}$	$H_0 : \mu + \frac{1}{a}\sum_{i=1}^{a}\alpha_i + \frac{1}{b}\sum_{j=1}^{b}\beta_j + \frac{1}{a\,b}\sum_{i=1}^{a}\sum_{j=1}^{b}\gamma_{ij} = 0$
$= a\,b\,n\,\bar{Y}_{\dots}^2$	$\left(H_0 : \frac{1}{a\,b}\sum_{i=1}^{a}\sum_{j=1}^{b}\mu_{ij} = 0\right)$
$R(\boldsymbol{\alpha}\|\mu) = R(\boldsymbol{\alpha}\|\mu, \boldsymbol{\beta})$	$H_0 : \alpha_i + \frac{1}{b}\sum_{j=1}^{b}(\beta_j + \gamma_{ij})$ are equal
$= n\,b\sum_{i=1}^{a}(\bar{Y}_{i..} - \bar{Y}_{\dots})^2$	$\left(H_0 : \frac{1}{b}\sum_{j=1}^{b}\mu_{ij} \text{ are equal}\right)$
$R(\boldsymbol{\beta}\|\mu) = R(\boldsymbol{\beta}\|\mu, \boldsymbol{\alpha})$	$H_0 : \beta_j + \frac{1}{a}\sum_{i=1}^{a}(\alpha_i + \gamma_{ij})$ are equal
$= n\,a\sum_{j=1}^{b}(\bar{Y}_{.j.} - \bar{Y}_{\dots})^2$	$\left(H_0 : \frac{1}{a}\sum_{i=1}^{a}\mu_{ij} \text{ are equal}\right)$

$$R(\boldsymbol{\gamma}|\mu, \boldsymbol{\alpha}, \boldsymbol{\beta}) = n \sum_{i=1}^{a} \sum_{j=1}^{b} (\bar{Y}_{ij.} - \bar{Y}_{i..} - \bar{Y}_{.j.} + \bar{Y}_{...})^2$$

$$H_0 : \gamma_{ij} - \gamma_{kj} - \gamma_{i\ell} + \gamma_{k\ell} = 0 \quad \text{for all} \quad (i,j) \quad \text{and} \quad (k,\ell)$$

$$\left(H_0 : \mu_{ij} - \mu_{kj} - \mu_{i\ell} + \mu_{k\ell} = 0 \quad \text{for all} \quad (i,j) \quad \text{and} \quad (k,\ell) \right)$$

Refer Slide7_r1.pdf and Slide7_ r2

Two factor experiments with empty cells:

Data from Littell, Freund, and Spector, 1991,

SAS System for Linear Models, 3rd edition, SAS Institute, Cary, N.C.

	Factor B		
Factor A	$j = 1$	$j = 2$	$j = 3$
$i = 1$	$Y_{111} = 5$ $Y_{112} = 6$	$Y_{121} = 2$ $Y_{122} = 3$ $Y_{123} = 5$ $Y_{124} = 6$ $Y_{125} = 7$	–
$i = 2$	$Y_{211} = 2$ $Y_{212} = 3$	$Y_{221} = 8$ $Y_{222} = 8$ $Y_{223} = 9$	$Y_{231} = 4$ $Y_{232} = 4$ $Y_{233} = 6$ $Y_{234} = 6$ $Y_{235} = 7$

Sample sizes:

Factor A	Factor B $j=1$	$j=2$	$j=3$
$i=1$	$n_{11}=2$	$n_{12}=5$	–
$i=2$	$n_{21}=2$	$n_{22}=3$	$n_{23}=5$

Effects model:

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}, \quad \text{for} \quad (i,j) \neq (1,3) \quad \text{and} \quad k = 1, \ldots, n_{ij}$$

$$\mu_{ij} = E(\bar{Y}_{ij.}) = \mu + \alpha_i + \beta_j + \gamma_{ij}$$

is estimable for all $(i,j) \neq (1,3)$.

Functions of parameters that are <u>not</u> estimable include:

$$\begin{aligned}
\mu_{13} &= \mu + \alpha_1 + \beta_3 + \gamma_{13} \\
\bar{\mu}_{..} &= \frac{1}{6}\sum_{i=1}^{2}\sum_{j=1}^{3}\mu_{ij} = \mu + \frac{1}{2}(\alpha_1+\alpha_2) + \frac{1}{3}(\beta_1+\beta_2+\beta_3) \\
&\quad + \frac{1}{6}(\gamma_{11}+\gamma_{12}+\gamma_{13}+\gamma_{21}+\gamma_{22}+\gamma_{23}).
\end{aligned}$$

$$\bar{\mu}_{1.} = \frac{1}{3}\sum_{j=1}^{3}\mu_{1j}, \qquad \bar{\mu}_{.3} = \frac{1}{2}(\mu_{13}+\mu_{23})$$

Two factor classifications with empty cells:

- No single *best* or *correct* analysis.
- Analysis of variance
  - Test for interaction is useful
  - Use SSE to estimate the error variance σ^2.
  - Tests for *main effects* may not be meaningful, especially in the presence of interaction.
- Compute F-tests and sums of squares for meaningful contrasts.
- Compare estimated means for different combinations of factor levels.
- It may be most convenient to consider the various combinations of factor levels as levels of a single *combined* factor.
  - one-way ANOVA
  - contrasts
  - compare means

Refer Slide7_ r3