Example 1). X_i's $\overset{iid}{\sim} U(0,\theta)$, $T(X) = X_{(n)}$.

Then $P(X_{(n)} \le t) = P(X_1 \le t, \cdots, X_n \le t) \overset{iid}{=} P(X_1 \le t)^n$

$$= \left(\frac{t}{\theta}\right)^n.$$

$$P(T=t) = n t^{n-1}/\theta^n, \quad 0<t<\theta$$

$$\frac{\prod_{i=1}^n f(X_i|\theta)}{g(T=t|\theta)} = \frac{\prod_{i=1}^n \frac{1}{\theta} I(0<X_i<\theta)}{n t^{n-1}/\theta^n I(0<t<\theta)} = \frac{1}{n t^{n-1}}$$

$T(X) = X_{(n)}$ is a sufficient statistics for θ.

Example 2).

$$\frac{\prod_{i=1}^n \frac{1}{\pi(X_i-\theta)^2}}{n! \prod_{i=1}^n \frac{1}{\pi(X_{(i)}-\theta)^2}} = \frac{1}{n!}.$$

joint pdf of $X_{(1)}, \cdots, X_{(n)}$. $\to$ $n! \prod_{i=1}^n \frac{1}{\pi(X_{(i)}-\theta)^2}$

$T(X) = (X_{(1)}, \cdots, X_{(n)})$ is a sufficient statistics for θ.

<Slide #6>. - Factorization Theorem

From Example 2) above, we have

$$\prod_{i=1}^n f(x_i|\theta) = \prod_{i=1}^n f(X_{(i)}|\theta) = \underbrace{\prod_{i=1}^n \frac{1}{\pi(X_{(i)}-\theta)^2}}_{g(t(X)|\theta)} \cdot \underbrace{\prod_{i=1}^n I(-\infty<X_i<\infty)}_{=1.\ h(X)}$$

By Factorization theorem, $T(X) = (X_{(1)}, \cdots, X_{(n)})$ is a S.S. for θ.

<Slide #8>

$$\prod_{i=1}^n f(x_i|\theta) = \frac{1}{(2\theta)^n} I(-\theta<X_1<\theta)\cdots I(-\theta<X_n<\theta) = \frac{1}{(2\theta)^n} I(-\theta<X_{(1)}) I(X_{(n)}<\theta)$$

$\therefore$ $(X_{(1)}, X_{(n)})$ is a S.S. for θ.

Another S.S : $\prod_{i=1}^n f(x_i|\theta) = \frac{1}{(2\theta)^n} \prod_{i=1}^n I(|X_i|<\theta) = \frac{1}{(2\theta)^n} I(\max_{1\le i\le n}|X_i|<\theta)$

$\therefore$ $\max_{1\le i\le n}|X_i|$ is a S.S. for θ.

X_i's $\overset{iid}{\sim} U(\theta, \theta+1)$.

$$\prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} I(\theta < X_i < \theta+1) = I(\theta < X_{(1)})\, I(X_{(n)} < \theta+1).$$

$$\frac{\prod_{i=1}^{n} f(x_i|\theta)}{\prod_{i=1}^{n} f(y_i|\theta)} = \frac{I(\theta < X_{(1)})I(X_{(n)} < \theta+1)}{I(\theta < Y_{(1)})\, I(Y_{(n)} < \theta+1)} = C \text{ iff } X_{(1)} = Y_{(1)} \text{ and } X_{(n)} = Y_{(n)}.$$

Thus $(X_{(1)}, X_{(n)})$ is a minimal sufficient statistic for θ.

* $(X_{(n)} - X_{(1)},\ X_{(n)} + X_{(1)})$ is a one-to-one transformation of $(X_{(1)}, X_{(n)})$. Thus, it is also a minimal s.s. for θ.

* In Slide #13, we will show $X_{(n)} - X_{(1)}$ is an ancillary statistic. Thus, ancillary statistic and the minimal s.s. are not unrelated even if minimal s.s. removes all information in the sample except for θ and ancillary stat. is not dependent on θ.

* If a statistic is complete and minimal s.s. then it is independent of "every" ancillary statistic. [Basu's theorem].

※ Sufficient, but not minimal sufficient.

$X_i \overset{iid}{\sim} U(-\theta, \theta)$. By factorization theorem, $T(X) = (X_{(1)}, X_{(n)})$ is s.s.

Now, consider $T(y) = \max_{1 \le i \le n} |Y_i| \overset{def}{\equiv} \begin{cases} |X_{(1)}| & \text{if } |X_{(1)}| > X_{(n)} \text{ - case①} \\ X_{(n)} & \text{if } |X_{(1)}| \le X_{(n)} \text{ - case②} \end{cases}$

$$\text{Then } \frac{\prod_{i=1}^{n} f(x_i|\theta)}{\prod_{i=1}^{n} f(y_i|\theta)} = \frac{I(-\theta < X_{(1)})\, I(X_{(n)} < \theta)}{I(\max_{1\le i\le n} |y_i| < \theta)} = \begin{cases} \dfrac{I(|X_{(1)}| < \theta)}{I(\max_i |y_i| < \theta)} = C & \text{case①} \\[2ex] \dfrac{I(X_{(n)} < \theta)}{I(\max_i |y_i| < \theta)} = C & \text{case②} \end{cases}$$

T(X) is not min. s.s. $\Leftarrow$ $T(X) \neq T(Y)$ whereas the ratio is free from θ

X_i's $\overset{iid}{\sim} U(\theta, \theta+1)$. Then $(X_{(n)} - X_{(1)}, X_{(n)} + X_{(1)})$ is a minimal suff. stat. for θ. [See page 2 of hand-written, Chap 6]

Let $R = X_{(n)} - X_{(1)}$. Then $R \sim Beta(n-1, 2)$

[See page 10 of hand-written, Chap5]

Then the distribution of R is free from θ, thus R is an ancillary statistic.

But, it is a part of minimal sufficient statistic for θ.

$$S_X^2 = \sum_{i=1}^{n} (X_i - \bar{X}_n)^2/(n-1) \quad \text{where } \bar{X}_n = \sum_{i=1}^{n} X_i/n.$$

$$= \sum_{i=1}^{n} \left((X_i - \theta) - (\bar{X}_n - \theta)\right)^2/(n-1), \text{ set } Z_i = X_i - \theta.$$

$$= \sum_{i=1}^{n} (Z_i - \bar{Z}_n)^2/(n-1).$$

As $f_X(x-\theta|\theta) = f_Z(z)$ is independent from θ, (location family)

S_X^2 is an ancillary statistic.

X_i's $\overset{iid}{\sim}$ Poisson(λ), $0<\lambda<\infty$, Then $T(X)=\sum X_i$ is a complete stat.

Let $Y=\sum_{i=1}^{n} X_i$, then $Y \sim$ Poisson$(n\lambda)$.

$$E[g(y)] = \sum_{y=0}^{\infty} g(y) \cdot \frac{e^{-n\lambda}(n\lambda)^y}{y!} = 0$$

iff $\sum_{y=0}^{\infty} g(y) \frac{(n\lambda)^y}{y!} = 0$ iff $\sum_{y=0}^{\infty} g(y)\lambda^y = 0$ since $n^y>0$, $y!>0$.

iff $g(0)+g(1)\lambda+g(2)\lambda^2+\cdots=0$.

iff $g(0)=0, g(1)=0, g(2)=0, \cdots$ for all $0<\lambda<\infty$.

Thus $\sum X_i$ is a complete statistic.

X_i's $\overset{iid}{\sim} U(\theta,\theta+1)$. $T(X)=(X_{(1)},X_{(n)})$. $F_X(x)=x-\theta$; cdf.

$$f_{X_{(1)},X_{(n)}}(y_1,y_n) = \frac{n!}{(1-1)!(n-1-1)!(n-n)!}\{(y_n-\theta)-(y_1-\theta)\}^{n-2}$$

$$= n(n-1)(y_n-y_1)^{n-2}, \quad \theta<y_1<y_n<\theta+1.$$

$$E[g(T)] = \int_\theta^{\theta+1}\int_\theta^{y_n} g(t)\, n(n-1)(y_n-y_1)^{n-2}\, dy_1\, dy_n = 0 \text{ iff } \int_\theta^{\theta+1}\int_\theta^{y_n} g(t)\, dy_1\, dy_n = 0$$

iff $g(t)=0$ $\forall\, 0<\theta<\infty$.

Thus $T(X)$ is complete. It is also minimal sufficient.

Thus, by the Basu's theorem, $T(X)$ is independent of any ancillary stat.

* As $R=X_{(n)}-X_{(1)}$ is an ancillary stat, $T(X)=(X_{(1)},X_{(n)}) \perp\!\!\!\perp R$.