X_i's iid Uniform$(0, \theta)$. ① $I_1(X) = [aX_{(n)}, bX_{(n)}]$, $1 \le a < b$. $X_{(n)} = \max_{1 \le i \le n} X_i$

$$P[\theta \in I_1(X)] = P[aX_{(n)} \le \theta \le bX_{(n)}]$$
$$= P[X_{(n)} \le \frac{\theta}{a} \text{ and } \frac{\theta}{b} \le X_{(n)}]$$
$$= P[\theta/b \le X_{(n)} \le \theta/a]$$
$$= P[X_{(n)} \le \theta/a] - P[X_{(n)} \le \theta/b]$$
$$= \left(\frac{1}{a}\right)^n - \left(\frac{1}{b}\right)^n$$

(= coverage probability & "confidence coefficient").

$$\boxed{* \; P[X_{(n)} \le t] = P(X_1 \le t)^n = (t/\theta)^n.}$$

② $I_2(X) = [X_{(n)} + c, \infty)$.

$$P[\theta \in I_2(X)] = P[X_{(n)} + c \le \theta < \infty] = P[X_{(n)} \le \theta - c] = \left[1 - \frac{c}{\theta}\right]^n.$$

(= coverage probability).

$$\lim_{\theta \to c} \left(1 - \frac{c}{\theta}\right)^n = 0. \text{ (= confidence coefficient)}$$

③ $I_3(X) = [X_{(n)} + a, X_{(n)} + b]$.

$$P[\theta \in I_3(X)] = P[X_{(n)} + a \le \theta \le X_{(n)} + b] = P[\theta - b \le X_{(n)} \le \theta - a]$$
$$= \left(1 - \frac{a}{\theta}\right)^n - \left(1 - \frac{b}{\theta}\right)^n \quad \text{(= coverage probability)}.$$

$$\lim_{n \to \infty} \left\{ (1 - a/\theta)^n - (1 - b/\theta)^n \right\} = 0. \text{ (= confidence coefficient)}.$$

<Slide #5> Inverting Test. ②

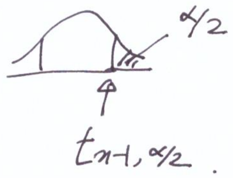
EX1). X_i's $\overset{iid}{\sim} N(\mu, \sigma^2)$, $i=1,\cdots,n$, μ & σ^2 unknown.

Find ① $1-\alpha$ 2-sided C.I. for μ ② $1-\alpha$ 1-sided $\underset{\text{lower}}{\text{C.I}}$ for μ. $I(X) = [L(X), \infty]$

sol). ① As σ^2 is unknown, we use its estimate S^2. Consider $H_0: \mu = \mu_0$ vs $H_a: \mu \neq \mu_0$.

$$1-\alpha = P\left(\left| \frac{\bar{X}-\mu_0}{S/\sqrt{n}} \right| \le t_{n-1,\alpha/2} \right) = P\left(\bar{X} - \frac{S}{\sqrt{n}} t_{n-1,\alpha/2} \le \mu_0 \le \bar{X} + \frac{S}{\sqrt{n}} t_{n-1,\alpha/2} \right)$$

$$\therefore I_1(X) = \left[\bar{X} - \frac{S}{\sqrt{n}} t_{n-1,\alpha/2},\ \bar{X} + \frac{S}{\sqrt{n}} t_{n-1,\alpha/2} \right].$$



② Consider $H_0: \mu = \mu_0$ vs $H_a: \mu > \mu_0$.

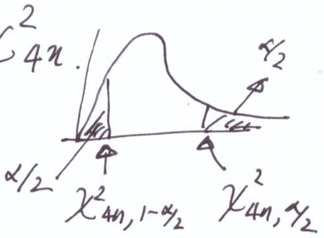
$$1-\alpha = P\left(\frac{\bar{X}-\mu_0}{S/\sqrt{n}} \le t_{n-1,\alpha} \right) = P\left(\mu_0 \ge \bar{X} - \frac{S}{\sqrt{n}} t_{n-1,\alpha} \right)$$

$$\therefore I_2(X) = \left[\bar{X} - \frac{S}{\sqrt{n}} t_{n-1,\alpha},\ \infty \right).$$

EX2). X_i's $\overset{iid}{\sim}$ Gamma$(2, \frac{1}{\theta})$, $i=1,\cdots,n$. Find $1-\alpha$ confidence set of θ.

sol) ① Exact. Consider $H_0: \theta = \theta_0$ vs $H_a: \theta \neq \theta_0$.

$$\Sigma X_i \sim \text{Gamma}(2n, \tfrac{1}{\theta}) \Rightarrow 2\theta \Sigma X_i \sim \text{Gamma}(4n/2, 2) \sim \chi^2_{4n}.$$



$$1-\alpha \overset{H_0}{=} P\left(\chi^2_{4n,1-\alpha/2} \le 2\theta_0 \Sigma X_i \le \chi^2_{4n,\alpha/2} \right)$$

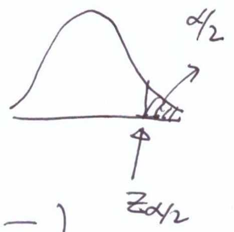
$$= P\left(\frac{1}{2\Sigma X_i} \chi^2_{4n,1-\alpha/2} \le \theta_0 \le \frac{1}{2\Sigma X_i} \chi^2_{4n,\alpha/2} \right)$$

$$\therefore C(X) = \left[\frac{1}{2\Sigma X_i} \chi^2_{4n,1-\alpha/2},\ \frac{1}{2\Sigma X_i} \chi^2_{4n,\alpha/2} \right].$$

② Approximate. $E[\Sigma X_i] = 2n/\theta$, $Var[\Sigma X_i] = 2n/\theta^2$.

Then
$$\frac{\Sigma X_i - 2n/\theta_0}{\sqrt{2n/\theta_0^2}} = \frac{\bar{X} - 2/\theta_0}{\sqrt{2/(n\theta_0^2)}} \xrightarrow{D} N(0,1). \text{ under } H_0.$$

Thus
$$1-\alpha \overset{H_0}{=} P\left(-z_{\alpha/2} \le \frac{\theta_0 \bar{X} - 2}{\sqrt{2/n}} \le z_{\alpha/2} \right)$$



$$= P\left(\frac{2}{\bar{X}} - \sqrt{\frac{2}{n}}\, z_{\alpha/2}/\bar{X} \le \theta_0 \le \frac{2}{\bar{X}} + \sqrt{\frac{2}{n}}\, z_{\alpha/2}/\bar{X} \right).$$

$$\therefore \text{Approx. } C(X) = \left[\frac{2}{\bar{X}} - \frac{1}{\bar{X}}\sqrt{\frac{2}{n}} \cdot z_{\alpha/2},\ \frac{2}{\bar{X}} + \frac{1}{\bar{X}}\sqrt{\frac{2}{n}}\, z_{\alpha/2} \right].$$

EX1). X_i's $\overset{iid}{\sim} Exp(\lambda)$, $i=1,\cdots,n$, Then $\frac{2}{\lambda}\sum X_i \sim \chi^2_{2n}$.

Thus $1-\alpha = P\left[\chi^2_{2n,1-\alpha/2} \le \frac{2}{\lambda}\sum X_i \le \chi^2_{2n,\alpha/2}\right]$

$= P\left[\frac{2\sum X_i}{\chi^2_{2n,\alpha/2}} \le \lambda \le \frac{2\sum X_i}{\chi^2_{2n,1-\alpha/2}}\right]$

$\therefore$ $(1-\alpha)$ C.I. for λ is $\left[\frac{2\sum X_i}{\chi^2_{2n,\alpha/2}}, \frac{2\sum X_i}{\chi^2_{2n,1-\alpha/2}}\right]$.

EX 9.2.15). $X_i \overset{iid}{\sim} Poisson(\lambda)$, $i=1,\cdots,n$. Find C.I. for λ.

Let $Y=\sum_{i=1}^n X_i$. $Y \sim Poisson(n\lambda)$. Let y_0 is the observed value of Y.

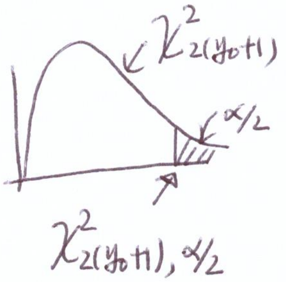
To find C.I. for λ, we set ① $\sum_{y=0}^{y_0} P(Y=y) = \frac{\alpha}{2}$ & ② $\sum_{y=y_0}^{\infty} P(Y=y) = \frac{\alpha}{2}$.

①: $P(Y \le y_0) = 1 - P(Y \ge y_0+1)$,

$= 1 - P(U \le 2n\lambda)$

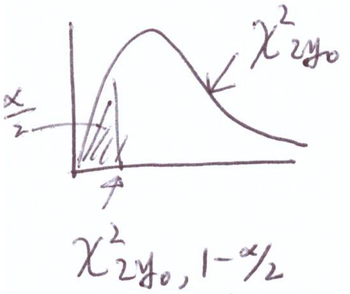
$= P(U \ge 2n\lambda) = \alpha/2$.

* Let $U \sim Gamma(y_0+1, 2) \sim \chi^2_{2(y_0+1)}$ & $Y \sim Poisson\left(\frac{2n\lambda}{2}\right)$



$\Leftrightarrow \chi^2_{2(y_0+1),\alpha/2} = 2n\lambda \Leftrightarrow \lambda = \frac{1}{2n}\chi^2_{2(y_0+1),\alpha/2}$

②: $P(Y \ge y_0) = P(V \le 2n\lambda) = \frac{\alpha}{2}$ Let $V \sim Gamma(y_0, 2) \sim \chi^2_{2y_0}$ & $Y \sim Poisson(2n\lambda/2)$



$\Leftrightarrow \chi^2_{2y_0,1-\alpha/2} = 2n\lambda$

$\Leftrightarrow \lambda = \frac{1}{2n}\chi^2_{2y_0,1-\alpha/2}$

$\therefore$ $(1-\alpha)$ C.I. for λ is $\left[\frac{1}{2n}\chi^2_{2y_0,1-\alpha/2}, \frac{1}{2n}\chi^2_{2(y_0+1),\alpha/2}\right]$.

<slide #9>

X_i's $\overset{iid}{\sim} f(x;\mu) = e^{-(x-\mu)}$, $x > \mu$, $i = 1, \cdots, n$. $T(X) = X_{(1)}$.

Find $(1-\alpha)$ C.I. for μ.

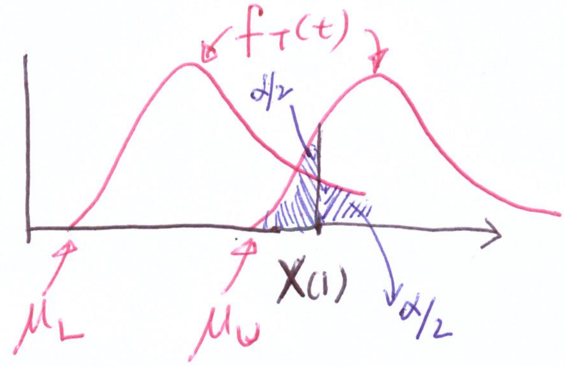
pdf of $X_{(1)}$; $f_T(t) = \frac{n!}{(1-1)!(n-1)!}[F_X(t)]^{1-1}[1-F_X(t)]^{n-1} f_X(t)$

$= n e^{-n(t-\mu)}$, $t > \mu$

* $F_X(t) = \int_\mu^t e^{-(x-\mu)} dx = 1 - e^{-(t-\mu)}$

Then cdf of $X_{(1)}$; $F_T(t) = 1 - e^{-n(t-\mu)}$, $t > \mu$.

Let μ_L and μ_U be the lower and upper bound of μ, respectively.



*Note $X_{(1)} \geq \mu_L$ & $X_{(1)} > \mu_U$

For μ_L, $F_T(t) = 1 - e^{-n(t-\mu_L)} = 1 - \frac{\alpha}{2}$ ... ⓐ

For μ_U, $F_T(t) = 1 - e^{-n(t-\mu_U)} = \alpha/2$ ... ⓑ

From ⓑ, $\mu_U = t + \frac{1}{n}\log(1-\alpha/2)$ & from ⓐ, $\mu_L = t + \frac{1}{n}\log \alpha/2$.

$\therefore$ $(1-\alpha)$ C.I. for μ is $[X_{(1)} + \frac{1}{n}\log(\alpha/2), X_{(1)} + \frac{1}{n}\log(1-\alpha/2)]$

<Slide #10> Bayesian Interval.

X_i's $\overset{iid}{\sim} N(\theta, \sigma^2)$, $\theta \sim N(\mu, \tau^2)$. $i = 1, \cdots, n$

First, find the posterior distribution $[\theta | \underset{\sim}{x}] \sim N\left(\underbrace{\frac{\sigma^2\mu + n\tau^2\bar{x}}{\sigma^2 + n\tau^2}}_{E[\theta|\underset{\sim}{x}]}, \underbrace{\frac{\sigma^2\tau^2}{\sigma^2 + n\tau^2}}_{Var[\theta|\underset{\sim}{x}]}\right)$

try this!

Then, $\frac{\theta - E[\theta|\underset{\sim}{x}]}{\sqrt{Var[\theta|\underset{\sim}{x}]}} \sim N(0,1)$

$\Rightarrow 1-\alpha = P\left[\left|\frac{\theta - E[\theta|\underset{\sim}{x}]}{\sqrt{Var[\theta|\underset{\sim}{x}]}}\right| < Z_{\alpha/2}\right] = P\left[\overbrace{E(\theta|\underset{\sim}{x}) - \sqrt{Var(\theta|\underset{\sim}{x})} Z_{\alpha/2}}^{ⓐ} \leq \theta \leq \underbrace{E[\theta|\underset{\sim}{x}] + \sqrt{Var(\theta|\underset{\sim}{x})} Z_{\alpha/2}}_{ⓑ}\right]$

$\Rightarrow$ [ⓐ, ⓑ] is a $(1-\alpha)$ credible set.