< Chapter 8 > slide #5.

X_i's $\overset{iid}{\sim} N(\theta, \sigma^2)$, σ^2 known, $i=1,\cdots,n$. , $H_0: \theta=\theta_0$ vs $H_a: \theta \neq \theta_0$

$$\lambda(x) = \frac{f(\underset{\sim}{x} \mid \hat{\theta}_0)}{f(\underset{\sim}{x} \mid \hat{\theta})} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\Sigma(X_i-\theta_0)^2}{2\sigma^2}\right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\Sigma(X_i-\bar{X})^2}{2\sigma^2}\right\}}$$

$$= \exp\left\{\frac{\Sigma X_i}{\sigma^2}(\theta_0-\bar{X}) - \frac{n}{2\sigma^2}(\theta_0^2-\bar{X}^2)\right\}$$

$$\overset{\Sigma X_i = n\bar{X}}{=} \exp\left\{\frac{n}{2\sigma^2}(2\bar{X}\theta_0 - 2\bar{X}^2 - \theta_0^2 + \bar{X}^2)\right\}$$

$$= \exp\left\{-\frac{n}{2\sigma^2}(\bar{X}^2 - 2\bar{x}\theta_0 + \theta_0^2)\right\} = \exp\left\{-\frac{n}{2\sigma^2}(\bar{X}-\theta_0)^2\right\}$$

Then, $$-2\log\lambda(x) = -2\left\{-\frac{n}{2\sigma^2}(\bar{X}-\theta_0)^2\right\}$$

$$= \left\{\frac{\bar{X}-\theta_0}{\sigma/\sqrt{n}}\right\}^2 \overset{\text{under } H_0}{\sim} \chi^2_{(1)}.$$

* When X_i's are normal, $-2\log\lambda(x)$ follow's $\chi^2_{(1)}$. * We reject H_0 if $\lambda(x) \le c$.

* Under certain conditions (conditions required for CRLB), $-2\log\lambda(x)$ approximately follows $\chi^2_{(1)}$.

< slide #6 >.

$$\lambda(x) = \frac{\prod_{i=1}^{n} e^{-(X_i-\theta_0)}}{\prod_{i=1}^{n} e^{-(X_i-\hat{\theta})}} = \frac{e^{-\Sigma X_i + n\theta_0}}{e^{-\Sigma X_i + nX_{(1)}}} = \exp\{n(\theta_0 - X_{(1)})\}$$

An LRT reject H_0 if $\lambda(x) \le c \Leftrightarrow \log\lambda(x) \le \log c$

$$\Leftrightarrow n(\theta_0 - X_{(1)}) \le \log c$$

$$\Leftrightarrow X_{(1)} \ge \theta_0 - \frac{\log c}{n}$$

The reject region depends on the sample only through the sufficient Statistic.

<Slide #12>.

X_i's $\overset{iid}{\sim}$ exponential(θ), $i=1,\cdots,n$. Consider a test fct $\phi(x) = \begin{cases} 1 & \bar{X} < c \\ 0 & o/w. \end{cases}$

$$\beta_\phi(\theta) = P_\theta[\phi(x)=1]$$
$$= P_\theta[\bar{X} < 1] = P_\theta[\Sigma X_i < n], \quad \Sigma X_i \sim Gamma(n, \theta)$$
$$= \int_0^n \frac{1}{\Gamma(n)\theta^n} t^{n-1} e^{-t/\theta} dt.$$; Power function.

<Slide #17>.

X_i's $\overset{iid}{\sim}$ Gamma$(3, \theta)$, $i=1,\cdots,n$. $H_0: \theta=\theta_0$ vs $H_1: \theta=\theta_1 \ (>\theta_0)$

Find MP-test of size α. $(=0.05)$

We reject H_0 if $L = \dfrac{f(\underset{\sim}{x}|\theta_1)}{f(\underset{\sim}{x}|\theta_0)} > k$.

$$L = \prod_{i=1}^n \frac{1}{\Gamma(3)\theta_1^3} x_i^{3-1} e^{-x_i/\theta_1} \Big/ \prod_{i=1}^n \frac{1}{\Gamma(3)\theta_0^3} x_i^{3-1} e^{-x_i/\theta_0}$$
$$= \left(\frac{\theta_0}{\theta_1}\right)^3 \exp\left\{-\Sigma X_i\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\right\} = \left(\frac{\theta_0}{\theta_1}\right)^3 \exp\left\{n\bar{X}\frac{(\theta_1-\theta_0)}{\theta_0\theta_1}\right\}.$$

$\Rightarrow$ Note L is an increasing function of $\bar{X}$ since $\theta_1 - \theta_0 > 0$.

Thus, rejeting H_0 if $L > k$ is the same as rejecting H_0 if $\bar{X} > C$.

Or, if $\Sigma X_i > C_1 = nC$. As $T = \Sigma X_i \overset{H_0}{\sim}$ Gamma$(3n, \theta_0)$,

C_1 can be found numerically s.t. $\displaystyle\int_{C_1}^\infty \frac{1}{\Gamma(3n)\theta_0^{3n}} t^{3n-1} e^{-t/\theta_0} dt = \alpha \ (=0.05)$;

$*$ When n is large, $(E[T] = 3n\theta_0$, $Var(T) = 3n\theta_0^2)$ Find c s.t.

$P\left(\dfrac{\Sigma X_i - 3n\theta_0}{\sqrt{3n\theta_0^2}} > c\right) = \alpha$. So, MP test of (approximate) size α $(=0.05)$ is

reject H_0 if $\Sigma X_i > 3n\theta_0 + c\sqrt{3n\theta_0^2}$ where $c = 1.645$ for $\alpha = 0.05$.

< Slide # 18 >

X_i's $\overset{iid}{\sim} N(\mu, \sigma^2)$, σ^2 known. $i=1,\cdots,n$.

$H_0 : \mu = \mu_0$ vs $H_1 : \mu = \mu_1 \ (>\mu_0)$.

Find MP-test of size α.

$$L = \exp\left\{-\frac{\Sigma(X_i-\mu_1)^2}{2\sigma^2}\right\} \Big/ \exp\left\{-\frac{\Sigma(X_i-\mu_0)^2}{2\sigma^2}\right\}$$

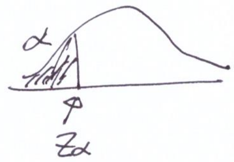
$$= \exp\left\{\frac{n\bar{X}}{2\sigma^2}(\mu_1-\mu_0) - \frac{n}{2\sigma^2}(\mu_1^2-\mu_0^2)\right\}.$$

As L is an increasing function of $\bar{X}$, reject H_0 if $L > k$ is the same as rejecting H_0 if $\bar{X} > C$. where c is determined as

$$P(\bar{X} > c) \overset{H_0}{=} \alpha \qquad \text{or} \qquad P\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > c_1\right) = \alpha.$$

$\therefore$ MP test : Reject H_0 if $\bar{X} > \mu_0 + C_1\frac{\sigma}{\sqrt{n}}$, where $C_1 = Z_{1-\alpha}$.

(*) $H_a : \mu = \mu_1 \ (< \mu_0)$.



Then, L is a decreasing function of $\bar{X}$.

$\Rightarrow$ MP test is "Reject H_0 if $\bar{X} < \mu_0 + C_1\frac{\sigma}{\sqrt{n}}$, where $C_1 = Z_\alpha$.

(*) $H_a : \mu = \mu_1 \ (\neq \mu_0)$

We have UMP test for the two cases of $(\mu_1 > \mu_0)$, $(\mu_1 < \mu_0)$, and they are not identical. So, for the hypothesis $H_0 : \mu = \mu_0$ vs $H_a : \mu \neq \mu_0$, we do NOT have a UMP test.

X_i's iid $f(x|\lambda) = \lambda e^{-\lambda x}$, $x>0$, $\lambda \geq 0$. Find UMP test of size α.

This is a scale parameter

$$\frac{L(X|\lambda_1)}{L(X|\lambda_0)} = \frac{\prod_{i=1}^{n} \lambda_1 e^{-\lambda_1 x_i}}{\prod_{i=1}^{n} \lambda_0 e^{-\lambda_0 x_i}} = \left(\frac{\lambda_1}{\lambda_0}\right)^n \exp\{-\Sigma X_i \underbrace{(\lambda_1-\lambda_0)}_{+}\} \geq k \quad \text{— ⓐ}$$

(*) $H_0: \lambda \leq \lambda_0$
$H_1: \lambda > \lambda_0$

$$\Rightarrow \exp\{-\Sigma X_i(\lambda_1-\lambda_0)\} \geq \left(\frac{\lambda_0}{\lambda_1}\right)^n k$$

$$\Rightarrow -\Sigma X_i(\lambda_1-\lambda_0) \geq \log\left(\frac{\lambda_0}{\lambda_1}\right)^n k \equiv k_1$$

$$\Rightarrow -\Sigma X_i(\lambda_1-\lambda_0) \geq k_1$$

$$\Rightarrow \Sigma X_i \leq c$$

By Neyman - Pearson Theorem, rejecting H_0 if $\Sigma X_i \leq c$ is UMP test. [Note; small "ΣX_i" indicates small mean or large scale λ]

(*) Alternative solution.

From ⓐ, we see the family has MLR in $T(X) = -\Sigma X_i$.

(Note ⓐ is a decreasing function of ΣX_i, so an increasing function of $-\Sigma X_i$)

Then, by Karlin-Rubin Thm, a UMP test is

rejecting H_0 if $-\Sigma X_i \geq k^{<0}$ or, equivalently $\Sigma X_i \leq c^{>0}$

When $n=100$ is given, then choose c s.t.

$$\int_0^c \frac{1}{\Gamma(n)(1/\lambda)^n} x^{n-1} e^{-\lambda x} dx = 0.05.$$

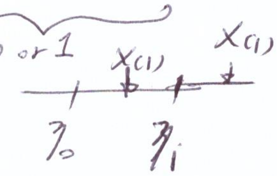
In R, c is found by
① > qgamma(0.05, 100, $1/\lambda_0$) [100 ↓ n; $1/\lambda_0$ ↓ rate or mean]
② > qgamma(0.05, 100, scale = λ_0)

Note that it makes $P(\Sigma X_i \leq c) = 0.05$ under H_0. ※ scale = 1/rate

EX) X_i's $\overset{iid}{\sim} f(x|\eta) = e^{-(x-\eta)}$, $x > \eta$, $\eta > 0$, $i = 1, \cdots, n$.

Find a UMP test of size α for $H_0 : \eta \le \eta_0$ vs $H_1 : \eta > \eta_0$.

$$\frac{L(x|\eta_1)}{L(x|\eta_0)} = \frac{e^{-\Sigma X_i + n\eta_1} I(X_{(1)} > \eta_1)}{e^{-\Sigma X_i + n\eta_0} I(X_{(1)} > \eta_0)} = \exp\{n\underbrace{(\eta_1 - \eta_0)}_{>0}\} \underbrace{\frac{I(X_{(1)} > \eta_1)}{I(X_{(1)} > \eta_0)}}_{0 \text{ or } 1}.$$

$\Rightarrow$ 

$\Rightarrow$ This is MLR in $T(x) = X_{(1)}$

By Karlin-Rubin Theorem, rejecting H_0 if $X_{(1)} > c$, where

$P(X_{(1)} > c) \overset{H_0}{=} \alpha$ is a UMP test of size α.

Now, find the value of c.

$$P(X_{(1)} > c) = P(X_1 > c)^n = \left[1 - \int_{\eta_0}^{c} e^{-(t-\eta_0)} dt\right]^n$$

This is the probability of rejecting H_0 when H_0 is true.

$$= \left[1 + (e^{c-\eta_0} - 1)\right]^n = e^{n(c-\eta_0)} = \alpha.$$

$\therefore$ $c = \frac{1}{n} \log \alpha + \eta_0$.