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Differential Equations
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Final Project: The Solow-Swan Model of Economic Growth

Introduction

How and why the economy changes over time and experiences shifts that lead to times of a prosperous economy and, arguably more important, times of a poor economy has constantly plagued economists. Knowing how inputs affect the output and productivity of the economy is central to understanding how an economy is growing and moving towards being efficient. The main way in which this is analyzed is through the infamous economic model known as the Solow-Swan model. The model attempts to utilize a system of equations and basic economic assumptions to identify how changes to the inputs of an economy will change the level of output. This model also is often utilized to measure a specific country's standard of living, indicating how, on average, its citizens are faring economically and in terms of monetary happiness. Commonly, the model employs labor, meaning the amount of labor those working in the economy are performing, and capital, meaning the amount of accumulated goods and services that exist in the economy, as its inputs. It also investigates the dynamics of consumption and investment and how investment is re-utilized as an input in the production function. The model exists in the long-run meaning that the variables are not fixed and each quantity can change. What is most interesting about this model is that it brings a linkage between mathematics and economics because the model is based on a series of differential equations, but these only exist based on economic principles. The big picture of the model is that it can help to predict how an economy will grow after large shocks and help policymakers understand how the inputs of the economy may need to change to better the output of production.

Mathematics

Background Information

To begin with, it is pertinent to understand the variables used in deriving this equation, thus each are defined below:

Q(t) = the amount of output

C(t) = amount of consumption

I(t) = amount of gross investment into capital

L(t) = amount of labor

K(t) = amount of capital

Each of these equations includes an unknown variables and are all based on time (t). To then build the relationship between these variables, we must utilize a couple of economic assumptions. These can be summarized as follows: 1) The level of output is determined by the production function which is based on capital and labor. 2) Once the level of output is determined it is then split into some proportion of consumption and investment. Investment will be returned into production capital for the next cycle. 3) The portion of output that is dedicated to investment must be between 0 and 1 and is a constant that will be referred to as the savings rate, or s(t). 4) The derivative of capital is directly proportional to investment, but inversely proportional to depreciation of capital at a rate greater than 0, labeled by μ. Depreciation means that capital will lose some of its value overtime and thus is not usable in production. 5) Labor grows exponentially starting at L_0 at a constant rate given by ƞ. These assumptions can be shown in the figure below.

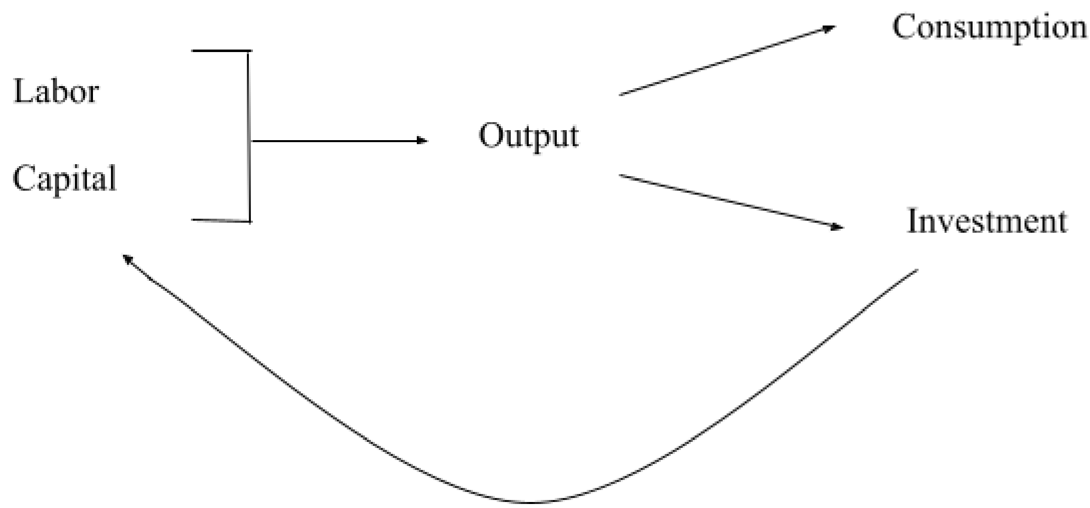


Figure 1: The flow of variables into and out of output, the arrows represent how the variables interact with each other.

Creation of the Model

With these assumptions, we can generate the five baseline equations that work in tandem to create the Solow-Swan model. The first equation represents the production function that generates the level of output. The second equation is how the output is split into consumption and investment. The third equation is how investment relates to the savings rate and output. The fourth equation represents how the derivative of capital changes with respect to investment and capital. Finally, the fifth equation represents how the derivative of labor changes at an exponential rate with an initial point of L_0. These equations were generated based on the assumptions given and utilizing the given flow of the variables based on the figure above.

(1) $Q(t) = F(K, L)$

(2) $Q(t) = k_1 C + k_2 I$

(3) $I(t) = s(t) \cdot Q(t)$

$$(4)\ K'(t) = I(t) - \mu K(t)$$

$$(5)\ L'(t) = L_0 e^{\eta t}$$

Refining of Production Function and Inclusion of Assumptions

With these equations expressing the Solow-Swan model, we must perform a steady-state analysis in search of a steady-state, which can be in two different forms. The first is either a stationary trajectory where the unknown variables are made to be constant at one specific time or a balanced growth trajectory where there is the same growth rate for all the variables along a balanced growth path. In this particular scenario, we will be assuming the same rate of growth for all the unknown variables in search of the balanced growth trajectory.

In order to achieve this we must first examine the assumption that the production function given by equation (1) is a linearly homogeneous function. This means that the highest power of the equation is one and that the right-hand side of the equation is equal to zero. With this information under this assumption, we can rearrange the original production function equation.

Given $F(K, L)$ is linearly homogeneous and $k = \frac{K}{L}$ is the capital-labor ratio

$$Z \cdot Q(t) = F(ZK, ZL)$$

$$\frac{1}{L}F = F(\frac{1}{L}K, \frac{1}{L}L)$$

$$\frac{F}{L} = F(\frac{K}{L}, 1)$$

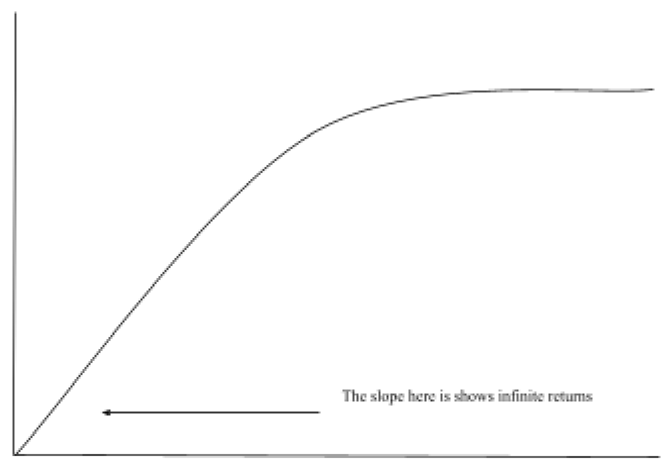
$$F = LF(\frac{K}{L}, 1)$$

If we say $k = \frac{K}{L}$ and $f(k) = F(k, 1)$ then $F(K, L) = Lf(k)$

Which under the given assumption has the following important properties:

$f(0) = 0$ $\qquad f'(k) > 0$ $\qquad \frac{lim}{k \to 0} f'(k) = \infty$ $\qquad \frac{lim}{k \to \infty} f'(k) = 0$

The first assumption stems from the fact that there can not be an output without any inputs. The second stems from the fact that this equation can never be negative because it reaches equilibrium, so the output will flatten out approaching zero but never having a negative slope. The third assumption is because the input increases from zero to something that is an infinite return so the slope approaches infinity. The final assumption is because if we increase the inputs to infinity, there are diminishing returns which approach zero, so the slope approaches zero. These limits can be shown below in the graphs of the production functions. Figure 2 depicts the third assumption, while figure 3 depicts the fourth assumption.



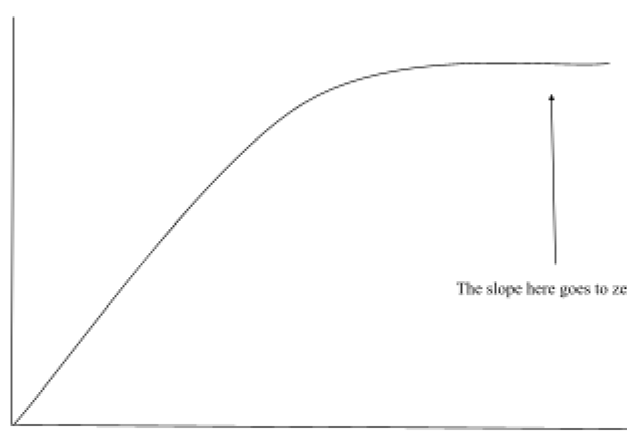


Figure 2: Simple depiction of the assumption that the limit will be infinity as k goes to 0.

Figure 3: A simple depiction of the above assumption that the limit will go to zero as k goes to infinity.

To continue on in finding the steady-state we must say that each of the unknown variables has the same rate of growth, which in this case will be the growth rate for labor, which is denoted by η. Thus we rewrite the beginning equations as follows:

$$Q(t) = \overline{Q}e^{\eta t} \qquad K(t) = \overline{K}e^{\eta t} \qquad I(t) = \overline{I}e^{\eta t} \qquad Q(t) = \overline{Q}e^{\eta t} \qquad C(t) = \overline{C}e^{\eta t}$$

The level variables, which are shown as the letters with a bar atop ($\overline{Q}$, $\overline{K}$, $\overline{I}$, $\overline{C}$) are simply the new unknown constant, changed to distinguish these from the previous unknown variables and to indicate that these values are based on one point in time.

Nonlinear Differential Equation

With these five basic equations that make-up the Solow-swan model and using the assumption that the production function is linear homogeneous and contains the four properties listed, we can generate one nonlinear differential equation.

$$F(K, L) = Lf(k) \xrightarrow{(1)} Q(t) = Lf(k) \xrightarrow{(3)} I(t) = s(t) \cdot Lf(k) \xrightarrow{(4)} K'(t) = s(t)Lf(k) - \mu K(t)$$

The arrows indicate which equation the former equation was substituted into beginning with the production function (F(K,L), for example this was substituted into equation (1), which was then substituted into equation (3). The final result is an equation for the derivative of capital over the equation for labor. It is important to note here, again, that $k(t) = \frac{K(t)}{L(t)}$ (8) is the unknown capital-labor ratio.

$$\frac{K'(t)}{L(t)} = s(t)f(k) - \mu\frac{K(t)}{L(t)} \qquad = s(t)f(k) - \mu k(t) \text{ (7)}$$

Then, we derive both sides of the capital-labor ratio equation (8).

$$\frac{d}{dt}k(t) = \frac{d}{dt}\frac{K(t)}{L(t)}$$

$$k'(t) = \frac{L(t)\frac{d}{dt}K(t) - K(t)\frac{d}{dt}L(t)}{(L(t))^2} = \frac{L(t)K'(t) - K(t)L'(t)}{L(t)^2}$$

$$= \frac{L(t)K'(t)}{L(t)^2} - \frac{K(t)L(t)'}{L(t)^2} = \frac{K(t)'}{L(t)} - \frac{K(t)}{L(t)} \cdot \frac{L'(t)}{L(t)}$$

$$= k'(t) = \frac{K'(t)}{L(t)} - \frac{K(t)}{L(t)} \cdot \frac{L'(t)}{L(t)} \quad (9)$$

Next, equation (5) is substituted into equation (9) to generate equation (10).

$$k'(t) = \frac{K'(t)}{L(t)} - \frac{K(t)}{L(t)} \cdot \frac{L_0 e^{\eta t}}{L(t)} \quad (10)$$

Equations (7) and (10) are then combined to make one equation, which excludes the first term of equation (10) and generates an equation with only one unknown variable.

$$k'(t) = s(t)f(k) - \mu k(t) - \frac{K(t)}{L(t)} \cdot \frac{L_0 e^{\eta t}}{L(t)} \quad (11)$$

This equation is considered the fundamental equation of the Solow-Swan model. It is a non-linear and autonomous equation. In words, this equation states that the derivative of the capital-labor ratio is equal to the savings rate multiplied by the production function of the capital-labor ratio minus depreciation multiplied by the capital labor ratio minus the capital labor ratio multiplied by the growth rate of labor divided by the growth rate of labor.

Steady-State Analysis

The next step is to decide whether a balanced growth trajectory exists for this system of equations. To do this, we say k(t) is a constant and thus Q(t), C(t), I(t), and K(t) grow at the same rate. We then substitute k(t) into equations (3), (4), and (5) to obtain equations (12).

$K(t) = kL(t)$, $\quad I(t) = (\mu + \eta)K(t)$, $\quad Q(t) = (\mu + \eta)K(t)/S(t)$, $\quad C(t) = Q(t) - I(t)$

All of these functions include the same growth rate as the labor function, which is represented by η, so we assume that k(t) is constant and, thus, its derivative will be equal to zero.

$$k'(t) = s(t)f(k) - \mu k(t) - \frac{K(t)}{L(t)} \cdot \frac{L_0 e^{\eta t}}{L(t)}$$

$$0 = s(t)f(k) - \mu k(t) - \frac{K(t)}{L(t)} \cdot \frac{L_0 e^{\eta t}}{L(t)}$$

$$s(t)f(k) = \mu k(t) + \frac{K(t)}{L(t)} \cdot \frac{L_0 e^{\eta t}}{L(t)}$$

From the assumptions made about the production function above, we can say the following:

$$f(0) = 0 \qquad f'(k) > 0 \qquad \frac{lim}{k\to\infty} f'(k) = 0 \qquad \frac{lim}{k\to 0} f'(k) = \infty$$

And thus equation (13) has a unique solution $\hat{k} = \hat{k}(s)$ for any given value of s>0. This solution $\widehat{(k)}$ is the intersection of f(k)s and (μ+η)k.

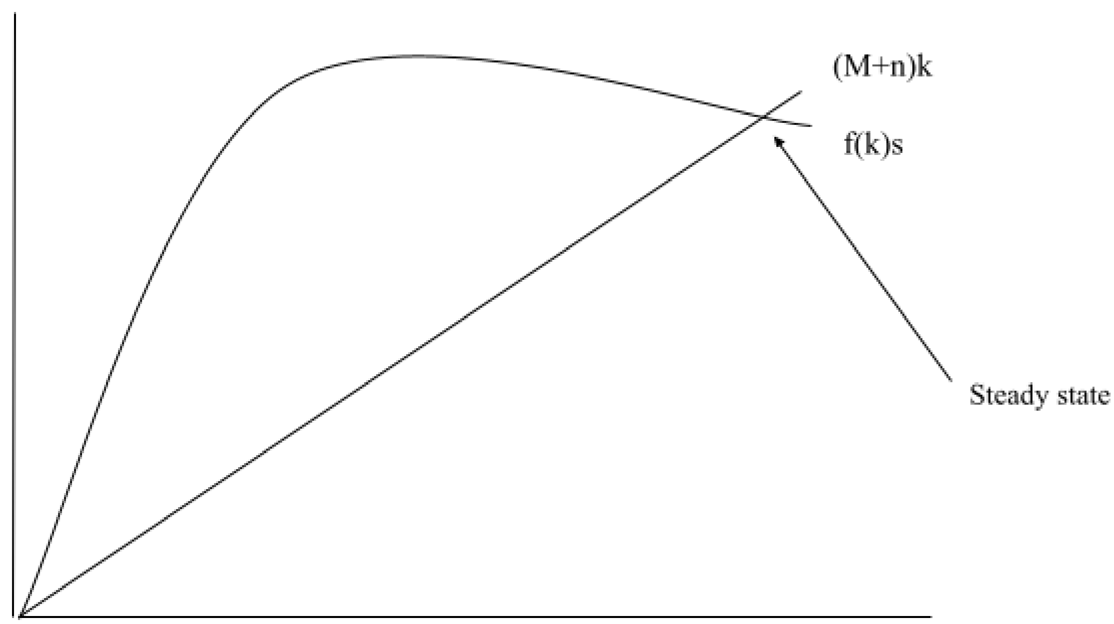


Figure 4: Depicts the above equation and the steady state being the intersection point of the straight-line (μ+η)k and f(k)s.

Simple Numerical Example

As I mentioned below in the "what to do when you do not know what to do" it was extremely beneficial to my understanding of the interpretation of the Solow-Swan model to look at the model in general terms. Thus I include a simple numerical example to emphasize how some of the variables interact with each other. I start with a production function that is based off of only capital and not labor and is given by:

$Y(t) = 5\sqrt{K}$ where Y(t) is amount of output and K is capital

I also utilize that the imaginary country in which we are examining has an initial amount of capital that is given by 10,000, the proportion of output that goes to investment is 0.25, and the depreciation rate is 1%.

To solve whether the overall output will be decreasing, increasing, or at steady state, we must first determine the numerical amount of investment that is being made. From equation (3) we see that the amount of investment is equal to the savings rate multiplied by the production function based on the initial level of output, and we can plug in the specifics given by this problem to solve for the level of investment, which is equal to 125. We also solve for the level of output based on the level of capital, which will be 500.

Investment: $I(t) = s(t) \cdot Q(t)$

$$I(t) = 0.25 \cdot 5\sqrt{K}$$

$$I(t) = 0.25 \cdot 5\sqrt{10,000} \quad = 125$$

Output: $Y(t) = 5\sqrt{10,000} = 500$

In relating back to the original five equations, we can also say that from equation (2) we know that the level of consumption for this example will be 375 because the level of output is split between consumption and investment and we have solved for investment.

Next, we can solve for the amount of depreciation based on the initial amount of output. This is done by using the economic assumptions, which say that the level of depreciation is equal to the rate of depreciation multiplied by the level of output.

Depreciation: $D(t) = 0.01(10,000) = 100$

With this information we can see that investment is greater than depreciation, thus the capital will grow within this time period (by 25) and thus the level of output is increasing. Allow this is a simple example and does not utilize all of the equations in the model from the module, it is pertinent in understanding how some of the variables interact with each other and is a less theoretical example of how economists actually use this Solow-Swan model to look at the overall economy.

Future Direction

With any assignment or new skill we learn in school it is natural to wonder whether we will ever utilize it later in our lives. This project mainly made me question how relevant this model is and how many people (outside of mathematicians or economists) actually see this model in their lives. With this question in mind I did a little research into the Solow-Swan model to see how common it really is. What I found was that it is not necessarily an everyday occurrence, but its real-world applications including its ability to be an accepted measure of a country's well-being is quite significant. Furthermore, I found an interesting connection between the Solow-Swan model and the economic growth of Japan and Germany after World War II. Wartime and post-war economies are a very common subject for economists as they present a large strain on the everyday person, a huge change to the budget, and most relevant, a large amount of fluctuations to the inputs of economic output. Simply, what my research found was that the economies of these two countries experienced record high levels of growth after World

War II and the growth originally stemmed from a substantial loss of capital stock during the war. This low level of capital caused the capital to be more productive and generate more output and that the capital was producing a large amount of investment, so that in the next time frame the capital stock was even larger, further boosting the output. In addition, the low capital led to less depreciation and less capital needed to be invested into combatting depreciation and could be used as investment into new capital and output. This growth was also strongly representative of the Solow-Swan model and even highlighted the important principle of the model that catching-up growth (when a country's economy is way behind other countries in terms of development) will show high growth rates in the beginning and will reach an equilibrium (or steady-state) as output reaches a limit. [5]

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What to do when you do not know what to do

In going about researching and writing up this project, there were many times when I was stuck on how to move forward. One particular time when I found myself unsure of what to do and how to continue was when I first began this project and was attempting to follow the section of the module that required me to write the five beginning equations. I felt as though I was not given enough information and was worried my equations were wrong, which would cause issues for the entirety of the project. To work through this problem I began researching the Solow-Swan model in more general terms in order to get a better sense of how the model was actually constructed and how each of the variables related to one another. This allowed me to take a step back from the specific module I was working through and come back to it with a new perspective. I also found it extremely helpful to talk through specific sections with my classmates or with other mathematics students. I found that even if they were not sure on exactly how to help with the module I was working on, it was extremely beneficial to have some to talk through my thoughts with and to bounce ideas off of. In addition, when I was still quite worried about my initial equations being incorrect and was not sure what to do, I decided to charge forward with generating the differential equations and found that I was able to check my work and see if the equations made sense in terms of what calculations the module was asking me to perform in later steps. Overall, for me, the best things to do “when you do not know what to do” was to talk through my thoughts with other students and to not freak out when I was not positive about what I was doing.