Quantum Mechanics I

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Semester I 2024/25

1 Schrodinger Wave Equation

Consider a spinless particle with time-dependent Hamiltonian:

$$H = \frac{\vec{P}^2}{2m} + V(\vec{X}, t) \tag{1}$$

However the discussion below can apply to an Hamiltonian.
In Schrodinger picture the state vector $|\psi\rangle$ evolves as

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = (\frac{\vec{P}^2}{2} + V(\vec{X}, t)|\psi(t)\rangle \tag{2}$$

and in the cooridinate representation we get **The Schrodinger Wave Equation**

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi(\vec{x}, t) + V(\vec{x}, t)\psi(\vec{x}, t) \tag{3}$$

Where $\nabla = \frac{\partial^2}{\partial \vec{x}^2}$ is the Laplacian.
If the potenital is time-independent the energy is therefore consrved. Writing the wave function as

$$\psi(\vec{x}, t) = e^{-\frac{1}{\hbar}Et}\psi(\vec{x}) \tag{4}$$

We then the time-independent Schrodinger equation for stationary states

$$-\frac{\hbar^2}{2m}\nabla^2 \psi(\vec{x}) + V(\vec{x}, t)\psi(\vec{x}) = E\psi(\vec{x}) \tag{5}$$

2 Probability Current

Probability Density of finding a particle near x at time t is

$$\rho(\vec{x}, t) \equiv |\psi(\vec{x}, t)|^2 \tag{6}$$

and must satisfy $\int d^3\vec{x}\rho(\vec{x}, t) = 1$
Intgerating The probabililty Density wrt time and using the wave function gives

$$\frac{\partial p}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t}\psi = ... = \frac{i\hbar}{2m}(\psi^* \vec{\nabla}\psi - \vec{\nabla}\psi^* \psi) \tag{7}$$

which is the **Probability Current**

$$\vec{J}(\vec{x}, t) \equiv \frac{i\hbar}{2m}(\psi^* \vec{\nabla}\psi - \vec{\nabla}\psi^* \psi) \tag{8}$$

This leads to the **Continuity Equation**

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{J} \tag{9}$$

which implies the conservation of probability is conservedd
In the coordinate representation $\vec{P} = -i\hbar\vec{\nabla}$

The wave function in polar coordinates is

$$\psi(\vec{x}, t) = \sqrt{\rho(\vec{x}, t)}\, \exp(\frac{i}{\hbar} S(\vec{x}, t)) \tag{10}$$

where S is real and ρ¿0. we know the meaning of ρ But not of S. To understand the physical meaning of S we can express the probability current J in terms of ρ and S.

$$\vec{J}(\vec{x}, t) = ... = \frac{\rho(\vec{x}, t)}{m} \vec{\nabla} S(\vec{x}, t) \tag{11}$$

If ψ is a pane was $\implies$

$$\psi(\vec{x}, t) \sim exp(\frac{i}{\hbar} \vec{p} \cdot \vec{x} - \frac{i}{\hbar} Et) \tag{12}$$

then

$$\vec{\nabla} S(\vec{x}, t) = \vec{p} \tag{13}$$

which is normal to the plane $\vec{p} \cdot \vec{x} = const$

3 Classical Limit

The polar form can be helpful in discussing the classical limit of wave mechanics. Rewriting the Schrodinger equation in terms of ρ and S lead to this relation for ρ

$$\frac{\partial\ \log\sqrt{\rho}}{\partial t} = -\frac{1}{2m} (\nabla^2(S) + 2\vec{\nabla}(log\sqrt{\rho}) \cdot \vec{\nabla}(S)) \tag{14}$$

which is simply the continuity equation from before, for the probability density and current. the second equation for just the phase S is

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \vec{\nabla}(S) \cdot \vec{\nabla}(S) + V - \frac{\hbar^2}{2m} \frac{\hbar^2}{2m} \frac{\nabla^2(\sqrt{\rho})}{\sqrt{\rho}} = 0 \tag{15}$$