ASRV

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1 Convergent sequences in $\mathbb{R}$

Definition 1

A sequence of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$. Represented as $x_1, x_2, x_3, \ldots$ or $\{x_n\}_{n=1}^{\infty}$, or more simply $\{x_n\}$.

Definition 2

A sequence $\{x_n\}$ of real numbers converges to a limit $L \in \mathbb{R}$, if for all $\varepsilon > 0$, there exists an integer N such that $|x_n - L| < \varepsilon$ for all $n \geq N$.

- We call a sequence *convergent* if it converges to a finite limit L, and denote this as $\{x_n\} \to L$ or $\lim_{n\to\infty} x_n = L$.

Definition 3

- A sequence is *bounded from above* if $\exists B \in \mathbb{R}$ such that $B \geq x_n$ for all $n \geq 1$.
- A sequence is *bounded from below* if $\exists A \in \mathbb{R}$ such that $A \leq x_n$ for all $n \geq 1$.

We say a sequence is bounded if it is bounded both above and below.

Lemma 1:

Every convergent sequence of real numbers is bounded.

Definition 4

A sequence is called increasing if $x_n \leq x_{n+1}$ for all $n \geq 1$.

- A sequence is called strictly increasing if $x_n < x_{n+1}$ for all $n \geq 1$.

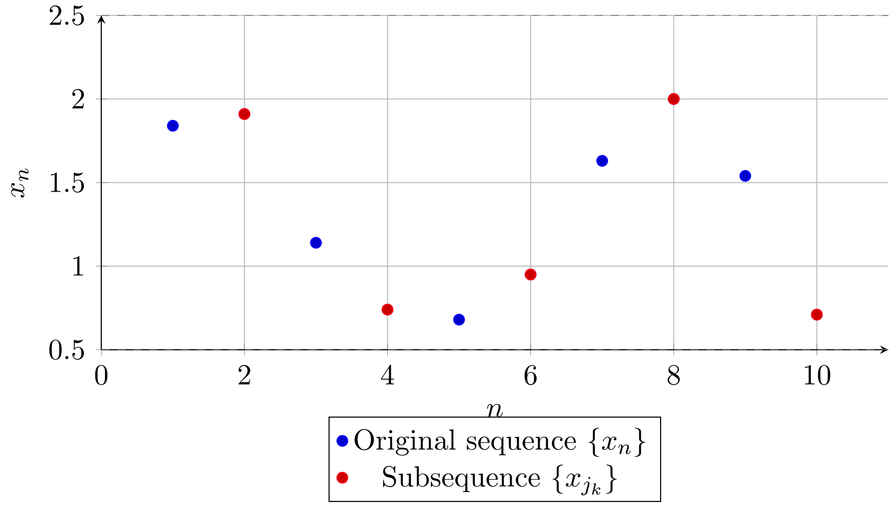
Decreasing and strictly decreasing are defined similarly.
A sequence is called monotonic if it is increasing or decreasing.

Theorem 1

- Any increasing sequence that is bounded above converges.
- Any decreasing sequence that is bounded below converges.

Theorem 2 (Bolzano–Weierstrass theorem in $\mathbb{R}$)

Let $\{x_n\}$ be a bounded sequence. Then there exists a subsequence $\{x_{j_k}\}_{k=1}^{\infty}$, with $j_k < j_{k+1}$, that converges to a limit.



2 Convergent sequences in $\mathbb{R}^n$

We will know begin a spooky wander into n dimensions. This can be quite abstract, but follows the same path as going from the visualisable 1 dimension to 2, and again from 2 dimensions to three, and so on. It can normally suffice to imagine in three dimensions, to begin to understand what is happening in more than three dimensions.

Definition 2.1

for a point $\vec{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ the euclidean norm can be defined as follows:

$$||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$$

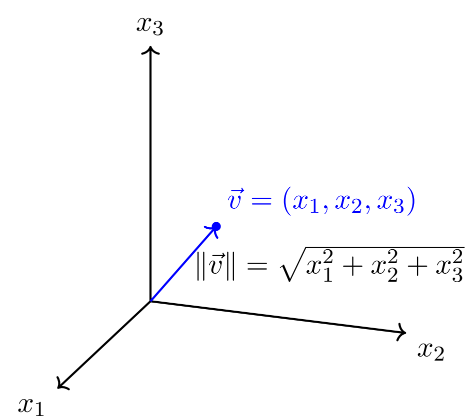


Figure 1: Illustration of the 3D Euclidean norm $\|\vec{v}\|$ of a vector from the origin to the point (x_1, yx_2, x_3).

and we can for points x and y, both of elements of $\mathbb{R}^n$, define their inner product will be defined as follows:

$$< x, y >= x_1y_1 + x_2y_2 + ... + x_ny_n$$