Supervised Regression with Multiple Variables

0. Import library

In [1]:

```
# Import libraries

# math library
import numpy as np

# visualization library
%matplotlib inline
from IPython.display import set_matplotlib_formats
set_matplotlib_formats('png2x','pdf')
import matplotlib.pyplot as plt

# machine learning library
from sklearn.linear_model import LinearRegression

# 3d visualization
from mpl_toolkits.mplot3d import axes3d

# computational time
import time
```

1. Load dataset

Load a set of data points $\{(x_i, y_i, z_i)\}_{i=1}^n$ where x_i and y_i are considered as an input and z_i is considered as an output for i-th data point.

In [2]:

```
# import data with numpy
data_clean = np.genfromtxt('data_clean.txt', delimiter=',')
data_noisy = np.genfromtxt('data_noisy.txt', delimiter=',')

data_clean = data_clean[:,0:3]  # do not change it
data_noisy = data_noisy[:,0:3]  # do not change it

# number of training data
n_clean = data_clean.shape
n_noisy = data_noisy.shape
print(n_clean,n_noisy)
```

```
(3600, 3) (3600, 3)
```

2. Explore the dataset distribution

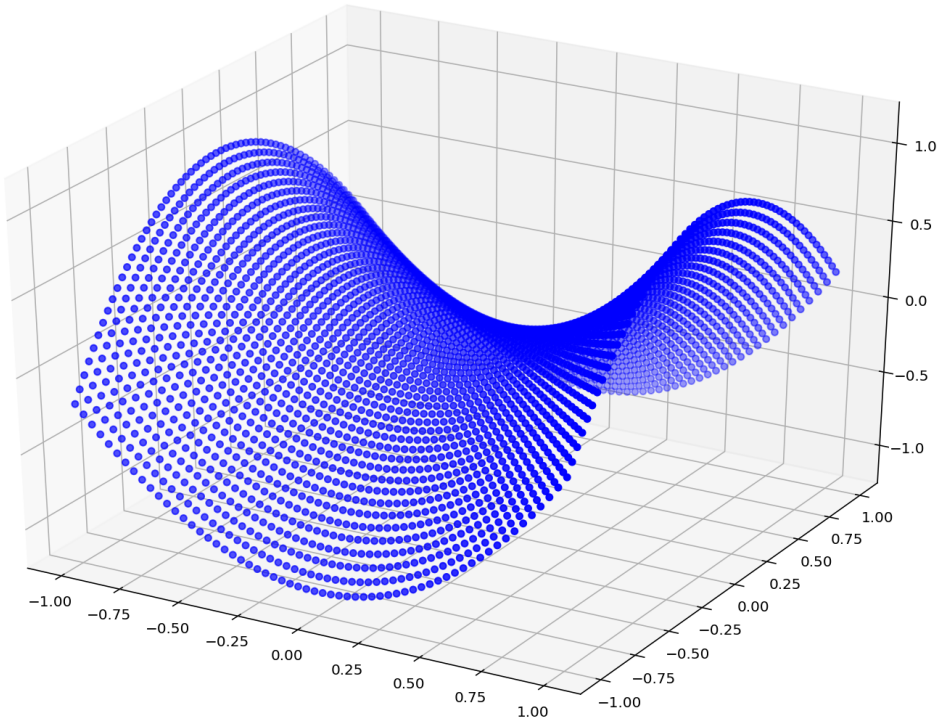
Plot the training data points in 3D cartesian coordinate system. (You may use matplotlib function `scatter3D()`.)

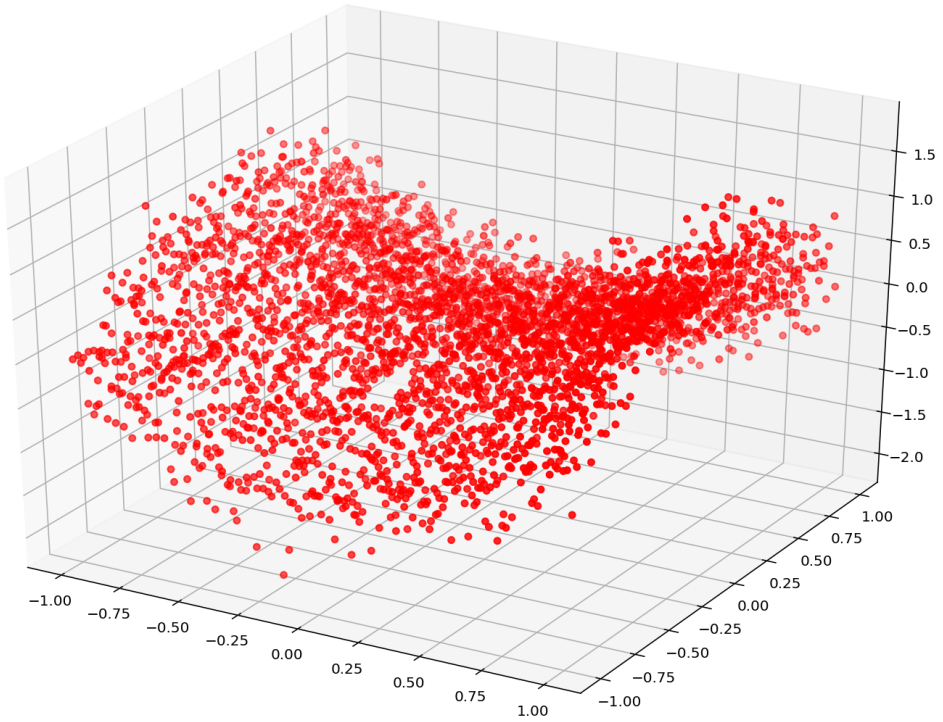
In [3]:

```
x_clean = data_clean[:,0]
y_clean = data_clean[:,1]
z_clean = data_clean[:,2]
fig1=plt.figure(figsize=(14,10))
ax1=plt.axes(projection="3d")
ax1.scatter3D(x_clean,y_clean,z_clean,c="blue")


x_train = data_noisy[:,0]
y_train = data_noisy[:,1]
z_train = data_noisy[:,2]
fig2=plt.figure(figsize=(14,10))
ax2=plt.axes(projection="3d")
ax2.scatter3D(x_train,y_train,z_train,c="red")
```

Out[3]:

```
<mpl_toolkits.mplot3d.art3d.Path3DCollection at 0x21f21506e50>
```





3. Define the prediction function

$$f_w(x,y) = w_0 f_0(x,y) + w_1 f_1(x,y) + w_2 f_2(x,y) + w_3 f_3(x,y) + w_4 f_4(x,y) + w_5 f_5(x,y) + w_6 f_6(\\ + w_9 f_9(x,y)$$

where feature function f is defined as follows:

$$f_0(x,y) = \\ f_1(x,y) = \\ f_2(x,y) = \\ f_3(x,y) = \\ f_4(x,y) = \\ f_5(x,y) = \\ f_6(x,y) = \\ f_7(x,y) = \\ f_8(x,y) = \\ f_9(x,y) =$$

Vectorized implementation:

$$f_w(x) = Xw$$

with

$$X = \begin{bmatrix} f_0(x_1,y_1) & f_1(x_1,y_1) & f_2(x_1,y_1) & f_3(x_1,y_1) & f_4(x_1,y_1) & f_5(x_1,y_1) & f_6(x_1,y_1) & f_7 \\ f_0(x_2,y_2) & f_1(x_2,y_2) & f_2(x_2,y_2) & f_3(x_2,y_2) & f_4(x_2,y_2) & f_5(x_2,y_2) & f_6(x_2,y_2) & f_7 \\ \vdots & & & & & & & \\ f_0(x_n,y_n) & f_1(x_n,y_n) & f_2(x_n,y_n) & f_3(x_n,y_n) & f_4(x_n,y_n) & f_5(x_n,y_n) & f_6(x_n,y_n) & f_7 \end{bmatrix}$$

$$w = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{bmatrix}$$

Implement the vectorized version of the predictive function.

In [34]:

```
x_train.shape
```

Out[34]:

```
(3600,)
```

In [6]:

```
# construct data matrix


n = data_noisy.shape[0]
X = np.ones([n,10])
#bias term
#X[:,1]
# X[:,2] = x_train
# X[:,3] = y_train
# X[:,4] = x_train**2
# X[:,5] = x_train*y_train
# X[:,6] = y_train**2
# X[:,7] = x_train**3
# X[:,8] = y_train**3
# X[:,9] = x_train**2*y_train
X[:,2] = x_train
X[:,3] = y_train
X[:,4] = x_train**2
X[:,5] = y_train**2
X[:,6] = y_train**3
X[:,7] = x_train**3
X[:,8] = x_train**4
X[:,9] = y_train**4

print(X.shape)
# print(X)
# parameters vector
w = np.array([1,1,1,1,1,1,1,1,1,1])[:,None] # [:,None] adds a singleton dimension

# predictive function definition
def f_pred(X,w):

    f = np.matmul(X,w)

    return f

# Test predicitive function
z_pred = f_pred(X,w)
print(z_pred)
print(z_pred.shape)
print(n)
```

```
(3600, 10)
[[ 2.        ]
 [ 1.93668129]
 [ 1.88187349]
 ...
 [ 9.3664312 ]
 [ 9.67229863]
 [10.        ]]
(3600, 1)
3600
```

4. Define the regression loss

$$L(w) = \frac{1}{n}\sum_{i=1}^{n}\left(f_w(x_i, y_i) - z_i\right)^2$$

Vectorized implementation:

$$L(w) = \frac{1}{n}(Xw - z)^T(Xw - z)$$

with

$$Xw = \begin{bmatrix} f_w(x_1, y_1) \\ f_w(x_2, y_2) \\ \vdots \\ f_w(x_n, y_n) \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

Implement the vectorized version of the regression loss function.

In [7]:

```
# loss function definition
def loss_mse(z_pred,z):

    loss = np.matmul((z_pred-z).T,z_pred-z)/z.size

    return loss.item()


# Test loss function
z = z_train.reshape(-1,1)# label
z_pred = f_pred(X,w)# prediction
print(z_pred.T.shape)
loss = loss_mse(z_pred,z)
print(loss)
print(z.size)
```

```
(1, 3600)
12.381734305974865
3600
```

5. Define the gradient of the regression loss

- Vectorized implementation: Given the loss

$$L(w) = \frac{1}{n}(Xw - z)^T(Xw - z)$$

The gradient is given by

$$\frac{\partial}{\partial w}L(w) = \frac{2}{n}X^T(Xw - z)$$

Implement the vectorized version of the gradient of the regression loss function.

In [8]:

```
# gradient function definition
def grad_loss(z_pred,z,X):

    grad = np.matmul(X.T,z_pred-z)/z.size*2

    return grad


# Test grad function
z_pred = f_pred(X,w)
grad = grad_loss(z_pred,z,X)
print(grad)
```

```
[[6.24051498]
 [6.24051498]
 [1.06259554]
 [1.07123348]
 [2.32119669]
 [2.69335635]
 [0.7086127 ]
 [0.70459986]
 [1.49744836]
 [1.82770809]]
```

6. Implement the gradient descent algorithm

Vectorized implementation:

$$w^{k+1} = w^k - \tau\frac{2}{n}X^T(Xw^k - z)$$

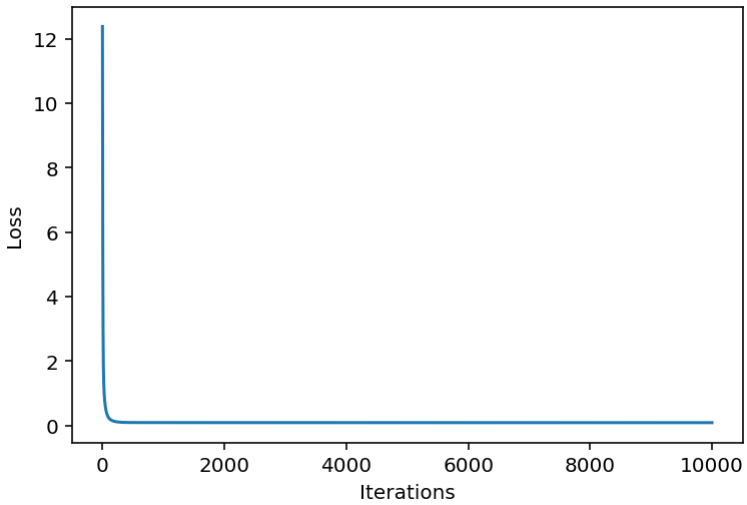
Implement the vectorized version of the gradient descent function.

Plot the loss values $L(w^k)$ with respect to iteration k the number of iterations.

In [40]:

```
# gradient descent function definition
def grad_desc(X, z , w_init=np.array([0,0])[:,None] ,tau=0.01, max_iter=500):

    L_iters = np.zeros([max_iter]) # record the loss values
    w_iters = np.zeros([max_iter,10]) # record the loss values
    w = w_init # initialization

    for i in range(max_iter): # loop over the iterations

        z_pred = f_pred(X,w)# linear predicition function
        grad_f = grad_loss(z_pred,z,X)# gradient of the loss
        w = w-tau*grad_f# update rule of gradient descent
        L_iters[i] =  loss_mse(z_pred,z)# save the current loss value
        w_iters[i,:] = w.reshape(10)# save the current w value

    return w, L_iters, w_iters


# run gradient descent algorithm
start = time.time()
w_init =  np.array([1,1,1,1,1,1,1,1,1,1])[:,None]
tau = 0.02
max_iter = 10000

w, L_iters, w_iters = grad_desc(X,z,w_init,tau,max_iter)

print('Time=',time.time() - start) # plot the computational cost
print(L_iters[max_iter-1] ) # plot the last value of the loss
print(w) # plot the last value of the parameter w


# plot
plt.figure(2)
plt.plot(L_iters) # plot the loss curve
plt.xlabel('Iterations')
plt.ylabel('Loss')
plt.show()
```

```
Time= 0.7300176620483398
0.08848371416691528
[[-0.00170002]
 [-0.00170002]
 [ 0.02053483]
 [-0.00852468]
 [ 0.84019494]
 [-0.87015522]
 [ 0.11996304]
 [ 0.09331026]
 [ 0.16034914]
 [-0.11304247]]
```



7. Plot the prediction function

$$f_w(x,y) = w_0 f_0(x,y) + w_1 f_1(x,y) + w_2 f_2(x,y) + w_3 f_3(x,y) + w_4 f_4(x,y) + w_5 f_5(x,y) + w_6 f_6(\\ + w_9 f_9(x,y)$$

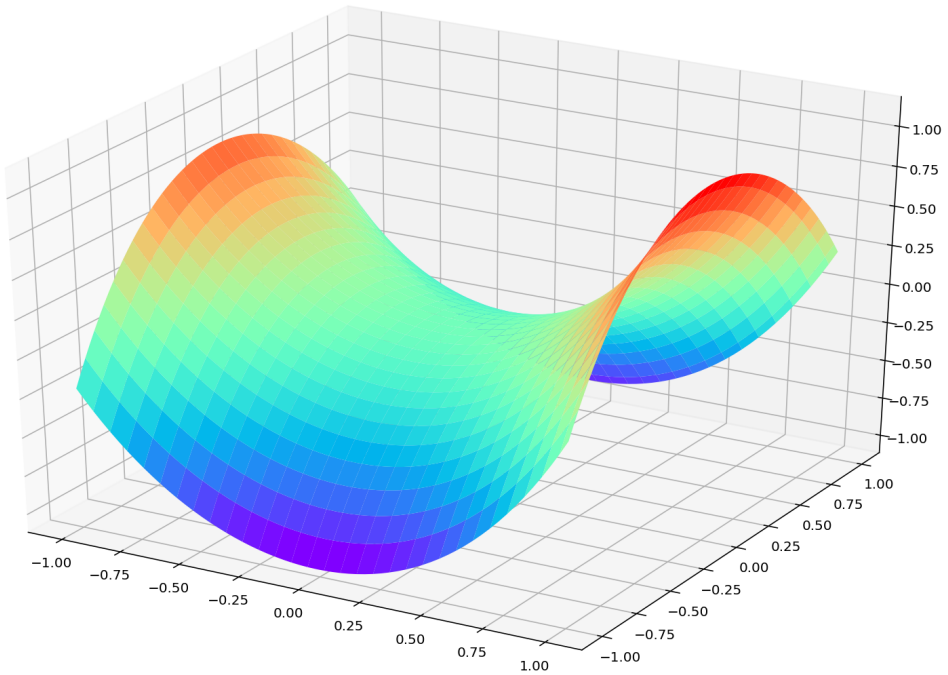
(You may use numpy function `meshgrid` and `plot_surface` for plot the linear prediction function.)

In [41]:

```
x_coordinate = np.linspace(-1, 1, 60)
y_coordinate = np.linspace(-1, 1, 60)
def data_to_X(x,y):
    n=x.size
    X = np.ones([n,10])
    #bias term
    #X[:,1]
    X[:,2] = x
    X[:,3] = y
    X[:,4] = x**2
    X[:,5] = y**2
    X[:,6] = x**3
    X[:,7] = y**3
    X[:,8] = x**4
    X[:,9] = y**4
    return X
x_pred, y_pred = np.meshgrid(x_coordinate, y_coordinate, indexing='xy')
x_flat=x_pred.reshape(1,-1)
y_flat=y_pred.reshape(1,-1)
X_flat=data_to_X(x_flat,y_flat)
z_pred=f_pred(X_flat,w)
# plot
fig = plt.figure(figsize=(15,10))
ax = fig.add_subplot(1, 1, 1, projection='3d')
ax.plot_surface(x_pred,y_pred,z_pred.reshape(60,60),cmap=plt.cm.rainbow)
```

Out[41]:

```
<mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x21f2d864220>
```



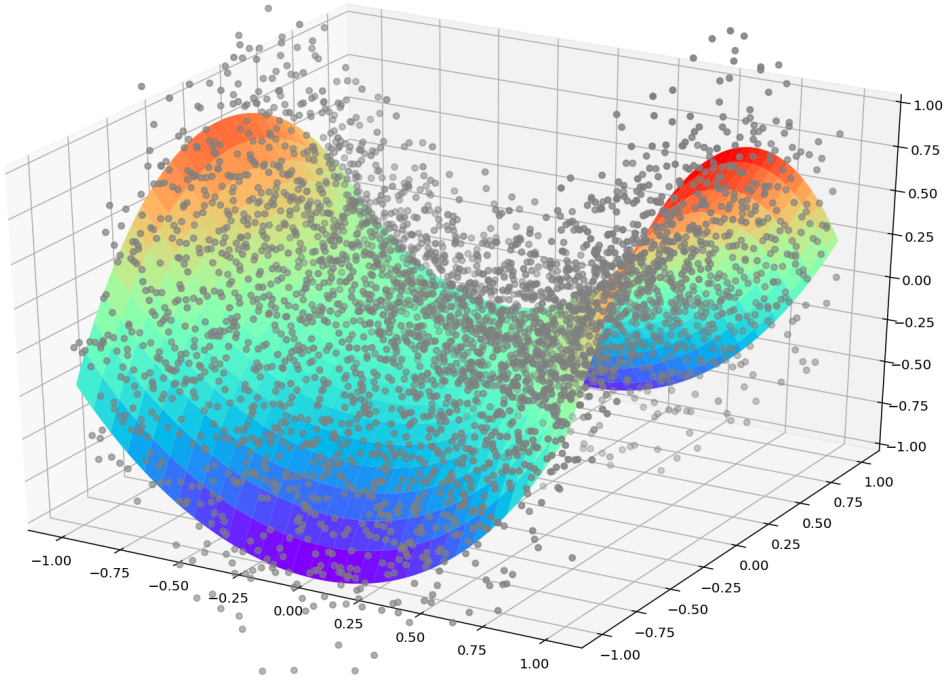
8. Plot the prediction function superimposed on the training data

In [42]:

```
x_coordinate = np.linspace(-1, 1, 60)
y_coordinate = np.linspace(-1, 1, 60)

x_pred, y_pred = np.meshgrid(x_coordinate, y_coordinate, indexing='xy')

# plot
fig = plt.figure(figsize=(15,10))
ax = fig.add_subplot(1, 1, 1, projection='3d')
ax.plot_surface(x_pred,y_pred,z_pred.reshape(60,60),cmap=plt.cm.rainbow)
ax.scatter3D(x_train,y_train,z_train,c="gray")
ax.set_zlim(-1,1)
```

Out[42]:

(-1.0, 1.0)



Output results

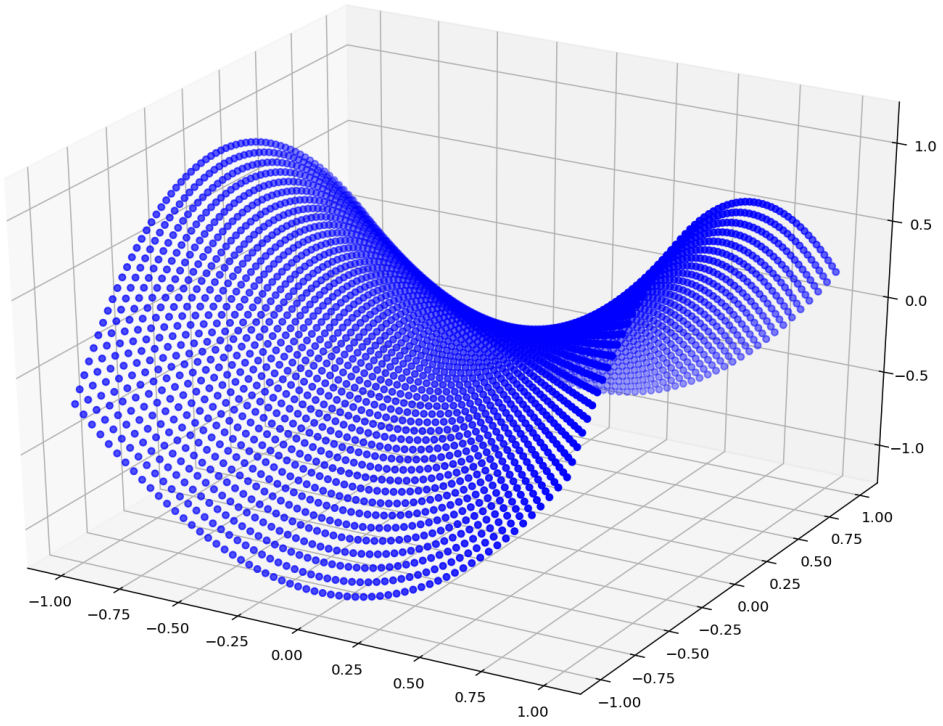
1. Plot the clean data in 3D cartesian coordinate system (1pt)

In [4]:

```
fig1=plt.figure(figsize=(14,10))
ax1=plt.axes(projection="3d")
ax1.scatter3D(x_clean,y_clean,z_clean,c="blue")
```

Out[4]:

```
<mpl_toolkits.mplot3d.art3d.Path3DCollection at 0x21f23f7eb80>
```



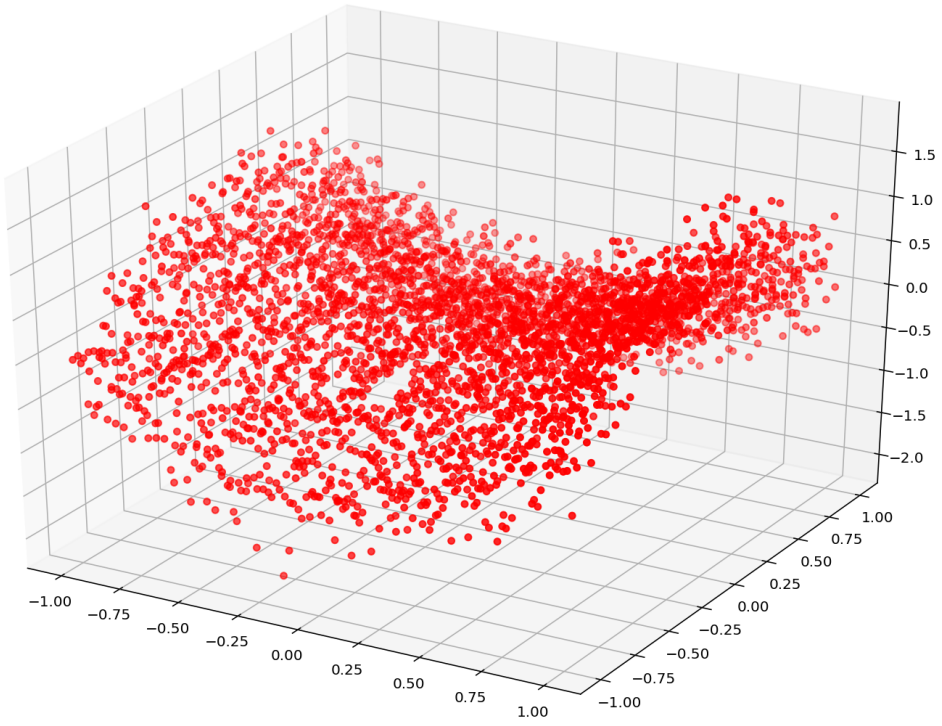
2. Plot the noisy data in 3D cartesian coordinate system (1pt)

In [5]:

```
fig2=plt.figure(figsize=(14,10))
ax2=plt.axes(projection="3d")
ax2.scatter3D(x_train,y_train,z_train,c="red")
```

Out[5]:

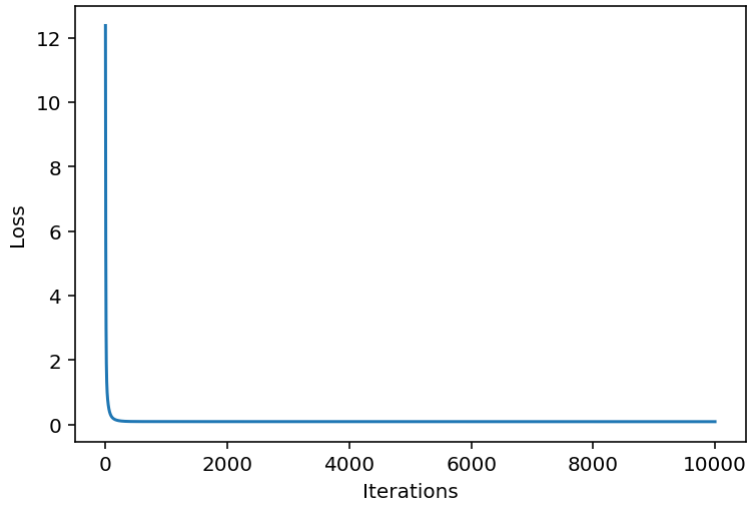
```
<mpl_toolkits.mplot3d.art3d.Path3DCollection at 0x21f241b1790>
```



3. Plot the loss curve in the course of gradient descent (2pt)

In [45]:

```
plt.figure(2)
plt.plot(L_iters) # plot the loss curve
plt.xlabel('Iterations')
plt.ylabel('Loss')
plt.show()
```



4. Print out the final loss value at convergence of the gradient descent (1pt)

In [43]:

```
print("loss at convergence = ",L_iters[max_iter-1] ) # plot the last value of the loss
```

```
loss at convergence =  0.08848371416691528
```

5. Print out the final model parameter values at convergence of the gradient descent (1pt)

In [46]:

```
for i,w_ in enumerate(w):
    print("model parameter: w_{0} = {1}".format(i,w_))
```

```
model parameter: w_0 = [-0.00170002]
model parameter: w_1 = [-0.00170002]
model parameter: w_2 = [0.02053483]
model parameter: w_3 = [-0.00852468]
model parameter: w_4 = [0.84019494]
model parameter: w_5 = [-0.87015522]
model parameter: w_6 = [0.11996304]
model parameter: w_7 = [0.09331026]
model parameter: w_8 = [0.16034914]
model parameter: w_9 = [-0.11304247]
```

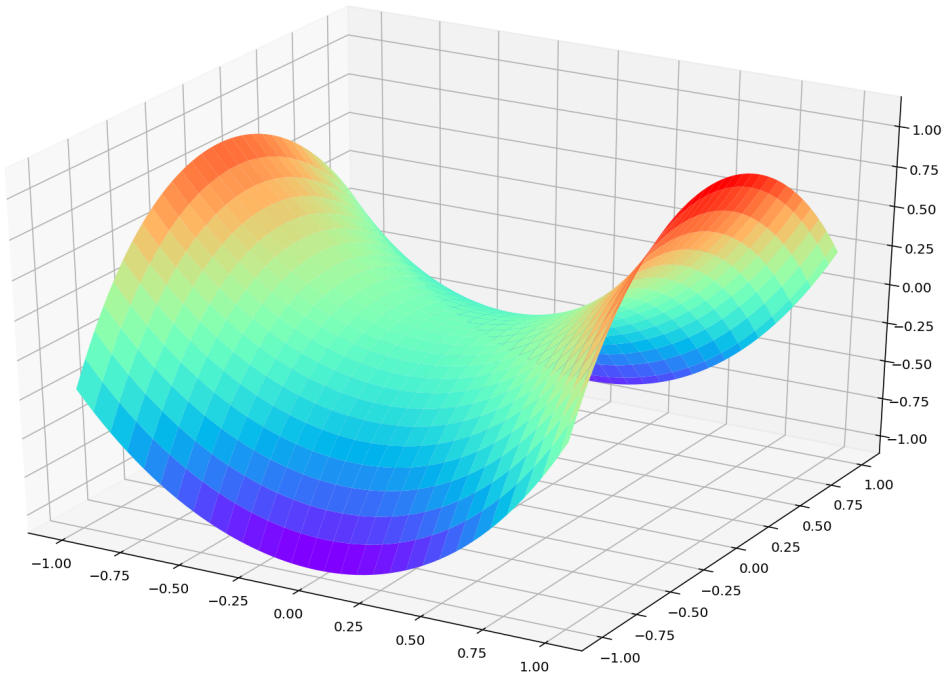
6. Plot the prediction function in 3D cartesian coordinate system (2pt)

In [47]:

```
fig = plt.figure(figsize=(15,10))
ax = fig.add_subplot(1, 1, 1, projection='3d')
ax.plot_surface(x_pred,y_pred,z_pred.reshape(60,60),cmap=plt.cm.rainbow)
```

Out[47]:

```
<mpl_toolkits.mplot3d.art3d.Poly3DCollection at 0x21f27fcca90>
```



7. Plot the prediction functions superimposed on the training data (2pt)

In [48]:

```
fig = plt.figure(figsize=(15,10))
ax = fig.add_subplot(1, 1, 1, projection='3d')
ax.plot_surface(x_pred,y_pred,z_pred.reshape(60,60),cmap=plt.cm.rainbow)
ax.scatter3D(x_train,y_train,z_train,c="gray")
ax.set_zlim(-1,1)
```

Out[48]:

(-1.0, 1.0)