THOMASON-TROBAUGH LOCALIZATION

Abstract. These are notes on Thomason's paper [TT90], titled "Higher Algebraic K-theory of Schemes and of Derived Categories".

I. Introduction

Today we will discuss a localization theorem for the K-theory of schemes due to Thomason. This was a remarkable result and extremely influential in the development of K-theory; Quillen's original definition of the higher K-groups based on exact categories did not lend itself to descent-theoretic statements. Following Waldhausen's generalization of K-theory, Thomason instead defines the K-theory of schemes with respect to the derived category of perfect complexes, i.e. the derived category of bounded complexes of algebraic vector bundles, which has the advantage of better global descent properties.

Example 1. Let X be a Noetherian scheme, Z a closed subscheme, and $U = X \setminus Z$.

$$Z \xrightarrow{j} X \xleftarrow{i} U$$

Recalling the notion of a coherent sheaf, we have a diagram

$$\mathrm{Coh}(Z) \xrightarrow{j_*} \mathrm{Coh}(X) \xrightarrow{i^*} \mathrm{Coh}(U) \to 0 \tag{1}$$

where the first arrow is extension by zero on U and the second arrow is restriction.

Remark 2. For those unfamiliar with (quasi-)coherent sheaves, we can assume all schemes are affine; let $X = \operatorname{Spec} A$, $Z = \operatorname{Spec} A/(f)$, and $U = \operatorname{Spec} A_f$. For any Notherian commutative ring R, the following are equivalences of categories

$$\mathrm{QCoh}(\operatorname{Spec} R) \cong R\text{-mod}$$
$$\mathrm{Coh}(\operatorname{Spec} R) \cong R\text{-mod}^{\text{f.g.}}$$

Then sequence (1) becomes

$$A/(f)\text{-mod}^{\text{f.g.}} \to A\text{-mod}^{\text{f.g.}} \to A_f\text{-mod}^{\text{f.g.}} \to 0$$

It is an exercise in Hartshorne ([Har77], II.6.10) to show that this induces an exact sequence

$$G_0(Z) \to G_0(X) \to G_0(U) \to 0$$

where $G_0(X) = K_0(\mathrm{Coh}(X))$.

Caution 3. Why did we use coherent sheaves instead of locally free sheaves? If $\mathscr{E}$ is a locally free $\mathscr{O}_Z$-module, then $j_*\mathscr{E}$ will generally not be a locally free $\mathscr{O}_X$-module.

Exercise 4. Let $X = \mathbb{A}^1_k = \operatorname{Spec} k[x]$,$Z = \operatorname{Spec} k[x]/(x)$, and $U = \mathbb{G}_m = \operatorname{Spec} k[x, x^{-1}]$. Work out the details of the exact sequence

$$G_0(\operatorname{Spec} k) \to G_0(\mathbb{A}^1) \to G_0(\mathbb{G}_m) \to 0$$

and prove all G_0-groups are infinite cyclic. *Hint: use the structure theorem for finitely generated modules over PIDs and prove that torsion classes vanish.*

For a a regular scheme X (i.e. X is the prime spectrum of a regular ring), Quillen proved that K-theory coincides with G-theory [Qui73]. Thus we might hope for (i) a purely K-theoretic localization theorem, and (ii) to extend the above sequence to higher K-groups. This is precisely the main theorem of Thomason-Trobaugh: for all n there is an exact sequence

$$\cdots \to K_n(X \text{ on } Z) \to K_n(X) \to K_n(U) \to \cdots$$

related the K-theory of a scheme, a closed subscheme, and its compliment.

The plan for this lecture is to (i) first present a more modern formulation of K-theory as a localizing invariant, (ii) show how this implies the main theorem of Thomason-Trobaugh, and (iii) then return to fill in necessary background on perfect complexes.

II. Localization

Let $\mathscr{C}$ be a small, stable ∞-category. We define the idempotent-completion $(\mathscr{C})^{\text{idem}}$ to be the full subcategory of presheaves on $\mathscr{C}$ generated by the representables under direct summands. This is equivalent to the full subcategory of compact objects in the ind-completion,

$$(\mathscr{C})^{\text{idem}} = (\text{Ind}\mathscr{C})^{\omega}$$

In particular, there is a canonical functor $\mathscr{C} \to (\mathscr{C})^{\text{idem}}$ and $\mathscr{C}$ is said to be idempotent-complete if it is an equivalence. In particular, we have the following.

Example 5. For any stable presentable ∞-category $\mathscr{C}$, the full subcategory of compact objects $\mathscr{C}^{\omega}$ is idempotent-complete

Define $\text{Cat}_{\infty}^{\text{perf}}$ be the ∞-category of small stable idempotent-complete ∞-categories. Idempotent-completion is closely related to the concept of categorical retraction. Given objects X, Y, we say that Y is a *retract* of X provided that there exists morphisms $i : R \to X$ and $r : X \to R$ such that the following diagram commutes.

$$\begin{tikzcd} & X \arrow[dr, "r"] & \\ Y \arrow[ur, "i"] \arrow[rr, "\mathrm{id}_Y"'] & & Y \end{tikzcd}$$

Note that $i \circ r : X \to X$ is an idempotent. In 1-categories, $\mathscr{C}$ is idempotent complete if and only if every idempotent map $X \to X$ comes from a retract as above.

Proposition 6 ([Lur09], 5.1.4.1)**.** *Let $f : \mathscr{C} \to \mathscr{D}$ be fully faithful with $\mathscr{D}$ idempotent complete. Then f exhibits $\mathscr{D}$ as an idempotent completion if every object of $\mathscr{D}$ is a retract of an object of $f(C)$ for some object of C.*

In particular, this is the same as asserting that every object of $\mathscr{D}$ is a direct summand of $f(C)$ for some C.

Definition 7. $\mathscr{D} \xrightarrow{i} \mathscr{C} \xrightarrow{j} \mathscr{E}$ in $\text{Cat}_{\infty}^{\text{perf}}$ is said to be a Karoubi sequence if

(a) $j \circ i = 0$,

(b) i is fully faithful,

(c) The Verdier quotient $\mathscr{C}/\mathscr{D} \to \mathscr{E}$ is an idempotent completion.

A functor $\mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$ (the ∞-category of spectra) is said to be a Karoubi localizing invariant if it sends Karoubi sequences to fiber sequences of spectra.

Remark 8. Condition (c) is essential because $\mathscr{C}/\mathscr{D}$ may not be idempotent-complete. In light of proposition 5, it suffices to check that every object E of $\mathscr{E}$ can be lifted to an object C of $\mathscr{C}$ with E a summand of $j(C)$.

As will be shown in the forthcoming lectures, K-theory can be extended to a functor

$$\mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$$

Thus far, we've only defined K-groups K_i for non-negative integers $i \geq 0$. Often, it is convenient to work with *non-connective* K-theory, which is an extension of K-theory to where $K_i \neq 0$ for $i < 0$. This frequently has better functorial properties. We will denote non-connective K-theory with $\mathbb{K}$ and assert that this functor exists satisfying $\pi_i(\mathbb{K}(X)) = K_i(X)$ for all $i \geq 0$. The main result of [BGT13] asserts the following:

Theorem 9 (Blumberg-Gepner-Tabuada)**.** *Non-connective K-theory*

$$\mathbb{K} : \mathrm{Cat}_\infty^{\mathrm{perf}} \to \mathrm{Sp}$$

is a Karoubi localizing invariant.

As a corollary, we have the celebrated Thomason-Trobaugh localization theorem.

Corollary 10 (Thomason-Trobaugh)**.** *Let X be a qcqs scheme, U a qc open subscheme, and $Z = X \setminus U$. Then*

$$\mathbb{K}(X \textit{ on } Z) \to \mathbb{K}(X) \to \mathbb{K}(U)$$

is a fiber sequence of K-theory spectra.

Remark 11. Taking homotopy groups, one establishes the long exact sequence

$$\cdots \to K_n(X \text{ on } Z) \to K_n(X) \to K_n(U) \to \cdots$$

In general, $K_0(X) \to K_0(U)$ is not surjective and thus the exact sequence ends awkwardly in connective K-theory. Building on the work of Bass, Thomason defines $K_{-1}(Z \text{ on } X)$ to be precisely the obstruction for this, and continues on in this fashion. The resulting K-theory spectra $\mathbb{K}(X)$ is said to be *non-connective* since it has non-vanishing homotopy groups in negative degrees.

Proof of 10. $\mathrm{Perf}(X \text{ on } Z) \to \mathrm{Perf}(X) \to \mathrm{Perf}(U)$ is a Karoubi sequence. $\square$

For the remainder of the lecture we wish to develop enough of the theory of perfect complexes to explain this result.

III. Perfect Complexes

Let X be a quasi-compact and quasi-separated scheme (qcqs) which admits an ample family of line bundles $(*)$[1], c.f. [TT90] for a precise definition. This last hypothesis is not strictly necessary, but it simplifies our definitions. Recall that an algebraic vector bundle on X is a locally free $\mathcal{O}_X$-module of finite rank.

Definition 12. A perfect complex on X is a bounded complex of algebraic vector bundles on X. Define $\mathrm{Perf}(X)$ to be the derived category of perfect complexes on X. For a closed subscheme $Z \subset X$, $\mathrm{Perf}(X \text{ on } Z)$ is the category of perfect complexes which are acyclic on $X \setminus Z$.

$\mathrm{Perf}(X)$ admits the structure of a (complicial bi)Waldhausen category with the cofibrations are degree-wise split monomorphisms and the weak equivalences are the quasi-isomorphisms (c.f [TT90] for details).

Proposition 13. $\mathrm{Perf}(X)$ *compactly generates* $\mathrm{QCoh}(X)$. *That is,* $\mathrm{QCoh}(X)^\omega \cong \mathrm{Perf}(X)$ *and* $\mathrm{Perf}(X)$ *generates* $\mathrm{QCoh}(X)$ *under colimits. Likewise for* $\mathrm{Perf}(X \text{ on } Z)$.

Remark 14. By 5 this means that $\mathrm{Perf}(X)$ is idempotent-complete.

Now we make explicit the connection between Quillen and Thomason-Trobaugh K-theory. Recall that Quillen defines the K-theory of X to be the K-theory of the exact category of algebraic vector bundles

$$K^Q(X) := K(\mathrm{Vect}(X))$$

The Thomason-Trobaugh K-theory is defined to be the Waldhausen K-theory

$$K^{TT}(X) := K^W(\mathrm{Perf}(X))$$

of perfect complexes. The two K-theory spectra are related by the following result.

Proposition 15 (Gillet-Waldhausen)**.** *Let $\mathscr{E}$ be an exact category which admits a fully-faithful functor $\mathscr{E} \to \mathscr{A}$ to an abelian category. Then*

$$K^Q(\mathscr{E}) \simeq K^W(C^b(\mathscr{E}))$$

is a homotopy equivalence of spectra.

Corollary 16. *There is a homotopy equivalence $K^Q(X) \simeq K^{TT}(X)$.*

Caution 17. This only holds for $(*)$-schemes. In general, one works with a more flexible definition of perfect complexes. For an arbitrary scheme, we consider complexes which are only locally quasi-isomorphic to a bounded complex of algebraic vector bundles, and with a Tor-amplitude condition.

[1]This condition is satisfied, for instance, if there exists an ample line bundle $\mathscr{L}$ over X.

Now, we come to the key technical lemma on the extension of perfect complexes in Thomason-Trobaugh. The key inspiration for this proof was apparently presented to Thomason by Trobaugh in a dream:

Tom's simulacrum remarked, "The direct limit characterization of perfect complexes shows that they extend, just as one extends a coherent sheaf."

Lemma 18 (Trobaugh)**.** *Let X be qcqs and let $i : U \to X$ be an open immersion with U qc. Then for every perfect complex $\mathscr{F}$ on U, there exists a perfect complex $\mathscr{E}$ on X such that $\mathscr{F}$ is quasi-isomorphic to a summand of $i^*\mathscr{E}$.*

Proof. Consider $Ri_*\mathscr{F}$. This is a quasi-coherent sheaf, and therefore quasi-isomorphic to a colimit of perfect complexes on $\varinjlim \mathscr{E}_\alpha$ by 13. Then,

$$\varinjlim i^*\mathscr{E}_\alpha = i^*(\varinjlim \mathscr{E}_\alpha) \cong i^* Ri_*\mathscr{F} \cong \mathscr{F}$$

But since $\mathscr{F}$ is a compact object in $\mathrm{QCoh}(U)$,

$$\varinjlim \mathrm{Hom}_{\mathrm{QCoh}(U)}(\mathscr{F}, i^*\mathscr{E}_\alpha) \cong \mathrm{Hom}_{\mathrm{QCoh}(U)}(\mathscr{F}, \varinjlim i^*\mathscr{E}_\alpha)$$

and therefore the isomorphism $\mathscr{F} \to \varinjlim i^*\mathscr{E}_\alpha$ must factor through $i^*\mathscr{E}_\alpha$ for some α.

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[dr] & \varinjlim i^*\mathscr{E}_\alpha \\ & i^*\mathscr{E}_\alpha \arrow[u] \end{tikzcd}$$

Therefore $\mathscr{F}$ is quasi-isomorphic to a summand of $i^*\mathscr{E}_\alpha$. $\square$

Thomason writes "*despite the flagrant triviality of the proof, this result is the key point in the paper*". He then goes on to use this lemma to prove a sequence of technical results about the extension of perfect complexes, after which the desired localization lemma falls. However, interpreted in the formalism of stable ∞-categories, the above lemma states every object of $\mathrm{Perf}(U)$ is a retract of an object of $\mathrm{Perf}(X)/\mathrm{Perf}(X \text{ on } Z)$. Once more, consider the diagram

$$\mathrm{Perf}(X \text{ on } Z) \to \mathrm{Perf}(X) \xrightarrow{i^*} \mathrm{Perf}(U)$$

Then i^* is an idempotent completion by 6. It's clear the first map is fully faithful and that the composite vanishes. Thus the above sequence is Karoubi in $\mathrm{Cat}_\infty^{\mathrm{perf}}$.

References

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